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Fixed Point in Double Controlled and Double Controlled Supra Metric Spaces with Rational Contraction

by

Ahsan Saghir

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degree of Master of Philosophy

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CERTIFICATE OF APPROVAL

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Abstract

This research focuses on fixed point results in extended b-suprametric spaces. The study is based on the fixed point theorem presented by Ahmad et al. for extended b-suprametric spaces. Their result is reviewed and used as a starting point for further development. In this work, the existing result is first extended from the single control setting to a double controlled metric space by using two independent control functions. After that, the obtained result is further extended to a double controlled suprametric space, where the double control functions are included in the suprametric structure. Fixed point results for admissible mappings under a rational type contraction are established in these new settings. Some examples are also given to explain the main results.

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Abbreviations

BCP	Banach Contraction Principle
b-SMS	b -Suprametric Space
CMS	Complete Metric Space
DCMS	Double Controlled Metric Space
DCSMS	Double Controlled Supra-Metric Space
Extended b-SMS	Extended b -Suprametric Space
Extended SMS	Extended Suprametric Space
FP	Fixed Point
MS	Metric Space
SMS	Suprametric Space

Symbols

ξ, η	Admissibility (screening) functions
$\bar{b}$	$\bar{b}$ -suprametric scaling constant, $\bar{b} \geq 1$
λ	Contraction constant for extended $\bar{b}$ -suprametric spaces
κ	Contraction constant for DCMS/DCSMS
$\delta, \delta_c, \delta_{cs}, \delta_{E\bar{b}s}$	Distance / generalized distance functions
p, q	Double control functions, $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$
ν, ζ, η	Points in the underlying space
$\mathcal{M}(\nu, \zeta)$	Generalized rational contractive functional
ν_1, ν_2	Integral-equation parameters
K_1, K_2	Kernel functions
ℓ_1, ℓ_2	Lipschitz constants for kernels
$\mathcal{X}$	Non-empty underlying set
$\mathcal{T}$	Self-mapping on the underlying set
σ	Suprametric (cross-term) constant, $\sigma \geq 0$
θ	Suprametric control function, $\theta : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$
$C([a, b], \mathbb{R})$	Space of continuous real-valued functions on $[a, b]$
$\ell^p(\mathbb{R})$	Space of p -summable real sequences

Chapter 1

Introduction

Fixed point theory (FPT) deals with mappings and the points that stay fixed under them. If T is a mapping from a space X into itself, a point $x \in X$ is called a fixed point (FP) of T when $T(x) = x$. This area is useful because it gives a way to show that solutions to a problem exist and are unique. It shows up in functional analysis, differential equations, integral equations, optimization, and basically any problem that can be written as a mapping.

A large part of FPT is built around the Banach contraction principle (BCP) [1]. It says that on a complete metric space (CMS), a mapping satisfying a certain contraction condition has exactly one FP, and this FP can be found just by iterating the mapping starting from any point. This is what makes the BCP so useful in practice, it does not just prove that a solution exists, it also tells you how to get to it.

The BCP is powerful but its assumptions are often too strong for real problems, so a lot of work has gone into generalizing it. Some of this work generalizes the contraction condition while keeping the metric space as it is. Other work changes the space itself by relaxing the triangle inequality. This second direction has produced several new spaces: b -metric spaces [2, 3], controlled metric spaces [4], suprametric spaces (SMS) [5, 6], and extended or double-controlled versions of these [7–9].

Relaxing the triangle inequality gives more freedom in how distance is defined. In controlled metric spaces, for example, the triangle inequality depends on control functions instead of a fixed constant, as in the work of Sagheer *et al.* [10].

Double controlled metric spaces (DCMS) take this further by using two control functions instead of one. SMS generalize the triangle inequality in a different way, by adding an extra product term to it.

Combining both ideas control functions and the suprametric product term gives an even more general structure, one that can handle FP problems that do not fit into ordinary metric spaces (MS).

Another direction in FPT comes from admissibility conditions [11, 12] combined with generalized contraction functions. Instead of requiring the contraction inequality to hold for every pair of points, these conditions only require it for a restricted class of pairs.

Rational-type contractions add more flexibility on top of this, since they mix distances between points with distances between points and their images in the same inequality. Put together, admissibility conditions and rational contractions have led to many FP results in both classical and generalized metric spaces.

This thesis works in that setting. It looks at FP results in extended $\bar{b}$ -suprametric spaces, DCMS, and double controlled suprametric spaces (DCSMS), for single-valued mappings that satisfy generalized rational contractions and are ξ -admissible with respect to η .

Results are first proved for extended $\bar{b}$ -suprametric spaces, and several known spaces are recovered as special cases. The same approach is then carried over to DCMS and DCSMS.

The goal is to prove FP results under conditions that are more general than what is already available, by combining generalized contractions with flexible distance structures. At the end, these results are applied to a nonlinear Volterra–Fredholm integral equation, to show that the theory also works on a concrete problem.

1.1 Literature Review

FPT has developed mainly through the study of self-mappings on different kinds of spaces. It started with Brouwer's FPT, which guarantees a FP for continuous mappings on compact, convex sets. Banach [1] then gave the contraction principle: in a CMS (X, δ) , if a self-mapping T satisfies

$$\delta(Tx, Ty) \leq k \delta(x, y), \quad 0 \leq k < 1,$$

then T has a FP which is unique, and iterating T from any starting point converges to it. These two results set the direction for later work in FPT: either generalize the space, or relax the condition on the mapping.

1.1.1 Generalizations of the Space

MS are the standard setting for FP results, where the distance function satisfies the usual triangular inequality

$$\delta(x, z) \leq \delta(x, y) + \delta(y, z).$$

Several generalized spaces have since relaxed this in different ways. Bakhtin [2] introduced the b -metric space, later developed by Czerwik [3], where a constant $s \geq 1$ scales the triangle inequality:

$$\delta(x, z) \leq s[\delta(x, y) + \delta(y, z)].$$

Replacing this constant with a control function gives extended b -metric spaces, studied by Kamran et al. [7], where the coefficient now depends on the points involved rather than being fixed.

Suprametric structures are a separate line of work. Berzig [5, 6, 13] introduced SMS and b -SMS, where the usual triangle inequality is replaced with a relation that includes an extra distance term. Panda et al. [14] added a control function

to get extended SMS, and in later work [15] combined this with b -scaling to get

$$\delta_s(x, y) \leq b[\delta_s(x, z) + \delta_s(z, y)] + \gamma(x, y)\delta_s(x, z)\delta_s(z, y).$$

This was studied further by Ahmad and Albargi [16].

Control functions have also been added directly to metric type spaces. Mlaiki et al. [4] introduced controlled MS, and Abdeljawad et al. [8] extended this to a double controlled version using two control functions. Double controlled metric-like spaces [9] extended this further. This body of work provides the background needed for the double controlled suprametric spaces used later in this thesis.

1.1.2 Generalizations of the Contraction Condition

Banach's condition relates the distance between two points directly to the distance between their images. Several later contractions add in distances involving each point and its own image. Kannan-type contractions [17], for example, use

$$\delta(Tx, Ty) \leq k[\delta(x, Tx) + \delta(y, Ty)], \quad 0 \leq k < \frac{1}{2},$$

while the Chatterjea condition [18] uses the cross-distances instead:

$$\delta(Tx, Ty) \leq k[\delta(x, Ty) + \delta(y, Tx)], \quad 0 \leq k < \frac{1}{2}.$$

Reich combined several distances in one condition, using weighted contributions from $\delta(x, y)$, $\delta(x, Tx)$, and $\delta(y, Ty)$, with $a + b + c < 1$.

'Ciri'c [19] went further and took the maximum over several distances:

$$\delta(Tx, Ty) \leq k \max\{\delta(x, y), \delta(x, Tx), \delta(y, Ty), \delta(x, Ty), \delta(y, Tx)\}, \quad 0 \leq k < 1.$$

These generalized conditions make it possible to prove FP results even when the direct Banach contraction does not apply.

The rest of the thesis has four more chapters.

Chapter 2 covers the background needed later, going from MS and CMS through SMS, DCMS, and DCSMS, with a formal definition and worked example for each.

Chapter 3 proves FP theorems for single-valued mappings that are ξ -admissible with respect to η and satisfy a generalized rational contraction, in extended $\bar{b}$ -suprametric spaces. The extended suprametric, $\bar{b}$ -suprametric, and suprametric cases follow as corollaries, and the results are applied to a nonlinear Volterra-Fredholm integral equation.

Chapter 4 carries the same approach into DCMS and introduces DCSMS. FP theorems are proved here as well, with a ratio-limit condition for convergence, backed up by concrete examples.

Chapter 5 closes the thesis by summarizing the contributions and pointing to future work.

The bibliography at the end lists all the references used.

Chapter 2

Preliminaries

This chapter puts together the background needed for the rest of the thesis. It starts with Fréchet's MS and completeness, then covers the BCP, which is one of the central results in FPT. After that we go through the generalizations in the order they were introduced: SMS, $\bar{b}$ -suprametric spaces ($\bar{b}$ -SMS), extended suprametric spaces (extended SMS), extended $\bar{b}$ -suprametric spaces (extended $\bar{b}$ -SMS), and finally DCMS.

Each of these spaces relaxes the triangle inequality in a different way, and that is what makes them useful for FP results that do not fit the classical setting. For each one we give a short note on where it came from, its formal definition, and a worked example.

Definition 2.1.

Take a set X that is non-empty, together with a distance function $\delta : X \times X \rightarrow [0, \infty)$. The pair (X, δ) is called a MS if δ obeys, for every $\nu, \zeta, \rho \in X$, the four requirements below:

- i. Non-negativity: $\delta(\nu, \zeta) \geq 0$.
- ii. Identity of indiscernible: $\delta(\nu, \zeta) = 0 \iff \nu = \zeta$.
- iii. Symmetry: $\delta(\nu, \zeta) = \delta(\zeta, \nu)$.
- iv. Triangle inequality: $\delta(\nu, \zeta) \leq \delta(\nu, \rho) + \delta(\rho, \zeta)$ [20].

Definition 2.2.

Suppose (X, δ) is a MS. A subset $G \subseteq X$ is called open when, for every $x \in G$, there is some $r > 0$ such that

$$B(x, r) \subseteq G,$$

where

$$B(x, r) = \{y \in X : \delta(x, y) < r\}$$

denotes the open ball centered at x with radius r [20].

Example 2.1. Take $X = C[a, b]$, the set of continuous real valued functions on $[a, b]$, with $\delta(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|$. We check the four axioms one by one.

- i. $\delta(f, g) \geq 0$ because it is a supremum of non-negative values.
- ii. $\delta(f, g) = 0$ means $|f(t) - g(t)| = 0$ for every t , i.e. $f = g$.
- iii. $|f(t) - g(t)| = |g(t) - f(t)|$ for every t , so $\delta(f, g) = \delta(g, f)$.
- iv. For any $h \in X$, $|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)| \leq \delta(f, h) + \delta(h, g)$ for all t , which gives $\delta(f, g) \leq \delta(f, h) + \delta(h, g)$.

So all four axioms hold, and $(C[a, b], \delta)$ is a MS.

Definition 2.3. Given a MS (X, δ) , a sequence $\{\nu_n\} \subset X$ is called Cauchy [20] if, for every $\varepsilon > 0$, there is an index $N \in \mathbb{N}$ such that

$$\delta(\nu_m, \nu_n) < \varepsilon \quad \text{for all } m, n \geq N.$$

Example 2.2. Take $(\mathbb{R}, \delta)$ with $\delta(\nu, \zeta) = |\nu - \zeta|$, and the sequence $\nu_n = \frac{1}{n}$. For any $\varepsilon > 0$, pick $N > \frac{1}{\varepsilon}$. Then for all $m, n \geq N$,

$$\left| \frac{1}{m} - \frac{1}{n} \right| < \varepsilon,$$

so $\{1/n\}$ is Cauchy.

Definition 2.4. A MS (X, δ) is complete if every Cauchy sequence drawn from X approaches a point that itself belongs to X [20].

Example 2.3. Take the non-empty set $X = [0, 1]$ with $\delta(\nu, \zeta) = |\nu - \zeta|$; this is complete. To see why, note the sequence $\nu_n = \frac{1}{n}$ stays inside $[0, 1]$ for every $n \geq 1$, is Cauchy by Example 2.2, and converges to $0 \in [0, 1]$. Any Cauchy sequence in $[0, 1]$ behaves the same way, which is exactly what completeness requires.

Definition 2.5. Consider a set X with a self map $T : X \rightarrow X$. A point $\nu \in X$ is called a FP of T if $T(\nu) = \nu$. Depending on how many such points exist, three cases can arise.

A FP is called **unique** if there is exactly one $\nu \in X$ satisfying $T(\nu) = \nu$. A mapping has **more than one fixed point** if there exist finitely many distinct points $\nu_1, \nu_2, \dots, \nu_n$, each satisfying $T(\nu_i) = \nu_i$. A mapping has **infinitely many fixed points** if the set of all points satisfying $T(\nu) = \nu$ is an infinite set.

Example 2.4. The three examples below show what each of these cases looks like in practice.

1. Let $T(\nu) = (\nu + 2)/3$. Setting $T(\nu) = \nu$ gives $\nu = (\nu + 2)/3$, so $3\nu = \nu + 2$, $2\nu = 2$, and $\nu = 1$. So $\nu = 1$ is the only FP of T .

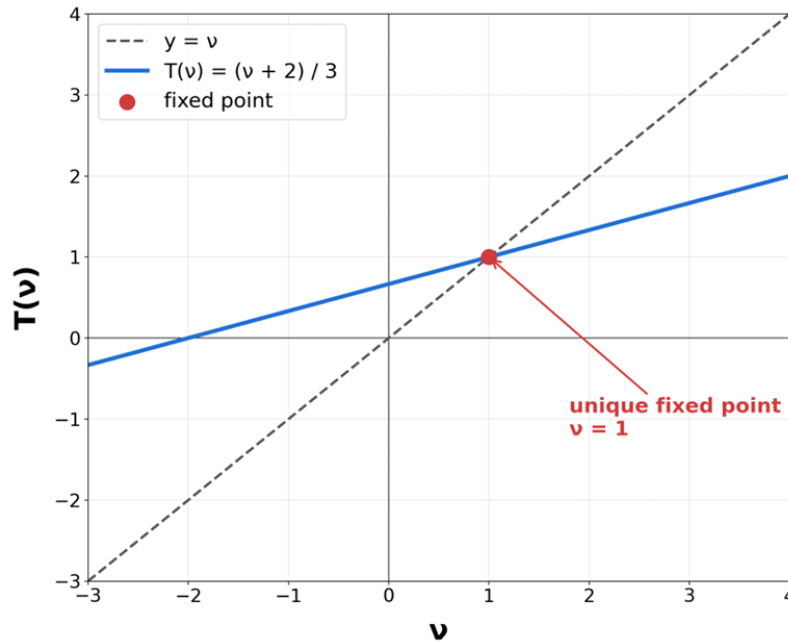
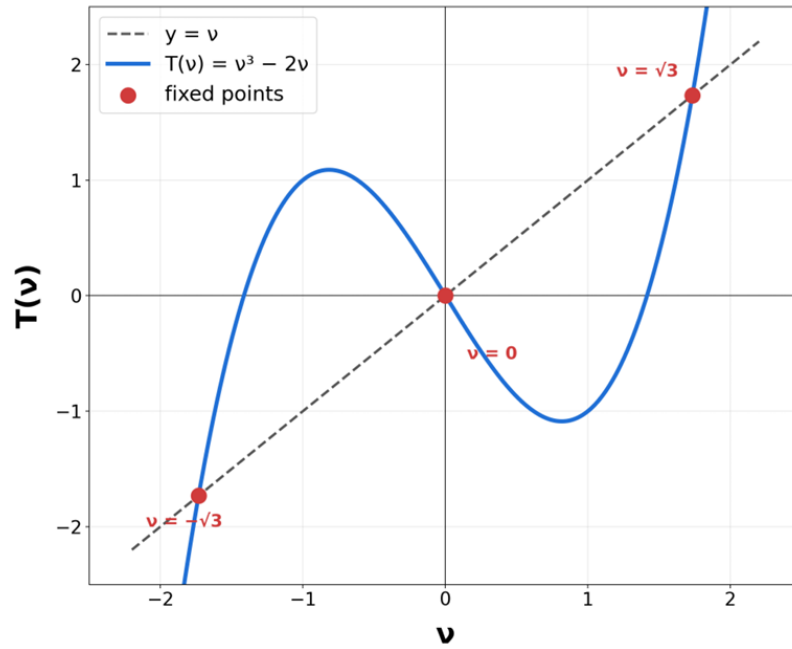
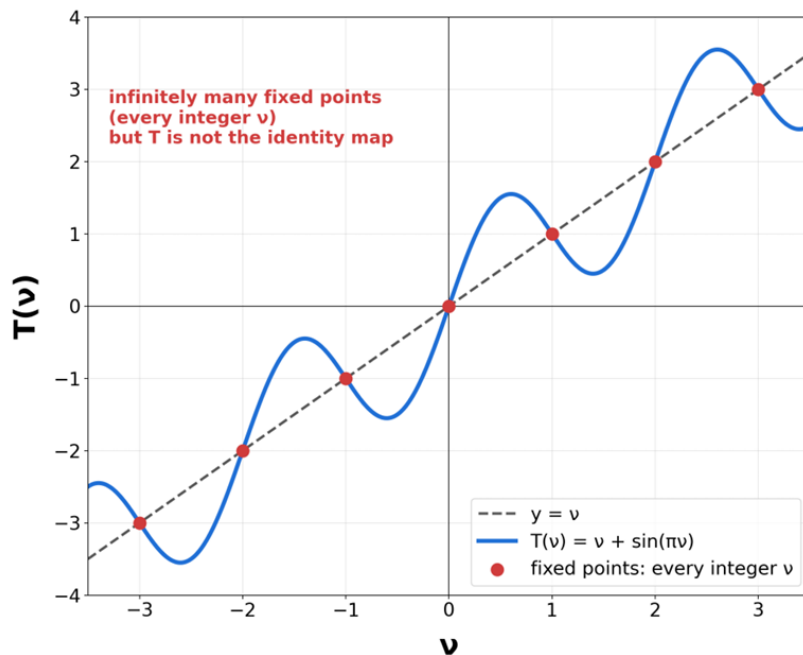


FIGURE 2.1: Unique fixed point: $T(\nu) = (\nu + 2)/3$

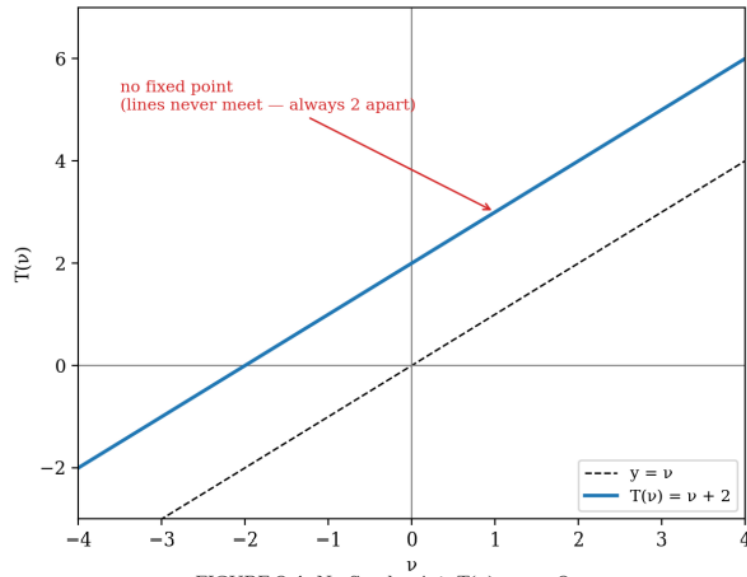
2. Let $T(\nu) = \nu^3 - 2\nu$. Setting $T(\nu) = \nu$ gives $\nu^3 - 3\nu = 0$, i.e. $\nu(\nu^2 - 3) = 0$, so $\nu = 0, \sqrt{3}, -\sqrt{3}$. So T has three distinct fixed points here.

FIGURE 2.2: Three fixed points: $T(v) = v^3 - 2v$

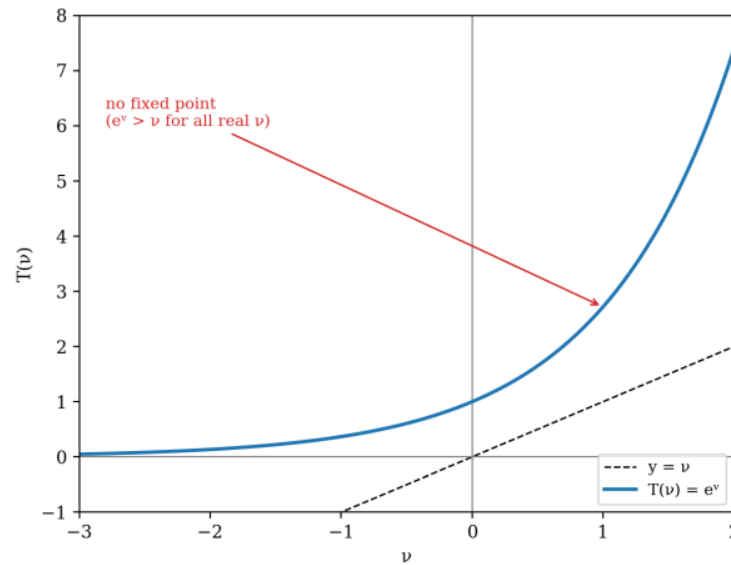
3. Let $T(v) = v + \sin(\pi v)$. The equation $T(v) = v$ forces $\sin(\pi v) = 0$, which holds for every integer v . So T has infinitely many fixed points.

FIGURE 2.3: Infinitely many fixed points: Identity map $T(v) = v$

4. Let $T(v) = v + 2$. Setting $T(v) = v$ gives $v + 2 = v$, i.e. $2 = 0$, which is never true. So T has no fixed point.

FIGURE 2.4: No fixed point: $T(v) = v + 2$ FIGURE 2.4: Infinitely many fixed points: Identity map $T(v) = v$

5. Let $T(v) = e^v$, defined for all real v . Setting $T(v) = v$ gives $e^v = v$. But $e^v > v$ holds for every real v , so the exponential curve always stays above the line $y = v$ and never intersects it. Hence T has no fixed point.

FIGURE 2.5: No fixed point: $T(v) = e^v$ FIGURE 2.5: Infinitely many fixed points: Identity map $T(v) = v$

Theorem 2.1 (BCP).

Suppose (X, δ) is a complete MS and T is a self map for which a constant $\lambda \in [0, 1)$

exists satisfying

$$\delta(T\nu, T\zeta) \leq \lambda \delta(\nu, \zeta) \quad \text{for all } \nu, \zeta \in X.$$

Such a T is called a contraction mapping, then T has exactly one FP $\nu^* \in X$, and for any starting point $\nu_0 \in X$ the Picard sequence $\nu_n = T^n \nu_0$ converges to ν^* [1].

Definition 2.6.

Take a MS (X, δ) , a self map $T : X \rightarrow X$, and a function $\xi : X \times X \rightarrow [0, \infty)$. We say T is ξ -admissible if, for every $\nu, \zeta \in X$,

$$\xi(\nu, \zeta) \geq 1 \implies \xi(T\nu, T\zeta) \geq 1,$$

that is, once $\xi(\nu, \zeta) \geq 1$ holds it continues to hold after T has been applied [11].

Example 2.5.

Let $X = [0, \infty)$ with the usual metric $\delta(\nu, \zeta) = |\nu - \zeta|$, and define $T : X \rightarrow X$ by

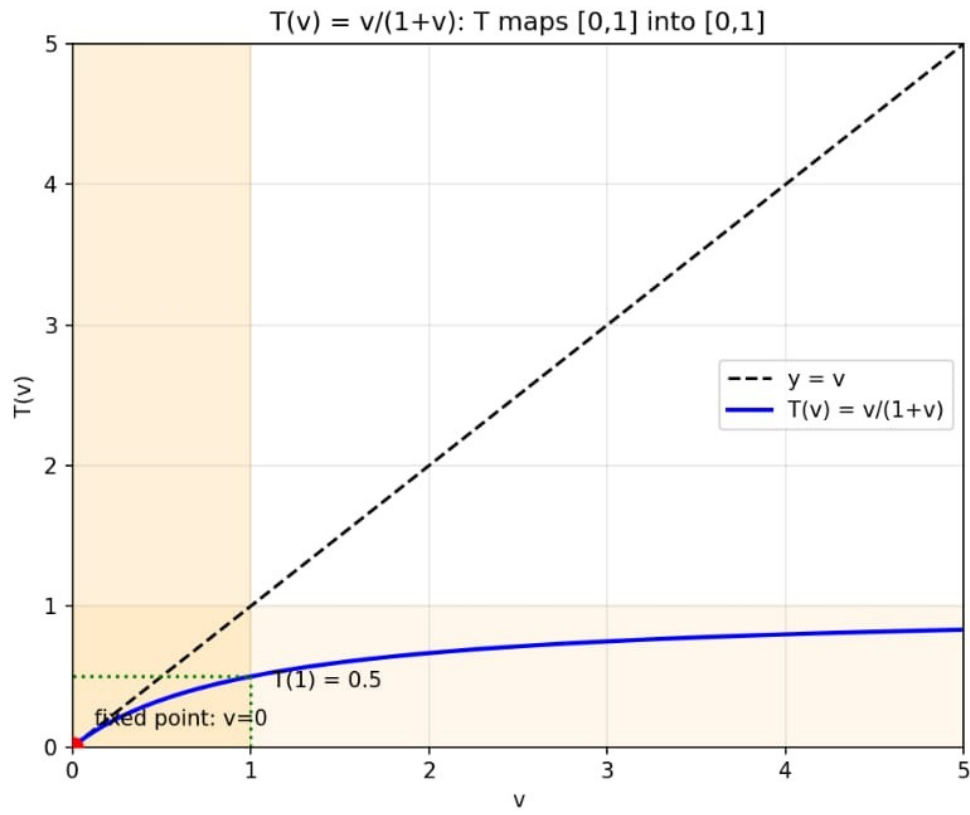
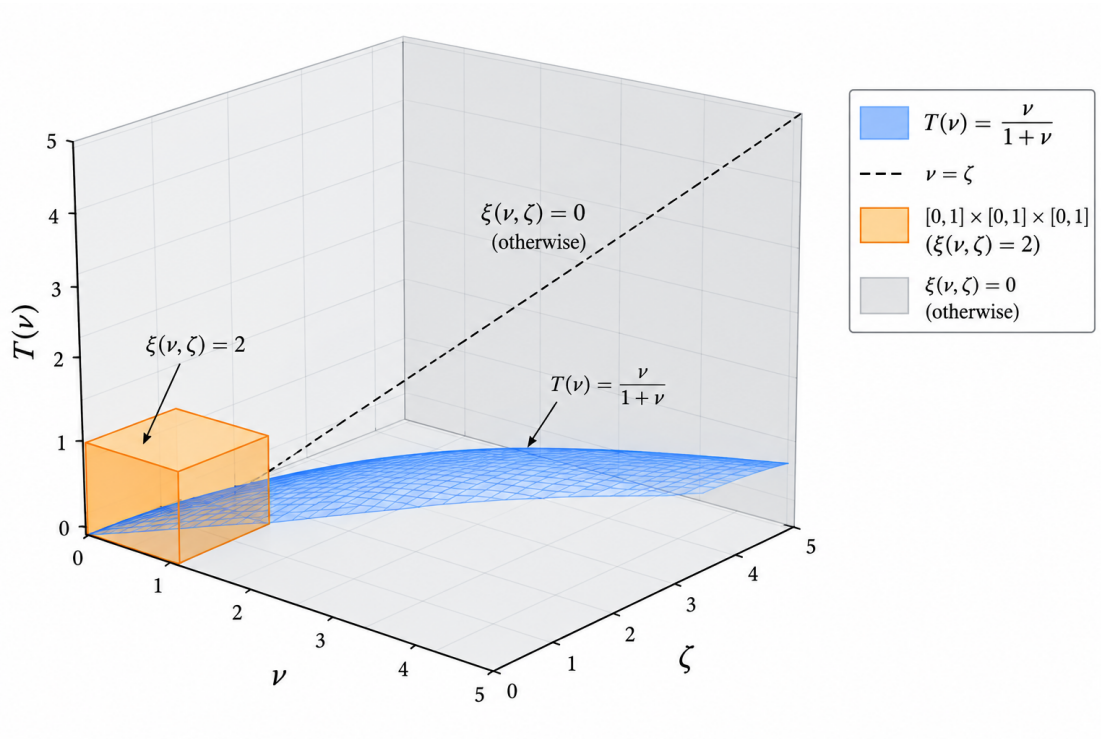
$$T(\nu) = \frac{\nu}{1 + \nu}, \quad \nu \in X.$$

Define $\xi : X \times X \rightarrow [0, \infty)$ by

$$\xi(\nu, \zeta) = \begin{cases} 2, & \text{if } \nu, \zeta \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Suppose $\xi(\nu, \zeta) \geq 1$ for some $\nu, \zeta \in X$; then $\nu, \zeta \in [0, 1]$.

Since T maps $[0, 1]$ into itself, $T\nu, T\zeta \in [0, 1]$ too, so $\xi(T\nu, T\zeta) = 2 \geq 1$. Hence T is ξ -admissible.

FIGURE 2.6: ξ -admissibleFIGURE 2.7: 3D view of ξ -admissibility

2.1 Generalizations of Metric Spaces

The BCP above was stated for an ordinary MS. But as noted in Chapter 1, the triangle inequality can be relaxed to get more general kinds of metric spaces, and this section goes through them in the order they appeared historically. We start with the SMS introduced by Berzig, move to its $\bar{b}$ -scaled and control function variants, and end with the DCMS, which is used directly in Chapter 4.

2.1.1 Suprametric Space

Berzig [5] introduced the SMS in 2022 as a common setting for nonlinear metric equations and error perturbation problems. The one thing that separates a SMS from an ordinary MS is an extra term added to the right side of the triangle inequality: a multiplicative cross term with a fixed weight $\sigma \geq 0$, built from the product of the two intermediate distances instead of their sum.

Definition 2.7.

Suppose X is a non-empty set and $\delta_s : X \times X \rightarrow [0, \infty)$. We call the pair (X, δ_s) a SMS if some constant $\sigma \geq 0$ exists such that, for all $\nu, \zeta, \rho \in X$:

- i. $\delta_s(\nu, \zeta) = 0 \iff \nu = \zeta$;
- ii. $\delta_s(\nu, \zeta) = \delta_s(\zeta, \nu)$;
- iii. $\delta_s(\nu, \zeta) \leq \delta_s(\nu, \rho) + \delta_s(\rho, \zeta) + \sigma \delta_s(\nu, \rho) \delta_s(\rho, \zeta)$ [5].

Example 2.6.

Take any MS (X, δ) and define

$$\delta_s(\nu, \zeta) = \delta(\nu, \zeta)(\delta(\nu, \zeta) + 2), \quad \nu, \zeta \in X.$$

We claim (X, δ_s) is a SMS. Axioms (i) and (ii) carry over directly from δ . For (iii), write $\delta_s(\nu, \zeta) = \delta^2(\nu, \zeta) + 2\delta(\nu, \zeta)$. Since $\delta(\nu, \zeta) \leq \delta(\nu, \rho) + \delta(\rho, \zeta)$, squaring both

sides and rearranging gives

$$\begin{aligned}
\delta_s(\nu, \zeta) &= \delta^2(\nu, \zeta) + 2\delta(\nu, \zeta) \\
&\leq (\delta(\nu, \rho) + \delta(\rho, \zeta))^2 + 2(\delta(\nu, \rho) + \delta(\rho, \zeta)) \\
&= \delta^2(\nu, \rho) + \delta^2(\rho, \zeta) + 2\delta(\nu, \rho)\delta(\rho, \zeta) + 2\delta(\nu, \rho) + 2\delta(\rho, \zeta) \\
&= (\delta^2(\nu, \rho) + 2\delta(\nu, \rho)) + (\delta^2(\rho, \zeta) + 2\delta(\rho, \zeta)) + 2\delta(\nu, \rho)\delta(\rho, \zeta) \\
&= \delta_s(\nu, \rho) + \delta_s(\rho, \zeta) + 2\delta_s(\nu, \rho)\delta_s(\rho, \zeta).
\end{aligned}$$

So Definition 2.7 holds with $\sigma = 2$.

2.1.2 $\bar{b}$ -Suprametric Space

Berzig [13] later added a second relaxation on top of the suprametric cross term, by scaling the two additive parts of the triangle inequality with a fixed constant $\bar{b} \geq 1$, the same idea used when passing from a MS to a b -metric space. Putting this scalar together with the suprametric term gives the $\bar{b}$ -SMS.

Definition 2.8.

Fix a non-empty set X , a constant $\bar{b} \geq 1$, and a fixed number $\theta \geq 0$. A map $\delta_{\bar{b}s} : X \times X \rightarrow [0, \infty)$ is called a $\bar{b}$ -suprametric if, for every $\nu, \zeta, \rho \in X$:

- i. $\delta_{\bar{b}s}(\nu, \zeta) \geq 0$, with equality to 0 precisely when $\nu = \zeta$;
- ii. $\delta_{\bar{b}s}(\nu, \zeta) = \delta_{\bar{b}s}(\zeta, \nu)$;
- iii. $\delta_{\bar{b}s}(\nu, \zeta) \leq \bar{b}(\delta_{\bar{b}s}(\nu, \rho) + \delta_{\bar{b}s}(\rho, \zeta)) + \theta \delta_{\bar{b}s}(\nu, \rho) \delta_{\bar{b}s}(\rho, \zeta)$.

Then $(X, \delta_{\bar{b}s})$ is called a $\bar{b}$ -SMS [13].

2.1.3 Extended Suprametric Space

Instead of using a fixed number $\bar{b}$ to scale the triangle inequality, Panda et al. [14] proposed in 2023 letting the scaling factor vary with the pair of points, the

same idea behind the extended b -metric spaces mentioned in Chapter 1. This is what gives the extended SMS, where the fixed weight σ in Definition 2.7 becomes a proper control function $\theta(\nu, \zeta) \geq 1$.

Definition 2.9.

Fix a non-empty set X and a control function $\theta : X \times X \rightarrow [1, \infty)$. A map $\delta_{Es} : X \times X \rightarrow [0, \infty)$ is called an extended suprametric provided the following three properties hold for all $\nu, \zeta, \rho \in X$:

- i. $\delta_{Es}(\nu, \zeta) \geq 0$, with $\delta_{Es}(\nu, \zeta) = 0$ exactly when $\nu = \zeta$;
- ii. $\delta_{Es}(\nu, \zeta) = \delta_{Es}(\zeta, \nu)$;
- iii. $\delta_{Es}(\nu, \zeta) \leq \delta_{Es}(\nu, \rho) + \delta_{Es}(\rho, \zeta) + \theta(\nu, \zeta) \delta_{Es}(\nu, \rho) \delta_{Es}(\rho, \zeta)$.

In that case (X, δ_{Es}) is referred to as an extended SMS [14].

Example 2.7.

Take $X = \ell^\infty(\mathbb{R}) = \{\{\nu_n\}_{n \geq 1} : \nu_n \text{ is bounded real sequence}\}$, with

$$\delta(\nu, \zeta) = \sup_{n \geq 1} |\nu_n - \zeta_n|^p, \quad p \in (0, 1),$$

and control function

$$\theta(\nu, \zeta) = 1 + \sup_{n \geq 1} |\nu_n| + \sup_{n \geq 1} |\zeta_n|.$$

It is straightforward to check that (X, δ_{Es}) built this way is an extended SMS.

2.1.4 Extended $\bar{b}$ -Suprametric Space

Combining a fixed scalar $\bar{b}$ with a variable control function θ gives the most general space in this chain: the extended $\bar{b}$ -SMS, put forward by Panda et al. [15] in 2024. It brings together every ingredient seen so far, the cross term from the SMS, the scalar $\bar{b}$ from the $\bar{b}$ -SMS, and the control function θ from the extended SMS, into one structure. Choosing $\bar{b}$ and θ appropriately recovers each of the earlier three spaces as a special case, which is used repeatedly in Chapter 3.

Remark 2.2. $(X, \delta_{E\bar{b}s})$ is an extended $\bar{b}$ -SMS if $X \neq \emptyset$, $\bar{b} \geq 1$, $\theta : X \times X \rightarrow [1, \infty)$, and $\delta_{E\bar{b}s} : X \times X \rightarrow [0, \infty)$ satisfies, for all $\nu, \zeta, \rho \in X$, non-negativity with $\delta_{E\bar{b}s}(\nu, \zeta) = 0 \iff \nu = \zeta$, symmetry, and

$$\delta_{E\bar{b}s}(\nu, \zeta) \leq \bar{b}(\delta_{E\bar{b}s}(\nu, \rho) + \delta_{E\bar{b}s}(\rho, \zeta)) + \theta(\nu, \zeta) \delta_{E\bar{b}s}(\nu, \rho) \delta_{E\bar{b}s}(\rho, \zeta).$$

This gives the following special cases:

1. If θ is constant, we get back the $\bar{b}$ -SMS of Definition 2.8.
2. If $\bar{b} = 1$, we get back the extended SMS of Definition 2.9.
3. If both hold at once, we get back the plain SMS of Definition 2.7.

This structure, and the FP theorem built on it, is worked out in full in Chapter 3, following [15] and [16].

2.1.5 Double Controlled Metric Space

DCMS extends the earlier extended $\bar{b}$ -metric spaces from Chapter 1. Abdeljawad et al. introduced it in their 2018 paper [8]. The main idea is that the two terms in the triangle inequality are each controlled by their own function, rather than a single constant, so the amount of relaxation can depend on which points are being compared.

Note that, unlike the SMS discussed above, DCMS has no multiplicative cross term. It simply controls the two triangle inequality terms independently. The formal definition and worked examples for DCMS, along with DCSMS (which combines this idea with Berzig's suprametric term), come in Chapter 4.

Chapter 3

A Study of Fixed Point Theorems in Extended $\bar{b}$ Suprametric Spaces

This chapter is built around the work of Ahmad et al. [16]. We use their FP theorem for extended $\bar{b}$ SMS and add a few more examples to make the results easier to follow. We first go over the basic notions needed for the main result, MS, $\bar{b}$ SMS, extended SMS, and extended $\bar{b}$ SMS, and then state a FP result (Theorem 3.4) for single valued self mappings that are ξ admissible with respect to η and satisfy a rational type contraction built from the functional $\mathcal{M}(\nu, \zeta)$.

Choosing $\bar{b}$ and the control function θ in the right way, the main theorem also gives us the same kind of result for extended SMS, $\bar{b}$ SMS, and plain SMS.

3.1 Preliminaries

Definition 3.1. [15]

Fix a nonempty set $\mathcal{X}$, a constant $\bar{b} \geq 1$, and a prescribed function $\theta : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$. A mapping $\delta_{E\bar{b}s} : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is called an extended $\bar{b}$ suprametric if the following conditions hold for all $\nu, \zeta, \rho \in \mathcal{X}$:

$$\delta_{E\bar{b}s}(1) : \quad \delta_{E\bar{b}s}(\nu, \zeta) \geq 0, \text{ and } \delta_{E\bar{b}s}(\nu, \zeta) = 0 \text{ if and only if } \nu = \zeta.$$

$$\delta_{E\bar{b}s}(2) : \quad \delta_{E\bar{b}s}(\nu, \zeta) = \delta_{E\bar{b}s}(\zeta, \nu).$$

$$\delta_{E\bar{b}s}(3) : \quad \delta_{E\bar{b}s}(\nu, \zeta) \leq \bar{b}(\delta_{E\bar{b}s}(\nu, \rho) + \delta_{E\bar{b}s}(\rho, \zeta)) + \theta(\nu, \zeta) \delta_{E\bar{b}s}(\nu, \rho) \delta_{E\bar{b}s}(\rho, \zeta).$$

The pair $(\mathcal{X}, \delta_{E\bar{b}s})$ is then called an extended $\bar{b}$ SMS.

Example 3.1.

Consider $\mathcal{X} = \ell^p(\mathbb{R})$ with $p \in (0, 1)$, where

$$\ell^p(\mathbb{R}) = \left\{ \{\nu_n\} \subset \mathbb{R} \mid \sum_{n=1}^{\infty} |\nu_n|^p < \infty \right\},$$

and define $\delta_{E\bar{b}s} : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ by

$$\delta_{E\bar{b}s}(\{\nu_n\}, \{\zeta_n\}) = \left(\sum_{n=1}^{\infty} |\nu_n - \zeta_n|^p \right)^{1/p} \quad \text{for all } \{\nu_n\}, \{\zeta_n\} \in \mathcal{X}.$$

Then $(\mathcal{X}, \delta_{E\bar{b}s})$ constitutes an extended $\bar{b}$ SMS with $\bar{b} = 2^{1/p}$ [15].

Example 3.2.

Let $\mathcal{X} = C[0, 1]$ denote the space of continuous real valued functions on $[0, 1]$, and define

$$\delta_{E\bar{b}s}(\phi, \psi) = \left(\int_0^1 |\phi(t) - \psi(t)|^p dt \right)^{1/p}, \quad p \in (0, 1).$$

Then $(\mathcal{X}, \delta_{E\bar{b}s})$ is an extended $\bar{b}$ SMS with $\bar{b} = 2^{1/p}$.

Remark 3.1.

Each of the spaces discussed earlier turns out to be a special case of Definition 3.1, once the parameters are restricted suitably:

- (i) If $\theta(\nu, \zeta) \geq 1$ is held constant, the space reduces to a $\bar{b}$ SMS;
- (ii) If $\bar{b} = 1$, we get back an extended SMS;
- (iii) If both hold at once, we get back an ordinary SMS.

Panda et al. [15] proved the following BCP type result for complete extended $\bar{b}$ SMS.

Theorem 3.2.

Suppose $(\mathcal{X}, \delta_{E\bar{b}s})$ is a complete extended $\bar{b}$ SMS with $\bar{b} \geq 1$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ is a self map satisfying, for some $\lambda \in [0, 1)$,

$$\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{\lambda}{\bar{b}} \delta_{E\bar{b}s}(\nu, \zeta)$$

for all $\nu, \zeta \in \mathcal{X}$. Then $\mathcal{T}$ possesses a unique FP, and for any initial $\nu_0 \in \mathcal{X}$, the Picard sequence $\nu_n = \mathcal{T}\nu_{n-1}$, $n \in \mathbb{N}$, converges to it [15].

We also recall below how convergence, Cauchy sequences, and completeness are defined for this class of spaces [15].

Lemma 3.3.

Suppose $(\mathcal{X}, \delta_{E\bar{b}s})$ is an extended $\bar{b}$ SMS and $\{\nu_n\}_{n \in \mathbb{N}} \subset \mathcal{X}$. Then:

- (i) $\{\nu_n\}$ converges to ν if, for every $\varepsilon > 0$, all but finitely many terms lie in the open ball $B(\nu, \varepsilon)$; this is denoted $\lim_{n \rightarrow \infty} \delta_{E\bar{b}s}(\nu_n, \nu) = 0$.
- (ii) $\{\nu_n\}$ is Cauchy if for every $\varepsilon > 0$ there exists an index k such that $\delta_{E\bar{b}s}(\nu_n, \nu_m) < \varepsilon$ whenever $n, m \geq k$.
- (iii) $(\mathcal{X}, \delta_{E\bar{b}s})$ is complete if every Cauchy sequence in $\mathcal{X}$ converges to a limit within $\mathcal{X}$ [15].

3.2 Main Results

We now state the main theorem of this chapter. It concerns a single valued self mapping satisfying a rational type contraction on a complete, continuous extended $\bar{b}$ SMS. The theorem and its proof below follow Ahmad et al. [16]. The corollaries afterward show that several known FP theorems in SMS fall out of this one as special cases.

Theorem 3.4.

Suppose $(\mathcal{X}, \delta_{E\bar{b}s})$ is a complete, continuous extended $\bar{b}$ SMS with $\bar{b} \geq 1$, and

$\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ is a mapping. Assume there exist functions $\xi, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ and a constant $\lambda \in [0, 1)$ such that whenever

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta),$$

one has

$$\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta), \quad (3.1)$$

where

$$\mathcal{M}(\nu, \zeta) = \max \left(\delta_{E\bar{b}s}(\nu, \zeta), \right. \quad (3.2)$$

$$\min \left(\frac{\delta_{E\bar{b}s}(\nu, \mathcal{T}\nu) \delta_{E\bar{b}s}(\zeta, \mathcal{T}\zeta)}{1 + \delta_{E\bar{b}s}(\nu, \zeta)}, \right. \quad (3.3)$$

$$\left. \frac{\delta_{E\bar{b}s}(\nu, \mathcal{T}\zeta) \delta_{E\bar{b}s}(\zeta, \mathcal{T}\nu)}{1 + \delta_{E\bar{b}s}(\nu, \zeta)} \right),$$

for all $\nu, \zeta \in \mathcal{X}$. Additionally, suppose:

- (i) $\mathcal{T}$ is ξ admissible with respect to η ;
- (ii) there exists $\nu_0 \in \mathcal{X}$ with $\xi(\nu_0, \mathcal{T}\nu_0) \geq \eta(\nu_0, \mathcal{T}\nu_0)$;
- (iii) either $\mathcal{T}$ is continuous, or for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu^*$ and $\xi(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$, we have $\xi(\nu^*, \mathcal{T}\nu^*) \geq \eta(\nu^*, \mathcal{T}\nu^*)$.

Then $\mathcal{T}$ has a unique FP.

Proof.

Select $\nu_0 \in \mathcal{X}$ satisfying $\xi(\nu_0, \nu_1) = \xi(\nu_0, \mathcal{T}\nu_0) \geq \eta(\nu_0, \mathcal{T}\nu_0) = \eta(\nu_0, \nu_1)$, and generate the sequence which is iterative $\{\nu_n\} \subset \mathcal{X}$ by $\nu_{n+1} = \mathcal{T}^n \nu_0 = \mathcal{T}\nu_n$ for each $n \in \mathbb{N}$. Since $\mathcal{T}$ is ξ admissible relative to η , we obtain

$$\xi(\nu_1, \nu_2) = \xi(\mathcal{T}\nu_0, \mathcal{T}\nu_1) \geq \eta(\mathcal{T}\nu_0, \mathcal{T}\nu_1) = \eta(\nu_1, \nu_2).$$

Repeating this argument inductively yields

$$\xi(\nu_{n-1}, \nu_n) = \xi(\mathcal{T}\nu_{n-2}, \mathcal{T}\nu_{n-1}) \geq \eta(\mathcal{T}\nu_{n-2}, \mathcal{T}\nu_{n-1}) = \eta(\nu_{n-1}, \nu_n),$$

and

$$\xi(\nu_n, \nu_{n+1}) = \xi(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n) \geq \eta(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n) = \eta(\nu_n, \nu_{n+1}), \quad \forall n \in \mathbb{N}.$$

These may be rewritten equivalently as

$$\xi(\nu_{n-1}, \nu_n) = \xi(\nu_{n-1}, \mathcal{T}\nu_{n-1}) \geq \eta(\nu_{n-1}, \mathcal{T}\nu_{n-1}) = \eta(\nu_{n-1}, \nu_n), \quad (3.4)$$

and

$$\xi(\nu_n, \nu_{n+1}) = \xi(\nu_n, \mathcal{T}\nu_n) \geq \eta(\nu_n, \mathcal{T}\nu_n) = \eta(\nu_n, \nu_{n+1}), \quad \forall n \in \mathbb{N}. \quad (3.5)$$

Multiplying (3.4) and (3.5) gives

$$\xi(\nu_{n-1}, \mathcal{T}\nu_{n-1}) \xi(\nu_n, \mathcal{T}\nu_n) \geq \eta(\nu_{n-1}, \mathcal{T}\nu_{n-1}) \eta(\nu_n, \mathcal{T}\nu_n), \quad \forall n \in \mathbb{N}. \quad (3.6)$$

If for some $n_0 \in \mathbb{N}$ we have $\nu_{n_0+1} = \nu_{n_0}$, then ν_{n_0} is already a FP of $\mathcal{T}$, and the proof is complete.

We therefore assume $\nu_{n+1} \neq \nu_n$, i.e., $\delta_{E\bar{b}s}(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n) > 0$, for all $n \in \mathbb{N}$. Applying (3.1) and (3.2) yields

$$\delta_{E\bar{b}s}(\nu_n, \nu_{n+1}) = \delta_{E\bar{b}s}(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu_{n-1}, \nu_n), \quad (3.7)$$

where a direct computation gives

$$\begin{aligned} \mathcal{M}(\nu_{n-1}, \nu_n) = & \max \left(\delta_{E\bar{b}s}(\nu_{n-1}, \nu_n), \right. \\ & \left. \min \left(\frac{\delta_{E\bar{b}s}(\nu_{n-1}, \mathcal{T}\nu_{n-1}) \delta_{E\bar{b}s}(\nu_n, \mathcal{T}\nu_n)}{1 + \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n)}, \frac{\delta_{E\bar{b}s}(\nu_{n-1}, \mathcal{T}\nu_n) \delta_{E\bar{b}s}(\nu_n, \mathcal{T}\nu_{n-1})}{1 + \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n)} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \max \left(\delta_{E\bar{b}s}(\nu_{n-1}, \nu_n), \right. \\
 &\quad \min \left(\frac{\delta_{E\bar{b}s}(\nu_{n-1}, \nu_n) \delta_{E\bar{b}s}(\nu_n, \nu_{n+1})}{1 + \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n)}, \right. \\
 &\quad \left. \left. \frac{\delta_{E\bar{b}s}(\nu_{n-1}, \nu_{n+1}) \delta_{E\bar{b}s}(\nu_n, \nu_n)}{1 + \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n)} \right) \right) \\
 &= \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n), \quad \forall n \in \mathbb{N}.
 \end{aligned}$$

Consequently, from (3.7),

$$\delta_{E\bar{b}s}(\nu_n, \nu_{n+1}) \leq \frac{\lambda}{\bar{b}} \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n) < \delta_{E\bar{b}s}(\nu_{n-1}, \nu_n).$$

Thus the sequence $\{\delta_{E\bar{b}s}(\nu_n, \nu_{n+1})\}$ is strictly decreasing, and for any fixed index κ ,

$$\delta_{E\bar{b}s}(\nu_n, \nu_{n+1}) \leq \left(\frac{\lambda}{\bar{b}} \right)^{n-\kappa} \delta_{E\bar{b}s}(\nu_\kappa, \nu_{\kappa+1}), \quad \forall n > \kappa. \quad (3.8)$$

In particular, $\lim_{n \rightarrow \infty} \delta_{E\bar{b}s}(\nu_n, \nu_{n+1}) = 0$, existence of κ in $\mathbb{N}$ depends on every $\varepsilon > 0$ such that

$$\delta_{E\bar{b}s}(\nu_n, \nu_{n+1}) < \varepsilon \quad (3.9)$$

for all $n \geq \kappa$. It remains to establish that $\{\nu_n\}$ is Cauchy. Using (3.8), (3.9), and the triangle inequality for the extended $\bar{b}$ suprametric,

$$\begin{aligned}
 \delta_{E\bar{b}s}(\nu_p, \nu_q) &\leq \bar{b} [\delta_{E\bar{b}s}(\nu_p, \nu_{p+1}) + \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q)] \\
 &\quad + \theta(\nu_p, \nu_q) \delta_{E\bar{b}s}(\nu_p, \nu_{p+1}) \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q) \\
 &\leq \bar{b} \left[\left(\frac{\lambda}{\bar{b}} \right)^{p-k} \delta_{E\bar{b}s}(\nu_k, \nu_{k+1}) + \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q) \right] \\
 &\quad + \theta(\nu_p, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^{p-k} \delta_{E\bar{b}s}(\nu_k, \nu_{k+1}) \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q) \\
 &\leq \bar{b} \left[\left(\frac{\lambda}{\bar{b}} \right)^{p-k} \varepsilon + \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q) \right] \\
 &\quad + \theta(\nu_p, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^{p-k} \varepsilon \cdot \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q)
 \end{aligned}$$

$$\begin{aligned} &\leq \bar{b} \left(\frac{\lambda}{\bar{b}} \right)^{p-k} \varepsilon \\ &\quad + \left[\bar{b} + \theta(\nu_p, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^{p-k} \varepsilon \right] \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q). \end{aligned} \quad (3.10)$$

Similarly,

$$\begin{aligned} \delta_{E\bar{b}s}(\nu_{p+1}, \nu_q) &\leq \bar{b} [\delta_{E\bar{b}s}(\nu_{p+1}, \nu_{p+2}) + \delta_{E\bar{b}s}(\nu_{p+2}, \nu_q)] \\ &\quad + \theta(\nu_{p+1}, \nu_q) \delta_{E\bar{b}s}(\nu_{p+1}, \nu_{p+2}) \delta_{E\bar{b}s}(\nu_{p+2}, \nu_q) \\ &\leq \bar{b} \left[\left(\frac{\lambda}{\bar{b}} \right)^{p-k+1} \varepsilon + \delta_{E\bar{b}s}(\nu_{p+2}, \nu_q) \right] + \theta(\nu_{p+1}, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^{p-k+1} \varepsilon \delta_{E\bar{b}s}(\nu_{p+2}, \nu_q) \\ &\leq \bar{b} \left(\frac{\lambda}{\bar{b}} \right)^{p-k+1} \varepsilon + \left[\bar{b} + \theta(\nu_{p+1}, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^{p-k+1} \varepsilon \right] \delta_{E\bar{b}s}(\nu_{p+2}, \nu_q). \end{aligned} \quad (3.11)$$

Substituting (3.11) into (3.2) and iterating this process, while applying (3.9) at each stage, yields

$$\delta_{E\bar{b}s}(\nu_p, \nu_q) \leq \bar{b} \left(\frac{\lambda}{\bar{b}} \right)^{p-k} \varepsilon \sum_{i=0}^{q-p-1} \left(\frac{\lambda}{\bar{b}} \right)^i \prod_{j=0}^{i-1} \left[\bar{b} + \varepsilon \theta(\nu_{p+j}, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^{p-k+j} \right].$$

Since $\frac{\lambda}{\bar{b}} \in [0, 1)$, the series $\sum_{i=0}^{\infty} U_i$ converges by the ratio test, where

$$U_i = \left(\frac{\lambda}{\bar{b}} \right)^i \prod_{j=0}^{i-1} \left[\bar{b} + \varepsilon \theta(\nu_{p+j}, \nu_q) \left(\frac{\lambda}{\bar{b}} \right)^j \right].$$

Therefore $\delta_{E\bar{b}s}(\nu_p, \nu_q) \rightarrow 0$ as $p, q \rightarrow \infty$, establishing that $\{\nu_n\}$ is Cauchy. By completeness of $(\mathcal{X}, \delta_{E\bar{b}s})$, there exists $\nu^* \in \mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \nu_n = \nu^*.$$

If $\mathcal{T}$ is continuous (case (iii)), then passing to the limit in $\nu_{n+1} = \mathcal{T}\nu_n$ gives $\nu^* = \mathcal{T}\nu^*$. In the alternative case, the convergence $\nu_n \rightarrow \nu^*$ together with $\xi(\nu_n, \nu_{n+1}) \geq$

$\eta(\nu_n, \nu_{n+1})$ yields $\xi(\nu^*, \mathcal{T}\nu^*) \geq \eta(\nu^*, \mathcal{T}\nu^*)$, and consequently

$$\xi(\nu^*, \mathcal{T}\nu^*) \xi(\nu_n, \mathcal{T}\nu_n) \geq \eta(\nu^*, \mathcal{T}\nu^*) \eta(\nu_n, \mathcal{T}\nu_n).$$

Appealing to (3.1) and (3.2),

$$\delta_{E\bar{b}s}(\nu_{n+1}, \nu^*) = \delta_{E\bar{b}s}(\mathcal{T}\nu_n, \mathcal{T}\nu^*) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu_n, \nu^*), \quad (3.12)$$

$$\begin{aligned} \mathcal{M}(\nu_n, \nu^*) &= \max \left(\delta_{E\bar{b}s}(\nu_n, \nu^*), \right. \\ &\quad \left. \min \left(\frac{\delta_{E\bar{b}s}(\nu_n, \mathcal{T}\nu_n) \delta_{E\bar{b}s}(\nu^*, \mathcal{T}\nu^*)}{1 + \delta_{E\bar{b}s}(\nu_n, \nu^*)}, \frac{\delta_{E\bar{b}s}(\nu_n, \mathcal{T}\nu^*) \delta_{E\bar{b}s}(\nu^*, \mathcal{T}\nu_n)}{1 + \delta_{E\bar{b}s}(\nu_n, \nu^*)} \right) \right) \\ &= \max \left(\delta_{E\bar{b}s}(\nu_n, \nu^*), \right. \\ &\quad \left. \min \left(\frac{\delta_{E\bar{b}s}(\nu_n, \nu_{n+1}) \delta_{E\bar{b}s}(\nu^*, \mathcal{T}\nu^*)}{1 + \delta_{E\bar{b}s}(\nu_n, \nu^*)}, \frac{\delta_{E\bar{b}s}(\nu_n, \mathcal{T}\nu^*) \delta_{E\bar{b}s}(\nu^*, \nu_{n+1})}{1 + \delta_{E\bar{b}s}(\nu_n, \nu^*)} \right) \right). \end{aligned} \quad (3.13)$$

Taking the limit $n \rightarrow \infty$ in (3.12) and (3.13) forces $\nu^* = \mathcal{T}\nu^*$, so ν^* is a FP of $\mathcal{T}$.

Assume that there is another FP $\nu' \neq \nu^*$ satisfying

$$\xi(\nu^*, \mathcal{T}\nu^*) \xi(\nu', \mathcal{T}\nu') \geq \eta(\nu^*, \mathcal{T}\nu^*) \eta(\nu', \mathcal{T}\nu').$$

Then (3.1) and (3.2) give

$$\delta_{E\bar{b}s}(\nu^*, \nu') = \delta_{E\bar{b}s}(\mathcal{T}\nu^*, \mathcal{T}\nu') \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu^*, \nu'), \quad (3.14)$$

where

$$\begin{aligned} \mathcal{M}(\nu^*, \nu') &= \max \left(\delta_{E\bar{b}s}(\nu^*, \nu'), \right. \\ &\quad \left. \min \left(\frac{\delta_{E\bar{b}s}(\nu^*, \mathcal{T}\nu^*) \delta_{E\bar{b}s}(\nu', \mathcal{T}\nu')}{1 + \delta_{E\bar{b}s}(\nu^*, \nu')}, \frac{\delta_{E\bar{b}s}(\nu^*, \mathcal{T}\nu') \delta_{E\bar{b}s}(\nu', \mathcal{T}\nu^*)}{1 + \delta_{E\bar{b}s}(\nu^*, \nu')} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \max \left(\delta_{E\bar{b}s}(\nu^*, \nu'), \right. \\
&\quad \left. \min \left(\frac{\delta_{E\bar{b}s}(\nu^*, \nu^*) \delta_{E\bar{b}s}(\nu', \nu')}{1 + \delta_{E\bar{b}s}(\nu^*, \nu')}, \frac{\delta_{E\bar{b}s}(\nu^*, \nu') \delta_{E\bar{b}s}(\nu', \nu^*)}{1 + \delta_{E\bar{b}s}(\nu^*, \nu')} \right) \right) \\
&= \delta_{E\bar{b}s}(\nu^*, \nu').
\end{aligned}$$

Thus $\mathcal{M}(\nu^*, \nu') = \delta_{E\bar{b}s}(\nu^*, \nu')$, and (3.14) becomes

$$\delta_{E\bar{b}s}(\nu^*, \nu') \leq \frac{\lambda}{\bar{b}} \delta_{E\bar{b}s}(\nu^*, \nu') < \delta_{E\bar{b}s}(\nu^*, \nu'),$$

a contradiction since $\frac{\lambda}{\bar{b}} < 1$. Therefore $\nu^* = \nu'$, guarantees unique FP. $\square$

Example 3.3. Let $\mathcal{X} = \mathbb{R}$ and define $\delta_{E\bar{b}s} : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ by the squared Euclidean distance

$$\delta_{E\bar{b}s}(\nu, \zeta) = |\nu - \zeta|^2, \quad \nu, \zeta \in \mathcal{X},$$

with control function $\theta : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$ given by

$$\theta(\nu, \zeta) = 1 + |\nu - \zeta|.$$

Define $\xi, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\xi(\nu, \zeta) = 1 + \frac{1}{1 + |\nu - \zeta|}, \quad \eta(\nu, \zeta) = \frac{1}{2} \xi(\nu, \zeta),$$

and

$$\mathcal{T}(\nu) = \frac{\nu}{3} + 2.$$

Then

$$\xi(\nu, \mathcal{T}\nu) = 1 + \frac{1}{1 + |\nu - \mathcal{T}\nu|} = 1 + \frac{1}{1 + \left| \nu - \left(\frac{\nu}{3} + 2 \right) \right|} = 1 + \frac{1}{1 + \left| \frac{2\nu}{3} - 2 \right|},$$

and similarly $\eta(\nu, \mathcal{T}\nu) = \frac{1}{2} \xi(\nu, \mathcal{T}\nu)$.

The admissibility condition $\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta)$ reduces to

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \frac{1}{4} \xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta),$$

which holds trivially since $\xi(\nu, \mathcal{T}\nu) \geq 1$ and $\xi(\zeta, \mathcal{T}\zeta) \geq 1$. Moreover, whenever $\xi(\nu, \zeta) \geq \eta(\nu, \zeta)$,

$$\xi(\mathcal{T}\nu, \mathcal{T}\zeta) = 1 + \frac{1}{1 + \frac{1}{3}|\nu - \zeta|} \geq \frac{1}{2} \left(1 + \frac{1}{1 + \frac{1}{3}|\nu - \zeta|} \right) = \eta(\mathcal{T}\nu, \mathcal{T}\zeta),$$

so $\mathcal{T}$ is ξ admissible with respect to η . Taking $\nu_0 = 0$ gives $\mathcal{T}\nu_0 = \mathcal{T}(0) = 2$ and

$$\xi(\nu_0, \mathcal{T}\nu_0) = \xi(0, 2) = \frac{4}{3} \geq \frac{2}{3} = \eta(0, 2) = \eta(\nu_0, \mathcal{T}\nu_0),$$

so hypothesis (ii) is satisfied. Since $\mathcal{T}$ is linear, it is continuous.

Finally,

$$\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) = \left| \frac{\nu}{3} + 2 - \left(\frac{\zeta}{3} + 2 \right) \right|^2 = \frac{|\nu - \zeta|^2}{9} = \frac{\lambda}{\bar{b}} \delta_{E\bar{b}s}(\nu, \zeta),$$

with $\lambda = \frac{2}{9}$ and $\bar{b} = 2$. Since $\delta_{E\bar{b}s}(\nu, \zeta)$ dominates the other terms in $\mathcal{M}(\nu, \zeta)$, we have $\mathcal{M}(\nu, \zeta) = \delta_{E\bar{b}s}(\nu, \zeta)$. Thus conditions (3.1) and (3.2) of Theorem 3.4 are satisfied with $\lambda = \frac{2}{9} \in [0, 1)$. Consequently, $\mathcal{T}$ has the unique FP $\nu^* = 3$.

Three corollaries follow from Theorem 3.4, either by fixing particular functions or by rewriting the contraction condition in a different form. Throughout, $\mathcal{M}(\nu, \zeta)$ keeps the meaning given in (3.2).

Remark 3.5.

When $\eta(\nu, \zeta) = 1$ then admissible condition becomes

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq 1 \text{ and } 1 \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) \text{ when } \xi(\nu, \zeta) = 1$$

Corollary 3.6.

Suppose $(\mathcal{X}, \delta_{E\bar{b}s})$ is a complete, continuous extended $\bar{b}$ SMS with $\bar{b} \geq 1$ and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$, where the contractive condition is

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq 1 \quad \Rightarrow \quad \delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta),$$

then by hypotheses (i)–(iii) of Theorem 3.4 $\mathcal{T}$ has a unique FP.

Proof.

With $\eta(\nu, \zeta) = 1$ the contraction condition in Theorem 3.4 retains the form

$$\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta).$$

Existence and uniqueness of the FP are then inherited directly from Theorem 3.4. $\square$

Corollary 3.7.

Take $(\mathcal{X}, \delta_{E\bar{b}s})$ to be a complete, continuous extended $\bar{b}$ SMS with $\bar{b} \geq 1$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$. Assume there exist a map $\xi : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$, a constant $\lambda \in [0, 1)$, and a number $a > 0$ such that

$$(\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) + a)^{\xi(\nu, \mathcal{T}\nu)\xi(\zeta, \mathcal{T}\zeta)} \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta) + a \quad (3.15)$$

then, under hypotheses (i)–(iii) of Theorem 3.4, $\mathcal{T}$ has a unique FP.

Proof.

The contraction condition from (3.15), we obtain

$$2^{\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta)} \leq (\xi(\nu, \mathcal{T}\nu)\xi(\zeta, \mathcal{T}\zeta) + 1)^{\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta)} \leq 2^{\frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta)}.$$

Since the exponential function is strictly increasing, this implies

$$\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta).$$

Therefore, the condition of Corollary 3.6 is satisfied, ensuring that $\mathcal{T}$ has a unique FP. $\square$

Corollary 3.8.

Consider $(\mathcal{X}, \delta_{E\bar{b}s})$, a complete, continuous extended $\bar{b}$ SMS with $\bar{b} \geq 1$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$. Suppose there exist a map $\eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ and a constant $a \in [0, 1)$

such that whenever $\eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) \leq 1$,

$$(\eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) + 1)^{\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta)} \leq 2^{\frac{a}{\bar{b}}} \mathcal{M}(\nu, \zeta), \quad (3.16)$$

then, under the hypotheses (i)–(iii) of Theorem 3.4, $\mathcal{T}$ has a unique FP.

Proof.

(3.16) yields

$$2^{\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta)} \leq (\eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) + 1)^{\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta)} \leq 2^{\frac{a}{\bar{b}}} \mathcal{M}(\nu, \zeta).$$

Again, strict monotonicity of the exponential function gives

$$\delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{a}{\bar{b}} \mathcal{M}(\nu, \zeta).$$

then from Theorem 3.4 $\mathcal{T}$ has a unique FP.

□

3.3 Consequences in Extended Supra MS

By Remark 3.1(ii), setting $\bar{b} = 1$ in Definition 3.1 turns an extended $\bar{b}$ SMS into an extended SMS. So the result below follows directly from Theorem 3.4: $\frac{\lambda}{\bar{b}}$ just becomes λ , while $\mathcal{M}(\nu, \zeta)$ keeps the form given in (3.2).

Corollary 3.9.

Suppose $(\mathcal{X}, \delta_{E\bar{b}s})$ is a complete, continuous extended SMS and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$. If the hypotheses of Theorem 3.4 are imposed with $\bar{b} = 1$, so that

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) \quad \Rightarrow \quad \delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \lambda \mathcal{M}(\nu, \zeta),$$

then $\mathcal{T}$ has a unique FP.

3.4 Existence of Fixed Points in $\bar{b}$ Suprametric Spaces

By Remark 3.1(i), holding the control function fixed at $\theta(\nu, \zeta) \geq 1$ in Definition 3.1 turns an extended $\bar{b}$ SMS into a plain $\bar{b}$ SMS. The corollary below is what Theorem 3.4 looks like in that case, with $\mathcal{M}(\nu, \zeta)$ still given by (3.2).

Corollary 3.10.

Take $(\mathcal{X}, \delta_{E\bar{b}s})$ to be a complete, continuous $\bar{b}$ SMS with $\bar{b} \geq 1$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$. Applying Theorem 3.4 with constant control function yields

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) \quad \Rightarrow \quad \delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \frac{\lambda}{\bar{b}} \mathcal{M}(\nu, \zeta),$$

then $\mathcal{T}$ has a unique FP.

Proof.

Holding the control function constant, $\theta(\nu, \zeta) \geq 1$ for all $\nu, \zeta \in \mathcal{X}$ and under the hypotheses (i)–(iii) of Theorem 3.4 $\mathcal{T}$ has a unique FP. $\square$

3.5 Fixed Point Findings in Suprametric Spaces

By Remark 3.1(iii), fixing both $\theta(\nu, \zeta) \geq 1$ and $\bar{b} = 1$ in Definition 3.1 brings an extended $\bar{b}$ SMS all the way down to a plain SMS. Below is the matching version of Theorem 3.4 for this case, with $\mathcal{M}(\nu, \zeta)$ still as in (3.2).

Corollary 3.11.

Consider $(\mathcal{X}, \delta_{E\bar{b}s})$, a complete, continuous SMS, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$. Applying Theorem 3.4 with $\bar{b} = 1$ and constant control function gives

$$\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta) \quad \Rightarrow \quad \delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \lambda \mathcal{M}(\nu, \zeta),$$

and $\mathcal{T}$ has a unique FP.

Proof.

Imposing $\bar{b} = 1$ together with a constant control $\theta(\nu, \zeta) \geq 1$ then $\mathcal{T}$ has a unique FP. $\square$

3.6 Applications

With Theorem 3.4 and its special cases now in place, we close the chapter with an application that shows the extended $\bar{b}$ SMS framework at work on a concrete problem.

Nonlinear Volterra-Fredholm integral equations turn up often in applied science, in areas such as physics, engineering, mathematical biology, and economics. Whenever a system carries some kind of memory effect or nonlocal interaction, population dynamics, heat conduction with aftereffects, or the response of viscoelastic media are typical examples, this is the natural type of equation to model it with.

FPT gives a useful way to handle such equations: once the integral equation is rewritten as a FP equation, the machinery built up so far can be used directly, giving existence, uniqueness, and even a way to approximate the solution iteratively. With this in mind, we look at the nonlinear integral equation known as Volterra-Fredholm.

$$\nu(t) = \phi(t) + \nu_1 \int_a^t K_1(t, s, \nu(s)) ds + \nu_2 \int_a^b K_2(t, s, \nu(s)) ds, \quad (3.18)$$

where

- $\nu(t)$ is the unknown function on $[a, b]$,
- $\phi(t)$ is a given function on $[a, b]$,
- ν_1, ν_2 are prescribed scalar parameters,
- a and b are the limits of integration,
- $K_1(t, s, \nu(s))$ and $K_2(t, s, \nu(s))$ are kernels depending on t, s , and the unknown $\nu(t)$.

Let $\mathcal{X} = C([a, b], \mathbb{R})$ denote the space of continuous real valued functions on $[a, b]$, equipped with

$$\delta_{E\bar{b}s}(\nu, \zeta) = \max_{t \in [a, b]} |\nu(t) - \zeta(t)|^2,$$

and with $\theta : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$ defined by

$$\theta(\nu, \zeta) = 1 + |\nu(t)| + |\zeta(t)|.$$

With these choices, $(\mathcal{X}, \delta_{E\bar{b}s})$ is a complete extended $\bar{b}$ SMS with $\bar{b} = 2$.

Theorem 3.12.

Consider the operator

$$\mathcal{T}(\nu)(t) = \phi(t) + \nu_1 \int_a^t K_1(t, s, \nu(s)) ds + \nu_2 \int_a^b K_2(t, s, \nu(s)) ds,$$

and suppose the following conditions hold:

- (i) The kernels $K_1(t, s, \nu(s))$ and $K_2(t, s, \nu(s))$ are continuous in t and s and bounded:

$$|K_1(t, s, \nu(s))| \leq C_1, \quad |K_2(t, s, \nu(s))| \leq C_2,$$

for some $C_1, C_2 > 0$ and all $t, s \in [a, b]$, $\nu \in \mathcal{X}$;

- (ii) Both kernels are Lipschitz in their last argument:

$$|K_1(t, s, \nu(s)) - K_1(t, s, \zeta(s))| \leq \ell_1 |\nu(s) - \zeta(s)|,$$

$$|K_2(t, s, \nu(s)) - K_2(t, s, \zeta(s))| \leq \ell_2 |\nu(s) - \zeta(s)|,$$

for some $\ell_1, \ell_2 > 0$ and all $t, s \in [a, b]$, $\nu, \zeta \in \mathcal{X}$;

- (iii) The constant $\lambda = 2(|\nu_1| \ell_1 (b - a) + |\nu_2| \ell_2 (b - a))^2$ satisfies $\lambda < 1$;

- (iv) There exist $\xi, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ such that

- $\xi(\nu, \mathcal{T}\nu) \xi(\zeta, \mathcal{T}\zeta) \geq \eta(\nu, \mathcal{T}\nu) \eta(\zeta, \mathcal{T}\zeta),$
- $\mathcal{T}$ is ξ admissible with respect to $\eta,$

- there exists $\nu_0 \in \mathcal{X}$ with $\xi(\nu_0, \mathcal{T}\nu_0) \geq \eta(\nu_0, \mathcal{T}\nu_0)$,
- either $\mathcal{T}$ is continuous, or for every sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu^*$ and $\xi(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$, one has $\xi(\nu^*, \mathcal{T}\nu^*) \geq \eta(\nu^*, \mathcal{T}\nu^*)$.

Then $\mathcal{T}$ has a unique FP $\nu^* \in \mathcal{X}$, which solves the nonlinear Volterra-Fredholm integral equation (3.18).

Proof.

Fix $\nu, \zeta \in \mathcal{X}$. Then

$$\begin{aligned}
 |\mathcal{T}\nu(t) - \mathcal{T}\zeta(t)| &= \left| \nu_1 \int_a^t K_1(t, s, \nu(s)) ds + \nu_2 \int_a^b K_2(t, s, \nu(s)) ds \right. \\
 &\quad \left. - \nu_1 \int_a^t K_1(t, s, \zeta(s)) ds - \nu_2 \int_a^b K_2(t, s, \zeta(s)) ds \right| \\
 &\leq |\nu_1| \int_a^t |K_1(t, s, \nu(s)) - K_1(t, s, \zeta(s))| ds \\
 &\quad + |\nu_2| \int_a^b |K_2(t, s, \nu(s)) - K_2(t, s, \zeta(s))| ds.
 \end{aligned}$$

Using the Lipschitz bounds on K_1 and K_2 gives

$$|\mathcal{T}\nu(t) - \mathcal{T}\zeta(t)| \leq |\nu_1| \ell_1 \int_a^t |\nu(s) - \zeta(s)| ds + |\nu_2| \ell_2 \int_a^b |\nu(s) - \zeta(s)| ds.$$

Since $|\nu(s) - \zeta(s)| \leq \sqrt{\delta_{E\bar{b}s}(\nu, \zeta)}$, we obtain

$$|\mathcal{T}\nu(t) - \mathcal{T}\zeta(t)| \leq |\nu_1| \ell_1 \sqrt{\delta_{E\bar{b}s}(\nu, \zeta)} (t - a) + |\nu_2| \ell_2 \sqrt{\delta_{E\bar{b}s}(\nu, \zeta)} (b - a).$$

For $t \in [a, b]$, this yields

$$\begin{aligned}
 \delta_{E\bar{b}s}(\mathcal{T}\nu, \mathcal{T}\zeta) &= \max_{t \in [a, b]} |\mathcal{T}\nu(t) - \mathcal{T}\zeta(t)|^2 \leq (|\nu_1| \ell_1 (b - a) + |\nu_2| \ell_2 (b - a))^2 \delta_{E\bar{b}s}(\nu, \zeta) \\
 &\leq \frac{\lambda}{2} \delta_{E\bar{b}s}(\nu, \zeta).
 \end{aligned}$$

Since $\delta_{E\bar{b}s}(\nu, \zeta) \leq \mathcal{M}(\nu, \zeta)$, with

$$\mathcal{M}(\nu, \zeta) = \max \left(\delta_{E\bar{b}s}(\nu, \zeta), \right. \\ \min \left(\frac{\delta_{E\bar{b}s}(\nu, \mathcal{T}\nu) \delta_{E\bar{b}s}(\zeta, \mathcal{T}\zeta)}{1 + \delta_{E\bar{b}s}(\nu, \zeta)}, \right. \\ \left. \left. \frac{\delta_{E\bar{b}s}(\nu, \mathcal{T}\zeta) \delta_{E\bar{b}s}(\zeta, \mathcal{T}\nu)}{1 + \delta_{E\bar{b}s}(\nu, \zeta)} \right) \right),$$

the contractive estimate of Theorem 3.4 is satisfied, and the remaining hypotheses follow from assumption (iv). Therefore $\mathcal{T}$ admits a unique FP $\nu^* \in \mathcal{X}$, which is precisely the solution of (3.18). $\square$

Example 3.4.

Let $\mathcal{X} = C([0, 1], \mathbb{R})$, the space of continuous real valued functions on $[0, 1]$, with $\delta_{E\bar{b}s} : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ given by

$$\delta_{E\bar{b}s}(\nu, \zeta) = \max_{t \in [0, 1]} |\nu(t) - \zeta(t)|^2,$$

and $\theta : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$ given by

$$\theta(\nu, \zeta) = 1 + |\nu(t)| + |\zeta(t)|,$$

so that $(\mathcal{X}, \delta_{E\bar{b}s})$ is a complete extended $\bar{b}$ SMS with $\bar{b} = 2$. Consider the operator

$$\mathcal{T}(\nu)(t) = \phi(t) + \nu_1 \int_0^t K_1(t, s, \nu(s)) ds + \nu_2 \int_0^1 K_2(t, s, \nu(s)) ds,$$

where

- $\phi(t) = t$ on $[0, 1]$,
- $\nu_1 = \frac{1}{4}$ and $\nu_2 = \frac{1}{4}$,
- the limits of integration are $a = 0$ and $b = 1$,
- the kernels are $K_1(t, s, \nu(s)) = \frac{\nu(s)}{1 + \nu^2(s)}$ and $K_2(t, s, \nu(s)) = \frac{\nu(s)}{2 + \nu^2(s)}$.

Since

$$|K_1(t, s, \nu(s))| = \left| \frac{\nu(s)}{1 + (\nu(s))^2} \right| \leq \frac{1}{2}, \quad |K_2(t, s, \nu(s))| = \left| \frac{\nu(s)}{2 + (\nu(s))^2} \right| \leq \frac{1}{2},$$

hypothesis (i) holds with $C_1 = \frac{1}{2}$ and $C_2 = \frac{1}{2}$. Differentiating K_1 and K_2 with respect to $\nu(s)$,

$$\frac{d}{d\nu(s)} \frac{\nu(s)}{1 + \nu^2(s)} = \frac{1 - \nu^2(s)}{(1 + \nu^2(s))^2}, \quad \frac{d}{d\nu(s)} \frac{\nu(s)}{2 + \nu^2(s)} = \frac{2 - \nu^2(s)}{(2 + \nu^2(s))^2}.$$

Their suprema on $[0, 1]$ are

$$\max_{s \in [0,1]} \left| \frac{1 - \nu^2(s)}{(1 + \nu^2(s))^2} \right| = 1, \quad \max_{s \in [0,1]} \left| \frac{2 - \nu^2(s)}{(1 + \nu^2(s))^2} \right| = \frac{1}{2},$$

so hypothesis (ii) holds with $\ell_1 = 1$ and $\ell_2 = \frac{1}{2}$. Consequently,

$$\lambda = 2 \left(\frac{1}{4} \cdot 1 \cdot (1 - 0) + \frac{1}{4} \cdot \frac{1}{2} \cdot (1 - 0) \right)^2 = 2 \left(\frac{1}{4} + \frac{1}{8} \right)^2 = 2 \left(\frac{3}{8} \right)^2 = \frac{9}{32} < 1.$$

Now define $\xi, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ by

$$\xi(\nu, \zeta) = 1 + \|\nu(t)\| + \|\zeta(t)\|,$$

where $\|\nu(t)\| = \max_{t \in [0,1]} |\nu(t)|$, and

$$\eta(\nu, \zeta) = \frac{1}{2} \xi(\nu, \zeta). \tag{3.19}$$

Then $\mathcal{T}$ is ξ admissible with respect to η , since

$$\xi(\nu, \mathcal{T}\nu) = 1 + \|\nu\| + \|\mathcal{T}\nu\| \geq \frac{1}{2}(1 + \|\nu\| + \|\mathcal{T}\nu\|) = \eta(\nu, \mathcal{T}\nu).$$

With $\nu_0(t) = 0$, we have

$$\xi(\nu_0, \mathcal{T}\nu_0) = 1 + \|\nu_0\| + \|\mathcal{T}\nu_0\| = 1 + 0 + \|\mathcal{T}\nu_0\| \geq \frac{1}{2}(1 + \|\nu_0\| + \|\mathcal{T}\nu_0\|) = \eta(\nu_0, \mathcal{T}\nu_0).$$

Since $\mathcal{T}$ is built from integrals of continuous functions, it is continuous. Alternatively, for any sequence $\{\nu_n\} \subset \mathcal{X}$ with $\nu_n \rightarrow \nu^*$ and $\xi(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$ for

all n , since $\xi(\nu, \zeta) \geq \eta(\nu, \zeta)$ everywhere on $\mathcal{X} \times \mathcal{X}$, we have $\xi(\nu^*, \mathcal{T}\nu^*) \geq \eta(\nu^*, \mathcal{T}\nu^*)$. All hypotheses of Theorem [3.12](#) are satisfied, so $\mathcal{T}$ has a unique FP $\nu^* \in \mathcal{X}$, which is the unique solution of the Volterra-Fredholm equation under consideration.

Chapter 4

Results in Double Controlled and Double Controlled Suprametric Spaces

This chapter puts together the ξ admissibility framework and the rational contraction function $\mathcal{M}(\nu, \zeta)$ for DCMS and DCSMS.

We introduce a new space, DCSMS, built with a modified structure, and prove a matching FP theorem for it.

Convergence of the Picard iteration is guaranteed through an explicit ratio limit condition placed on the two control functions, and a few examples are worked out to illustrate how the results play out.

We then combine the ξ admissibility condition with the rational function $\mathcal{M}(\nu, \zeta)$ to get a more general FP theorem.

This theorem rests on hypotheses about the control functions together with a ratio limit test, and we finish with a worked example showing it in action.

4.1 Double Controlled Metric Spaces

Definition 4.1. Fix a nonempty set $\mathcal{X}$ together with two control functions $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$. A mapping $\delta_c : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is called a *double controlled metric* if the following conditions hold for all $\nu, \zeta, \rho \in \mathcal{X}$:

$$(\delta_c 1): \quad \delta_c(\nu, \zeta) = 0 \iff \nu = \zeta.$$

$$(\delta_c 2): \quad \delta_c(\nu, \zeta) = \delta_c(\zeta, \nu).$$

$$(\delta_c 3): \quad \delta_c(\nu, \zeta) \leq p(\nu, \rho) \delta_c(\nu, \rho) + q(\rho, \zeta) \delta_c(\rho, \zeta).$$

Example 4.1. Let $\mathcal{X} = \{1, 2, 3\}$ and define a symmetric map $\delta_c : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ by

$$\delta_c(1, 2) = 5, \quad \delta_c(1, 3) = 2, \quad \delta_c(2, 3) = 1,$$

with $\delta_c(\nu, \nu) = 0$ for all $\nu \in \mathcal{X}$. Taking $\rho = 3$ as the intermediate point, we verify the double controlled triangle inequality for the pair $(1, 2)$.

Note that the standard triangle inequality fails since

$$\delta_c(1, 3) + \delta_c(3, 2) = 2 + 1 = 3 < 5 = \delta_c(1, 2).$$

This confirms that $(\mathcal{X}, \delta_c)$ is not an ordinary MS, so the relaxed inequality $(\delta_c 3)$ is genuinely needed here rather than being a formality.

For $(\delta_c 3)$, choose the control functions

$$p(\nu, \zeta) = \nu + \zeta, \quad q(\nu, \zeta) = 2.$$

Then

$$\begin{aligned} \delta_c(1, 2) &\leq p(1, 3) \delta_c(1, 3) + q(3, 2) \delta_c(3, 2) \\ &= (1 + 3)(2) + 2(1) \\ &= 8 + 2 = 10 \geq 5 \\ &= \delta_c(1, 2). \end{aligned}$$

Thus the double controlled triangle inequality is satisfied. Since $(\delta_c 1)$ and $(\delta_c 2)$ are also clearly satisfied, $(\mathcal{X}, \delta_c)$ is a DCMS.

Theorem 4.1. Take $(\mathcal{X}, \delta_c)$ to be a complete, continuous DCMS with controls $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ a map for which there exist functions $\xi, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [1, +\infty)$ and a constant $\kappa \in [0, 1)$ satisfying

$$\xi(\nu, \zeta) \eta(\nu, \zeta) \delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \mathcal{M}(\nu, \zeta), \quad (4.1)$$

where

$$\mathcal{M}(\nu, \zeta) = \max \left\{ \delta_c(\nu, \zeta), \min \left\{ \frac{\delta_c(\nu, \mathcal{T}\nu) \delta_c(\zeta, \mathcal{T}\zeta)}{1 + \delta_c(\nu, \zeta)}, \frac{\delta_c(\nu, \mathcal{T}\zeta) \delta_c(\zeta, \mathcal{T}\nu)}{1 + \delta_c(\nu, \zeta)} \right\} \right\} \quad (4.2)$$

for every $\nu, \zeta \in \mathcal{X}$. Suppose further that:

- i. $\mathcal{T}$ is ξ admissible with respect to η ;
- ii. there exists $\nu_0 \in \mathcal{X}$ satisfying $\xi(\nu_0, \mathcal{T}\nu_0) \geq \eta(\nu_0, \mathcal{T}\nu_0)$;
- iii. $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$, one has $\xi(\nu, \mathcal{T}\nu) \geq \eta(\nu, \mathcal{T}\nu)$;
- iv.

$$\lim_{r \rightarrow \infty} \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \cdot q(\nu_{r+1}, \nu_m) < \frac{1}{\kappa}, \quad \forall \kappa \in (0, 1). \quad (4.3)$$

Then $\mathcal{T}$ possesses a unique FP in $\mathcal{X}$.

Proof. Select an arbitrary initial point $\nu_0 \in \mathcal{X}$ and generate the sequence $\{\nu_n\}$ by repeated application of $\mathcal{T}$:

$$\nu_1 = \mathcal{T}\nu_0, \quad \nu_2 = \mathcal{T}\nu_1, \quad \dots, \quad \nu_{n+1} = \mathcal{T}\nu_n, \quad \forall n \in \mathbb{N}.$$

Since both ξ and η assume values in $[1, +\infty)$, we have

$$\xi(\nu, \zeta) \geq 1 \quad \text{and} \quad \eta(\nu, \zeta) \geq 1, \quad \forall \nu, \zeta \in \mathcal{X}.$$

Consequently, for every consecutive pair,

$$\xi(\nu_n, \nu_{n+1}) \geq 1 \quad \text{and} \quad \eta(\nu_n, \nu_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}.$$

Substituting $\nu = \nu_{n-1}$ and $\zeta = \nu_n$ into the contraction inequality yields

$$\xi(\nu_{n-1}, \nu_n) \eta(\nu_{n-1}, \nu_n) \delta_c(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n) \leq \kappa \mathcal{M}(\nu_{n-1}, \nu_n).$$

Thus

$$\xi(\nu_{n-1}, \nu_n) \eta(\nu_{n-1}, \nu_n) \delta_c(\nu_n, \nu_{n+1}) \leq \kappa \mathcal{M}(\nu_{n-1}, \nu_n). \quad (4.4)$$

Since $\xi \geq 1$ and $\eta \geq 1$, their product is at least 1; therefore,

$$\delta_c(\nu_n, \nu_{n+1}) \leq \kappa \mathcal{M}(\nu_{n-1}, \nu_n). \quad (4.5)$$

Now

$$\begin{aligned} \mathcal{M}(\nu_{n-1}, \nu_n) &= \max \left\{ \delta_c(\nu_{n-1}, \nu_n), \right. \\ &\quad \min \left\{ \frac{\delta_c(\nu_{n-1}, \nu_n) \delta_c(\nu_n, \nu_{n+1})}{1 + \delta_c(\nu_{n-1}, \nu_n)}, \right. \\ &\quad \left. \left. \frac{\delta_c(\nu_{n-1}, \nu_{n+1}) \delta_c(\nu_n, \nu_n)}{1 + \delta_c(\nu_{n-1}, \nu_n)} \right\} \right\} \\ &= \max\{\delta_c(\nu_{n-1}, \nu_n), 0\}. \end{aligned}$$

Hence

$$\mathcal{M}(\nu_{n-1}, \nu_n) = \delta_c(\nu_{n-1}, \nu_n). \quad (4.6)$$

Substituting this back yields $\delta_c(\nu_n, \nu_{n+1}) \leq \kappa \delta_c(\nu_{n-1}, \nu_n)$.

Iterating this inequality gives

$$\delta_c(\nu_n, \nu_{n+1}) \leq \kappa^n \delta_c(\nu_0, \nu_1), \quad \forall n \in \mathbb{N}. \quad (4.7)$$

To establish that $\{\nu_n\}$ is Cauchy, take indices $m, n \in \mathbb{N}$ with $m > n$. Applying the triangle inequality (δ_c3) repeatedly:

$$\begin{aligned}
\delta_c(\nu_n, \nu_m) &\leq p(\nu_n, \nu_{n+1})\delta_c(\nu_n, \nu_{n+1}) + q(\nu_{n+1}, \nu_m)\delta_c(\nu_{n+1}, \nu_m) \\
&\leq p(\nu_n, \nu_{n+1})\delta_c(\nu_n, \nu_{n+1}) + q(\nu_{n+1}, \nu_m) \left[p(\nu_{n+1}, \nu_{n+2})\delta_c(\nu_{n+1}, \nu_{n+2}) \right. \\
&\quad \left. + q(\nu_{n+2}, \nu_m)\delta_c(\nu_{n+2}, \nu_m) \right].
\end{aligned}$$

Continuing this expansion up to index $m - 1$ yields

$$\delta_c(\nu_n, \nu_m) \leq p(\nu_n, \nu_{n+1})\delta_c(\nu_n, \nu_{n+1}) + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i q(\nu_j, \nu_m) \right) p(\nu_i, \nu_{i+1})\delta_c(\nu_i, \nu_{i+1}).$$

Since each control value is at least 1, extending the product from $j = 0$ only increases the right hand side:

$$\delta_c(\nu_n, \nu_m) \leq p(\nu_n, \nu_{n+1})\delta_c(\nu_n, \nu_{n+1}) + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i q(\nu_j, \nu_m) \right) p(\nu_i, \nu_{i+1})\delta_c(\nu_i, \nu_{i+1}).$$

Using the estimate $\delta_c(\nu_i, \nu_{i+1}) \leq \kappa^i \delta_c(\nu_0, \nu_1)$, we obtain

$$\delta_c(\nu_n, \nu_m) \leq \delta_c(\nu_0, \nu_1) \left[p(\nu_n, \nu_{n+1})\kappa^n + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i q(\nu_j, \nu_m) \right) p(\nu_i, \nu_{i+1})\kappa^i \right].$$

Define the partial sums

$$R_r = \sum_{i=1}^r \left(\prod_{j=0}^i q(\nu_j, \nu_m) \right) p(\nu_i, \nu_{i+1})\kappa^i.$$

The convergence of these sums is governed by the ratio $\frac{R_{r+1}}{R_r}$:

$$\lim_{r \rightarrow \infty} \frac{R_{r+1}}{R_r} = \lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \kappa.$$

By condition (iv),

$$\lim_{r \rightarrow \infty} \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} q(\nu_{r+1}, \nu_m) < \frac{1}{\kappa}.$$

Therefore,

$$\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \kappa < 1.$$

By the ratio test, the series converges absolutely. Letting $n, m \rightarrow \infty$ gives

$$\lim_{n, m \rightarrow \infty} \delta_c(\nu_n, \nu_m) = 0,$$

so $\{\nu_n\}$ is Cauchy. Completeness of $(\mathcal{X}, \delta_c)$ ensures the existence of $\nu^* \in \mathcal{X}$ such that

$$\nu_n \longrightarrow \nu^* \quad \text{as } n \rightarrow \infty.$$

From the inequality $\delta_c(\nu_{n+1}, \nu_{n+2}) \leq \kappa^n \delta_c(\nu_0, \nu_1)$ and the fact that $\kappa \in [0, 1)$, we have $\kappa^n \rightarrow 0$ as $n \rightarrow \infty$.

If $\mathcal{T}$ is continuous (condition iii), then

$$\mathcal{T}\nu^* = \mathcal{T}\left(\lim_{n \rightarrow \infty} \nu_n\right) = \lim_{n \rightarrow \infty} \nu_{n+1} = \nu^*.$$

Thus ν^* is a FP.

For uniqueness, suppose ν^* and ζ^* are distinct FPs with $\mathcal{T}\nu^* = \nu^*$, $\mathcal{T}\zeta^* = \zeta^*$, and $\delta_c(\nu^*, \zeta^*) > 0$.

From the contraction inequality,

$$\delta_c(\nu^*, \zeta^*) = \delta_c(\mathcal{T}\nu^*, \mathcal{T}\zeta^*) \leq \kappa \mathcal{M}(\nu^*, \zeta^*),$$

where

$$\mathcal{M}(\nu^*, \zeta^*) = \max \left\{ \delta_c(\nu^*, \zeta^*), \min \left\{ \frac{\delta_c(\nu^*, \nu^*) \delta_c(\zeta^*, \zeta^*)}{1 + \delta_c(\nu^*, \zeta^*)}, \frac{\delta_c(\zeta^*, \nu^*) \delta_c(\nu^*, \zeta^*)}{1 + \delta_c(\nu^*, \zeta^*)} \right\} \right\}.$$

Since $\delta_c(\nu^*, \nu^*) = 0$, we have $\mathcal{M}(\nu^*, \zeta^*) = \delta_c(\nu^*, \zeta^*)$. Thus

$$\delta_c(\nu^*, \zeta^*) \leq \kappa \delta_c(\nu^*, \zeta^*).$$

As $\kappa < 1$, this forces $\delta_c(\nu^*, \zeta^*) = 0$, hence $\nu^* = \zeta^*$. Therefore the FP is unique. $\square$

Example 4.2. Let $\mathcal{X} = \{1, 2, 3\}$. Define a symmetric distance function

$$\delta_c : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$$

such that

$$\delta_c(\nu, \nu) = 0$$

and

$$\delta_c(1, 2) = 3, \quad \delta_c(2, 3) = 4, \quad \delta_c(1, 3) = \frac{1}{2}.$$

Define the control functions

$$p, q : \mathcal{X} \times \mathcal{X} \longrightarrow [1, \infty)$$

by

$$p(\nu, \zeta) = \begin{cases} 2.0, & \text{if } (\nu, \zeta) = (1, 2), \\ 1.0, & \text{otherwise,} \end{cases}$$

and

$$q(\nu, \zeta) = \begin{cases} 2.5, & \text{if } (\nu, \zeta) = (2, 3), \\ 1.0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be defined by

$$\mathcal{T}(1) = 2, \quad \mathcal{T}(2) = 2, \quad \mathcal{T}(3) = 1.$$

Take

$$\xi(\nu, \zeta) = 1.2, \quad \eta(\nu, \zeta) = 1.1.$$

We verify that $(\mathcal{X}, \delta_c)$ is a DCMS.

Since $\xi(\nu, \zeta) = 1.2 \geq 1.1 = \eta(\nu, \zeta)$ for all ν, ζ , we have $\xi(\nu, \zeta) \geq \eta(\nu, \zeta)$. Consequently,

$$\xi(\mathcal{T}\nu, \mathcal{T}\zeta) \geq \eta(\mathcal{T}\nu, \mathcal{T}\zeta),$$

as $1.2 \geq 1.1$ holds universally. Thus $\mathcal{T}$ is ξ admissible with respect to η .

The contractive condition takes the form

$$\xi(\nu, \zeta)\eta(\nu, \zeta)\delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \mathcal{M}(\nu, \zeta), \quad (4.8)$$

for some $\kappa \in [0, 1)$.

Consider the pair $(\nu, \zeta) = (2, 3)$. The left hand side is

$$\xi(2, 3)\eta(2, 3)\delta_c(\mathcal{T}2, \mathcal{T}3) = (1.2)(1.1)\delta_c(2, 1) = (1.32)(3) = 3.96.$$

For the right hand side,

$$\mathcal{M}(2, 3) = \max \left\{ \delta_c(2, 3), \min \left\{ \frac{\delta_c(2, \mathcal{T}2)\delta_c(3, \mathcal{T}3)}{1 + \delta_c(2, 3)}, \frac{\delta_c(3, \mathcal{T}2)\delta_c(2, \mathcal{T}3)}{1 + \delta_c(2, 3)} \right\} \right\}.$$

Using $\delta_c(2, 3) = 4$, $\mathcal{T}(2) = 2$, $\delta_c(2, 2) = 0$, $\mathcal{T}(3) = 1$, and $\delta_c(3, 1) = \frac{1}{2}$, we get

$$\begin{aligned} \mathcal{M}(2, 3) &= \max \left\{ 4, \min \left\{ 0, \frac{4 \cdot 3}{5} \right\} \right\} \\ &= 4. \end{aligned}$$

The contraction inequality requires $3.96 \leq 4\kappa$, so $\kappa \geq 0.99$. Taking $\kappa \in [0.99, 1)$ satisfies all conditions of Theorem 4.1. The point $\nu^* = 2$ is the unique FP.

Theorem 4.1 gives rise to five corollaries below. Each one loosens or specializes a different piece of it, the rational functional, the admissibility pair, the space itself, or the way the contraction constant is expressed, so that someone coming from any of these neighboring theories can see where their own result fits within this framework.

In each of the following corollaries, the function η is taken to be identically 1, i.e. $\eta(\nu, \zeta) = 1$ for all $\nu, \zeta \in \mathcal{X}$. Accordingly, the symbol η is omitted from the statements and proofs below.

Corollary 4.2. Take $(\mathcal{X}, \delta_c)$ to be a complete DCMS with control functions $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ a mapping for which there exists a constant $\kappa \in [0, 1)$ satisfying:

$$\xi(\nu, \zeta) \delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \delta_c(\nu, \zeta), \quad \forall \nu, \zeta \in \mathcal{X}.$$

If, in addition:

- $\mathcal{T}$ is ξ admissible;
- there exists $\nu_0 \in \mathcal{X}$ satisfying $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;
- $\lim_{r \rightarrow \infty} \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \cdot q(\nu_{r+1}, \nu_m) < \frac{1}{\kappa}$ for every $\kappa \in (0, 1)$;

then $\mathcal{T}$ has a unique FP in $\mathcal{X}$.

Proof. Since $\delta_c(\nu, \zeta) \leq \mathcal{M}(\nu, \zeta)$ by definition, the hypothesised bound on $\delta_c(\nu, \zeta)$ automatically forces

$$\xi(\nu, \zeta) \delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \mathcal{M}(\nu, \zeta),$$

so the claim reduces to an instance of Theorem 4.1 with $\eta \equiv 1$. □

Corollary 4.3. Suppose $(\mathcal{X}, \delta_b)$ is a complete b metric space with parameter $s \geq 1$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ is a mapping for which there exists a function $\xi : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$ and a constant $\kappa \in [0, 1)$ satisfying:

$$\xi(\nu, \zeta) \delta_b(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \mathcal{M}(\nu, \zeta),$$

where $\mathcal{M}(\nu, \zeta)$ is defined using the b metric δ_b in place of δ_c . If:

- $\mathcal{T}$ is ξ admissible;
- there exists $\nu_0 \in \mathcal{X}$ satisfying $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;

- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;
- $s < \frac{1}{\sqrt{\kappa}}$;

then $\mathcal{T}$ has a unique FP in $\mathcal{X}$.

Proof. Taking both controls equal to the constant s , i.e. $p(\nu, \zeta) = q(\nu, \zeta) = s$, collapses the double controlled metric of Theorem 4.1 to an ordinary b metric. Under this identification, the ratio limit hypothesis reads $s^2 < \frac{1}{\kappa}$, equivalently $s < \frac{1}{\sqrt{\kappa}}$, and the stated conclusion is then immediate from Theorem 4.1 with $\eta \equiv 1$. $\square$

Corollary 4.4. Consider $(\mathcal{X}, \delta_c)$, a complete DCMS with control functions $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ a mapping for which there exists a function $\xi : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$ and a constant $\kappa \in [0, 1)$ satisfying:

$$\xi(\nu, \zeta) \delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \mathcal{M}(\nu, \zeta), \quad \forall \nu, \zeta \in \mathcal{X},$$

where $\mathcal{M}(\nu, \zeta)$ is as in Theorem 4.1. Suppose additionally that:

- $\mathcal{T}$ is ξ admissible (i.e., $\xi(\nu, \zeta) \geq 1 \Rightarrow \xi(\mathcal{T}\nu, \mathcal{T}\zeta) \geq 1$);
- there exists $\nu_0 \in \mathcal{X}$ with $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;
- $\lim_{r \rightarrow \infty} \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \cdot q(\nu_{r+1}, \nu_m) < \frac{1}{\kappa}$ for every $\kappa \in (0, 1)$.

Then $\mathcal{T}$ has a unique FP in $\mathcal{X}$.

Proof. This is the special case of Theorem 4.1 in which $\eta \equiv 1$, so the contractive hypothesis reduces to the form stated here, since $\xi(\nu, \zeta) \cdot 1 = \xi(\nu, \zeta)$. $\square$

Corollary 4.5. Suppose $(\mathcal{X}, \delta_c)$ is a complete controlled MS with a single control function $\theta : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$ satisfying

$$\delta_c(\nu, \zeta) \leq \theta(\nu, \rho)\delta_c(\nu, \rho) + \theta(\rho, \zeta)\delta_c(\rho, \zeta).$$

Let $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a mapping satisfying:

$$\xi(\nu, \zeta)\delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa\mathcal{M}(\nu, \zeta), \quad \forall \nu, \zeta \in \mathcal{X},$$

where:

- $\mathcal{T}$ is ξ admissible;
- there exists $\nu_0 \in \mathcal{X}$ satisfying $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$.

If the Picard iterates $\nu_n = \mathcal{T}^n\nu_0$ satisfy:

$$\lim_{n \rightarrow \infty} \frac{\theta(\nu_{n+1}, \nu_{n+2})}{\theta(\nu_n, \nu_{n+1})} \cdot \theta(\nu_{n+1}, \nu_n) < \frac{1}{\kappa},$$

then $\mathcal{T}$ has a unique FP in $\mathcal{X}$.

Proof. Identifying both controls with a single function, $p(\nu, \zeta) = q(\nu, \zeta) = \theta(\nu, \zeta)$, reduces the double controlled metric of Theorem 4.1 to a controlled MS, under which the ratio limit hypothesis becomes precisely the condition stated here, and the result follows from Theorem 4.1 with $\eta \equiv 1$. $\square$

Corollary 4.6. Take $(\mathcal{X}, \delta_c)$ to be a complete DCMS with control functions $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ a mapping. Suppose there exists a function which is continuous strictly increasing $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega(t) = 0 \iff t = 0$, and a constant $\kappa \in [0, 1)$ such that:

$$\xi(\nu, \zeta)\omega(\delta_c(\mathcal{T}\nu, \mathcal{T}\zeta)) \leq \kappa\omega(\mathcal{M}(\nu, \zeta)),$$

where $\mathcal{M}(\nu, \zeta)$ is as in Theorem 4.1. If, in addition:

- $\mathcal{T}$ is ξ admissible;
- there exists $\nu_0 \in \mathcal{X}$ satisfying $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;
- $\lim_{r \rightarrow \infty} \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \cdot q(\nu_{r+1}, \nu_m) < \frac{1}{\kappa}$ for every $\kappa \in (0, 1)$;

then $\mathcal{T}$ contains a unique FP in $\mathcal{X}$.

Proof. Because ω is continuous, strictly increasing, and vanishes only at 0, it admits a continuous inverse ω^{-1} ; applying ω^{-1} throughout the hypothesised inequality gives

$$\xi(\nu, \zeta) \delta_c(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \omega^{-1}(\kappa \omega(\mathcal{M}(\nu, \zeta))).$$

Setting $\tilde{\kappa}(t) = \omega^{-1}(\kappa \omega(t))$, strict monotonicity of ω guarantees $\tilde{\kappa}(t) < t$ for all $t > 0$, and Theorem 4.1 applies through this comparison function, with $\eta \equiv 1$, to yield the claim. $\square$

4.2 Double Controlled Suprametric Spaces

Definition 4.2. Fix a nonempty set $\mathcal{X}$, control functions $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, \infty)$, and a constant $\sigma \geq 0$. A mapping $\delta_{cs} : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty)$ is called a *DCSMS* if, for all $\nu, \zeta, \rho \in \mathcal{X}$:

1. $\delta_{cs}(\nu, \zeta) = 0 \iff \nu = \zeta$.
2. $\delta_{cs}(\nu, \zeta) = \delta_{cs}(\zeta, \nu)$.
3. $\delta_{cs}(\nu, \zeta) \leq p(\nu, \rho) \delta_{cs}(\nu, \rho) + q(\rho, \zeta) \delta_{cs}(\rho, \zeta) + \sigma \delta_{cs}(\nu, \rho) \delta_{cs}(\rho, \zeta)$.

The idea here is simple: take the two independent control functions p, q from a DCMS, and add on top of them the multiplicative cross term $\sigma \delta_{cs}(\nu, \rho) \delta_{cs}(\rho, \zeta)$ that Berzig used when defining SMS. If $\sigma = 0$, condition (3) collapses back to $(\delta_c 3)$, so every DCMS is automatically a DCSMS with $\sigma = 0$, the case worth paying attention to is when $\sigma > 0$ actually matters.

Example 4.3. Let $\mathcal{X} = [0, 1]$ and define $\delta_{cs}(\nu, \zeta) = (\nu - \zeta)^2$. Take $p(\nu, \rho) = 2, q(\rho, \zeta) = 3$ and $\sigma = 2$. Then $(\mathcal{X}, \delta_{cs})$ satisfies the double controlled suprametric axioms.

Conditions (1) and (2) follow immediately from the properties of the squared difference. For (3), expanding the square gives

$$\begin{aligned} \delta_{cs}(\nu, \zeta) &= (\nu - \zeta)^2 = ((\nu - \rho) + (\rho - \zeta))^2 \\ &\leq (\nu - \rho)^2 + (\rho - \zeta)^2 + 2(\nu - \rho)(\rho - \zeta) \\ &\leq \delta_{cs}(\nu, \rho) + \delta_{cs}(\rho, \zeta) + 2\sqrt{\delta_{cs}(\nu, \rho)}\sqrt{\delta_{cs}(\rho, \zeta)}. \end{aligned}$$

With $p = 2, q = 3$ and $\sigma = 2$, we obtain

$$\delta_{cs}(\nu, \zeta) \leq 2\delta_{cs}(\nu, \rho) + 3\delta_{cs}(\rho, \zeta) + 2\delta_{cs}(\nu, \rho)\delta_{cs}(\rho, \zeta).$$

Thus $(\mathcal{X}, \delta_{cs})$ is a DCSMS.

Theorem 4.7. Consider $(\mathcal{X}, \delta_{cs})$, a complete DCSMS with independent controls $p, q : \mathcal{X} \times \mathcal{X} \rightarrow [1, +\infty)$ and a structural parameter $\sigma \geq 0$, and let $\xi, \eta : \mathcal{X} \times \mathcal{X} \rightarrow [1, +\infty)$ be two controlled functions. Suppose $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ is a self map fulfilling, for all $\nu, \zeta \in \mathcal{X}$, the unified contractive condition

$$\xi(\nu, \zeta) \cdot \eta(\nu, \zeta) \cdot \delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \kappa \cdot \mathcal{M}(\nu, \zeta), \quad (4.9)$$

where $\kappa \in [0, 1)$ and

$$\mathcal{M}(\nu, \zeta) = \max \left\{ \delta_{cs}(\nu, \zeta), \min \left\{ \frac{\delta_{cs}(\nu, \mathcal{T}\nu) \delta_{cs}(\zeta, \mathcal{T}\zeta)}{1 + \delta_{cs}(\nu, \zeta)}, \frac{\delta_{cs}(\nu, \mathcal{T}\zeta) \delta_{cs}(\zeta, \mathcal{T}\nu)}{1 + \delta_{cs}(\nu, \zeta)} \right\} \right\}. \quad (4.10)$$

Presume moreover that, for each $m \geq 1$, the dynamic control bounds obey the ratio limit requirement, and that:

1.

$$\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \cdot \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} < \frac{1}{\kappa} \quad (4.11)$$

2. $\mathcal{T}$ is ξ –admissible with respect to η ;

3. some $\nu_0 \in \mathcal{X}$ satisfies $\xi(\nu_0, \mathcal{T}\nu_0) \geq \eta(\nu_0, \mathcal{T}\nu_0)$;

4. $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq \eta(\nu_n, \nu_{n+1})$, one has $\xi(\nu, \mathcal{T}\nu) \geq \eta(\nu, \mathcal{T}\nu)$.

Then $\mathcal{T}$ owns a unique FP in $\mathcal{X}$.

Proof. Fix any starting point $\nu_0 \in \mathcal{X}$ and iterate $\mathcal{T}$:

$$\nu_1 = \mathcal{T}\nu_0, \quad \nu_2 = \mathcal{T}\nu_1, \quad \dots, \quad \nu_{n+1} = \mathcal{T}\nu_n, \quad \forall n \in \mathbb{N}.$$

As ξ and η take values ≥ 1 throughout:

$$\xi(\nu, \zeta) \geq 1 \quad \text{and} \quad \eta(\nu, \zeta) \geq 1, \quad \forall \nu, \zeta \in \mathcal{X}.$$

Hence along every consecutive pair,

$$\xi(\nu_n, \nu_{n+1}) \geq 1 \quad \text{and} \quad \eta(\nu_n, \nu_{n+1}) \geq 1, \quad \forall n \in \mathbb{N}.$$

Set $\nu = \nu_{n-1}$, $\zeta = \nu_n$ in the governing contractive inequality:

$$\begin{aligned} \xi(\nu_{n-1}, \nu_n) \cdot \eta(\nu_{n-1}, \nu_n) \cdot \delta_{cs}(\mathcal{T}\nu_{n-1}, \mathcal{T}\nu_n) &\leq \kappa \cdot \mathcal{M}(\nu_{n-1}, \nu_n), \\ \xi(\nu_{n-1}, \nu_n) \cdot \eta(\nu_{n-1}, \nu_n) \cdot \delta_{cs}(\nu_n, \nu_{n+1}) &\leq \kappa \cdot \mathcal{M}(\nu_{n-1}, \nu_n). \end{aligned} \quad (4.12)$$

As $\xi \geq 1$ and $\eta \geq 1$, the product $\xi(\nu_{n-1}, \nu_n) \eta(\nu_{n-1}, \nu_n) \geq 1$; therefore

$$\delta_{cs}(\nu_n, \nu_{n+1}) \leq \kappa \cdot \mathcal{M}(\nu_{n-1}, \nu_n). \quad (4.13)$$

$$\begin{aligned}
\mathcal{M}(\nu_{n-1}, \nu_n) &= \max \left\{ \delta_{cs}(\nu_{n-1}, \nu_n), \right. \\
&\quad \min \left\{ \frac{\delta_{cs}(\nu_{n-1}, \nu_n) \delta_{cs}(\nu_n, \nu_{n+1})}{1 + \delta_{cs}(\nu_{n-1}, \nu_n)}, \right. \\
&\quad \left. \frac{\delta_{cs}(\nu_{n-1}, \nu_{n+1}) \delta_{cs}(\nu_n, \nu_n)}{1 + \delta_{cs}(\nu_{n-1}, \nu_n)} \right\} \left. \right\}. \\
&\quad \min \left\{ \frac{\delta_{cs}(\nu_{n-1}, \nu_n) \delta_{cs}(\nu_n, \nu_{n+1})}{1 + \delta_{cs}(\nu_{n-1}, \nu_n)}, 0 \right\} = 0, \\
\mathcal{M}(\nu_{n-1}, \nu_n) &= \max\{\delta_{cs}(\nu_{n-1}, \nu_n), 0\} = \delta_{cs}(\nu_{n-1}, \nu_n). \tag{4.14}
\end{aligned}$$

Replace this in the previous display:

$$\delta_{cs}(\nu_n, \nu_{n+1}) \leq \kappa \cdot \delta_{cs}(\nu_{n-1}, \nu_n).$$

Inducting over the preceding steps again and again, we obtain the following:

$$\delta_{cs}(\nu_n, \nu_{n+1}) \leq \kappa^n \delta_{cs}(\nu_0, \nu_1), \quad \forall n \in \mathbb{N}. \tag{4.15}$$

To confirm the Cauchy property of $\{\nu_n\}$, for indices $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
\delta_{cs}(\nu_n, \nu_m) &\leq p(\nu_n, \nu_{n+1})\delta_{cs}(\nu_n, \nu_{n+1}) + q(\nu_{n+1}, \nu_m)\delta_{cs}(\nu_{n+1}, \nu_m) \\
&\quad + \sigma \delta_{cs}(\nu_n, \nu_{n+1})\delta_{cs}(\nu_{n+1}, \nu_m).
\end{aligned}$$

Collecting the common factor $\delta_{cs}(\nu_{n+1}, \nu_m)$,

$$\begin{aligned}
&= p(\nu_n, \nu_{n+1})\delta_{cs}(\nu_n, \nu_{n+1}) \\
&\quad + [q(\nu_{n+1}, \nu_m) + \sigma \delta_{cs}(\nu_n, \nu_{n+1})]\delta_{cs}(\nu_{n+1}, \nu_m).
\end{aligned}$$

Expanding $\delta_{cs}(\nu_{n+1}, \nu_m)$ in its turn,

$$\begin{aligned} \delta_{cs}(\nu_n, \nu_m) &\leq p(\nu_n, \nu_{n+1})\delta_{cs}(\nu_n, \nu_{n+1}) \\ &\quad + [q(\nu_{n+1}, \nu_m) + \sigma \delta_{cs}(\nu_n, \nu_{n+1})] \\ &\quad \times \left(p(\nu_{n+1}, \nu_{n+2})\delta_{cs}(\nu_{n+1}, \nu_{n+2}) \right. \\ &\quad \left. + [q(\nu_{n+2}, \nu_m) + \sigma \delta_{cs}(\nu_{n+1}, \nu_{n+2})]\delta_{cs}(\nu_{n+2}, \nu_m) \right). \end{aligned}$$

Expanding once more, the term $\delta_{cs}(\nu_{n+2}, \nu_m)$ delivers

$$\begin{aligned} \delta_{cs}(\nu_n, \nu_m) &\leq p(\nu_n, \nu_{n+1})\delta_{cs}(\nu_n, \nu_{n+1}) \\ &\quad + [q(\nu_{n+1}, \nu_m) + \sigma \delta_{cs}(\nu_n, \nu_{n+1})]p(\nu_{n+1}, \nu_{n+2})\delta_{cs}(\nu_{n+1}, \nu_{n+2}) \\ &\quad + [q(\nu_{n+1}, \nu_m) + \sigma \delta_{cs}(\nu_n, \nu_{n+1})] \\ &\quad \times [q(\nu_{n+2}, \nu_m) + \sigma \delta_{cs}(\nu_{n+1}, \nu_{n+2})]\delta_{cs}(\nu_{n+2}, \nu_m). \end{aligned}$$

Extending the expansion up to index $m - 1$ furnishes

$$\begin{aligned} \delta_{cs}(\nu_n, \nu_m) &\leq p(\nu_n, \nu_{n+1})\delta_{cs}(\nu_n, \nu_{n+1}) \\ &\quad + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i [q(\nu_j, \nu_m) + \sigma \delta_{cs}(\nu_{j-1}, \nu_j)] \right) \\ &\quad \times p(\nu_i, \nu_{i+1})\delta_{cs}(\nu_i, \nu_{i+1}). \end{aligned}$$

Inserting the term-by-term bound $\delta_{cs}(\nu_i, \nu_{i+1}) \leq \kappa^i \delta_{cs}(\nu_0, \nu_1)$, the full expansion transforms into

$$\begin{aligned} \delta_{cs}(\nu_n, \nu_m) &\leq \delta_{cs}(\nu_0, \nu_1) \left[p(\nu_n, \nu_{n+1})\kappa^n \right. \\ &\quad + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i [q(\nu_j, \nu_m) + \sigma \kappa^{j-1} \delta_{cs}(\nu_0, \nu_1)] \right) \\ &\quad \left. \times p(\nu_i, \nu_{i+1})\kappa^i \right]. \end{aligned}$$

Let the partial sum

$$R_r = \prod_{j=n+1}^r [q(\nu_j, \nu_m) + \sigma \kappa^{j-1} \delta_{cs}(\nu_0, \nu_1)] p(\nu_r, \nu_{r+1}) \kappa^r.$$

Now applying the ratio test using condition (1),

$$\lim_{r \rightarrow \infty} \frac{R_{r+1}}{R_r} = \lim_{r \rightarrow \infty} [q(\nu_{r+1}, \nu_m) + \sigma \kappa^r \delta_{cs}(\nu_0, \nu_1)] \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \kappa.$$

As $0 \leq \kappa < 1$, we have $\kappa^r \rightarrow 0$ as $r \rightarrow \infty$, and the ratio reduces to

$$\lim_{r \rightarrow \infty} \frac{R_{r+1}}{R_r} = \left(\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} \right) \kappa.$$

$\Rightarrow$

$$\lim_{r \rightarrow \infty} \frac{R_{r+1}}{R_r} < \frac{1}{\kappa} \cdot \kappa = 1.$$

The series, therefore converging absolutely to a finite value, and passing to the limit $n, m \rightarrow \infty$, $\lim_{n, m \rightarrow \infty} \delta_{cs}(\nu_n, \nu_m) = 0$, hence $\{\nu_n\}$ is Cauchy. By completeness, this Cauchy sequence possesses a unique limit in the space:

$$\lim_{n \rightarrow \infty} \nu_n = \nu^* \in \mathcal{X}.$$

Using condition (2) of Theorem 4.7, since $\delta_{cs}(\nu_{n+1}, \mathcal{T}\nu^*) = \delta_{cs}(\mathcal{T}\nu_n, \mathcal{T}\nu^*)$, the contractive condition furnishes

$$\xi(\nu_n, \nu^*) \eta(\nu_n, \nu^*) \delta_{cs}(\mathcal{T}\nu_n, \mathcal{T}\nu^*) \leq \kappa \cdot \mathcal{M}(\nu_n, \nu^*),$$

where

$$\mathcal{M}(\nu_n, \nu^*) = \max \left\{ \delta_{cs}(\nu_n, \nu^*), \right. \\ \min \left\{ \frac{\delta_{cs}(\nu_n, \nu_{n+1}) \delta_{cs}(\nu^*, \mathcal{T}\nu^*)}{1 + \delta_{cs}(\nu_n, \nu^*)}, \right. \\ \left. \frac{\delta_{cs}(\nu_n, \mathcal{T}\nu^*) \delta_{cs}(\nu^*, \nu_{n+1})}{1 + \delta_{cs}(\nu_n, \nu^*)} \right\} \left. \right\}.$$

Passing to the limit $n \rightarrow \infty$, both $\delta_{cs}(\nu_n, \nu^*) \rightarrow 0$ and $\delta_{cs}(\nu_n, \nu_{n+1}) \rightarrow 0$, hence

$$\mathcal{M}(\nu_n, \nu^*) = \max\{0, \min\{0, 0\}\} = 0.$$

Thus $\xi(\nu_n, \nu^*)\eta(\nu_n, \nu^*)\delta_{cs}(\mathcal{T}\nu_n, \mathcal{T}\nu^*) \geq 0$; this compels $\delta_{cs}(\mathcal{T}\nu_n, \mathcal{T}\nu^*) = 0$ in the limit, that is, $\delta_{cs}(\nu_{n+1}, \mathcal{T}\nu^*) \rightarrow 0$. This establishes that ν^* is a FP of $\mathcal{T}$.

For uniqueness, let $\zeta^* \in \mathcal{X}$ be another FP with $\mathcal{T}\zeta^* = \zeta^*$ and $\delta_{cs}(\nu^*, \zeta^*) > 0$. Then

$$\xi(\nu^*, \zeta^*)\eta(\nu^*, \zeta^*)\delta_{cs}(\mathcal{T}\nu^*, \mathcal{T}\zeta^*) \leq \kappa \cdot \mathcal{M}(\nu^*, \zeta^*).$$

Then

$$\delta_{cs}(\nu^*, \zeta^*) \leq \kappa \cdot \mathcal{M}(\nu^*, \zeta^*).$$

Now

$$\mathcal{M}(\nu^*, \zeta^*) = \max \left\{ \delta_{cs}(\nu^*, \zeta^*), \min \left\{ \frac{\delta_{cs}(\nu^*, \mathcal{T}\nu^*) \delta_{cs}(\zeta^*, \mathcal{T}\zeta^*)}{1 + \delta_{cs}(\nu^*, \zeta^*)}, \frac{\delta_{cs}(\nu^*, \mathcal{T}\zeta^*) \delta_{cs}(\zeta^*, \mathcal{T}\nu^*)}{1 + \delta_{cs}(\nu^*, \zeta^*)} \right\} \right\},$$

so, as $\mathcal{T}\nu^* = \nu^*$ and $\mathcal{T}\zeta^* = \zeta^*$,

$$\mathcal{M}(\nu^*, \zeta^*) = \max \left\{ \delta_{cs}(\nu^*, \zeta^*), \min \left\{ 0, \frac{\delta_{cs}(\nu^*, \zeta^*)^2}{1 + \delta_{cs}(\nu^*, \zeta^*)} \right\} \right\} = \delta_{cs}(\nu^*, \zeta^*).$$

Hence

$$\delta_{cs}(\nu^*, \zeta^*) \leq \kappa \cdot \delta_{cs}(\nu^*, \zeta^*),$$

and as $0 \leq \kappa < 1$, this is impossible unless $\delta_{cs}(\nu^*, \zeta^*) = 0$, compelling $\nu^* = \zeta^*$.

The uniqueness of the FP is thereby established. $\square$

Example 4.4. Let $\mathcal{X} = \{1, 2, 3\}$. Define $\delta_{cs} : \mathcal{X} \times \mathcal{X} \rightarrow [0, +\infty)$ by

$$\delta_{cs}(\nu, \nu) = 0, \quad \delta_{cs}(1, 2) = \delta_{cs}(2, 1) = 2,$$

$$\delta_{cs}(2, 3) = \delta_{cs}(3, 2) = 3, \quad \delta_{cs}(1, 3) = \delta_{cs}(3, 1) = 25.$$

Take $\sigma = 1$, and define

$$p(1, 2) = p(2, 3) = 5, \quad p(\nu, \zeta) = 1 \text{ otherwise,}$$

$$q(1, 2) = q(2, 3) = 5, \quad q(\nu, \zeta) = 1 \text{ otherwise.}$$

Let $\xi(\nu, \zeta) = \eta(\nu, \zeta) = 1$ for all ν, ζ , and define $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\mathcal{T}(1) = \mathcal{T}(2) = \mathcal{T}(3) = 1.$$

Then $(\mathcal{X}, \delta_{cs})$ is a DCSMS.

The corollaries below do for Theorem 4.7 what the Boyd-Wong, Geraghty, and Matkowski-type theorems do for the classical Banach principle: in each case, the fixed contraction constant κ is replaced by a weaker, function based comparison condition, and each turns out to be a special case of the main theorem.

In each of the following corollaries, the function η is taken to be identically 1, i.e. $\eta(\nu, \zeta) = 1$ for all $\nu, \zeta \in \mathcal{X}$. Accordingly, the symbol η is omitted from the statements and proofs below.

Corollary 4.8. Consider $(\mathcal{X}, \delta_{cs})$, a complete DCSMS. Let $\mathcal{S}$ be the class of functions $\theta : [0, \infty) \rightarrow [0, 1)$ with the property that $\limsup_{n \rightarrow \infty} \theta(t_n) = 1$ implies $t_n \rightarrow 0$. If $\mathcal{T}$ satisfies the Geraghty-type contraction

$$\xi(\nu, \zeta) \delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \theta(\mathcal{M}(\nu, \zeta)) \mathcal{M}(\nu, \zeta)$$

for $\nu, \zeta \notin \text{FP}(\mathcal{T})$, and:

- $\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \cdot \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} < \frac{1}{\kappa}$;
- $\mathcal{T}$ is ξ admissible;
- some $\nu_0 \in \mathcal{X}$ satisfies $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;

then $\mathcal{T}$ has a unique FP.

Proof. Because θ belongs to the class $\mathcal{S}$, it satisfies $\theta(t) < 1$ for every $t > 0$; substituting this bound gives

$$\xi(\nu, \zeta) \delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) < \mathcal{M}(\nu, \zeta)$$

whenever $\mathcal{M}(\nu, \zeta) > 0$, which is exactly the contractive form treated in Theorem 4.7 once κ is allowed to vary as $\theta(\mathcal{M}(\nu, \zeta))$ and $\eta \equiv 1$; the stated existence and uniqueness therefore carry over directly. $\square$

Corollary 4.9. Let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be upper semi continuous with $\Phi(t) < t$ for $t > 0$. If $\mathcal{T}$ satisfies the Boyd-Wong-type contraction

$$\xi(\nu, \zeta) \delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \Phi(\mathcal{M}(\nu, \zeta))$$

and:

- $\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \cdot \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} < \frac{1}{\kappa}$;
- $\mathcal{T}$ is ξ admissible;
- some $\nu_0 \in \mathcal{X}$ satisfies $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;

then $\mathcal{T}$ has a unique FP.

Proof. Since $\Phi(t) < t$ holds for every $t > 0$, the successive distances produced by the Picard iteration decrease strictly, which is enough to force the sequence to be Cauchy by the same estimate used in Theorem 4.7. Upper semi continuity of Φ then guarantees that the sequence's limit is fixed by $\mathcal{T}$, so existence and uniqueness follow as in Theorem 4.7 with $\eta \equiv 1$. $\square$

Corollary 4.10. Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be non-decreasing with $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for $t > 0$. If $\mathcal{T}$ satisfies the Matkowski-type contraction

$$\xi(\nu, \zeta) \delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) \leq \varphi(\mathcal{M}(\nu, \zeta))$$

and:

- $\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \cdot \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} < \frac{1}{\kappa}$;
- $\mathcal{T}$ is ξ admissible;
- some $\nu_0 \in \mathcal{X}$ satisfies $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;

then $\mathcal{T}$ has a unique FP.

Proof. As φ is non-decreasing and $\varphi^n(t) \rightarrow 0$ for every $t > 0$, the distances $\delta_{cs}(\nu_n, \nu_{n+1})$ generated by the Picard sequence are driven to zero in the limit, and the remaining argument mirrors Theorem 4.7 with $\eta \equiv 1$. $\square$

Corollary 4.11. If $\mathcal{T}$ satisfies the Edelstein-type strict contraction

$$\xi(\nu, \zeta) \delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) < \mathcal{M}(\nu, \zeta)$$

for $\nu, \zeta \notin \text{FP}(\mathcal{T})$ with $\mathcal{M}(\nu, \zeta) > 0$, and:

- $\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \cdot \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} < \frac{1}{\kappa}$;
- $\mathcal{T}$ is ξ admissible;
- some $\nu_0 \in \mathcal{X}$ satisfies $\xi(\nu_0, \mathcal{T}\nu_0) \geq 1$;
- $\mathcal{T}$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}\nu) \geq 1$;

then $\mathcal{T}$ has a unique FP.

Proof. Whenever $\mathcal{M}(\nu, \zeta) > 0$, the strict inequality $\delta_{cs}(\mathcal{T}\nu, \mathcal{T}\zeta) < \mathcal{M}(\nu, \zeta)$ leaves each Picard iterate strictly closer than the last, and the existence and uniqueness of the FP again follow by the reasoning of Theorem 4.7 with $\eta \equiv 1$. $\square$

Corollary 4.12. If for some $k \geq 1$, $\mathcal{T}^k$ satisfies the Sehgal-Guseman-type contraction

$$\xi(\nu, \zeta) \delta_{cs}(\mathcal{T}^k \nu, \mathcal{T}^k \zeta) \leq \kappa \mathcal{M}_k(\nu, \zeta)$$

where

$$\mathcal{M}_k(\nu, \zeta) = \max \left\{ \delta_{cs}(\nu, \zeta), \min \left\{ \frac{\delta_{cs}(\nu, \mathcal{T}^k \nu) \delta_{cs}(\zeta, \mathcal{T}^k \zeta)}{1 + \delta_{cs}(\nu, \zeta)}, \frac{\delta_{cs}(\nu, \mathcal{T}^k \zeta) \delta_{cs}(\zeta, \mathcal{T}^k \nu)}{1 + \delta_{cs}(\nu, \zeta)} \right\} \right\},$$

and:

- $\lim_{r \rightarrow \infty} q(\nu_{r+1}, \nu_m) \cdot \frac{p(\nu_{r+1}, \nu_{r+2})}{p(\nu_r, \nu_{r+1})} < \frac{1}{\kappa}$;
- $\mathcal{T}^k$ is ξ admissible;
- some $\nu_0 \in \mathcal{X}$ satisfies $\xi(\nu_0, \mathcal{T}^k \nu_0) \geq 1$;
- $\mathcal{T}^k$ is continuous, or else, for any sequence $\{\nu_n\}$ in $\mathcal{X}$ with $\nu_n \rightarrow \nu$ and $\xi(\nu_n, \nu_{n+1}) \geq 1$, one has $\xi(\nu, \mathcal{T}^k \nu) \geq 1$;

then $\mathcal{T}$ guarantees a unique FP.

Proof. Use Theorem 4.7 with $\eta \equiv 1$; $\mathcal{T}^k$ has a unique FP $\nu^* \in \mathcal{X}$. From $\mathcal{T}(\mathcal{T}^k \nu^*) = \mathcal{T}(\nu^*) = \mathcal{T}^k(\mathcal{T} \nu^*)$, the point $\mathcal{T} \nu^*$ is seen to also solve $\mathcal{T}^k \nu = \nu$; uniqueness for $\mathcal{T}^k$ then forces $\mathcal{T} \nu^* = \nu^*$, so ν^* is fixed by $\mathcal{T}$ itself, and its uniqueness is inherited from that of $\mathcal{T}^k$. $\square$

Chapter 5

Conclusion and Future Work

5.1 Conclusion

This thesis looked at FPT across several generalizations of classical MS. There were two goals behind this: to pull together results that already exist in the literature, and to push the range of settings where contraction based arguments still work.

We started from Fréchet's MS and the idea of completeness, then went through how the triangle inequality has been loosened step by step, first into SMS, then DCMS, and finally into their combination, DCSMS. Each of these spaces came with its own definition and a worked example to show what makes it different from the others.

One of the main results here is a FP theorem for single valued self mappings that are ξ admissible with respect to η and satisfy a generalized rational contraction written in terms of $\mathcal{M}(\nu, \zeta)$. This theorem is general enough that, by choosing the constant $\bar{b}$ and the control function θ appropriately, it reduces to the corresponding results for extended SMS, $\bar{b}$ SMS, and ordinary SMS. To show this is not just an abstract exercise, we used it to prove existence and uniqueness of solutions for a nonlinear Volterra-Fredholm integral equation, backed up by a concrete numerical example.

We carried the same admissibility idea over to the controlled metric setting as well. The main result there establishes FPs in DCMS, where the single scalar used in the extended $\bar{b}$ SMS setting is now split into two independent control functions, p and q . We also introduced DCSMS, a space that combines the DCMS structure with the multiplicative perturbation term from SMS, and proved a FP theorem for it too. Convergence of the Picard iterates in this setting is guaranteed by an explicit ratio limit condition, which gives a practical way to check the hypotheses when applying the theorem.

Taken together, this thesis both brings several existing FP results under one roof and introduces new spaces that connect generalization programs that had, until now, developed mostly separately.

5.2 Future Work

There are a few directions worth pursuing from here. Right now the screening functions ξ and η are assumed to take values in $[1, \infty)$. Allowing them to take values in $[0, \infty)$ instead would make the admissibility condition do more work, and might bring a wider range of known results under this same framework.

The ratio limit condition used for convergence could probably be relaxed or made more general. It would be worth looking for weaker convergence conditions that do not demand such strict asymptotic behaviour from the control functions, since this would widen where the theorems can be applied.

The distance functions used in controlled and SMS are not required to be continuous. A closer look at when continuity does hold, or what extra assumptions are needed to guarantee it, would put the limiting arguments used in the proofs on firmer ground.

Moving from single valued to multivalued mappings would also be a natural extension, and would open the door to applications in areas like differential inclusions and optimization.

The framework built here is not limited to the integral equation used as an example, it could just as easily be applied to fractional differential equations, boundary value problems, or more general integral equations. Each such application would both test the theory further and possibly point to new directions for extending it.

Finally, building numerical algorithms based on these FP theorems, along with an analysis of how fast they converge, would help connect this theoretical work to actual computation.

This thesis adds to the broader effort of generalizing MS and extending FPT. DCSMS in particular gives a flexible setting that draws together two lines of generalization that had mostly been developed on their own. Hopefully the results here can serve as a starting point for further work, both theoretical and applied, in nonlinear analysis.

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