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Steady-State Spherically Symmetric Accretion onto a Bardeen Black Hole

by

Laiba Mughal

A thesis submitted in partial fulfillment for the
degree of Master of Philosophy

in the

Faculty of Computing

Department of Mathematics

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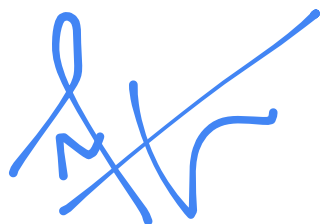
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List of Publications

It is certified that following publication(s) have been made out of the research work that has been carried out for this thesis:-

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Abstract

This thesis investigates steady-state, spherically symmetric accretion of a relativistic perfect fluid onto a Bardeen regular black hole. The Bardeen black hole, characterized by a magnetic charge parameter g , represents a nonsingular spacetime geometry that avoids the central curvature singularity present in classical black hole solutions. By employing the framework of general relativity and relativistic hydrodynamics, we derive the fundamental conservation equations governing the accretion flow, including the continuity equation and the relativistic Bernoulli equation. The critical point conditions, which determine the location where the fluid transitions from subsonic to supersonic motion, are obtained and analyzed. For a polytropic equation of state, we derive an explicit expression for the mass accretion rate and study the thermodynamic properties of the fluid near the outer horizon. Our analysis reveals that the magnetic charge parameter significantly modifies the critical radius, the accretion rate, and the near-horizon gas compression and temperature profiles. The results demonstrate that regular black hole geometries can produce observable imprints on accretion dynamics, with the compression ratio and temperature ratio increasing monotonically with the magnetic charge parameter. These findings suggest that accretion processes may serve as a useful probe for distinguishing regular black holes from their singular counterparts and provide insights into the observational signatures of Bardeen-type black holes.

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Symbols

$g_{\mu\nu}$	Metric tensor
$G_{\mu\nu}$	Einstein tensor
$R_{\mu\nu}$	Ricci tensor
R	Ricci scalar
$T_{\mu\nu}$	Energy–momentum tensor
Λ	Cosmological constant
∇_μ	Covariant derivative
Q	Electric charge
g	Bardeen magnetic charge parameter
r_c	Critical radius
U^μ	Four-velocity of the fluid
u	Radial fluid velocity
a_s	Local sound speed
a_∞	Sound speed at infinity
ρ	Energy density
P	Fluid pressure
n	Particle number density
γ	Adiabatic index
K	Polytropic constant
$\dot{M}$	Mass accretion rate

Chapter 1

Introduction

1.1 Newtonian Concept of Gravity

The study of gravity has a long history in physics and astronomy. In classical mechanics, gravity was first described in a precise mathematical form by Newton through the law of universal gravitation [1]. In classical mechanics, gravity is described by Newton's law of universal gravitation, which states that any two massive objects attract each other with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. Mathematically, it is expressed as

$$F = \frac{Gm_1m_2}{r^2}, \tag{1.1}$$

where G denotes the universal gravitational constant, m_1 and m_2 represent the masses of the two bodies, and r is the distance between the centers of two objects.

Newtonian gravity successfully explains a wide range of physical phenomena. It describes the motion of planets around the sun, the bodies falling near the surface of the earth, and the orbital motion of many celestial objects. For weak gravitational fields and for objects moving with speeds much smaller than the speed of light c , Newton's theory gives very accurate results. Therefore, it remains an

important approximation in classical mechanics, astronomy, and many practical applications [2].

However, Newtonian gravity also has some important limitations. In this theory, gravity is treated as an instantaneous force acting at a distance. This means that if the position of one mass changes, its gravitational influence on another mass is assumed to change immediately, regardless of how far away the second mass is. This concept is inconsistent with special relativity, which prohibits the propagation of physical signals or interactions at speeds exceeding the speed of light, c . Furthermore, Newtonian gravity was unable to provide a complete explanation for several precise astronomical observations, one notable example being the anomalous advance of Mercury's perihelion [3].

These limitations showed that gravity required a more general theoretical description. Einstein resolved these difficulties by developing the general theory of relativity. In this theory, gravity is not treated as an ordinary force. Instead, it is understood as a consequence of the curvature of spacetime produced by matter and energy.

1.2 The Concept of Spacetime

In Newtonian mechanics, space and time are regarded as separate and absolute quantities. Space provides a fixed background in which physical events take place, while time is assumed to flow uniformly for all observers. This classical view was changed by the theory of relativity. Special relativity unifies space and time into a single four-dimensional continuum known as spacetime. General relativity extends this framework further by allowing spacetime to be curved in the presence of matter and energy [4, 5].

Mathematically, spacetime is represented by a four-dimensional differentiable manifold, whose points correspond to physical events. Its geometry is described by the

metric tensor $g_{\mu\nu}$. The infinitesimal interval between two neighboring events is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (1.2)$$

where ds^2 is the spacetime interval and dx^μ denotes the coordinate differentials. The metric tensor determines the geometrical properties of spacetime, including distances and time intervals.

The sign of the interval determines the causal relationship between events. For the metric signature adopted in this thesis, $ds^2 < 0$ corresponds to a timelike interval, $ds^2 = 0$ to a null interval, and $ds^2 > 0$ to a spacelike interval. Timelike intervals can describe the trajectories of massive particles, whereas null intervals describe the propagation of light. Spacelike-separated events cannot be connected by a causal signal. This causal structure plays an important role in the study of black holes, where the geometry of spacetime can produce regions from which no signal can escape.

1.3 Overview of General Relativity

General relativity (GR) is Einstein's relativistic theory of gravitation in which gravitational effects are described in terms of the curvature of spacetime. The theory provides a geometric description of gravity in which freely falling test particles follow geodesics determined by the spacetime metric.

The development of the modern theory of gravitation can be viewed through three important stages. Newtonian gravity describes gravitation as a force between masses. Special relativity subsequently established the relativistic framework of space and time and demonstrated that the laws of physics are compatible with the finite speed of light. General relativity extended these ideas to gravitation by replacing the Newtonian concept of gravitational force with a geometric description based on spacetime curvature.

In the absence of non-gravitational forces, a freely falling test particle follows a geodesic, which represents its natural trajectory through curved spacetime. Thus, gravitational effects are manifested through the geometry and curvature of spacetime rather than through a gravitational force in the Newtonian sense. The curvature is determined by the metric tensor and its derivatives and is described mathematically by curvature tensors introduced in the formal development of general relativity. The relationship between the geometry of spacetime and its physical sources is governed by the Einstein field equations. These fundamental equations relate the curvature of spacetime to the distribution of matter and energy. When the cosmological constant Λ is included, they take the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1.3)$$

where $G_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light, and $T_{\mu\nu}$ is the energy-momentum tensor of matter and fields. The Einstein tensor is constructed from the Ricci curvature according to

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (1.4)$$

where $R_{\mu\nu}$ is the Ricci tensor and R denotes the Ricci scalar. Substituting this expression into the field equations yields the equivalent representation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \quad (1.5)$$

The LHS represents the spacetime geometry, while the RHS contains the matter that causes the curvature. This relationship indicates that the presence of matter and energy shapes the geometry of spacetime, while the curvature causes the motion of particles and electromagnetic radiation.

In vacuum

$$T_{\mu\nu} = 0. \quad (1.6)$$

If the cosmological constant is also neglected, that is,

$$\Lambda = 0, \tag{1.7}$$

then the Einstein field equations take their vacuum form

$$R_{\mu\nu} = 0. \tag{1.8}$$

Several important black hole spacetimes, including the Schwarzschild black hole, arise as exact solutions of these vacuum field equations [6, 7, 12].

1.4 Black Holes

A black hole is a region of spacetime in which gravity is sufficiently strong that no physical signal can propagate from the interior to distant observers. In general relativity, this behavior is not understood as a conventional gravitational force that becomes infinitely strong. Instead, it arises from the causal structure of curved spacetime. The geometry produced by a sufficiently compact concentration of mass and energy can form a boundary beyond which future-directed causal trajectories cannot reach the external universe.

The basic idea of a black hole can first be illustrated using the Newtonian concept of escape velocity. For an object of mass M and radius R , the escape velocity is given by

$$v_{\text{esc}} = \sqrt{\frac{2M}{R}}. \tag{1.9}$$

If the radius of the object is reduced while its mass is kept fixed, the escape velocity increases. Formally, setting the escape velocity equal to the speed of light c gives

$$c = \sqrt{\frac{2M}{R}}, \tag{1.10}$$

and hence

$$R = 2M. \tag{1.11}$$

This radius is the Schwarzschild radius. Although this Newtonian argument is only a heuristic picture and does not provide a relativistic description of a black hole, it gives an intuitive indication of why sufficiently compact objects can possess a region from which light cannot escape.

1.5 Schwarzschild Black Hole

Exact solutions of the Einstein field equations provide the mathematical description of black hole spacetimes. The first such solution was obtained by Karl Schwarzschild in 1916, shortly after Einstein introduced the general relativity [7]. Schwarzschild was a German theoretical physicist and astronomer who, remarkably, derived this solution during his military service in the German Army in the First World War. Despite the harsh conditions at the front, he was able to solve Einstein's field equations for the simplest nontrivial case, assuming a static, spherically symmetric spacetime in vacuum.

The Schwarzschild solution describes the gravitational field of a non-rotating, uncharged, spherically symmetric massive object. It introduced the concept of the Schwarzschild radius, now interpreted as the event horizon of a black hole, marking the boundary beyond which no signals can escape to infinity. This work laid the foundation for the modern theory of black holes and remains one of the most important exact solutions in general relativity.

In standard coordinates, the Schwarzschild line element is given by

$$ds^2 = - \left[1 - \frac{2MG}{rc^2} \right] c^2 dt^2 + \left[1 - \frac{2MG}{rc^2} \right]^{-1} dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\phi^2]. \quad (1.12)$$

where M is the mass of the gravitating object. In geometrized units, where $G = c = 1$, the above metric takes the simpler form

$$ds^2 = - \left[1 - \frac{2M}{r} \right] dt^2 + \left[1 - \frac{2M}{r} \right]^{-1} dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\phi^2]. \quad (1.13)$$

The Schwarzschild geometry contains a special radius

$$r_s = 2M, \quad (1.14)$$

which is known as the Schwarzschild radius. This radius represents the event horizon of the Schwarzschild black hole. Once a particle or light ray crosses this surface, it cannot return to the exterior region.

The Schwarzschild spacetime also contains a central singularity at $r = 0$. At this point, curvature quantities become infinite, indicating that the classical description of spacetime breaks down. This singular behavior is one of the major difficulties in classical general relativity and provides an important motivation for the study of regular black holes [8, 9].

1.6 Deriving the Schwarzschild Solution

The Schwarzschild spacetime is the unique static and spherically symmetric vacuum solution of Einstein's field equations. It describes the exterior gravitational field generated by an isolated, non-rotating, and electrically neutral massive body. Owing to its simplicity and physical significance, this solution provides the foundation for the study of black holes, relativistic stellar models, and gravitational phenomena in general relativity.

1.6.1 Symmetry Assumptions

To determine the spacetime geometry, the following assumptions are made:

- i. The spacetime is static so that all metric components are independent of the temporal coordinate.
- ii. The spacetime possesses spherical symmetry, implying invariance under arbitrary spatial rotations.

- iii. Outside the gravitating object no matter or radiation is present; consequently the energy-momentum tensor vanishes,

$$T_{\alpha\beta} = 0. \quad (1.15)$$

Under these conditions Einstein's field equations reduce to

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0. \quad (1.16)$$

The most general line element compatible with these symmetries may be written in Schwarzschild coordinates (t, r, θ, ϕ) as

$$ds^2 = e^{A(r)}dt^2 - e^{B(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \quad (1.17)$$

where the unknown functions $A(r)$ and $B(r)$ depend only upon the radial coordinate. The corresponding metric tensor assumes the matrix form in (t, r, θ, ϕ) coordinates

$$g_{\alpha\beta} = \begin{pmatrix} e^{A(r)} & 0 & 0 & 0 \\ 0 & -e^{B(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \quad (1.18)$$

Since the exterior region is empty, Einstein's equations become

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 0. \quad (1.19)$$

To simplify these equations, we contract Eq. (1.19) with the inverse metric $g^{\alpha\beta}$. Making use of the identities

$$g^{\alpha\beta}R_{\alpha\beta} = R, \quad g^{\alpha\beta}g_{\alpha\beta} = n,$$

where n denotes the spacetime dimension, yields

$$\begin{aligned} g^{\alpha\beta} \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) &= 0, \\ R - \frac{n}{2} R &= 0, \\ \left(1 - \frac{n}{2} \right) R &= 0. \end{aligned} \tag{1.20}$$

For four-dimensional spacetime ($n = 4$), the coefficient multiplying the Ricci scalar is non-zero, and therefore

$$R = 0. \tag{1.21}$$

Substituting this result into Eq. (1.19) immediately gives

$$R_{\alpha\beta} = 0. \tag{1.22}$$

Hence the exterior geometry is characterized by a Ricci-flat spacetime.

1.6.2 Evaluation of the Ricci Tensor Components

Having established that the exterior spacetime satisfies the vacuum condition

$$R_{\alpha\beta} = 0,$$

our next objective is to determine the unknown metric functions $A(r)$ and $B(r)$ appearing in the line element. This requires evaluating the Ricci tensor associated with the static, spherically symmetric metric. The Christoffel symbols are first computed from the metric coefficients using

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} (\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\mu\sigma} - \partial_{\sigma} g_{\mu\nu}). \tag{1.23}$$

Substituting these connection coefficients into the definition of the Ricci tensor,

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\mu\nu}^\lambda \Gamma_{\lambda\sigma}^\sigma - \Gamma_{\mu\lambda}^\sigma \Gamma_{\nu\sigma}^\lambda, \quad (1.24)$$

one obtains the independent non-vanishing components

$$R_{tt} = \frac{1}{2} e^{A-B} \left[A'' + \frac{1}{2} A' (A' - B') + \frac{2A'}{r} \right], \quad (1.25)$$

$$R_{rr} = -\frac{1}{2} \left[A'' + \frac{1}{2} A' (A' - B') - \frac{2B'}{r} \right], \quad (1.26)$$

$$R_{\theta\theta} = (-r e^{-B})' - \frac{1}{2} r (A' + B') e^{-B} + 1, \quad (1.27)$$

$$R_{\phi\phi} = \sin^2 \theta R_{\theta\theta}. \quad (1.28)$$

Since the vacuum Einstein equations require every Ricci tensor component to vanish, the field equations reduce to

$$R_{tt} = 0, \quad R_{rr} = 0, \quad R_{\theta\theta} = 0, \quad R_{\phi\phi} = 0. \quad (1.29)$$

However, the last equation is not independent because

$$R_{\phi\phi} = \sin^2 \theta R_{\theta\theta},$$

which means that satisfying the angular equation for $R_{\theta\theta}$ automatically guarantees $R_{\phi\phi} = 0$.

Consequently, only three independent differential equations must be solved.

Setting $R_{tt} = 0$ gives

$$A'' + \frac{1}{2} A' (A' - B') + \frac{2A'}{r} = 0. \quad (1.30)$$

Likewise, imposing the condition $R_{rr} = 0$ leads to

$$A'' + \frac{1}{2}A'(A' - B') - \frac{2B'}{r} = 0. \quad (1.31)$$

Finally, the angular component produces

$$(-re^{-B})' - \frac{1}{2}r(A' + B')e^{-B} + 1 = 0. \quad (1.32)$$

A useful relation between the metric functions can be obtained directly from Eqs. (1.30) and (1.31). Subtracting Eq. (1.31) from Eq. (1.30) removes the common second-order and nonlinear terms, leaving

$$A' + B' = 0, \quad (1.33)$$

or

$$A' = -B'. \quad (1.34)$$

Equation (1.34) considerably simplifies the remaining field equations because it establishes that the two metric potentials differ only by an additive constant.

1.6.3 Determination of the Metric Functions

The relation obtained in Eq. (1.34),

$$A' = -B',$$

considerably simplifies the remaining field equations. Substituting this result into the angular equation (1.32) removes the second term immediately, since

$$A' + B' = 0.$$

Consequently, Eq. (1.32) reduces to

$$\begin{aligned} (-re^{-B})' - \frac{1}{2}r(0)e^{-B} + 1 &= 0, \\ (-re^{-B})' + 1 &= 0. \end{aligned} \quad (1.35)$$

Hence the governing differential equation becomes

$$(-re^{-B})' = -1. \quad (1.36)$$

Integrating Eq. (1.36) with respect to the radial coordinate gives

$$-re^{-B} = -r + C_2, \quad (1.37)$$

where C_2 is an arbitrary integration constant. Rearranging the above expression yields

$$re^{-B} = r - C_2. \quad (1.38)$$

Dividing both sides by r gives

$$e^{-B} = 1 - \frac{C_2}{r}. \quad (1.39)$$

Thus, the radial metric coefficient is completely determined,

$$g_{rr} = -e^B = -\left(1 - \frac{C_2}{r}\right)^{-1}.$$

The temporal metric function follows directly from the relation

$$A' = -B'.$$

Integrating once more with respect to r produces

$$A = -B + C_3, \quad (1.40)$$

where C_3 denotes another integration constant.

Exponentiating both sides gives

$$e^A = e^{C_3} e^{-B}. \quad (1.41)$$

To simplify the notation, we introduce

$$C_1 = e^{C_3}, \quad (1.42)$$

which is simply a positive constant. Using Eq. (1.39), the temporal metric function becomes

$$e^A = C_1 \left(1 - \frac{C_2}{r} \right). \quad (1.43)$$

Therefore, both unknown metric potentials have now been determined.

1.6.4 General Static Vacuum Metric

Substituting Eqs. (1.43) and (1.39) into the original line element gives the most general static, spherically symmetric vacuum solution,

$$ds^2 = C_1 \left(1 - \frac{C_2}{r} \right) dt^2 - \left(1 - \frac{C_2}{r} \right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2. \quad (1.44)$$

Introducing the standard solid-angle element,

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (1.45)$$

the metric can be written in the compact form

$$ds^2 = C_1 \left(1 - \frac{C_2}{r}\right) dt^2 - \left(1 - \frac{C_2}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \quad (1.46)$$

1.6.5 Interpretation of the Integration Constants

The two constants appearing in the metric acquire physical significance through suitable boundary conditions.

First, consider the limit $r \rightarrow \infty$. In this regime,

$$\frac{C_2}{r} \rightarrow 0,$$

and Eq. (1.46) approaches

$$ds^2 = C_1 dt^2 - dr^2 - r^2 d\Omega^2. \quad (1.47)$$

For an isolated gravitating object, spacetime must become Minkowskian far from the source. This requirement, known as asymptotic flatness, is satisfied by rescaling the time coordinate,

$$t \rightarrow \frac{t}{\sqrt{C_1}},$$

which allows one to choose

$$C_1 = 1. \quad (1.48)$$

The remaining constant, C_2 , cannot be fixed solely from Einstein's equations. Its physical meaning is established later by comparing the weak-field limit of the relativistic equations of motion with Newtonian gravity, where it will be identified with the mass of the central object.

1.7 Geodesic Motion in Schwarzschild Spacetime

The physical meaning of the remaining integration constant C_2 can be established by examining the motion of freely falling test particles. Such particles travel along geodesics, which represent the natural trajectories determined solely by the spacetime curvature.

For an affine parameter λ (or equivalently the proper time for massive particles), the geodesic equation is

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0. \quad (1.49)$$

Throughout this discussion, an overdot denotes differentiation with respect to λ , namely,

$$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}.$$

The individual equations corresponding to each spacetime coordinate are obtained by assigning $\mu = t, r, \theta$, and ϕ in Eq. (1.49).

1.7.1 Temporal Equation

For the temporal coordinate ($x^0 = t$), the geodesic equation becomes

$$\ddot{t} + \Gamma_{\alpha\beta}^t \dot{x}^\alpha \dot{x}^\beta = 0. \quad (1.50)$$

Because the metric is diagonal and static, the only non-zero Christoffel symbols involving the upper index t are

$$\Gamma_{tr}^t = \Gamma_{rt}^t = \frac{1}{2}A'.$$

Consequently,

$$\begin{aligned}\ddot{t} + 2\Gamma_{tr}^t \dot{t}\dot{r} &= 0, \\ \ddot{t} + 2\left(\frac{1}{2}A'\right)\dot{t}\dot{r} &= 0.\end{aligned}\tag{1.51}$$

Hence the time equation simplifies to

$$\ddot{t} + A'\dot{t}\dot{r} = 0.\tag{1.52}$$

1.7.2 Radial Equation

The radial coordinate satisfies

$$\ddot{r} + \Gamma_{\alpha\beta}^r \dot{x}^\alpha \dot{x}^\beta = 0.\tag{1.53}$$

Substituting the non-zero Christoffel symbols for the Schwarzschild metric yields

$$\ddot{r} + \Gamma_{tt}^r \dot{t}^2 + \Gamma_{rr}^r \dot{r}^2 + \Gamma_{\theta\theta}^r \dot{\theta}^2 + \Gamma_{\phi\phi}^r \dot{\phi}^2 = 0.\tag{1.54}$$

Using the explicit expressions,

$$\Gamma_{tt}^r = \frac{1}{2}e^{A-B}A',$$

$$\Gamma_{rr}^r = \frac{1}{2}B',$$

$$\Gamma_{\theta\theta}^r = -re^{-B},$$

and

$$\Gamma_{\phi\phi}^r = -r \sin^2 \theta e^{-B},$$

the radial equation becomes

$$\ddot{r} + \frac{1}{2}e^{A-B}A'\dot{t}^2 + \frac{1}{2}B'\dot{r}^2 - re^{-B}\dot{\theta}^2 - re^{-B}\sin^2 \theta \dot{\phi}^2 = 0. \quad (1.55)$$

For the Schwarzschild solution obtained previously,

$$A = -B,$$

which implies

$$e^{-B} = e^A, \quad B' = -A'.$$

Substituting these identities into the above equation gives

$$\ddot{r} + \frac{1}{2}e^{2A}A'\dot{t}^2 - \frac{1}{2}A'\dot{r}^2 - re^A\dot{\theta}^2 - re^A\sin^2 \theta \dot{\phi}^2 = 0. \quad (1.56)$$

1.7.3 Polar Equation

For the polar angle θ , Eq. (1.49) gives

$$\ddot{\theta} + \Gamma_{\alpha\beta}^{\theta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0. \quad (1.57)$$

The only non-zero contributions are

$$\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \frac{1}{r},$$

and

$$\Gamma_{\phi\phi}^\theta = -\sin\theta \cos\theta.$$

Therefore,

$$\begin{aligned}\ddot{\theta} + 2\Gamma_{r\theta}^\theta \dot{r}\dot{\theta} + \Gamma_{\phi\phi}^\theta \dot{\phi}^2 &= 0, \\ \ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta \cos\theta \dot{\phi}^2 &= 0.\end{aligned}\tag{1.58}$$

Thus,

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta \cos\theta \dot{\phi}^2 = 0.\tag{1.59}$$

1.7.4 Azimuthal Equation

Finally, for the azimuthal coordinate ϕ , the geodesic equation reads

$$\ddot{\phi} + \Gamma_{\alpha\beta}^\phi \dot{x}^\alpha \dot{x}^\beta = 0.\tag{1.60}$$

The only non-zero Christoffel symbols entering this equation are

$$\Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \frac{1}{r},$$

and

$$\Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta.$$

Substituting these quantities leads to

$$\begin{aligned}\ddot{\phi} + 2\Gamma_{r\phi}^{\phi}\dot{r}\dot{\phi} + 2\Gamma_{\theta\phi}^{\phi}\dot{\theta}\dot{\phi} &= 0, \\ \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} &= 0.\end{aligned}\tag{1.61}$$

Therefore,

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0.\tag{1.62}$$

Equations (1.52), (1.56), (1.59), and (1.62) constitute the complete set of geodesic equations governing particle motion in the Schwarzschild spacetime.

1.7.5 Conservation of Energy

We first consider the temporal geodesic equation derived previously,

$$\ddot{t} + A'\dot{r}\dot{t} = 0.\tag{1.63}$$

To express this equation in an integrable form, multiply both sides by the factor e^A ,

$$e^A\ddot{t} + A'e^A\dot{r}\dot{t} = 0.\tag{1.64}$$

As the metric function is solely a function of the radial coordinate,

$$\frac{dA}{d\lambda} = A'\dot{r},$$

the above equation may be rewritten as

$$e^A \ddot{t} + \dot{A} e^A \dot{t} = 0. \quad (1.65)$$

Recognizing the LHS as the derivative of a product,

$$\frac{d}{d\lambda} (e^A \dot{t}) = 0, \quad (1.66)$$

integration immediately yields

$$e^A \dot{t} = E, \quad (1.67)$$

where E is a constant along the geodesic. Solving for the temporal velocity gives

$$\dot{t} = E e^{-A}. \quad (1.68)$$

The quantity E represents the conserved energy per unit mass of the particle. For trajectories beginning in the asymptotically flat region, it is customary to normalize the affine parameter so that

$$E = 1.$$

1.7.6 Conservation of Angular Momentum

The azimuthal equation obtained earlier is

$$\ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0. \quad (1.69)$$

Multiplying the entire equation by $r^2 \sin^2 \theta$ gives

$$\begin{aligned} r^2 \sin^2 \theta \ddot{\phi} + 2r \sin^2 \theta \dot{r} \dot{\phi} \\ + 2r^2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} = 0. \end{aligned} \quad (1.70)$$

The LHS is recognized as an exact derivative,

$$\frac{d}{d\lambda} \left(r^2 \sin^2 \theta \dot{\phi} \right) = 0. \quad (1.71)$$

Therefore,

$$r^2 \sin^2 \theta \dot{\phi} = L, \quad (1.72)$$

where L is another integration constant corresponding to the conserved angular momentum per unit mass.

The azimuthal velocity may therefore be written as

$$\dot{\phi} = \frac{L}{r^2 \sin^2 \theta}. \quad (1.73)$$

1.7.7 Equation for the Polar Angle

Substituting Eq. (1.73) into the polar geodesic equation gives

$$\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta \left(\frac{L}{r^2 \sin^2 \theta} \right)^2 = 0. \quad (1.74)$$

Simplifying,

$$\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \frac{L^2}{r^4} \cot \theta \csc^2 \theta = 0. \quad (1.75)$$

Multiplying by $r^2 \sin^2 \theta$ yields

$$r^2 \sin^2 \theta \ddot{\theta} + 2r\dot{r} \sin^2 \theta \dot{\theta} - \frac{L^2}{r^2} \cot \theta = 0. \quad (1.76)$$

Recognizing

$$\frac{d}{d\lambda}(r^2 \dot{\theta}) = r^2 \ddot{\theta} + 2r\dot{r} \dot{\theta},$$

the equation becomes

$$\sin^2 \theta \frac{d}{d\lambda}(r^2 \dot{\theta}) - \frac{L^2}{r^2} \cot \theta = 0. \quad (1.77)$$

Using

$$\frac{d}{d\lambda} = \frac{d\phi}{d\lambda} \frac{d}{d\phi},$$

together with Eq. (1.73), gives

$$\sin^2 \theta \frac{L}{r^2 \sin^2 \theta} \frac{d}{d\phi}(r^2 \dot{\theta}) - \frac{L^2}{r^2} \cot \theta = 0. \quad (1.78)$$

After dividing by L/r^2 ,

$$\frac{d}{d\phi}(r^2 \dot{\theta}) - L \cot \theta = 0. \quad (1.79)$$

Finally,

$$r^2 \dot{\theta} = r^2 \frac{d\theta}{d\phi} \frac{d\phi}{d\lambda},$$

and substituting Eq. (1.73) produces

$$\frac{d}{d\phi} \left(\frac{L}{\sin^2 \theta} \frac{d\theta}{d\phi} \right) - L \cot \theta = 0. \quad (1.80)$$

Dividing through by L gives the governing equation

$$\frac{d}{d\phi} \left(\csc^2 \theta \frac{d\theta}{d\phi} \right) - \cot \theta = 0. \quad (1.81)$$

1.8 Orbital Motion and the Weak-Field Approximation

1.8.1 Motion in the Equatorial Plane

The differential equation obtained for the polar coordinate is

$$\frac{d}{d\phi} \left(\csc^2 \theta \frac{d\theta}{d\phi} \right) - \cot \theta = 0. \quad (1.82)$$

Using the identity

$$\frac{d}{d\phi} (\cot \theta) = -\csc^2 \theta \frac{d\theta}{d\phi},$$

Eq. (1.82) becomes

$$\frac{d^2}{d\phi^2} (\cot \theta) + \cot \theta = 0. \quad (1.83)$$

The solution of this second-order equation is

$$\cot \theta = A_0 \cos(\phi + \phi_0), \quad (1.84)$$

where A_0 and ϕ_0 are constants of integration.

The spherical symmetry of the Schwarzschild spacetime allows the coordinates to be selected such that the motion of a test particle is confined to the equatorial plane. Therefore,

$$\theta = \frac{\pi}{2}, \quad \dot{\theta} = 0, \quad \ddot{\theta} = 0. \quad (1.85)$$

1.8.2 Radial Equation of Motion

Applying the conditions (1.85) to the radial geodesic equation yields

$$\ddot{r} - \frac{1}{2}A'\dot{r}^2 - re^A\dot{\phi}^2 + \frac{1}{2}e^{2A}A'\dot{t}^2 = 0. \quad (1.86)$$

Using the conserved quantities

$$\dot{t} = Ee^{-A},$$

and

$$\dot{\phi} = \frac{L}{r^2},$$

Eq. (1.86) becomes

$$\ddot{r} - \frac{1}{2}A'\dot{r}^2 - e^A\frac{L^2}{r^3} + \frac{1}{2}A'E^2 = 0. \quad (1.87)$$

Choosing the standard normalization $E = 1$ gives

$$\ddot{r} + \frac{1}{2}A'(1 - \dot{r}^2) - e^A\frac{L^2}{r^3} = 0. \quad (1.88)$$

1.8.3 First Integral of the Motion

An alternative relation follows directly from the spacetime interval.

For timelike particles,

$$ds^2 = d\tau^2, \quad (1.89)$$

and in the equatorial plane,

$$ds^2 = e^A dt^2 - e^{-A} dr^2 - r^2 d\phi^2. \quad (1.90)$$

Dividing by $d\tau^2$ gives

$$1 = e^A \dot{t}^2 - e^{-A} \dot{r}^2 - r^2 \dot{\phi}^2. \quad (1.91)$$

Substituting

$$\dot{t} = e^{-A}, \quad \dot{\phi} = \frac{L}{r^2},$$

yields

$$1 = e^{-A} - e^{-A} \dot{r}^2 - \frac{L^2}{r^2}. \quad (1.92)$$

Rearranging,

$$1 + \frac{L^2}{r^2} = e^{-A} (1 - \dot{r}^2). \quad (1.93)$$

1.8.4 Reduction to the Orbit Equation

Equation (1.93) implies

$$\left(1 + \frac{L^2}{r^2}\right) e^A = 1 - \dot{r}^2. \quad (1.94)$$

Substituting this result into Eq. (1.88) gives

$$\ddot{r} + \frac{1}{2}A' \left(1 + \frac{L^2}{r^2}\right) e^A - e^A \frac{L^2}{r^3} = 0. \quad (1.95)$$

Since

$$e^A = 1 - \frac{C_2}{r},$$

we obtain

$$e^A A' = \frac{d}{dr}(e^A) = \frac{C_2}{r^2}. \quad (1.96)$$

Therefore,

$$\ddot{r} + \frac{C_2}{2r^2} \left(1 + \frac{L^2}{r^2}\right) - \left(1 - \frac{C_2}{r}\right) \frac{L^2}{r^3} = 0. \quad (1.97)$$

1.8.5 Transformation to the Reciprocal Radius

Introduce the reciprocal variable

$$w = \frac{1}{r}. \quad (1.98)$$

From angular momentum conservation,

$$\frac{d\phi}{d\tau} = Lw^2. \quad (1.99)$$

Using the chain rule,

$$\dot{r} = \frac{dr}{d\phi} \frac{d\phi}{d\tau} = -L \frac{dw}{d\phi}. \quad (1.100)$$

Differentiating once more,

$$\ddot{r} = -L^2 w^2 \frac{d^2 w}{d\phi^2}. \quad (1.101)$$

Substituting into Eq. (1.97) gives

$$-L^2 w^2 \frac{d^2 w}{d\phi^2} + \frac{C_2}{2} w^2 (1 + L^2 w^2) - (1 - C_2 w) L^2 w^3 = 0. \quad (1.102)$$

Multiplying by -1 and dividing by $L^2 w^2$ leads to

$$\frac{d^2 w}{d\phi^2} + w = -\frac{C_2}{2L^2} + \frac{3C_2}{2} w^2. \quad (1.103)$$

1.8.6 Newtonian Limit

To determine the constant C_2 , Eq. (1.103) is compared with the Newtonian orbit equation,

$$\frac{d^2 w}{d\phi^2} + w = \frac{GM}{L^2 c^2}. \quad (1.104)$$

Matching the constant terms gives

$$-\frac{C_2}{2L^2} = \frac{GM}{L^2 c^2}, \quad (1.105)$$

from which

$$C_2 = -\frac{2GM}{c^2}. \quad (1.106)$$

1.8.7 The Schwarzschild Metric

Substituting this value into the metric gives

$$ds^2 = \left[1 - \frac{2MG}{c^2 r}\right] dt^2 - \left[1 - \frac{2MG}{c^2 r}\right]^{-1} dr^2 - r^2 d\Omega^2. \quad (1.107)$$

Likewise, the orbital equation becomes

$$\frac{d^2 w}{d\phi^2} + w = \frac{GM}{L^2 c^2} + \frac{3GM}{c^2} w^2. \quad (1.108)$$

The second term on the right-hand side is the relativistic correction responsible for the perihelion advance of planetary orbits.

1.8.8 Schwarzschild Radius

The escape velocity from a gravitating body is

$$V_e^2 = \frac{2GM}{R}. \quad (1.109)$$

When the escape velocity equals the speed of light,

$$V_e = c, \quad (1.110)$$

the corresponding radius satisfies

$$R_s = \frac{2GM}{c^2}, \quad (1.111)$$

which defines the Schwarzschild radius and marks the location of the event horizon of a non-rotating black hole.

1.9 Types of Singularities

In black hole physics, singularities are generally classified into two main types: coordinate singularities and curvature singularities. A coordinate singularity appears because of an unsuitable choice of coordinates. It does not represent a real physical divergence of spacetime curvature. Instead, the metric components may become zero or infinite only due to the coordinate system used to describe the geometry. Such singularities can be removed by introducing more appropriate coordinates. The event horizon of the Schwarzschild black hole provides the standard example of a coordinate singularity. In Schwarzschild coordinates, the metric seems to become ill-defined at $r = 2M$. However, curvature invariants remain finite at this surface, showing that the horizon is not a physical singularity. By using alternative coordinate systems, such as Eddington–Finkelstein coordinates, Kruskal–Szekeres coordinates, or Painlevé–Gullstrand coordinates, the geometry can be extended smoothly across the horizon [9–12]. On the other hand, a curvature singularity is a physical singularity. It occurs when curvature scalars, such as the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, become infinite. Such a divergence cannot be removed by changing coordinates. In the Schwarzschild spacetime, the point $r = 0$ is a curvature singularity because curvature invariants diverge there. This indicates the breakdown of the classical spacetime description [13]. The existence of curvature singularities in classical black hole solutions is one of the main motivations for studying regular black holes, where the central divergence is avoided and the curvature remains finite throughout the spacetime.

1.10 Other Important Black Hole Metrics

The Schwarzschild solution describes the simplest black hole spacetime corresponding to a static, spherically symmetric, and electrically neutral gravitational source. However, more general black hole solutions arise when additional physical properties such as electric charge, angular momentum, or a cosmological constant are incorporated into Einstein’s field equations. These generalized geometries play an

important role in relativistic astrophysics because realistic black holes are expected to rotate, and several theoretical models also include charge or cosmological effects. In this section, we briefly discuss some important black hole metrics beyond the Schwarzschild spacetime.

1.10.1 Reissner–Nordström Black Hole

The Reissner–Nordström black hole represents the spacetime surrounding a static and spherically symmetric charged mass. It is obtained as a solution of the coupled Einstein–Maxwell equations and extends the Schwarzschild geometry through the inclusion of an electric charge parameter Q [14, 15]. Historically, the solution was first derived independently by Hans Reissner in 1916 and Gunnar Nordström in 1918, during the early development of unified theories combining gravity and electromagnetism. Reissner, a German engineer and physicist, obtained the charged solution shortly after Schwarzschild’s work, while Nordström, a Finnish theoretical physicist, arrived at a similar result within his scalar theory of gravitation. Their contributions predate the modern formulation of Einstein–Maxwell theory but were later reinterpreted within general relativity. The Reissner–Nordström solution therefore describes the external gravitational and electromagnetic field of a non-rotating charged mass. In addition to the Schwarzschild radius, the presence of charge modifies the horizon structure, leading to the possibility of multiple horizons or an extremal black hole configuration when Q approaches the mass scale.

In geometrized units, the spacetime metric is given by

$$ds^2 = - \left[\frac{Q^2}{r^2} - \frac{2M}{r} + 1 \right] dt^2 + \left[\frac{Q^2}{r^2} - \frac{2M}{r} + 1 \right]^{-1} dr^2 + r^2 [d\theta^2 + \sin^2 \theta d\phi^2]. \quad (1.112)$$

where M denotes the mass of the black hole and Q represents its electric charge. The corresponding metric function is

$$f(r) = \frac{Q^2}{r^2} - \frac{2M}{r} + 1. \quad (1.113)$$

The horizons are determined from the condition $f(r) = 0$, yielding

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (1.114)$$

The quantity r_+ corresponds to the outer event horizon, whereas r_- represents the inner Cauchy horizon. When $M^2 > Q^2$, the spacetime contains two distinct horizons. The limiting case $M^2 = Q^2$ corresponds to an extremal black hole for which both horizons coincide. If $M^2 < Q^2$, no horizon exists and the singularity becomes naked, which is generally regarded as unphysical within classical general relativity. Compared with the Schwarzschild geometry, the additional charge term modifies the causal structure of spacetime and changes the gravitational behavior near the central region.

1.10.2 Kerr Black Hole

The Kerr black hole represents the spacetime geometry surrounding an uncharged rotating black hole. Because black holes are generally expected to acquire angular momentum during their formation and evolution, the Kerr metric is regarded as one of the most significant exact solutions to Einstein's field equations [16]. The solution was discovered by Roy P. Kerr in 1963, a New Zealand mathematician and general relativist, who remarkably found the exact rotating vacuum solution of Einstein's field equations. His result generalized the Schwarzschild spacetime by incorporating angular momentum, leading to a stationary and axisymmetric geometry. The Kerr solution describes the exterior spacetime of an uncharged rotating mass. Its geometry is uniquely determined by the mass (M) and the specific angular momentum ($a = J/M$). It exhibits rich geometrical features including frame dragging, an ergosphere, and an event horizon whose structure depends on the rotation parameter. Expressed in Boyer–Lindquist coordinates, the Kerr spacetime is described by the metric

$$\begin{aligned}
ds^2 = & - \left[1 - \frac{2Mr}{\Sigma} \right] dt^2 - \frac{4Mra \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\
& + \left[a^2 + r^2 + \frac{2Ma^2r \sin^2 \theta}{\Sigma} \right] \sin^2 \theta d\phi^2,
\end{aligned} \tag{1.115}$$

where

$$\Sigma = a^2 \cos^2 \theta + r^2, \quad \Delta = a^2 - 2Mr + r^2. \tag{1.116}$$

The horizons of the Kerr spacetime are obtained from the roots of $\Delta = 0$, namely

$$r_{\pm} = M \pm \sqrt{M^2 - a^2}. \tag{1.117}$$

The outer root r_+ corresponds to the event horizon and the inner root r_- represents the Cauchy horizon. The existence of horizons requires

$$a^2 \leq M^2. \tag{1.118}$$

The limiting case $a^2 = M^2$ describes an extremal Kerr black hole.

A distinctive characteristic of the Kerr spacetime is the presence of the ergosphere, a region surrounding the event horizon in which every observer is compelled to co-rotate with the black hole and therefore cannot remain at rest with respect to a distant observer. This effect results from the dragging of inertial frames produced by the rotation of the spacetime.

1.10.3 Kerr–Newman Black Hole

The Kerr–Newman black hole provides the most general asymptotically flat, stationary solution to the Einstein–Maxwell equations in four-dimensional spacetime, describing a black hole that possesses both angular momentum and electric charge

[17]. The solution was discovered in 1965 by Ezra T. Newman and his collaborators, who extended Kerr's rotating vacuum solution to include electromagnetic fields within the framework of the Einstein–Maxwell equations. This work completed the family of classical black hole solutions characterized by mass, charge, and angular momentum. Expressed in Boyer–Lindquist coordinates, the Kerr–Newman spacetime is described by the line element

$$ds^2 = - \left[1 - \frac{2Mr - Q^2}{\Sigma} \right] dt^2 - \frac{2a(2Mr - Q^2) \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left[a^2 + r^2 + \frac{a^2(2Mr - Q^2) \sin^2 \theta}{\Sigma} \right] \sin^2 \theta d\phi^2, \quad (1.119)$$

where

$$\Sigma = a^2 \cos^2 \theta + r^2, \quad \Delta = Q^2 + a^2 - 2Mr + r^2. \quad (1.120)$$

The quantities M , a , and Q correspond to the black hole mass, the rotation parameter, and the electric charge, respectively. The horizons are obtained from the condition $\Delta = 0$, which gives

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}. \quad (1.121)$$

The existence of horizons requires

$$M^2 \geq a^2 + Q^2. \quad (1.122)$$

If this condition is violated, the horizons disappear and the spacetime contains a naked singularity.

1.10.4 Black Holes with a Cosmological Constant

Black hole geometries may also be generalized through the inclusion of a cosmological constant Λ . The presence of Λ modifies the asymptotic structure of spacetime and becomes important in cosmological and large-scale gravitational models [18, 19].

For the Schwarzschild–de Sitter and Schwarzschild–anti-de Sitter spacetimes, the metric function is given by

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}. \quad (1.123)$$

The corresponding line element takes the form

$$ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1.124)$$

When $\Lambda > 0$, the spacetime becomes asymptotically de Sitter and may contain a cosmological horizon in addition to the black hole horizon. In contrast, for $\Lambda < 0$, the spacetime becomes asymptotically anti-de Sitter and plays an important role in black hole thermodynamics, quantum gravity, and gauge/gravity duality [20].

These solutions demonstrate that the global properties of spacetime significantly influence the horizon structure and physical behavior of black holes.

1.10.5 Regular Black Holes

The generalized black hole solutions discussed above illustrate how spacetime geometry depends strongly upon the physical properties of the gravitational source. The mass parameter determines the primary gravitational attraction, electric charge modifies the horizon structure, angular momentum introduces rotational effects and frame dragging, while the cosmological constant changes the asymptotic behavior of spacetime. Although these classical black hole geometries are of fundamental importance, most of them still possess curvature singularities. For example, the Schwarzschild and Reissner–Nordström spacetimes contain a central singularity at $r = 0$, whereas the Kerr and Kerr–Newman geometries possess a ring singularity. The existence of such singularities suggests that classical general relativity may become incomplete in extremely strong gravitational regimes. This motivates the study of regular black holes, which are modified geometries constructed to avoid curvature divergences at the center of spacetime. Regular black

holes are nonsingular black hole solutions that possess coordinate singularities while avoiding physical singularities throughout the spacetime. In such metrics, all curvature invariants remain finite everywhere, including the central region of the black hole [21–23]. The idea of regular black holes originated from the works of Sakharov and Gliner [24, 25], who suggested that spacetime singularities may be removed if the central vacuum region is replaced by a de Sitter-like core. This concept was later extended and developed in greater detail by Dymnikova, Gurevich, and Starobinsky [26–28].

Bardeen [29] proposed the first regular black hole solution in 1968. This solution represents a singularity-free black hole in which the central curvature singularity is avoided. Several decades after the original proposal by Bardeen, Ayón-Beato and García [22, 30] developed a field-theoretic interpretation of regular black hole geometries by coupling Einstein gravity to nonlinear electrodynamics. In this framework, the matter source is described by a nonlinear electromagnetic field, which can be interpreted as a magnetic monopole in suitable limits. This approach provides a physically motivated foundation for regular black hole spacetimes, showing that such solutions can arise as exact solutions of the Einstein field equations rather than being purely phenomenological constructions. The Ayón-Beato–García (ABG) black hole constitutes one of the most important examples obtained within this approach. It shares several features with the Reissner–Nordström geometry, including the presence of a charge parameter, but differs crucially in its regularity properties. Unlike the Reissner–Nordström spacetime, the ABG solution remains finite at the origin and all curvature invariants are nonsingular throughout the spacetime.

Thus, the ABG model demonstrates that black hole singularities can be avoided through nonlinear modifications of the matter sector while preserving the geometric structure of general relativity. This result strengthened the view that regular black holes may arise from well-defined classical field theories. Many regular black hole models have subsequently been constructed. In particular, the method has been further generalized to describe a wide class of spherically symmetric regular black hole solutions [31–36].

Another important regular black hole solution was introduced by Hayward [37]. This geometry was constructed specifically to provide a simple model of a nonsingular black hole while retaining many properties of classical black hole spacetimes.

The Hayward black hole possesses a metric function that smoothly interpolates between a Schwarzschild-like exterior and a de Sitter-like interior. At large radial distances, the spacetime behaves like an ordinary Schwarzschild black hole. Near the center, however, the metric approaches de Sitter spacetime and avoids curvature divergences.

The Hayward solution has been extensively used in studies of black hole thermodynamics, gravitational collapse, and quantum gravity motivated corrections to classical spacetime geometries. Because of its mathematical simplicity, it is often employed as a prototype regular black hole model in theoretical investigations [38–40].

1.10.6 Energy Conditions and Regular Black Holes

The formulation of regular black holes is closely tied to the role of energy conditions in classical general relativity. These conditions place constraints on the energy–momentum tensor and play a central role in the derivation of the singularity theorems [8, 41].

In a wide class of regular black hole solutions, one or more of these classical energy conditions is necessarily violated in the strong-field region near the center. Such violations relax the assumptions underlying the singularity theorems and thereby permit spacetime geometries that remain regular everywhere, avoiding the formation of curvature singularities [38].

From a physical perspective, these violations do not indicate inconsistency, but rather reflect effective descriptions that may arise in more fundamental theories. In particular, quantum field effects in curved spacetime, dark-energy-like sources in cosmology, and nonlinear electrodynamics models can all lead to effective departures from classical energy conditions [22, 42]. This makes regular black holes

a useful theoretical setting for exploring strong-gravity regimes where classical general relativity is expected to receive corrections.

Within this context, regular black hole spacetimes have become an active area of research, especially in studies related to gravitational collapse, thermodynamics, and particle dynamics in strong gravitational fields. Among the various proposed models, the Bardeen black hole is of particular importance, as it represents the first known regular black hole solution and serves as the background geometry considered in this thesis for investigating accretion phenomena.

1.11 Black Hole Accretion

Black holes are most commonly observed through the radiation emitted by infalling matter during the process of accretion. In this mechanism, gas drawn toward the compact object heats up due to gravitational energy conversion, and the efficiency of radiative cooling plays a crucial role in determining the structure and dynamics of the accretion flow. Depending on the physical conditions, both radiatively efficient thin accretion disks and radiatively inefficient thick disk configurations can form and are supported by astrophysical observations.

As the accreting material approaches the central region of the black hole, the emitted radiation becomes increasingly sensitive to the spacetime geometry, particularly the black hole spin and the presence of an event horizon. Consequently, observational studies of spectra and luminosity profiles from accreting systems provide indirect evidence for rotating black holes and the existence of event horizons.

Furthermore, numerical simulations suggest that relativistic jets associated with active black holes may extract part of its rotational energy from the black hole's spin of the black hole itself, indicating a possible coupling between accretion dynamics and spin-driven energy mechanisms. Accretion refers to the process by

which matter flows inward toward a central gravitationally bound object or toward the center of mass of an extended system. It is one of the most fundamental and widespread mechanisms in astrophysics, playing a key role in the formation and evolution of cosmic structures.

In the early universe, for example, galaxies were formed as baryonic matter gradually accumulated within the gravitational potential wells created by dark matter halos [43]. Star formation continues to occur within galaxies in the present epoch, where interstellar gas clouds undergo gravitational collapse and fragmentation under their own self-gravity. The resulting proto-stellar objects subsequently increase their mass by accreting material from the surrounding medium [44]. Planetary systems, including Earth, are formed through the gradual coalescence of gas and solid particles within the circumstellar disks surrounding newly born stars. In these environments, dust and rocky material collide and stick together, eventually leading to the formation of larger planetary bodies [45].

One of the most striking observational signatures of accretion, however, is seen in systems containing black holes, where infalling matter releases enormous amounts of energy in the process of being accreted by the central compact object.

1.12 Dissertation Contribution

This dissertation presents a comprehensive investigation of steady-state spherically symmetric accretion onto a Bardeen regular black hole. The Bardeen black hole, characterized by a magnetic charge parameter g , represents a nonsingular spacetime geometry that avoids the central curvature singularity present in classical black hole solutions. The primary contribution of this study lies in elucidating how the presence of the magnetic charge parameter g alters the fundamental properties of accretion dynamics in comparison with classical Schwarzschild and Reissner-Nordström black holes.

A systematic and detailed analysis of the accretion flow onto a Bardeen black hole is carried out. The fundamental conservation equations governing the accretion process are derived by employing the framework of relativistic hydrodynamics. Explicit expressions for the critical point conditions are obtained, allowing for a clear classification of transonic flow regimes. The analysis demonstrates that the Bardeen parameter g significantly modifies the critical radius, critical velocity, and critical sound speed, thereby altering the conditions under which the fluid transitions from subsonic to supersonic motion.

A key result is the derivation of the mass accretion rate for a polytropic fluid accreting onto a Bardeen black hole. Unlike the Schwarzschild case, where the accretion rate depends only on the black hole mass and asymptotic fluid parameters, our expression contains explicit corrections due to the magnetic charge parameter g . The dependence of the accretion rate on g is analyzed in detail, revealing that the regular geometry of the Bardeen black hole enhances the accretion rate compared to the singular Schwarzschild case. Furthermore, a comprehensive analysis of the near-horizon thermodynamic properties is presented, showing that the gas compression ratio and temperature ratio increase monotonically with the magnetic charge parameter. These results clarify the role of the Bardeen parameter in reshaping the accretion dynamics and establish clear constraints on physically admissible flow configurations.

A detailed study of the critical point structure is also undertaken. By analyzing the conditions for the existence of physically acceptable critical points, we identify the parameter space for which transonic accretion is possible. The analysis reveals that the Bardeen parameter g enlarges the region of parameter space where critical points exist, indicating that regular black hole geometries can support a wider range of accretion flows compared to singular black holes. The influence of the adiabatic index γ on the critical point location is also examined, showing that softer equations of state lead to critical points farther from the black hole.

Furthermore, a detailed analytical study of the near-horizon behavior of the accreting fluid is presented. By evaluating the Bernoulli equation at the outer horizon,

we derive explicit expressions for the horizon velocity, gas compression ratio, and temperature ratio. These results demonstrate that the thermodynamic properties of the accreting gas near the horizon are directly influenced by the horizon structure of the Bardeen black hole. The compression ratio and temperature ratio are shown to increase with increasing magnetic charge parameter, indicating that the regular geometry enhances the density and thermal energy of the gas in the strong-gravity region.

Overall, the results presented in this dissertation establish that the Bardeen black hole geometry significantly modifies accretion dynamics compared to classical black hole solutions. The magnetic charge parameter plays a crucial role in determining the critical point location, mass accretion rate, and near-horizon thermodynamic properties. These findings suggest that accretion processes may serve as a useful probe for distinguishing regular black holes from their singular counterparts and provide insights into the observational signatures of Bardeen-type black holes.

1.13 Organization of the Dissertation

This dissertation investigates steady-state spherically symmetric accretion onto a Bardeen regular black hole within the framework of general relativity and relativistic hydrodynamics. The central objective is to examine how the magnetic charge parameter of the Bardeen spacetime influences the accretion dynamics, critical point structure, mass accretion rate, and near-horizon thermodynamic properties of the accreting fluid. To this end, we systematically analyze the conservation equations, critical point conditions, and thermodynamic profiles of the accretion flow.

The dissertation is organized as follows:

- i. Chapter 2 introduces the mathematical preliminaries and relativistic fluid dynamics required for the analysis. It covers the metric tensor, curvature

tensors, Einstein field equations, and the basic concepts of relativistic hydrodynamics, including the four-velocity, perfect fluid, conservation laws, equation of state, specific enthalpy, sound speed, and the Bernoulli equation. The chapter also reviews Bondi accretion theory and Michel's relativistic generalization.

- ii. Chapter 3 presents the main results of the dissertation. The Bardeen space-time is introduced, and the fundamental conservation equations for the accretion flow are derived. The critical point conditions are obtained and analyzed for both extremal and non-extremal configurations. For a polytropic equation of state, the mass accretion rate is derived, and the near-horizon thermodynamic behavior is studied in detail. The influence of the magnetic charge parameter on the critical radius, accretion rate, gas compression, and temperature profiles is systematically examined.
- iii. Chapter 4 summarizes the main findings of the dissertation, discusses their physical implications for regular black hole physics and accretion theory, and outlines several directions for future research.

Chapter 2

Preliminaries

2.1 Introduction

This chapter establishes the mathematical and physical foundations required for the analysis of steady-state spherically symmetric accretion onto a Bardeen regular black hole. We systematically introduce the key concepts from general relativity, relativistic fluid dynamics, and accretion theory that form the basis of our investigation.

Throughout this thesis, we employ geometrized units in which the speed of light and Newton's gravitational constant are set to unity, i.e., $c = G = 1$, unless otherwise stated. We adopt the metric signature $(-, +, +, +)$ and follow the standard conventions of relativistic hydrodynamics.

2.2 General Relativity and Black Hole Spacetimes

2.2.1 Metric Tensor and Line Element

In general relativity, the gravitational field is described by the geometry of spacetime. The fundamental object encoding this geometry is the metric tensor $g_{\mu\nu}$, which defines the invariant spacetime interval ds^2 between two infinitesimally close events. In a four-dimensional spacetime with coordinates $x^\mu = (t, r, \theta, \phi)$, the line element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.1)$$

where repeated indices are summed over according to the Einstein summation convention. The metric tensor is symmetric, $g_{\mu\nu} = g_{\nu\mu}$, and its inverse $g^{\mu\nu}$ satisfies $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$.

The determinant of the metric tensor is denoted by $g = \det(g_{\mu\nu})$. For a spacetime with signature $(-, +, +, +)$, the determinant is negative, and we define the quantity $\sqrt{-g}$, which appears frequently in the covariant formulation of conservation laws [12]. The metric tensor determines the causal structure of spacetime. A vector V^μ is classified as:

- i) Timelike if $g_{\mu\nu} V^\mu V^\nu < 0$, corresponding to the trajectory of a massive particle,
- ii) Null (or lightlike) if $g_{\mu\nu} V^\mu V^\nu = 0$, corresponding to the trajectory of light,
- iii) Spacelike if $g_{\mu\nu} V^\mu V^\nu > 0$, which cannot be traversed by any physical signal [8].

This causal classification is essential for understanding the horizon structure of black hole spacetimes, as we shall see in Subsection 2.2.3.

2.2.2 Spherically Symmetric Static Spacetimes

A spacetime is said to be *static* if it possesses a timelike Killing vector field that is hypersurface-orthogonal. This implies that the metric components are independent of the time coordinate t , and there are no cross-terms between time and space coordinates after a suitable coordinate choice. A spacetime is *spherically symmetric* if it is invariant under the full rotation group $SO(3)$, meaning the geometry depends only on the radial coordinate r .

For a static, spherically symmetric spacetime, the most general line element can be written in the form [6]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.2)$$

where $f(r)$ is an arbitrary function of the radial coordinate r called the *lapse function* or *metric function*. The angular part of the metric can be conveniently written using the solid angle element

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (2.3)$$

so that the line element becomes

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \quad (2.4)$$

This representation of the metric is particularly useful for studying black hole spacetimes because it explicitly displays the radial dependence of the gravitational field. The metric components are given by

$$g_{\mu\nu} = \text{diag} \left(-f(r), \frac{1}{f(r)}, r^2, r^2 \sin^2 \theta \right), \quad (2.5)$$

with the corresponding determinant

$$g = -r^4 \sin^2 \theta, \quad (2.6)$$

and hence

$$\sqrt{-g} = r^2 \sin \theta. \quad (2.7)$$

For the Schwarzschild black hole, which describes the exterior gravitational field of a non-rotating, uncharged spherical mass, the metric function takes the form

$$f_{\text{Sch}}(r) = 1 - \frac{2M}{r}, \quad (2.8)$$

where M is the mass of the central object. The Schwarzschild metric will serve as the singular limit of the Bardeen spacetime studied in this thesis.

2.2.3 Horizon Structure

A fundamental feature of black hole spacetimes is the existence of horizons, which are hypersurfaces. The most important of these is the *event horizon*, which marks the boundary beyond which no information or signal can escape to infinity [9].

For a static, spherically symmetric spacetime of the form given in Eq. (2.2), the horizons are located at the roots of the equation

$$f(r) = 0. \quad (2.9)$$

At these radii, the g_{tt} component of the metric vanishes and the g_{rr} component diverges, indicating a coordinate singularity. However, as discussed in Chapter 1, this singularity is not physical; it arises from the choice of coordinates and can be removed by a suitable coordinate transformation.

The nature of the horizons depends on the number of roots of $f(r) = 0$:

- i) If $f(r)$ has two distinct positive roots, the spacetime possesses an *outer event horizon* r_+ and an *inner Cauchy horizon* r_- . The outer horizon is the boundary of the region from which no signals can escape to infinity.
- ii) If the two roots coincide, the black hole is said to be *extremal*, characterized by a degenerate horizon at which both $f(r) = 0$ and $f'(r) = 0$.
- iii) If $f(r)$ has no roots, the spacetime contains no horizon and corresponds to a naked singularity.

The horizon structure of the Bardeen black hole, which depends on the magnetic charge parameter g , will be discussed in detail in Chapter 3.

2.3 Relativistic Fluid Dynamics

2.3.1 Four-Velocity and Normalization

In relativistic fluid dynamics, the motion of a fluid is described by its *four-velocity* U^μ , which is the four-dimensional generalization of the ordinary velocity. For a fluid element moving along a worldline parameterized by the proper time τ , the four-velocity is defined as [46]

$$U^\mu = \frac{dx^\mu}{d\tau}. \quad (2.10)$$

The four-velocity satisfies the fundamental normalization condition

$$U^\mu U_\mu = g_{\mu\nu} U^\mu U^\nu = -1, \quad (2.11)$$

which follows from the definition of proper time, $d\tau^2 = -ds^2$. This normalization ensures that the four-velocity is a timelike vector, as required for a massive fluid.

For a fluid undergoing purely radial motion, which is the focus of this thesis, the four-velocity has the form

$$U^\mu = (U^0, U^1, 0, 0) = (U^0, u, 0, 0), \quad (2.12)$$

where $u = dr/d\tau$ is the radial component of the four-velocity. Using the normalization condition with the metric (2.2), we obtain

$$-f(r)(U^0)^2 + \frac{1}{f(r)}u^2 = -1. \quad (2.13)$$

Solving for U^0 , we find

$$(U^0)^2 = \frac{f(r) + u^2}{f(r)^2}. \quad (2.14)$$

This relation will be used extensively in Chapter 3 when deriving the accretion flow equations for the Bardeen black hole.

2.3.2 Energy-Momentum Tensor

The macroscopic properties of a fluid are encoded in its *energy-momentum tensor* $T^{\mu\nu}$, which describes the flux of energy and momentum through spacetime. For a *perfect fluid*, which is the simplest and most widely used model in relativistic astrophysics, the energy-momentum tensor takes the form [47]

$$T^{\mu\nu} = P g^{\mu\nu} + (P + \rho) U^\mu U^\nu, \quad (2.15)$$

where:

- i) ρ is the *energy density* of the fluid (including rest mass energy and internal energy),
- ii) P is the *pressure*,
- iii) U^μ is the four-velocity of the fluid,

iv) $g^{\mu\nu}$ is the inverse metric tensor.

The physical interpretation of the components of $T^{\mu\nu}$ is as follows:

- i) T^{00} represents the energy density,
- ii) T^{0i} represents the energy flux,
- iii) T^{ij} represents the momentum flux (or stress).

The perfect fluid approximation assumes that dissipative effects such as shear viscosity, bulk viscosity, and thermal conduction are negligible.

The energy-momentum tensor satisfies the conservation law

$$\nabla_\mu T^{\mu\nu} = 0, \quad (2.16)$$

where ∇_μ denotes the covariant derivative with respect to the spacetime metric. This equation represents the local conservation of energy and momentum [48].

2.3.3 Particle Number Conservation

In addition to energy and momentum conservation, the number of particles in the fluid is also conserved in the absence of particle creation or annihilation. This is expressed through the *particle number conservation equation*.

The particle current density is defined as [49]

$$J^\mu = nU^\mu, \quad (2.17)$$

where n is the *particle number density* measured in the fluid rest frame. The particle number conservation law is

$$\nabla_\mu J^\mu = \nabla_\mu (nU^\mu) = 0. \quad (2.18)$$

This equation states that the rate of change of the particle number density, including the effects of spacetime curvature and fluid expansion, is zero. For a static, spherically symmetric spacetime, the particle conservation equation can be written explicitly in terms of the radial coordinate. Using the identity

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu), \quad (2.19)$$

and substituting $\sqrt{-g} = r^2 \sin \theta$ from Eq. (2.7), we obtain

$$\frac{1}{r^2} \frac{d}{dr} (r^2 n u) = 0, \quad (2.20)$$

which upon integration gives

$$r^2 n u = C_1, \quad (2.21)$$

where C_1 is an integration constant. This equation is a central result that will be used in Chapter 3 to derive the mass accretion rate.

2.3.4 Energy-Momentum Conservation

The conservation of the energy-momentum tensor, Eq. (2.16), yields important dynamical equations for the fluid flow. For a perfect fluid, the projection of $\nabla_\mu T^{\mu\nu} = 0$ along the direction of the four-velocity gives the *energy conservation equation*, while the projection orthogonal to the four-velocity gives the *Euler equation* [49]. The component along the four-velocity is obtained by contracting with U_ν :

$$U_\nu \nabla_\mu T^{\mu\nu} = 0, \quad (2.22)$$

which yields

$$\nabla_\mu (\rho U^\mu) + P \nabla_\mu U^\mu = 0. \quad (2.23)$$

This equation describes the change in energy density due to fluid expansion and pressure work.

The component orthogonal to the four-velocity is obtained by projecting with the operator $h_{\alpha\nu} = g_{\alpha\nu} + U_\alpha U_\nu$, which yields the relativistic Euler equation

$$(\rho + P)U^\mu \nabla_\mu U^\alpha = -(g^{\alpha\mu} + U^\alpha U^\mu) \nabla_\mu P. \quad (2.24)$$

For the radial flow in a static, spherically symmetric spacetime, the conservation of the t -component of $T^{\mu\nu}$ leads to

$$r^2(\rho + P)u\sqrt{f(r) + u^2} = C_2, \quad (2.25)$$

where C_2 is another integration constant. This equation will be combined with the particle conservation equation in Chapter 3 to derive the relativistic Bernoulli equation.

2.3.5 Thermodynamic Relations

For an adiabatic fluid, which is the focus of this thesis, the thermodynamic variables satisfy a set of relations that close the hydrodynamic system. A fundamental relation for relativistic fluids is [50]

$$\frac{d\rho}{dn} = \frac{P + \rho}{n}, \quad (2.26)$$

which is obtained from the first law of thermodynamics in the absence of entropy production. This relation is equivalent to the statement that the specific entropy is constant along fluid streamlines.

The proof of Eq. (2.26) follows from the thermodynamic identity for a perfect fluid. The first law of thermodynamics for a relativistic fluid can be written as

$$d\rho = \frac{P + \rho}{n} dn + nT ds, \quad (2.27)$$

where T is the temperature and s is the specific entropy. For isentropic (adiabatic) flow, $ds = 0$, and we immediately obtain Eq. (2.26).

This relation is crucial because it connects the energy density ρ to the particle number density n , allowing the system of equations to be closed once an equation of state relating P and ρ is specified.

2.3.6 Specific Enthalpy

The *specific enthalpy* h of a relativistic fluid is defined as [51]

$$h = \frac{\rho + P}{n}. \quad (2.28)$$

This quantity represents the enthalpy per particle and is a measure of the energy content of the fluid. In the non-relativistic limit, where $\rho \approx mn$ (with m the particle mass) and $P \ll \rho$, the specific enthalpy reduces to the mass per particle $h \approx m$.

The specific enthalpy is a convenient quantity because it appears naturally in the relativistic Bernoulli equation. From Eq. (2.26), we can also write

$$\frac{d\rho}{dn} = h, \quad (2.29)$$

which shows that the specific enthalpy is the derivative of the energy density with respect to the particle number density.

Using the definition of the specific enthalpy, the energy-momentum tensor can be written in the alternative form

$$T^{\mu\nu} = nhU^\mu U^\nu + Pg^{\mu\nu}. \quad (2.30)$$

Similarly, the energy conservation equation (2.25) can be expressed as

$$r^2 n h u \sqrt{f(r) + u^2} = C_2, \quad (2.31)$$

which, when divided by the particle conservation Eq. (2.21), gives Bernoulli equation in terms of the specific enthalpy.

2.3.7 Sound Speed

The propagation of small perturbations in a fluid is governed by the sound speed, denoted by a . In relativistic fluid dynamics, the sound speed is defined as the square root of the derivative of pressure with respect to energy density at constant entropy [52]

$$a^2 = \left(\frac{\partial P}{\partial \rho} \right)_s, \quad (2.32)$$

where the subscript s is for sonic/critical point. For an adiabatic flow, where the entropy is constant, the sound speed can also be written as

$$a^2 = \frac{dP}{d\rho}. \quad (2.33)$$

Using the thermodynamic relation (2.26) and the chain rule, the sound speed can be expressed in terms of the particle number density:

$$a^2 = \frac{n}{\rho + P} \frac{dP}{dn} = \frac{n}{nh} \frac{dP}{dn} = \frac{1}{h} \frac{dP}{dn}. \quad (2.34)$$

Equivalently, using Eq. (2.28),

$$a^2 = \frac{n}{\rho + P} \frac{dP}{dn}. \quad (2.35)$$

The sound speed is a crucial parameter in accretion theory because the transition from subsonic to supersonic flow occurs when the fluid velocity equals the local sound speed. This occurs at the critical point, which will be discussed in detail in Sections 2.5 and 2.6. To ensure that sound waves do not propagate faster than light, the sound speed must satisfy

$$a^2 \leq 1, \quad (2.36)$$

since sound waves cannot propagate faster than the speed of light. This condition imposes important constraints on admissible equations of state.

2.4 Polytropic Fluids

2.4.1 Equation of State

A particularly useful and widely studied class of fluids are those satisfying a *polytropic equation of state*. In its simplest form, a polytropic fluid is defined by the relation [51, 53]

$$P = Kn^\gamma, \quad (2.37)$$

where K is a constant (the polytropic constant) and γ is the *adiabatic index*. The adiabatic index is a measure of the compressibility of the fluid and determines the relationship between pressure and density.

For physically realistic fluids, the adiabatic index satisfies

$$1 < \gamma \leq \frac{5}{3}. \quad (2.38)$$

The limiting cases are:

- i) $\gamma = 5/3$ corresponds to a non-relativistic monoatomic ideal gas,
- ii) $\gamma = 4/3$ corresponds to a radiation-dominated ultra-relativistic fluid,
- iii) $\gamma \rightarrow 1$ corresponds to an isothermal fluid.

The polytropic equation of state is particularly convenient because it leads to analytic solutions for the accretion flow, as will be demonstrated in Chapter 3.

2.4.2 Energy Density Relation

Substituting the polytropic equation of state (2.37) into the thermodynamic relation (2.26) and integrating yields an explicit expression for the energy density ρ in terms of the particle number density n . We start with

$$\frac{d\rho}{dn} = \frac{\rho + Kn^\gamma}{n}. \quad (2.39)$$

This is a first-order linear differential equation for $\rho(n)$. Rearranging,

$$\frac{d\rho}{dn} - \frac{1}{n}\rho = Kn^{\gamma-1}. \quad (2.40)$$

The integrating factor is $1/n$, so

$$\frac{d}{dn} \left(\frac{\rho}{n} \right) = Kn^{\gamma-2}. \quad (2.41)$$

Integrating with respect to n gives

$$\frac{\rho}{n} = \frac{K}{\gamma-1} n^{\gamma-1} + m, \quad (2.42)$$

where m is the integration constant, which physically represents the mass per particle. Therefore,

$$\rho = nm + \frac{K}{\gamma - 1} n^\gamma. \quad (2.43)$$

The first term, $Kn^\gamma/(\gamma - 1)$, represents the internal energy density, while the second term, mn , represents the rest mass energy density. This decomposition is particularly useful because it explicitly separates the contributions from rest mass and internal energy.

2.4.3 Sound Speed in Polytropic Fluids

For a polytropic fluid, the sound speed can be expressed explicitly in terms of the particle number density. Using Eq. (2.35) and the polytropic equation of state (2.37), we obtain

$$a^2 = \frac{n}{\rho + P} \frac{dP}{dn}. \quad (2.44)$$

Differentiating the polytropic equation of state gives

$$\frac{dP}{dn} = \gamma K n^{\gamma-1}. \quad (2.45)$$

Substituting this and the polytropic EOS into the sound speed expression yields

$$a^2 = \frac{n\gamma K n^{\gamma-1}}{\rho + K n^\gamma}. \quad (2.46)$$

Using the energy density expression (2.43), we have

$$\rho + P = \frac{K}{\gamma - 1} n^\gamma + mn + K n^\gamma = \frac{\gamma K}{\gamma - 1} n^\gamma + mn. \quad (2.47)$$

Therefore, the sound speed can be written as

$$a^2 = \frac{\gamma K n^\gamma}{\frac{\gamma K}{\gamma - 1} n^\gamma + mn}. \quad (2.48)$$

Dividing numerator and denominator by mn , we obtain

$$a^2 = \frac{\frac{\gamma K}{m} n^{\gamma-1}}{1 + \frac{\gamma K}{(\gamma-1)m} n^{\gamma-1}}. \quad (2.49)$$

Alternatively, using the polytropic equation of state, we can express the sound speed in terms of the pressure:

$$a^2 = \frac{\gamma P}{\rho + P}. \quad (2.50)$$

This relation is particularly useful because it shows that the sound speed depends only on the ratio of pressure to enthalpy density [54]. For future reference, we note that from Eq. (2.44) and the polytropic equation of state, we can derive the useful relation

$$\gamma K n^{\gamma-1} = \frac{ma^2}{1 - \frac{a^2}{\gamma-1}}. \quad (2.51)$$

This relation connects the polytropic constant K , the particle number density n , the particle mass m , and the sound speed a . It will be used in Chapter 3 to relate the critical sound speed to the asymptotic sound speed.

2.5 Bondi's Newtonian Accretion Model

2.5.1 Mass Conservation

The Newtonian theory of spherically symmetric accretion was developed by Bondi and serves as the foundation for all subsequent accretion models. In Bondi's approach, the accreting gas is assumed to be steady ($\partial/\partial t = 0$), spherically symmetric ($\partial/\partial \theta = \partial/\partial \phi = 0$), and non-relativistic.

Under these assumptions, the conservation of mass is expressed through the *continuity equation* [53]

$$\dot{M} = 4\pi r^2 \rho v, \quad (2.52)$$

where:

- i) $\dot{M}$ is the mass accretion rate (constant for steady flow),
- ii) ρ is the mass density of the gas,
- iii) v is the radial velocity of the gas (positive for inflow, so $v < 0$ for accretion),
- iv) r is the radial distance from the central object.

The constancy of $\dot{M}$ follows from the steady-state assumption: the same amount of mass crosses every spherical surface surrounding the central object per unit time.

It is conventional to define the accretion rate in terms of the inward velocity $u = -v > 0$, which is the magnitude of the radial velocity. In this convention, the continuity equation becomes

$$\dot{M} = 4\pi r^2 \rho u. \quad (2.53)$$

This form will be used in Chapter 3 when comparing the Newtonian results with the relativistic generalization.

2.5.2 Euler Equation

The dynamics of the fluid are governed by the Newtonian *Euler equation*, which expresses the conservation of momentum. For steady, spherically symmetric flow, the Euler equation takes the form [53]

$$v \frac{dv}{dr} = - \left[\frac{M}{r^2} + \frac{1}{\rho} \frac{dP}{dr} \right], \quad (2.54)$$

where

- i) P is the pressure of the gas,
- ii) M is the mass of the central gravitating object.

The two terms on the RHS correspond to the pressure gradient force and the gravitational force, respectively. In an accretion flow, gravity dominates the pressure gradient, driving the fluid inward.

Using the inward velocity $u = -v$, the Euler equation becomes

$$-u \frac{du}{dr} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{M}{r^2}, \quad (2.55)$$

or equivalently,

$$u \frac{du}{dr} - \frac{1}{\rho} \frac{dP}{dr} = \frac{M}{r^2}, \quad (2.56)$$

This form shows that the inward acceleration is driven by both the pressure gradient and the gravitational attraction.

2.5.3 Critical Point in Newtonian Flow

A fundamental feature of Bondi accretion is the existence of a *critical point* or *sonic point*, where the flow transitions from subsonic ($u < a_s$) to supersonic ($u > a_s$), with a_s being the local sound speed defined as

$$a_s^2 = \frac{dP}{d\rho}. \quad (2.57)$$

At the critical radius r_c , the fluid velocity coincides with the local sound speed, satisfying $u = a_s$.

To derive the critical point condition in Newtonian theory, we combine the continuity and Euler equations with the polytropic equation of state. Taking the logarithmic derivative of the continuity equation (2.53) gives

$$\frac{1}{\rho} \frac{d\rho}{dr} + \frac{1}{u} \frac{du}{dr} = -\frac{2}{r}. \quad (2.58)$$

Using the polytropic equation of state, the pressure gradient can be written as

$$\frac{1}{\rho} \frac{dP}{dr} = a_s^2 \frac{1}{\rho} \frac{d\rho}{dr}. \quad (2.59)$$

Substituting this into the Euler equation (2.56) yields

$$u \frac{du}{dr} = a_s^2 \frac{1}{\rho} \frac{d\rho}{dr} + \frac{M}{r^2}. \quad (2.60)$$

Using the continuity equation to eliminate $d\rho/dr$, we obtain the radial velocity gradient

$$\frac{du}{dr} = \frac{\frac{2a_s^2}{r} - \frac{M}{r^2}}{u^2 - a_s^2} u. \quad (2.61)$$

At the critical point $r = r_c$, the denominator vanishes, requiring the numerator to also vanish for the flow to be smooth:

$$\frac{2a_s^2}{r_c} - \frac{M}{r_c^2} = 0. \quad (2.62)$$

This gives the critical sound speed

$$a_{s,c}^2 = \frac{M}{2r_c}, \quad (2.63)$$

and using the polytropic equation of state, the critical radius can be expressed as

$$r_c = \frac{M}{2a_{s,\infty}^2} \left(\frac{5 - 3\gamma}{4} \right), \quad (2.64)$$

where $a_{s,\infty}$ is the sound speed at infinity. This result shows that the critical radius is proportional to the mass of the central object and inversely proportional to the square of the asymptotic sound speed [55, 56].

2.6 Michel's Relativistic Generalization

2.6.1 Relativistic Conservation Equations

Michel [51] extended Bondi's Newtonian theory to include relativistic effects by considering spherical accretion in the Schwarzschild spacetime. The foundation of Michel's analysis is the conservation of particle number and energy-momentum in a curved spacetime. As derived in Section 2.3, the conservation equations are

$$\nabla_\mu(nU^\mu) = 0, \quad (2.65)$$

and

$$\nabla_\mu T^{\mu\nu} = 0, \quad (2.66)$$

where $T^{\mu\nu} = Pg^{\mu\nu} + (P + \rho)U^\mu U^\nu$, denotes the energy-momentum tensor of a perfect fluid.

For a static and spherically symmetric spacetime described by the metric in Eq. (2.2), these equations reduce to two ordinary differential equations. The particle conservation equation gives

$$r^2 nu = C_1, \quad (2.67)$$

while the energy conservation equation gives

$$r^2(\rho + P)u\sqrt{f(r) + u^2} = C_2. \quad (2.68)$$

These equations are the relativistic generalizations of the Newtonian continuity equation (2.52) and the Bernoulli equation.

2.6.2 Bernoulli Equation for Relativistic Flow

The relativistic Bernoulli equation follows from combining the particle and energy conservation equations. Dividing Eq. (2.68) by Eq. (2.67) and using the definition of the specific enthalpy $h = (\rho + P)/n$, we obtain

$$h\sqrt{f(r) + u^2} = C, \quad (2.69)$$

where C is a constant. Evaluating this equation at infinity, where $f(\infty) = 1$ and $u(\infty) = 0$, gives

$$C = h_\infty = \left(\frac{\rho_\infty + P_\infty}{n_\infty} \right). \quad (2.70)$$

Therefore, the relativistic Bernoulli equation is

$$h^2 [f(r) + u^2] = h_\infty^2. \quad (2.71)$$

This is a fundamental result that will be used extensively in Chapter 3. It states that the specific enthalpy and the gravitational potential combine to give a constant along the flow.

Using the thermodynamic relation (2.29), the Bernoulli equation can also be written as

$$\left(\frac{\rho + P}{n} \right)^2 [f(r) + u^2] = \left(\frac{\rho_\infty + P_\infty}{n_\infty} \right)^2. \quad (2.72)$$

This form is particularly useful when the equation of state is specified, as it relates the fluid variables ρ and P to the radial velocity u and the metric function $f(r)$.

2.6.3 Critical Point Conditions

The critical point in relativistic accretion [57] is determined by the same mathematical structure as in the Newtonian case, but with important modifications due to the spacetime geometry. To derive the critical point conditions, we combine the Bernoulli equation with the radial component of the energy-momentum conservation equation.

From the energy-momentum conservation, the radial component of the Euler equation in a static, spherically symmetric spacetime is

$$u \frac{du}{dr} = - \left(\frac{f(r) + u^2}{\rho + P} \right) \frac{dP}{dr} - \frac{1}{2} f'(r). \quad (2.73)$$

Using the thermodynamic relation $dP = a^2 d\rho$ and the chain rule $\rho' = (\rho + P)n'/n$, we obtain

$$u \frac{du}{dr} = -a^2 \left(\frac{f(r) + u^2}{n} \right) n' - \frac{1}{2} f'(r). \quad (2.74)$$

Taking the logarithmic derivative of the particle conservation equation (2.67) gives

$$\frac{n'}{n} + \frac{u'}{u} = -\frac{2}{r}. \quad (2.75)$$

Eliminating n' between these two equations yields the radial velocity gradient

$$\frac{du}{dr} = \frac{\frac{2a^2(f+u^2)}{r} - \frac{1}{2}f'(r)u}{u^2 - a^2(f + u^2)}. \quad (2.76)$$

At the critical point $r = r_c$, the denominator vanishes, requiring the numerator to also vanish:

$$\frac{2a_c^2(f_c + u_c^2)}{r_c} - \frac{1}{2}f'(r_c)u_c = 0, \quad (2.77)$$

and

$$u_c^2 - a_c^2(f_c + u_c^2) = 0. \quad (2.78)$$

From Eq. (2.78), we obtain the critical sound speed

$$a_c^2 = \frac{u_c^2}{f_c + u_c^2}. \quad (2.79)$$

From Eq. (2.77), we obtain the critical velocity

$$u_c^2 = \frac{r_c}{4}f'(r_c). \quad (2.80)$$

These conditions reduce to the Newtonian result in the limit $f(r) \rightarrow 1$ and $f'(r) \rightarrow 2GM/r^2$. However, for black hole spacetimes, the metric function $f(r)$ and its derivative $f'(r)$ introduce important modifications that depend on the specific black hole geometry.

The critical point conditions derived here are completely general and apply to any static, spherically symmetric black hole spacetime. In Chapter 3, we will apply these conditions to the Bardeen black hole by substituting the explicit form of the metric function $f(r)$ for that spacetime.

Chapter 3

Accretion onto a Bardeen Black Hole

3.1 Introduction

This work examines accretion onto steady-state, spherically symmetric Bardeen black hole under the assumption that the self-gravity of the accreting fluid does not alter the background spacetime geometry. The governing equations of the flow are formulated, and the conditions required for the existence of critical points are established together with the corresponding mass accretion rate. The analysis further identifies the criteria necessary for physically admissible critical solutions. A detailed investigation of polytropic fluid accretion is also carried out to examine the behavior of gas compression and the temperature distribution in the vicinity of the outer horizon. The results indicate that the Bardeen magnetic charge parameter has a significant impact on the accretion process by shifting the critical conditions, modifying the mass accretion rate, and influencing both the compression and thermal profiles of the gas near the outer horizon. Overall, these findings highlight the important role of regular black-hole geometry in determining the properties of accretion flows.

3.2 Bardeen Spacetime and Basic Equations for Accretion

In this section, we investigate the accretion process onto a static and spherically symmetric Bardeen black hole. The spacetime geometry is described by the line element

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.1)$$

where the metric function $f(r)$ is given by

$$f(r) = 1 - \frac{2Mr^2}{(r^2 + g^2)^{3/2}}. \quad (3.2)$$

Here, M denotes the mass of the black hole and g represents the magnetic charge parameter. The Bardeen solution represents a regular spacetime provided that the magnetic charge parameter satisfies

$$g^2 \leq \frac{16}{27}M^2. \quad (3.3)$$

For $g^2 < \frac{16}{27}M^2$, the Bardeen spacetime possesses two distinct horizons. In the extremal case $g^2 = \frac{16}{27}M^2$, these two horizons coincide and merge into a single degenerate horizon.

To visually illustrate the impact of the magnetic charge parameter g on the spacetime geometry and horizon structure, we plot the metric function $f(r)$ for various parameter values in Fig. 3.1. For the classical Schwarzschild limit ($g = 0$, represented by the solid black curve), the metric function diverges to $-\infty$ as the radial coordinate approaches the origin ($r \rightarrow 0$), reflecting the presence of the physical central curvature singularity. Conversely, for any non-zero value of g , the metric remains completely finite everywhere, yielding $f(0) = 1$. This behavior demonstrates that the regular Bardeen geometry successfully removes the central singularity, replacing it with a de Sitter-like core. Additionally, the parameter g fundamentally determines the horizon configuration. When the regularity parameter satisfies the sub-extremal condition $g^2 < \frac{16}{27}M^2$ (modeled by $g = 0.40$, blue

dashed curve), the metric function possesses two real roots, which correspond to the inner Cauchy horizon and the outer event horizon. As the magnetic charge reaches the critical extremal value $g = g_{\text{ext}} = \sqrt{16/27}M \approx 0.77M$ (green dash-dotted curve), these two horizons coincide and merge into a single degenerate horizon where $f(r) = f'(r) = 0$. For super-extremal charges satisfying $g^2 > \frac{16}{27}M^2$ (modeled by $g = 0.95$, red dotted curve), the metric function remains strictly positive ($f(r) > 0$) for all $r \geq 0$. In this regime, the event horizons disappear completely, leaving behind a regular, horizonless ultra-compact object.

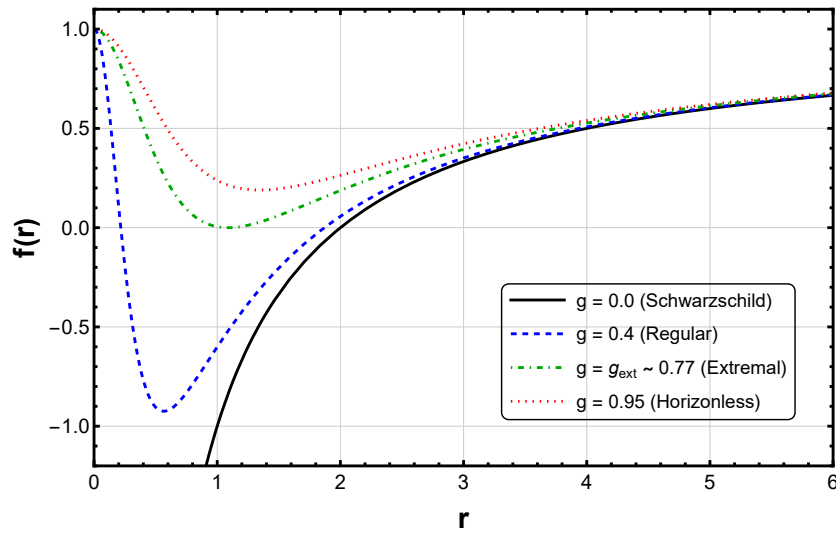


FIGURE 3.1: The metric function $f(r)$ as a function of radial distance r for $M = 1.0$ and different values of the Bardeen magnetic charge parameter g . The solid black line ($g = 0$) represents the singular Schwarzschild limit, the blue dashed line ($g = 0.4$) represents a regular black hole with two horizons, the green dash-dotted line ($g \approx 0.77$) represents the extremal case, and the red dotted line ($g = 0.95$) represents a regular horizonless compact core.

We model the accreting matter as a perfect fluid. The energy-momentum tensor of the fluid can be written as

$$T_{\alpha\beta} = P g_{\alpha\beta} + (P + \rho) U_{\alpha} U_{\beta}, \quad (3.4)$$

where P and ρ represent the fluid pressure and energy density, respectively. The corresponding four-velocity of the fluid is assumed to be

$$U^{\alpha} = (u^0, u, 0, 0), \quad (3.5)$$

consistent with radial flow. The four-velocity satisfies the normalization condition

$$U^\mu U_\mu = -1. \quad (3.6)$$

Using the metric components, this condition leads to

$$-f(r)(u^0)^2 + \frac{1}{f(r)}u^2 = -1. \quad (3.7)$$

Solving the above relation for $(u^0)^2$, we obtain

$$(u^0)^2 = \frac{f(r) + u^2}{f(r)^2}. \quad (3.8)$$

Substituting the explicit form of the lapse function, we get

$$(u^0)^2 = \frac{1 - \frac{2Mr^2}{(r^2+g^2)^{3/2}} + u^2}{(1 - \frac{2Mr^2}{(r^2+g^2)^{3/2}})^2}. \quad (3.9)$$

For an adiabatic fluid flow, the thermodynamic variables satisfy the relation

$$\frac{d\rho}{dn} = \frac{\rho + P}{n}, \quad (3.10)$$

where n denotes the particle number density. The sound speed within the fluid is defined as

$$a^2 = \frac{dP}{d\rho}, \quad (3.11)$$

which, by using the above relation, can also be written as

$$a^2 = \frac{n}{\rho + P} \frac{dP}{dn}. \quad (3.12)$$

The conservation of particle number is expressed through the continuity equation

$$J^\mu_{;\mu} = 0, \quad (3.13)$$

where the particle current density is defined as $J^\mu = nU^\mu$. Thus, the conservation law takes the form

$$(nU^\mu)_{;\mu} = 0. \quad (3.14)$$

The general metric is given by

$$g_{\mu\nu} = \text{diag} \left(-f(r), \frac{1}{f(r)}, r^2, r^2 \sin^2 \theta \right). \quad (3.15)$$

The determinant of the metric tensor is

$$g = -r^4 \sin^2 \theta, \quad (3.16)$$

which yields

$$\sqrt{-g} = r^2 \sin \theta. \quad (3.17)$$

Using the identity

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} J^\mu), \quad (3.18)$$

and assuming steady-state, spherically symmetric flow, the continuity equation reduces to

$$\frac{1}{r^2} \frac{d}{dr} (r^2 n u) = 0. \quad (3.19)$$

Integrating this equation, we obtain

$$r^2 n u = C_1, \quad (3.20)$$

where C_1 is a constant of integration. The mass accretion rate, which measures the rate at which matter falls into the black hole, is defined as [53]

$$\dot{M} = 4\pi r^2 m n u, \quad (3.21)$$

where m is the mass of each particle. The conservation of energy–momentum tensor is given by

$$T^\mu_{\nu;\mu} = 0. \quad (3.22)$$

For $\nu = 0$, this equation leads

$$r^2(\rho + P)u\sqrt{f(r) + u^2} = C_2, \quad (3.23)$$

where C_2 is another constant of integration. Substituting the explicit form of the metric function, we obtain

$$r^2(\rho + P)u\sqrt{1 - \frac{2Mr^2}{(r^2 + g^2)^{3/2}} + u^2} = C_2. \quad (3.24)$$

By dividing the above equation by the particle conservation relation and squaring the result, we obtain the relativistic Bernoulli equation

$$\left(\frac{\rho + P}{n}\right)^2 [f(r) + u^2] = \left(\frac{\rho_\infty + P_\infty}{n_\infty}\right)^2. \quad (3.25)$$

Explicitly, this can be written as

$$\left(\frac{\rho + P}{n}\right)^2 \left[1 - \frac{2Mr^2}{(r^2 + g^2)^{3/2}} + u^2\right] = \left(\frac{\rho_\infty + P_\infty}{n_\infty}\right)^2. \quad (3.26)$$

Eqs. (3.21) and (3.26) constitute the fundamental equations governing the accretion flow. Finally, the radial component of the conservation equation $T^\mu_{r;\mu} = 0$ provides the equation of motion for the fluid flow. This yields

$$u \frac{du}{dr} = - \left(\frac{f(r) + u^2}{\rho + P} \right) \frac{dP}{dr} - \frac{1}{2} \frac{df(r)}{dr}. \quad (3.27)$$

The derivative of the lapse function for the Bardeen black hole is

$$\frac{df}{dr} = \frac{2Mr(r^2 - 2g^2)}{(r^2 + g^2)^{5/2}}. \quad (3.28)$$

Substituting Eq. (3.28) into (3.27), we obtain the final form of the radial flow equation

$$u \frac{du}{dr} = - \left(\frac{f(r) + u^2}{\rho + P} \right) \frac{dP}{dr} - \frac{Mr(r^2 - 2g^2)}{(r^2 + g^2)^{5/2}}. \quad (3.29)$$

Eq. (3.29) governs the radial motion of the accreting fluid in the Bardeen space-time. Now the particle conservation equation (3.20) and the energy-momentum conservation (3.26) forms the fundamental system of equations describing the accretion flow. The critical points of the flow are obtained by analyzing these equations and determining the conditions under which the fluid undergoes a smooth transition from subsonic to supersonic motion.

3.3 Critical Point Conditions

To determine the location of critical points, we start from the conservation equations governing the flow. From the particle conservation equation

$$r^2 n u = C_1, \quad (3.30)$$

taking the radial derivative of the above expression yields

$$\frac{n'}{n} + \frac{u'}{u} = -\frac{2}{r}, \quad (3.31)$$

where the prime denotes differentiation with respect to r . Next, we consider the radial component of the energy-momentum conservation equation

$$u \frac{du}{dr} = - \left(\frac{f(r) + u^2}{\rho + P} \right) \frac{dP}{dr} - \frac{1}{2} f'(r). \quad (3.32)$$

Using the thermodynamic relations for adiabatic flow, the pressure gradient can be expressed through the chain rule as

$$\frac{dP}{dr} = \frac{dP}{d\rho} \frac{d\rho}{dn} \frac{dn}{dr} = a^2 \frac{\rho + P}{n} n'. \quad (3.33)$$

Substituting this expression into the above equation, we obtain

$$uu' = -a^2 \frac{f(r) + u^2}{n} n' - \frac{1}{2} f'(r). \quad (3.34)$$

Eqs. (3.31) and (3.34) constitute a coupled system for u' and n' . Solving Eq. (3.31) for n' and substituting into Eq. (3.34), we arrive at

$$uu' = a^2(f + u^2) \left(\frac{u'}{u} + \frac{2}{r} \right) - \frac{1}{2} f'(r). \quad (3.35)$$

After rearranging terms, the radial velocity gradient becomes

$$u' [u^2 - a^2(f + u^2)] = \frac{2a^2(f + u^2)}{r} u - \frac{1}{2} f'(r) u. \quad (3.36)$$

Thus,

$$u' = \frac{\frac{2a^2(f+u^2)}{r} - \frac{1}{2} f'(r)}{\frac{u^2 - a^2(f+u^2)}{u}}. \quad (3.37)$$

Similarly, the system can be expressed in compact form as

$$u' = \frac{N_1}{N}, \quad n' = -\frac{N_2}{N}, \quad (3.38)$$

where

$$N_1 = \frac{1}{n} \left[\frac{2}{r} (f + u^2) a^2 - \frac{1}{2} f'(r) \right], \quad (3.39)$$

$$N_2 = \frac{1}{u} \left[\frac{2u^2}{r} - \frac{1}{2} f'(r) \right], \quad (3.40)$$

$$N = \frac{u^2 - (f + u^2) a^2}{nu}. \quad (3.41)$$

For a physically meaningful and smooth transonic flow, the derivatives must remain finite. This requires that N_1 , N_2 , and N vanish simultaneously at the critical

point r_c . Hence, the critical conditions are

$$\frac{2}{r_c}(f_c + u_c^2)a_c^2 - \frac{1}{2}f'(r_c) = 0, \quad (3.42)$$

$$\frac{2u_c^2}{r_c} - \frac{1}{2}f'(r_c) = 0, \quad (3.43)$$

$$u_c^2 - (f_c + u_c^2)a_c^2 = 0. \quad (3.44)$$

Eq. (3.43) determines the critical radial velocity,

$$u_c^2 = \frac{r_c}{4}f'(r_c), \quad (3.45)$$

while Eq. (3.44) yields the corresponding critical sound speed

$$a_c^2 = \frac{u_c^2}{f_c + u_c^2}. \quad (3.46)$$

Physically, the critical point corresponds to the location where the flow transitions from subsonic to supersonic, i.e., where the fluid velocity becomes comparable to the local sound speed. Physically acceptable solutions of Eq. (3.38) exist if $u_c^2 > 0$ and $a_c^2 > 0$, therefore we easily get

$$u_c^2 = \frac{Mr_c^2(r_c^2 - 2g^2)}{2(r_c^2 + g^2)^{5/2}} > 0 \Rightarrow r_c > \sqrt{2}g, \quad (3.47)$$

$$a_c^2 = \frac{u_c^2}{f(r_c) + u_c^2} > 0 \Rightarrow (r_c^2 + g^2)^{5/2} > \frac{3}{2}Mr_c^2(r_c^2 + 2g^2). \quad (3.48)$$

Critical points must fulfill the above conditions.

The dependence of the critical accretion velocity squared u_c^2 on the critical radius r_c is analyzed in Fig. 3.2 for various choices of the magnetic charge parameter g . For a physical and stable transonic accretion flow, the critical velocity squared must remain positive ($u_c^2 > 0$). In the classical Schwarzschild case ($g = 0.0$), the critical velocity squared diverges to infinity as the radial coordinate approaches the origin. However, the regular geometry of the Bardeen spacetime significantly alters this

behavior. For any non-zero value of g , the critical velocity reaches a smooth, finite maximum peak and then sharply decreases, eventually crossing into the negative (unphysical) regime in the inner regions of the spacetime. Specifically, the physical transition boundary is reached when $u_c^2 = 0$, which yields the analytical constraint $r_c > \sqrt{2}g$, consistent with Eq. (3.47). Below this threshold, u_c^2 becomes negative, rendering a physical transition from subsonic to supersonic flow mathematically impossible. Physically, this indicates that the magnetic monopole charge core of the Bardeen geometry exerts a repulsive-like gravitational regularization effect at short distances. This strictly limits the maximum possible accretion velocity and forces the critical sonic horizon to form further out compared to standard singular black hole spacetimes, providing a clear observable hallmark that distinguishes regular geometries from classical singular ones.

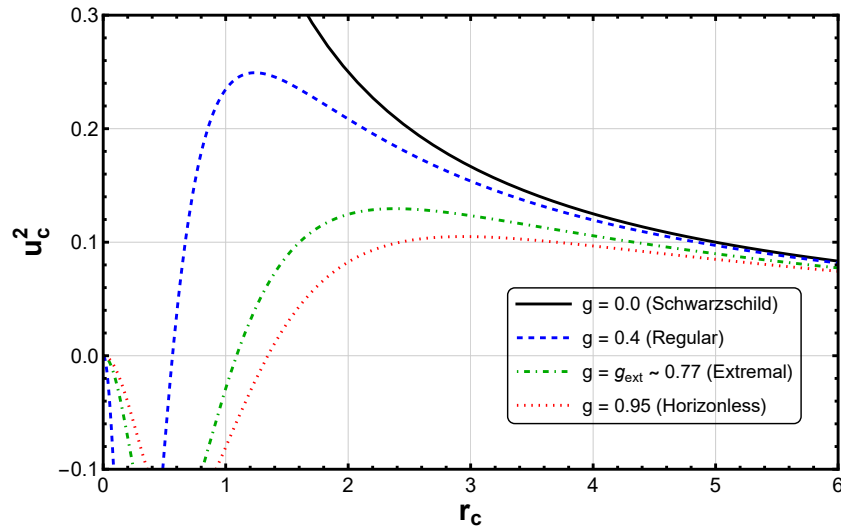


FIGURE 3.2: The critical accretion velocity squared $u_c^2(r_c)$ as a function of the critical radius r_c for $M = 1.0$ and various values of the magnetic charge parameter g . The curves denote different regimes: the solid black curve ($g = 0.0$), the dashed blue curve ($g = 0.40$), the green dash-dotted curve ($g = g_{\text{ext}} \approx 0.77$), and the red dotted curve ($g = 0.95$).

3.4 Polytropic solutions

In this section, we examine polytropic fluid accretion onto the Bardeen black hole. Our focus is to determine the mass accretion rate and to study the gas compression together with the adiabatic temperature profile near the outer horizon. We assume

that the accreting matter satisfies the polytropic equation of state [51, 53]

$$p = Kn^\gamma, \quad \gamma \in \left[1, \frac{5}{3}\right], \quad (3.49)$$

where K is a constant, $\gamma = 1$ corresponds to an isothermal fluid, while $\gamma = 5/3$ corresponds to an adiabatic non-relativistic ideal gas. Substituting this equation of state into the mass-energy conservation relation (3.10) and integrating, we obtain

$$\rho = \frac{K}{\gamma - 1} n^\gamma + mn, \quad (3.50)$$

where mn denotes the rest-mass energy density and m appears as an integration constant. Using the sound-speed definition in (3.12), the Bernoulli equation (3.26) takes the form

$$\left(1 + \frac{a^2}{\gamma - 1 - a^2}\right)^2 \left(1 - \frac{2Mr^2}{(r^2 + g^2)^{3/2}} + u^2\right) = \left(1 + \frac{a_\infty^2}{\gamma - 1 - a_\infty^2}\right)^2. \quad (3.51)$$

At the critical point, Eq. (3.51) must hold with $u = u_c$ and $a = a_c$. Substituting the critical point relations (3.45) and (3.46), we find

$$\left[1 - \frac{3Mr_c^2(r_c^2 + 2g^2)}{2(r_c^2 + g^2)^{5/2}}\right] \left(1 - \frac{a_\infty^2}{\gamma - 1}\right)^2 = \left(1 - \frac{a_c^2}{\gamma - 1}\right)^2. \quad (3.52)$$

In the domain $r_c \leq r < \infty$, the accreting baryonic matter may be treated as non-relativistic. Hence, $a^2 < a_c^2 \ll 1$, and Eq. (3.52) can be expanded up to first order in a_c^2 and a_∞^2 . Solving for a_c^2 we find

$$a_c^2 \approx \frac{a_\infty^2}{1 - \frac{3(\gamma - 1)}{2} \frac{r_c^2 + 2g^2}{r_c^2 - 2g^2}}. \quad (3.53)$$

For physical consistency, the denominator in Eq. (3.53) must remain positive.

Accordingly, the critical radius r_c may be written in terms of the boundary value a_∞ and the parameters M and g in the following form

$$a_\infty^2 \approx \frac{Mr_c^2(r_c^2 - 2g^2)}{2(r_c^2 + g^2)^{5/2}} \left[1 - \frac{3(\gamma - 1)}{2} \frac{r_c^2 + 2g^2}{r_c^2 - 2g^2}\right]. \quad (3.54)$$

This is an implicit equation for r_c . For $g = 0$, Eq. (3.54) reduces to

$$a_\infty^2 \approx \frac{M}{2r_c} \left[1 - \frac{3(\gamma - 1)}{2} \right], \quad (3.55)$$

or

$$r_c \approx \frac{M(5 - 3\gamma)}{4a_\infty^2}. \quad (3.56)$$

From Eqs. (3.12), (3.49) and (3.50), we obtain

$$\gamma K n^{\gamma-1} = \frac{ma^2}{1 - \frac{a^2}{\gamma-1}}. \quad (3.57)$$

Since $a^2 \ll 1$ for non-relativistic flow, we have $n \sim a^{2/(\gamma-1)}$ and therefore

$$\frac{n_c}{n_\infty} \approx \left(\frac{a_c}{a_\infty} \right)^{\frac{2}{\gamma-1}}. \quad (3.58)$$

As the mass accretion rate in Eq. (3.21) is conserved along the flow, it may be evaluated at the critical radius. Thus, at $r = r_c$, we obtain

$$\dot{M} = 4\pi r_c^2 m n_c u_c. \quad (3.59)$$

Substituting the critical velocity obtained from Eq. (3.47) and the density ratio given by Eq. (3.58), and using Eq. (3.53) to relate a_c to a_∞ , we obtain

$$\dot{M} = 4\pi r_c^2 m n_\infty \left[\frac{Mr_c^2(r_c^2 - 2g^2)}{2(r_c^2 + g^2)^{5/2}} \right]^{1/2} \left[1 - \frac{3(\gamma - 1)}{2} \frac{r_c^2 + 2g^2}{r_c^2 - 2g^2} \right]^{-\frac{1}{\gamma-1}}. \quad (3.60)$$

Eq. (3.60) represents the mass accretion rate of a polytropic fluid onto a Bardeen black hole. The accretion rate depends on three distinct factors. The first factor, $4\pi r_c^2 m n_\infty$, is associated with the effective area of the critical surface and the asymptotic properties of the fluid. The second factor, $\left[\frac{Mr_c^2(r_c^2 - 2g^2)}{2(r_c^2 + g^2)^{5/2}} \right]^{1/2}$, originates from the critical velocity of the flow and contains the geometric correction introduced by the Bardeen magnetic charge parameter g . The third factor, $\left[1 - \frac{3(\gamma-1)}{2} \frac{r_c^2 + 2g^2}{r_c^2 - 2g^2} \right]^{-\frac{1}{\gamma-1}}$, arises from the thermodynamic properties of the polytropic

gas and determines how the equation of state affects the accretion process. It follows from Eq. (3.60) that the mass accretion rate is controlled not only by the asymptotic fluid properties but also by the regular black-hole parameter g . Consequently, the magnetic charge modifies both the location of the critical point and the velocity of the accreting matter, leading to deviations from the corresponding accretion rate of singular black-hole spacetimes. Michel [51] studied steady-state spherical accretion onto a Schwarzschild black hole and obtained an accretion rate that depends only on the black-hole mass and the asymptotic fluid parameters. In contrast, Eq. (3.60) contains additional corrections due to the Bardeen parameter (g), indicating that the regular geometry modifies the critical conditions of the flow and hence the mass accretion rate.

Ahmed et al. [58] investigated accretion onto Reissner–Nordström black holes and showed that the electric charge affects the location of the critical point and the corresponding accretion rate. Our result is consistent with this general behavior, since the Bardeen parameter also modifies the critical radius and critical velocity. However, unlike the Reissner–Nordström spacetime, the Bardeen black hole remains free from curvature singularities, and the correction arises from nonlinear electrodynamics rather than a conventional Maxwell charge.

Accretion onto rotating black holes such as Kerr and Kerr–Newman geometries has been studied extensively in the literature [59–61]. In those cases, the accretion process is influenced by rotation and frame-dragging effects, leading to angular momentum transport and non-spherical flow configurations. By contrast, the present analysis is restricted to spherically symmetric accretion, and the modification of the accretion rate originates entirely from the regular magnetic-charge parameter (g). Therefore, the effect observed in Eq. (3.60) is fundamentally different from the rotational corrections encountered in Kerr and Kerr–Newman spacetimes. Therefore, the influence of the Bardeen parameter on the accretion flow represents a genuinely geometric effect associated with spacetime regularization rather than with electric charge or rotation. The present results indicate that regular black-hole geometries can alter the accretion dynamics in a manner similar to charged black holes while simultaneously avoiding the central curvature singularity. This

suggests that accretion may provide a useful probe for distinguishing regular black holes from conventional singular black-hole solutions.

A distinctive feature of the present analysis is that the magnetic charge parameter enters the critical point conditions and the mass accretion rate through the non-linear combination $(r_c^2 + g^2)^{3/2}$, which arises naturally from the regular Bardeen geometry. This dependence is fundamentally different from the corrections appearing in charged or rotating black-hole spacetimes, where the corresponding modifications originate from the electric charge or the rotation parameter. Consequently, the regularization of the spacetime influences the location of the critical point, the transonic flow conditions, and the mass accretion rate in a manner that cannot be reproduced simply by introducing charge or rotation into singular black-hole geometries. Therefore, the present results provide an analytical description of how spacetime regularity affects steady-state spherical accretion onto black holes.

3.4.1 Behavior of the Accreting Fluid at the Event Horizon

At the outer horizon $r = r_+$, where $f(r_+) = 0$, the Bernoulli equation can be written as

$$h^2 (f(r) + u^2) = h_\infty^2, \quad (3.61)$$

where $h = (\rho + p)/n$ is the specific enthalpy. Hence,

$$u^2 = \left(\frac{h_\infty}{h} \right)^2 - f(r). \quad (3.62)$$

At the outer horizon, this gives

$$u_h^2 = \left(\frac{h_\infty}{h_h} \right)^2. \quad (3.63)$$

Therefore, the horizon velocity depends on the ratio of the asymptotic and horizon enthalpies. In the cold-flow approximation, where $h_h \simeq h_\infty$, we obtain

$$u^2 \simeq 1 - f(r) = \frac{2Mr^2}{(r^2 + g^2)^{3/2}}. \quad (3.64)$$

Using the horizon condition

$$f(r_+) = 1 - \frac{2Mr_+^2}{(r_+^2 + g^2)^{3/2}} = 0, \quad (3.65)$$

we obtain

$$u_h^2 \simeq 1. \quad (3.66)$$

Using the mass accretion rate

$$\dot{M} = 4\pi r^2 m n u, \quad (3.67)$$

and evaluating it at the outer horizon $r = r_+$, we obtain

$$\dot{M} = 4\pi r_+^2 m n_h u_h. \quad (3.68)$$

Solving for n_h , we get

$$n_h = \frac{\dot{M}}{4\pi r_+^2 m u_h}. \quad (3.69)$$

Dividing by n_∞ , the gas compression ratio at the outer horizon becomes

$$\frac{n_h}{n_\infty} = \frac{\dot{M}}{4\pi m n_\infty r_+^2 u_h}. \quad (3.70)$$

Defining

$$A_B = \frac{\dot{M}}{4\pi m n_\infty}, \quad (3.71)$$

the gas compression ratio can be written as

$$\frac{n_h}{n_\infty} = \frac{A_B}{r_+^2 u_h}. \quad (3.72)$$

In the cold-flow approximation discussed above, the horizon velocity satisfies $u_h \simeq 1$. Therefore, the gas compression ratio simplifies to

$$\frac{n_h}{n_\infty} = \frac{A_B}{r_+^2}. \quad (3.73)$$

Fig. 3.3 shows the variation of the gas compression ratio n_h/n_∞ with the magnetic charge parameter g for different values of the asymptotic sound speed a_∞ . It is observed that the compression ratio increases as g increases, indicating that the regular geometry of the Bardeen black hole enhances the density of the accreting gas near the outer horizon. The effect becomes more pronounced for smaller values of a_∞ , corresponding to colder inflowing matter. Furthermore, the growth of the compression ratio accelerates as the magnetic charge approaches the extremal regime, highlighting the significant influence of the regular black-hole parameter on the near-horizon density profile.

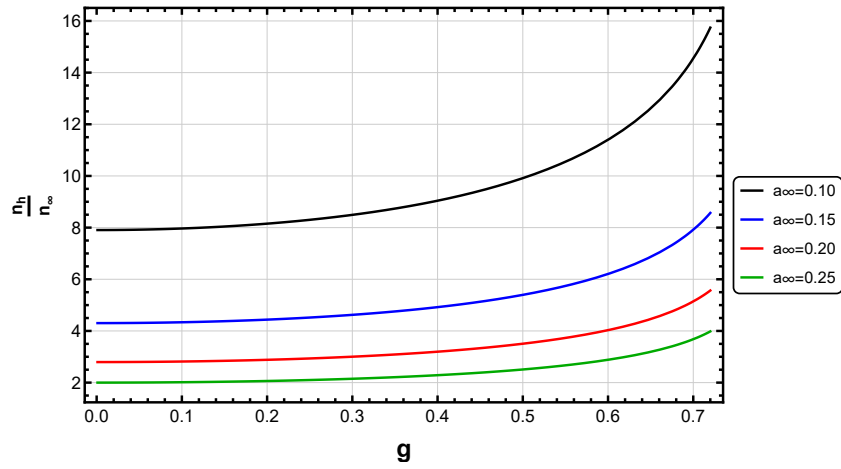


FIGURE 3.3: Gas compression ratio as a function of the Bardeen magnetic charge parameter g for different values of the asymptotic sound speed a_∞ .

Assuming the Maxwell–Boltzmann relation $p = nk_B T$ together with the polytropic equation of state $p = Kn^\gamma$, we obtain

$$T = \frac{K}{k_B} n^{\gamma-1}. \quad (3.74)$$

Therefore,

$$\frac{T_h}{T_\infty} = \left(\frac{n_h}{n_\infty} \right)^{\gamma-1}. \quad (3.75)$$

Substituting Eq. (3.72), we obtain

$$\frac{T_h}{T_\infty} = \left(\frac{A_B}{r_+^2 u_h} \right)^{\gamma-1}. \quad (3.76)$$

In the cold-flow approximation, where $u_h \simeq 1$, the above expression reduces to

$$\frac{T_h}{T_\infty} = \left(\frac{A_B}{r_+^2} \right)^{\gamma-1}. \quad (3.77)$$

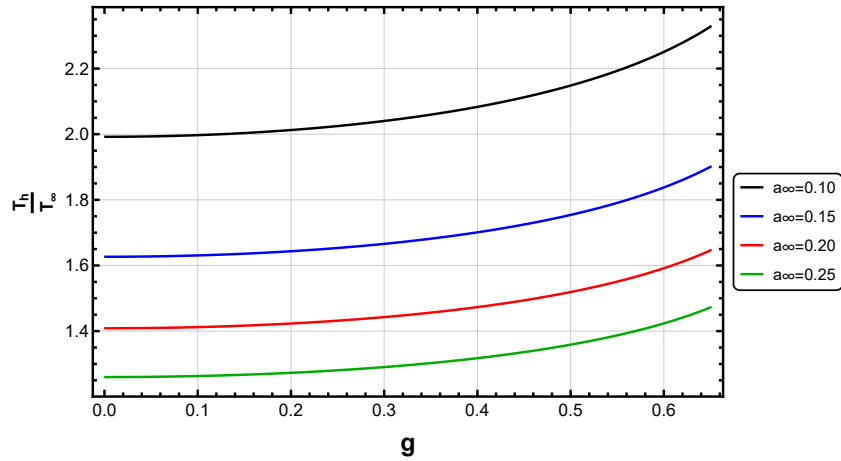


FIGURE 3.4: Temperature ratio as a function of the Bardeen magnetic charge parameter g for different values of the asymptotic sound speed a_∞ .

Fig. 3.4 illustrates the dependence of the temperature ratio T_h/T_∞ on the magnetic charge parameter g for several values of the asymptotic sound speed a_∞ . The temperature ratio increases monotonically with g , showing that the regular magnetic charge affects the thermodynamic properties of the accreting fluid near the outer horizon. As in the case of the compression ratio, smaller values of a_∞ lead to larger temperature enhancements. The figure demonstrates that the influence of the Bardeen parameter extends beyond the critical point and significantly modifies the thermal behavior of the fluid in the strong-gravity region.

The results obtained in Eqs. (3.73) and (3.77), together with Figs. 3.3 and 3.4, show that the thermodynamic properties of the accreting gas near the outer horizon are directly influenced by the horizon structure of the Bardeen black hole. Since the horizon radius depends on the magnetic charge parameter g , both the gas compression ratio and the temperature profile are modified by the regular

geometry. Consequently, the magnetic charge affects not only the critical properties of the flow but also the density and thermal behavior of the fluid in the strong-gravity region close to the event horizon.

Chapter 4

Conclusion

We studied steady-state spherical accretion onto a Bardeen black hole, which is characterized by a regular spacetime structure. The basic conservation laws for the accreting fluid were formulated by neglecting the back-reaction of the matter distribution on the background geometry. From these equations, we derived the critical point conditions and identified the constraints required for physically admissible transonic flow.

We further examined the case of a polytropic fluid. In this setting, the mass accretion rate was obtained, and the gas compression and adiabatic temperature profile near the outer horizon were discussed. The radial velocity and sound speed were also analyzed at both the critical point and the horizon.

The present analysis demonstrates that the magnetic charge parameter g modifies the accretion flow throughout the entire process, from the critical point to the horizon. In contrast to singular charged or rotating black-hole solutions, the Bardeen geometry introduces nonlinear corrections associated with spacetime regularization, leading to modifications in the critical radius, the mass accretion rate, the gas compression ratio, and the temperature profile of the accreting fluid. These analytical results indicate that regular black-hole geometries may exhibit distinctive accretion characteristics and provide a useful theoretical framework for investigating possible observational signatures of nonsingular black holes.

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