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Fixed Point Results and their Geometry in S_b-Metric Spaces

by

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CERTIFICATE OF APPROVAL

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Abstract

This thesis develops a unified $\mathcal{M}$-class contractive framework for fixed point and fixed figure results in S-metric spaces (SMS) and symmetric S_b-metric spaces (SbMS). The class $\mathcal{M}$ consists of continuous, positively homogeneous, monotone functions that unify a broad range of contractive conditions as special cases.

Using this framework, the thesis proves existence and uniqueness results for fixed points and for fixed circles, ellipses, and elliptic discs in SMS, based on Picard iteration and a lemma reducing the $\mathcal{M}$-class inequality to a single contractive constant.

The theory is extended to symmetric SbMS, where the triangle inequality carries a relaxation coefficient b, where b is any non-negative real number. Convergence and fixed circle results depend on conditions involving the parameters b and the contractive constant. Analogous fixed ellipse and fixed elliptic disc theorems are proved.

The thesis concludes with directions for further generalization, including non-symmetric SbMS, controlled metric-type spaces, and multi-valued contractive mappings.

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Abbreviations

BVP	Boundary Value Problem
CUST	Capital University of Science and Technology
FPT	Fixed Point Theory
HEC	Higher Education Commission
ODE	Ordinary Differential Equation

Symbols

S	S-metric
S_b	S_b-metric
b	Coefficient of S_b-metric space
η^*	Element of the space
θ^*	Element of the space
ϑ^*	Element of the space
λ	Contraction constant
Λ	$\mathcal{M}$-class function
$\mathcal{M}$	Family of $\mathcal{M}$-class functions
$\mathcal{M}^*$	Subfamily of $\mathcal{M}$-class functions
Φ_1^*, Φ_2^*	Focal points of ellipse/disc
κ	Radius of circle
τ	Parameter of ellipse/elliptic disc
σ, ψ	Contraction constants
ς	Auxiliary function
$C(\eta_0^*, \kappa)$	Circle with centre η_0^* and radius κ
$E(\Phi_1^*, \Phi_2^*, \tau)$	Ellipse in S_b-metric space
$D(\Phi_1^*, \Phi_2^*, \tau)$	Elliptic disc in S_b-metric space
$\mathcal{G}(t, \xi)$	Green's function
Γ	Self-mapping / coupling operator
T	Integral operator in application
μ	Satellite web coupling parameter
$\alpha^*, \beta^*, \gamma^*, \delta^*$	Coefficients in contractive conditions

Chapter 1

Introduction

The concept of a *metric space*, introduced by Maurice Fréchet in 1906, offers a precise mathematical setting in which distances, convergence and continuity can be systematically investigated. Its simplicity and versatility have made it a cornerstone of modern mathematics, with widespread applications in analysis and topology.

Building on this foundation, in 1922, Stefan Banach introduced the Banach Contraction Principle, establishing that a contraction defined on a complete metric space posses a unique fixed point. This result lies at the heart of fixed-point theory and has found extensive applications across mathematics, engineering, economics, and the physical sciences.

Subsequent developments in fixed-point theory have followed two main directions. The first concerns the refinement of contractive conditions, leading to a rich collection of generalizations including those of Kannan [1], Chatterjea [2], Ćirić [3], Hardy and Rogers [4], Reich [5], Meir–Keeler [6] and more recently, contractive conditions in fuzzy settings with directed graphs [7].. Rhoades [8] observed that all such contractive conditions guaranteeing uniqueness also force continuity of the mapping at its fixed point. This raised the possibility of obtaining even when continuity is absent, which was subsequently demonstrated by Pant [9], who presented discontinuous mappings that nevertheless satisfy a Meir–Keeler[6] type condition.

The second direction concerns the generalization of the underlying distance structure. Among these extensions, the *b-metric space* relaxes the triangle inequality by introducing a multiplicative constant $b \geq 1$, thereby accommodating a broader class of problems. Another notable generalization is the *S-metric space*, which employs a three-point distance function, offering greater geometric flexibility. The synthesis of these two ideas led to the *S-b-metric space*, which combines the three-point distance structure of S-metric spaces with the relaxed triangle inequality of b-metric spaces.

The search for more flexible distance structures has resulted in several generalizations of classical metric spaces. Dhage [10] introduced D-metric spaces, followed by the G-metric spaces developed by Mustafa and Sims [11], the D*-metric spaces proposed by Sedghi, Shobe, and Zhou [12], and the S-metric spaces developed by Sedghi, Shobe, and Aliouche [13]. In contrast to ordinary metrics, these generalized frameworks rely on three variables rather than two to define distance, thereby providing a richer setting for investigating nonlinear problems. Expanding upon $\mathcal{S}$-metric environments, Souayah and Mlaiki [14] formulated $\mathcal{S}_{\mathfrak{b}}$-metric spaces, in which a scaling parameter $\mathfrak{b} \geq 1$ moderates the triangle-like inequality. Extensive investigations in these spaces have been carried out by Phaneendra [15], Phaneendra and Swamy [16], Gupta and Mani [17], and Taş [18], who established several fundamental fixed point results within these generalized environments.

A unified contractive framework based on $\mathcal{M}-$class functions was introduced by Joshi and Tomar [19] and later generalized by Joshi, Tomar, and Abdeljawad [20]. This framework incorporates many classical contractions, including those of Banach, Kannan, Chatterjea, Ćirić, Hardy–Rogers, and Reich, as particular cases. This class comprises continuous mappings from a five-dimensional domain satisfying homogeneity, monotonicity, and certain normalization properties. Appropriate choices of parameters recover a variety of well-known contractive conditions. The versatility of this approach has led to unified existence results for fixed points, fixed circles, fixed ellipses, and fixed elliptic discs in both S-metric and S_b-metric spaces. The geometric viewpoint introduced by Özgür and Taş [21, 22] has considerably broadened the scope of fixed point theory by focusing on geometric sets of fixed points rather than isolated solutions. Subsequent developments established analogous results for fixed circles [23, 24], fixed ellipses [25, 26], and fixed elliptic discs [20], revealing deeper geometric properties of self-mappings.

The significance of fixed figure theory extends beyond pure mathematics to several applied disciplines. For instance, applications to neural networks [20] and satellite web coupling problems [20, 27] illustrate the usefulness of these techniques. In neural network models, elliptical structures naturally arise in connection with Mexican-hat and Gaussian-wavelet activation functions. Moreover, discontinuous activation functions often provide improved storage capabilities by processing different input regions independently. A further noteworthy application of fixed point theory is found in the mathematical modeling of a thin elastic web joining cylindrical satellites. The associated nonlinear boundary value problem admits a rigorous analytical treatment via fixed point methods, consequently verifying that its solutions are both unique and existent.

The primary focus of this thesis is the further development of fixed geometric structure theory within the setting of generalized metric spaces. To this end, the class of functions $\mathcal{M}$ is employed to formulate a unified framework encompassing a wide spectrum of existence and uniqueness results in $\mathcal{S}-$metric spaces. These results provide a common framework encompassing several well-known contraction principles. We then extend the theory to symmetric S_b-metric spaces, where additional analytical difficulties arise because of the relaxed triangle inequality. In this setting, the condition $b\lambda < 1$ plays a fundamental role in obtaining Cauchy estimates, whereas the inequality $\Lambda(1, 1, 0, 3b, 1) < 1$ becomes essential in proving fixed circle results. Moreover, the theoretical framework developed herein is applied to the satellite web coupling problem to derive existence and uniqueness results for its solutions in both $\mathcal{S}-$metric and symmetric $\mathcal{S}_\mathfrak{b}-$metric spaces. The study concludes with an analysis of the continuity and discontinuity characteristics of self-mappings at fixed ellipses, offering a novel perspective on Rhoades' unresolved problem within broader metric settings.

The thesis is structured as follows. Chapter 2 introduces the definitions and preliminary results on which the later chapters depend: $\mathcal{S}$-metric spaces, $\mathcal{S}_b-$metric spaces, symmetric $\mathcal{S}_b-$metric spaces, and $\mathcal{M}-$class functions, along with the geometric notions of circles, ellipses, and elliptic discs. With this groundwork in place, Chapter 3 establishes the fixed figure theory in $\mathcal{S}-$metric spaces through a a succession of theorems concerning fixed points, circles, ellipses, and elliptic discs. Applications to neural network activation functions and the satellite web coupling problem are also presented, along with continuity results for fixed ellipses. Chapter 4 extends these investigations to symmetric S_b-metric spaces, where the relaxed triangle inequality introduces new analytical challenges, particularly in deriving Cauchy estimates under the condition $b\lambda < 1$. Ultimately, Chapter 5 provides a summary of the main conclusions derived in this work and details of several possible directions for future research, including broader contractive frameworks, additional geometric fixed figures, and further applications in mathematics, science, and engineering.

Chapter 2

Basic Terminologies

The purpose of this chapter is to establish the essential definitions, notations, and mathematical structures that will be required later. We begin by examining the frameworks of S-metrics and S_b-metric spaces, which generalize the conventional concept of a metric by assessing distances using a triad of variables rather than pairs. The subsequent discussion is devoted to symmetric SbMS, with particular emphasis on the fundamental concepts of convergence, Cauchy sequences, and completeness. Subsequently, the class of $\mathcal{M}$-functions is presented as a unified tool for formulating a broad family of contractive conditions. Finally, the geometric objects that form the foundation of fixed figure theory, namely circles, ellipses, and the elliptic discs, used in later chapters, are introduced.

2.1 S-Metric Space and S_b-Metric Space

Our preliminary review begins with the concept of an S-metric space, a widely recognized adaptation that expands upon traditional metric theory. Unlike an ordinary metric, an S-metric evaluates the relationship among three points simultaneously, making it particularly suitable for many generalized FP investigations.

Definition 2.1.1. A mapping

$$S:\ U\times\ U\times\ U\to\ [0,\infty)$$

is referred to as an SMS provided that it fulfills each of the following axioms for arbitrary $\eta^*,\theta^*,\vartheta^*,a^*\in\ U$:

(S1): $S(\eta^*,\theta^*,\vartheta^*)\geq 0$

(S2): $S(\eta^*,\theta^*,\vartheta^*)\ =0\Rightarrow \eta^*=\theta^*=\vartheta^*.$

(S3): $S(\eta^*,\theta^*,\vartheta^*)\leq S(\eta^*,\eta^*,a^*)+S(\theta^*,\theta^*,a^*)+S(\vartheta^*,\vartheta^*,a^*)$ [13]

Example 2.1.2. Let $U=\mathbb{R}$ and define

$$S(\eta^*,\theta^*,\vartheta^*)=|\eta^*-\vartheta^*|+|\theta^*-\vartheta^*|.$$

(S1): $S(\eta^*,\theta^*,\vartheta^*)\geq 0.$

(S2): $S(\eta^*,\theta^*,\vartheta^*)=0\quad\text{iff}\quad \eta^*=\theta^*=\vartheta^*.$

(S3): We check

$$S(\eta^*,\theta^*,\vartheta^*)\leq b\Big[S(\eta^*,\eta^*,a^*)+S(\theta^*,\theta^*,a^*)+S(\vartheta^*,\vartheta^*,a^*)\Big].$$

Since

$$S(\eta^*,\theta^*,\vartheta^*)=|\eta^*-\vartheta^*|+|\theta^*-\vartheta^*|,$$

we have

$$\begin{aligned}S(\eta^*,\eta^*,a^*)&=|\eta^*-a^*|+|\eta^*-a^*|\\&=2|\eta^*-a^*|,\\S(\theta^*,\theta^*,a^*)&=2|\theta^*-a^*|,\\S(\vartheta^*,\vartheta^*,a^*)&=2|\vartheta^*-a^*|.\end{aligned}$$

now,

$$|\eta^* - \vartheta^*| \le |\eta^* - a^*| + |\vartheta^* - a^*|,$$
$$|\theta^* - \vartheta^*| \le |\theta^* - a^*| + |\vartheta^* - a^*|.$$

Adding the above inequalities, we get

$$\begin{aligned}|\eta^* - \vartheta^*| + |\theta^* - \vartheta^*| &\le |\eta^* - a^*| + |\vartheta^* - a^*| + |\theta^* - a^*| + |\vartheta^* - a^*| \\ &= |\eta^* - a^*| + |\theta^* - a^*| + 2|\vartheta^* - a^*|.\end{aligned}$$

Therefore,

$$S(\eta^*, \theta^*, \vartheta^*) \le \frac{1}{2}S(\eta^*, \eta^*, a^*) + \frac{1}{2}S(\theta^*, \theta^*, a^*) + S(\vartheta^*, \vartheta^*, a^*).$$

Also,

$$\begin{aligned}&S(\eta^*, \eta^*, a^*) + S(\theta^*, \theta^*, a^*) + S(\vartheta^*, \vartheta^*, a^*) \\ &\qquad = 2|\eta^* - a^*| + 2|\theta^* - a^*| + 2|\vartheta^* - a^*|.\end{aligned}$$

Hence,

$$S(\eta^*, \theta^*, \vartheta^*) \le S(\eta^*, \eta^*, a^*) + S(\theta^*, \theta^*, a^*) + S(\vartheta^*, \vartheta^*, a^*).$$

Thus, it $S(\eta^*, \theta^*, \vartheta^*) = |\eta^* - \vartheta^*| + |\theta^* - \vartheta^*|$ fulfills each of the following axioms of an S-metric space. Therefore, it $(\mathbb{R}, S)$ is an S-metric space.

We next consider SbMS, which further enlarges this setting by introducing a relaxation parameter into the triangle inequality.

Definition 2.1.3. Consider a set U together with a fixed real parameter $b \ge 1$. A mapping

$$S_b : U \times U \times U \to [0, \infty)$$

is regarded as a SbM $\Rightarrow$ it fulfills each of the following defining axioms for arbitrary $\eta^*, \theta^*, \vartheta^*, a^* \in U$:

(B1): Non-negativity: $S_b(\eta^*, \theta^*, \vartheta^*) \geq 0$.

(B2): Identity condition: $S_b(\eta^*, \theta^*, \vartheta^*) = 0$ iff $\eta^* = \theta^* = \vartheta^*$.

(B3): Relaxed triangle inequality: $S_b(\eta^*, \theta^*, \vartheta^*) \leq b[S_b(\eta^*, \eta^*, a^*) + S_b(\theta^*, \theta^*, a^*) + S_b(\vartheta^*, \vartheta^*, a^*)]$. [14]

The constant b serves as the relaxation parameter in the generalized triangle inequality. When $b = 1$, the condition reduces to the standard triangle inequality of the S-metric.

Example 2.1.4. Consider the set $U = \mathbb{R}$ and define it.

$$S(\eta^*, \theta^*, \vartheta^*) = \big(|\eta^* - \vartheta^*| + |\theta^* - \vartheta^*|\big)^2.$$

(B1): $S_b(\eta^*, \theta^*, \vartheta^*) \geq 0$.

(B2): $S_b(\eta^*, \theta^*, \vartheta^*) = 0$ if and only if $\eta^* = \theta^* = \vartheta^*$.

(B3): We verify the relaxed triangle inequality:

$$S_b(\eta^*, \theta^*, \vartheta^*) \leq b\Big[S_b(\eta^*, \eta^*, a^*) + S_b(\theta^*, \theta^*, a^*) + S_b(\vartheta^*, \vartheta^*, a^*)\Big].$$

Given that

$$S_b(\eta^*, \theta^*, \vartheta^*) = \big(|\eta^* - \vartheta^*| + |\theta^* - \vartheta^*|\big)^2,$$

and

$$S_b(\eta^*, \eta^*, a^*) = 4|\eta^* - a^*|^2,$$
$$S_b(\theta^*, \theta^*, a^*) = 4|\theta^* - a^*|^2,$$
$$S_b(\vartheta^*, \vartheta^*, a^*) = 4|\vartheta^* - a^*|^2.$$

Observe that,

$$|\eta^* - a^* + a^* - \vartheta^*| = |(\eta^* - a^*) - (\vartheta^* - a^*)| \leq |\eta^* - a^*| + |\vartheta^* - a^*|,$$

that is,

$$|\eta^* - \vartheta^*| \leq |\eta^* - a^*| + |\vartheta^* - a^*|.$$

Similarly,

$$|\theta^* - \vartheta^*| \leq |\theta^* - a^*| + |\vartheta^* - a^*|.$$

Adding the above inequalities, we obtain

$$|\eta^* - \vartheta^*| + |\theta^* - \vartheta^*| \leq |\eta^* - a^*| + |\theta^* - a^*| + 2|\vartheta^* - a^*|.$$

Consequently,

$$S_b(\eta^*, \theta^*, \vartheta^*) \leq \big(|\eta^* - a^*| + |\theta^* - a^*| + 2|\vartheta^* - a^*|\big)^2.$$

it follows that

$$S_b(\eta^*, \theta^*, \vartheta^*) \leq 3\Big(|\eta^* - a^*|^2 + |\theta^* - a^*|^2 + 4|\vartheta^* - a^*|^2\Big).$$

Moreover,

$$S_b(\eta^*, \eta^*, a^*) + S_b(\theta^*, \theta^*, a^*) + S_b(\vartheta^*, \vartheta^*, a^*) = 4|\eta^* - a^*|^2 + 4|\theta^* - a^*|^2 + 4|\vartheta^* - a^*|^2.$$

Therefore,

$$S_b(\eta^*, \theta^*, \vartheta^*) \leq 3\Big[S_b(\eta^*, \eta^*, a^*) + S_b(\theta^*, \theta^*, a^*) + S_b(\vartheta^*, \vartheta^*, a^*)\Big].$$

hence, it

$$S_b(\eta^*, \theta^*, \vartheta^*) = \big(|\eta^* - \vartheta^*| + |\theta^* - \vartheta^*|\big)^2$$

fulfills all the defining axioms of an SbMS with coefficient $b = 3$. However, the $S-$metric triangle inequality is not satisfied when $b = 1$. Consequently, this construction provides an example of an SbMS that does not, in general, constitute a SMS.

Definition 2.1.5. An SbMS (U, S_b) is termed symmetric if the condition

$$S_b(\eta^*, \eta^*, \theta^*) = S_b(\theta^*, \theta^*, \eta^*)$$

holds for all $\eta^*, \theta^* \in U$. [14]

We conclude this section by recalling the standard notions of convergence, Cauchy sequences, and completeness, which extend naturally from classical metric spaces to the symmetric SbM setting.

Theorem 2.1.6. The collection of SMS is contained within the collection of SbMSs.

Proof. For any arbitrary SMS, the following inequality holds:

$$S(\eta^*, \theta^*, \vartheta^*) \leq S(\vartheta^*, \vartheta^*, a^*) + S(\theta^*, \theta^*, a^*) + S(\eta^*, \eta^*, a^*).$$

Selecting $b = 1$, we obtain

$$S(\eta^*, \theta^*, \vartheta^*) \leq 1 \cdot \left[S(\vartheta^*, \vartheta^*, a^*) + S(\theta^*, \theta^*, a^*) + S(\eta^*, \eta^*, a^*) \right].$$

Consequently, the assertion holds for $b = 1$. $\square$

Definition 2.1.7. Given a sequence $\{\eta_n^*\}$ within a symmetric SbMS (U, S_b), the subsequent criteria apply:

i. **Convergence** $\{\eta_n^*\}$ to a target point $\eta^* \in U$ is established when the following limit condition holds true:

$$\lim_{n\to\infty} S_b(\eta_n^*, \eta_n^*, \eta^*) = 0$$

ii. The designation of $\{\eta_n^*\}$ a **Cauchy sequence** occurs when the distance metric approaches zero for arbitrarily large indices:

$$\lim_{n,m\to\infty} S_b(\eta_n^*, \eta_n^*, \eta_m^*) = 0, \qquad \forall\, n, m > N(\text{Natural number})$$

iii. The space (U, S_b) earns the label of being **complete** on the condition that any Cauchy sequence constructed within this domain possesses a valid limit belonging directly to U. [14]

2.2 The $\mathcal{M}$-Class Functions

To establish the unified fixed figure results presented in this thesis, we require a general class of contractive functions capable of representing numerous well-known contraction principles within a single mathematical framework. The family of $\mathcal{M}-$class functions [20] fulfils this objective by providing a systematic formulation that includes several classical contractions as particular instances.

Definition 2.2.1. Suppose $\mathcal{M}$ represents the class of continuous mappings $\Lambda : [0, \infty)^5 \to [0, \infty)$ satisfying the following three conditions:

($\mathcal{M}_1$): The value at the specific point $(1, 1, 0, 3, 1)$ lies in the interval $[0, 1)$, i.e., $\Lambda(1, 1, 0, 3, 1) \in [0, 1)$.

($\mathcal{M}_2$): Positive homogeneity of degree one: For every $\lambda \geq 0$ and every $(x_1, \ldots, x_5) \in [0, \infty)^5$, $\Lambda(\lambda x_1, \ldots, \lambda x_5) = \lambda\Lambda(x_1, \ldots, x_5)$.

($\mathcal{M}_3$): Monotonicity: Whenever $x_i \leq y_i$ for all $i = 1, \ldots, 5$, the inequality $\Lambda(x_1, \ldots, x_5) \leq \Lambda(y_1, \ldots, y_5)$ must hold.

Any mapping satisfying these three properties is referred to as an $\mathcal{M}$-class function. The particular evaluation $\Lambda(1, 1, 0, 3, 1)$ serves as a crucial factor in computing the constants of contraction [20].

The following examples demonstrate how several well-known contraction mappings arise naturally as particular members of the $\mathcal{M}$-class.

Example 2.2.2. In the following examples of $\mathcal{M}-$class function. i.e Λ_i : $[0, \infty)^5 \to [0, \infty)$ where $i \leqslant 9 \in N$(Natural number) is given as follows.

$$\Lambda_1(x_1, \ldots, x_5) = a^* x_1, \qquad a^* \in [0, 1).$$

(i) For $a^* \in \left[0, \frac{1}{3}\right)$, the function

$$\Lambda_2(x_1, \dots, x_5) = a^*(x_2 + x_5).$$

(ii) If $a^* \in \left[0, \frac{1}{3}\right)$, then

$$\Lambda_3(x_1, \dots, x_5) = a^* \max\{x_2, x_5\}$$

(iii) Let

$$\Lambda_4(x_1, \dots, x_5) = a^* x_1 + \beta^* x_2 + \gamma^* x_5,$$

where

$$a^* + \beta^* + \gamma^* \in [0, 1).$$

(iv) The mapping

$$\Lambda_5(x_1, \dots, x_5) = \max\{x_1, x_2, x_5\}$$

(v) For $a^* \in \left[0, \frac{1}{3}\right)$,

$$\Lambda_6(x_1, \dots, x_5) = a^*(x_3 + x_4)$$

(vi) Similarly,

$$\Lambda_7(x_1, \dots, x_5) = a^* \max\{x_3, x_4\}, \qquad a^* \in \left[0, \frac{1}{3}\right),$$

(vii) Let

$$\Lambda_8(x_1, \dots, x_5) = a^* x_1 + \beta^* x_2 + \gamma^*(x_3 + x_4) + \delta^* x_5,$$

where

$$a^* + \beta^* + 3\gamma^* + \delta^* \in [0, 1).$$

(viii) Finally, for $a^* \in \left[0, \frac{1}{3}\right)$,

$$\Lambda_9(x_1, \dots, x_5) = a^* \max\{x_1, x_2, x_3, x_4, x_5\}$$

2.3 Fixed Point and their Geometric Concepts

The principal objective of this thesis is the investigation of fixed geometric objects in generalized metric spaces. For this purpose, it is necessary to introduce the geometric structures that will be studied throughout the subsequent chapters.

Definition 2.3.1. Let (U, S_b) be a symmetric SbMS and let it $\Gamma : U \to U$ be a self-mapping. A point $\eta^* \in U$ is called an FP of Γ if

$$\Gamma(\eta^*) = \eta^*$$

$$Fix(\Gamma) = \{\eta^* \in U : \Gamma(\eta^*) = \eta^*\}.$$

Definition 2.3.2. it (U, S_b) is a symmetric SbMS. Given a center $\eta_0^* \in U$ and a radius $\kappa > 0$, the circle with the center η_0^* is defined as

$$C(\eta_0^*, \kappa) = \{\eta^* \in \ U : S_b(\eta^*, \eta^*, \eta_0^*) = \kappa\}.$$

follows: for a self-mapping $\Gamma : \ U \to \ U$, any circle $C(\eta_0^*, \kappa)$ that satisfies the FP criterion $\Gamma(\eta^*) = \eta^*$ for all $\eta^* \in C(\eta_0^*, \kappa)$ is designated as a fixed circle of Γ. [28]

Definition 2.3.3. Let (U, S_b) be a symmetric SbMS and let $\Phi_1^*, \Phi_2^* \in \ U$ be two distinct focal points. For a parameter $\tau \geq 0$, the associated ellipse is the set

$$E(\Phi_1^*, \Phi_2^*, \tau) = \{\eta^* \in \ U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) = 2\tau\}.$$

The corresponding elliptic disc is obtained by replacing equality with inequality:

$$D(\Phi_1^*, \Phi_2^*, \tau) = \{\eta^* \in \ U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) \leq 2\tau\}.$$

For a non-degenerate ellipse, the condition

$$S_b(\Phi_1^*, \Phi_1^*, \Phi_2^*) < 2\tau$$

is required. The quantity

$$2f_0 = S_b(\Phi_1^*, \Phi_1^*, \Phi_2^*)$$

is referred to as the *linear eccentricity* of the ellipse. The center is identified as the midpoint of the line segment connecting the focal points Φ_1^* and Φ_2^*. The straight line passing through these foci specifies the principal axis, whereas a centrally located line positioned perpendicular to the principal axis forms the shorter axis. [20]

Example 2.3.4. Let $U = \mathbb{R}$ and define $S_b(\eta^*, \theta^*, \vartheta^*) = \big(|\eta^* - \vartheta^*| + |\eta^* + \vartheta^* - 2\theta^*|\big)^2$, $\eta^*, \theta^*, \vartheta^* \in U$. Since $(a+b)^2 \leq 2(a^2+b^2)$, (U, S_b) is an SbMS with $b = 2$. Let

$$\Phi_1^* = -2, \qquad \Phi_2^* = 2.$$

$$S_b(-2,-2,2) = \big(|-2-2| + |-2+2+4|\big)^2 = (4+4)^2 = 64.$$

Choose $\tau = 34$ (so $2\tau = 68$). The ellipse is determined by the foci -2 and 2 is

$$E(-2,2,34) = \big\{\eta^* \in \mathbb{R} : S_b(-2,-2,\eta^*) + S_b(2,2,\eta^*) = 68\big\}.$$

Since

$$S_b(-2,-2,\eta^*) = 4|\eta^*+2|^2, \quad S_b(2,2,\eta^*) = 4|\eta^*-2|^2,$$

the ellipse equation $4|\eta^*+2|^2 + 4|\eta^*-2|^2 = 68$ gives $|\eta^*+2|^2 + |\eta^*-2|^2 = 17$. Using $(\eta^*+2)^2 + (\eta^*-2)^2 = 2\eta^{*2} + 8$, we obtain $\eta^{*2} = \frac{9}{2}$. Thus

$$E(-2,2,34) = \Big\{-\tfrac{3}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}\Big\}, \qquad D(-2,2,34) = \Big[-\tfrac{3}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}\Big].$$

Example 2.3.5. Let $U = \mathbb{R}^2$. For any points $\eta^* = (\eta_1^*, \eta_2^*)$, $\theta^* = (\theta_1^*, \theta_2^*)$, $\vartheta^* = (\vartheta_1^*, \vartheta_2^*) \in U$, define

$$S_b(\eta^*, \theta^*, \vartheta^*) = \left(\sum_{\mathrm{i}=1}^{2} \big(|\eta_{\mathrm{i}}^* - \vartheta_{\mathrm{i}}^*| + |\eta_{\mathrm{i}}^* + \vartheta_{\mathrm{i}}^* - 2\theta_{\mathrm{i}}^*|\big)\right)^2.$$

Then (U, S_b) is an SbMS. With $\Phi_1^* = (2,0)$ and $\Phi_2^* = (0,2)$,

$$S_b(\Phi_1^*, \Phi_1^*, \Phi_2^*) = \big(|2-0| + |2+0-4| + |0-2| + |0+2-0|\big)^2 = (2+2+2+2)^2 = 64.$$

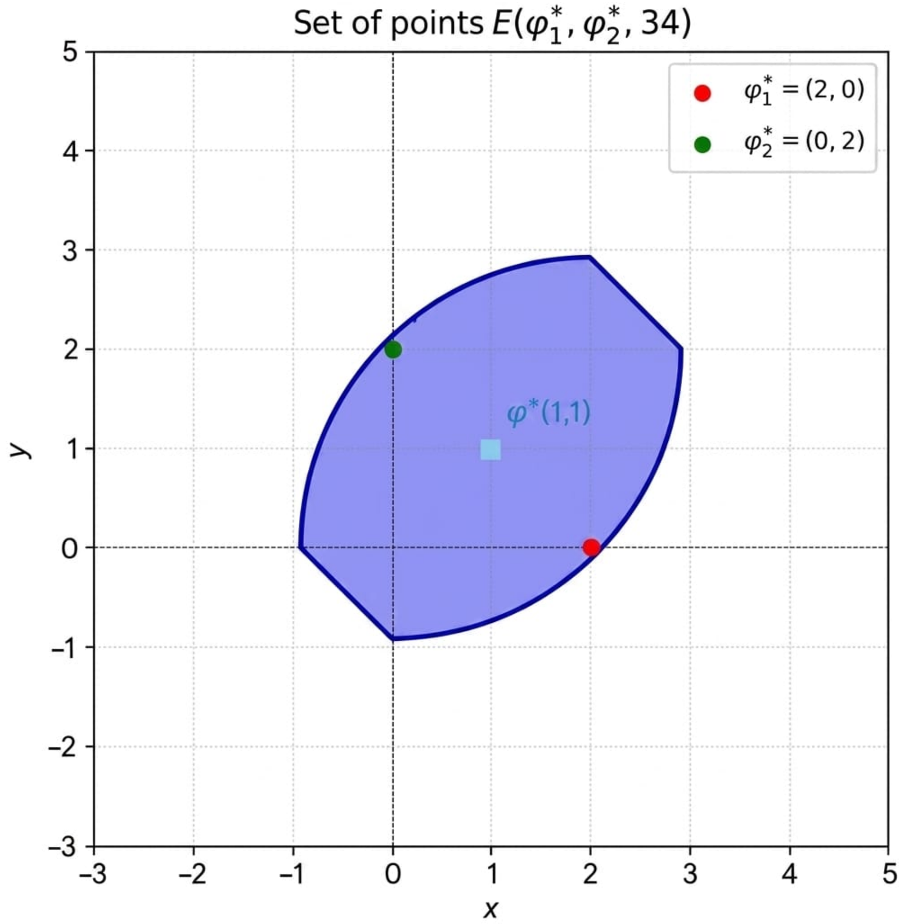

FIGURE 2.1: Visualization of the S_b-ellipse determined by the foci $\Phi_1^* = (2,0)$ and $\Phi_2^* = (0,2)$ with parameter $\tau = 34$. The blue curve represents the set of all points $\eta^* \in U$ satisfying $S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) = 2\tau$.

Take $\tau = 34$ (so $2\tau = 68$). Now,

$$E(\Phi_1^*, \Phi_2^*, 34) = \Big\{\eta^* \in U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) = 68\Big\}.$$

Let $\eta^* = (\eta_1^*, \eta_2^*)$. Then

$$\begin{aligned} S_b(\Phi_1^*, \Phi_1^*, \eta^*) &= \big(2|\eta_1^*| + 2|2 - \eta_2^*|\big)^2 \\ &= 4\big(|\eta_1^*| + |2 - \eta_2^*|\big)^2. \end{aligned}$$

Therefore,

$$E(\Phi_1^*, \Phi_2^*, 34) = \Big\{\eta^* \in U : 4\big(|2 - \eta_1^*| + |\eta_2^*|\big)^2 + 4\big(|\eta_1^*| + |2 - \eta_2^*|\big)^2 = 68\Big\}.$$

Equivalently,

$$E(\Phi_1^*,\Phi_2^*,34)=\Big\{\eta^*\in U:\big(|2-\eta_1^*|+|\eta_2^*|\big)^2+\big(|\eta_1^*|+|2-\eta_2^*|\big)^2=17\Big\}.$$

Hence, $E(\Phi_1^*,\Phi_2^*,34)$ represents an ellipse in the SbMS (U,S_b) with centre $(1,1)$ and foci $(2,0)$ and $(0,2)$. Similarly, the corresponding elliptic disc is

$$D(\Phi_1^*,\Phi_2^*,34)=\Big\{\eta^*\in U:\big(|2-\eta_1^*|+|\eta_2^*|\big)^2+\big(|\eta_1^*|+|2-\eta_2^*|\big)^2\leq 17\Big\}.$$

Example 2.3.6. Let $U=\mathbb{R}^2$ and define $S_b:U\times U\times U\to[0,\infty)$ by

$$S_b(\eta^*,\theta^*,\vartheta^*)=\big[(\eta_1^*-\theta_1^*)^2+(\eta_2^*-\theta_2^*)^2+(\theta_1^*-\vartheta_1^*)^2+(\theta_2^*-\vartheta_2^*)^2+(\vartheta_1^*-\eta_1^*)^2+(\vartheta_2^*-\eta_2^*)^2\big]^2,$$

where $\eta^*=(\eta_1^*,\eta_2^*)$, $\theta^*=(\theta_1^*,\theta_2^*)$, $\vartheta^*=(\vartheta_1^*,\vartheta_2^*)\in U$. Then (U,S_b) is an SbMS. With $\Phi_1^*=(2,0)$, $\Phi_2^*=(0,2)$,

$$S_b(\Phi_1^*,\Phi_1^*,\Phi_2^*)=\big[(2-0)^2+(0-2)^2+(0-2)^2+(2-0)^2\big]^2=16^2=256.$$

Choose $\tau=130$ (so $S_b(\Phi_1^*,\Phi_1^*,\Phi_2^*)<2\tau$). Now,

$$E(\Phi_1^*,\Phi_2^*,130)=\Big\{\eta^*\in U:S_b(\Phi_1^*,\Phi_1^*,\eta^*)+S_b(\Phi_2^*,\Phi_2^*,\eta^*)=260\Big\}.$$

For $\eta^*=(\eta_1^*,\eta_2^*)$,

$$\begin{aligned} S_b(\Phi_1^*,\Phi_1^*,\eta^*)&=\big[2(2-\eta_1^*)^2+2\eta_2^{*2}\big]^2,\\ \text{and}\quad S_b(\Phi_2^*,\Phi_2^*,\eta^*)&=\big[2\eta_1^{*2}+2(2-\eta_2^*)^2\big]^2.\end{aligned}$$

Therefore,

$$E(\Phi_1^*,\Phi_2^*,130)=\Big\{\eta^*\in U:\big[2(2-\eta_1^*)^2+2\eta_2^{*2}\big]^2+\big[2\eta_1^{*2}+2(2-\eta_2^*)^2\big]^2=260\Big\}.$$

Equivalently,

$$E(\Phi_1^*,\Phi_2^*,130)=\Big\{\eta^*\in U:4\big[(2-\eta_1^*)^2+\eta_2^{*2}\big]^2+4\big[\eta_1^{*2}+(2-\eta_2^*)^2\big]^2=260\Big\}.$$

Similarly,

$$D(\Phi_1^*, \Phi_2^*, 130) = \Big\{\eta^* \in U : 4\big[(2-\eta_1^*)^2 + \eta_1^{*2}\big]^2 + 4\big[\eta_1^{*2} + (2-\eta_2^*)^2\big]^2 \leq 260\Big\}.$$

Hence, $E(\Phi_1^*, \Phi_2^*, 130)$ and $D(\Phi_1^*, \Phi_2^*, 130)$ are, respectively, the ellipse and elliptic disc in the SbMS (U, S_b) having center $(1,1)$ and foci $(2,0)$ and $(0,2)$.

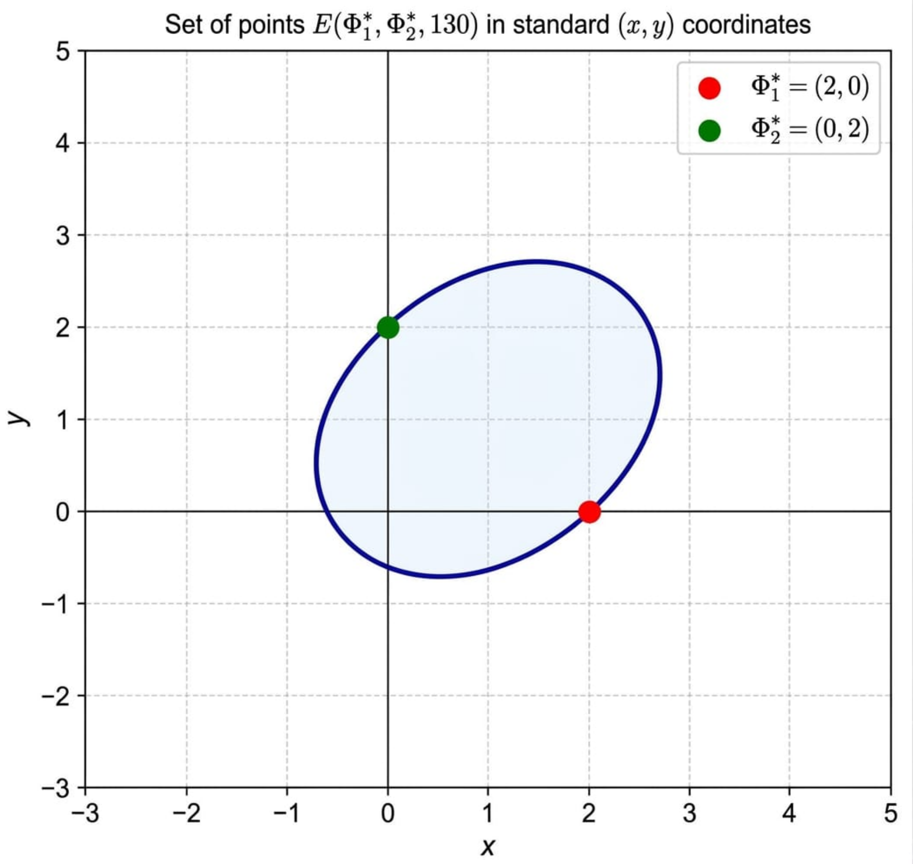

FIGURE 2.2: Visualization of the elliptic region $E(\Phi_1^*, \Phi_2^*, 130)$ under the S_b-metric, plotted in standard (η_1^*, η_2^*) coordinates with foci at $\Phi_1^* = (2,0)$ and $\Phi_2^* = (0,2)$.

Definition 2.3.7. Let (U, S_b) be an SbMS with a symmetric S_b given a point $\Phi^* \in U$, and a radius; $\tau \geq 0$ the disc centered at Φ^* is defined by

$$D(\Phi^*, \tau) = \{\eta^* \in U : S_b(\Phi^*, \Phi^*, \eta^*) \leq \tau\}.$$

if $\Gamma(\eta^*) = \eta^*$ for every $\eta^* \in S(\Phi^*, \tau)$. [28]

Definition 2.3.8. Let (U, S_b) be a symmetric SbMS with foci $\Phi_1^*, \Phi_2^* \in U$. The elliptic disc is the set

$$D(\Phi_1^*, \Phi_2^*, \tau) = \{\eta^* \in U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) \leq 2\tau\}.$$

A self-mapping $\Gamma : U \to U$ admits $S(\Phi_1^*, \Phi_2^*, \tau)$ as its fixed elliptic disc if the FP equation $\Gamma(\eta^*) = \eta^*$ remains valid for every point η^* belonging to $D(\Phi_1^*, \Phi_2^*, \tau)$. [28]

Example 2.3.9. Let $U = \mathbb{R}^2$ and define $S_b : U \times U \times U \to [0, \infty)$ by

$$\begin{aligned} S_b(\eta^*, \theta^*, \vartheta^*) = \Big[&(\eta_1^* - \theta_1^*)^2 + (\eta_2^* - \theta_2^*)^2 \\ &+ (\theta_1^* - \vartheta_1^*)^2 + (\theta_2^* - \vartheta_2^*)^2 \\ &+ (\vartheta_1^* - \eta_1^*)^2 + (\vartheta_2^* - \eta_2^*)^2\Big]^2, \end{aligned}$$

where $\eta^* = (\eta_1^*, \eta_2^*)$, $\theta^* = (\theta_1^*, \theta_2^*)$ and $\vartheta^* = (\vartheta_1^*, \vartheta_2^*) \in U$.
Then (U, S_b) is an SbMS.
Let $\Phi_1^* = (4, 5)$, $\Phi_2^* = (-2, -3)$.
Then $S_b(\Phi_1^*, \Phi_1^*, \Phi_2^*) = 40000$. Let $\tau = 30000$, so that $2\tau = 60,000$.
Define

$$E = \{\eta^* \in U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) = 60000\}$$

For $\eta^* = (\eta_1^*, \eta_2^*)$,

$$\begin{aligned} S_b(\Phi_1^*, \Phi_1^*, \eta^*) &= 4\left[(4 - \eta_1^*)^2 + (5 - \eta_2^*)^2\right]^2, \\ S_b(\Phi_2^*, \Phi_2^*, \eta^*) &= 4\left[(\eta_1^* + 2)^2 + (\eta_2^* + 3)^2\right]^2. \end{aligned}$$

thus,

$$E = \left\{\eta^* \in \mathbb{R}^2 : 4\left[(4 - \eta_1^*)^2 + (5 - \eta_2^*)^2\right]^2 + 4\left[(\eta_1^* + 2)^2 + (\eta_2^* + 3)^2\right]^2 = 60000\right\}.$$

Now define $\Gamma : U \to U$ by

$$\Gamma(\eta^*) = \begin{cases} \eta^* & \eta^* \in E, \\ (1, 1), & \eta^* \notin E. \end{cases}$$

If $\eta^* \in E$, then $\Gamma(\eta^*) = \eta^*$, so every point of it E is fixed. If $\eta^* \notin E$, then $\Gamma(\eta^*) = \eta^* \notin E$, no additional FPs arise. Hence,

$$\mathrm{Fix}(\Gamma) = E.$$

Thus, it Γ admits a unique fixed ellipse in the SbMS (U, S_b), given by E.

Chapter 3

Fixed-Point Results in S-Metric Spaces, their Geometries and an Application FPR in SMS

3.1 Introduction

FP theory occupies a central position in nonlinear analysis because of its strong theoretical foundation and wide-ranging applications. Building on recent progress in this area, Joshi, Tomar, and Abdeljawad [20] proposed a unified framework for investigating geometric fixed figures in generalized metric spaces. This study aligns with a major trend in current FP theory by widening the scope of traditional FP principles beyond ordinary metric environments. By relaxing one or more of the standard metric axioms while preserving the essential analytical properties, generalized distance structures provide an effective setting for studying nonlinear phenomena beyond the scope of classical metric spaces, thereby extending both the theory and its applications.

Accordingly, this chapter reviews the theory of geometric fixed figures in SMSs through the unified $\mathcal{M}-$class framework. The presentation begins with existence results for FPs and fixed circles, followed by fixed ellipses and fixed elliptic discs, with illustrative examples accompanying the main theorems. It further recalls the conditions under which a fixed elliptic disc coincides with the entire space except for its focal points, together with criteria ensuring that the corresponding self-mapping is nontrivial, that is, distinct from the identity mapping.

3.2 Main Results

Throughout this chapter, let

$$\Lambda \in \mathcal{M}$$

denote an $\mathcal{M}$-class function satisfying the properties stated in Chapter 2. Unless otherwise specified, every contractive condition is formulated with respect to this class

Lemma 3.2.1. Let $\Lambda \in \mathcal{M}$ and let $\eta^*, \theta^* \in [0, \infty)$ satisfy

$$\eta^* \leq \max\Big\{\Lambda(\theta^*, \eta^*, 0, \theta^* + 2\eta^*, \eta^*),\ \Lambda(\theta^* + 2\eta^*, \eta^*, 0, \theta^*, \theta^*),$$
$$\Lambda(\theta^*, \theta^* + 2\eta^*, 0, \theta^*, \eta^*),\ \Lambda(\eta^*, \theta^*, 0, \eta^*, \theta^* + 2\eta^*)\Big\}.$$

Then there exists a constant

$$\lambda = \Lambda(1, 1, 0, 3, 1) \in\ [0, 1)$$

such that

$$\eta^* \leq \lambda\theta^*.$$

Proof. Among the four quantities appearing inside the maximum, at least one dominates η^*. With no loss of generality, let us assume that

$$\eta^* \leq \Lambda(\theta^*, \eta^*, 0, \theta^* + 2\eta^*, \eta^*).$$

We first establish that

$$\eta^* \leq \theta^*.$$

To obtain a contradiction, let us assume that there exists an element that does not satisfy the stated property.

$$\theta^* < \eta^*.$$

Since Λ is monotone by $(\mathcal{M}_3)$, it shows that

$$\begin{aligned}\eta^* &\leq \Lambda(\theta^*, \eta^*, 0, \theta^* + 2\eta^*, \eta^*)\\ &\leq \Lambda(\eta^*, \eta^*, 0, 3\eta^*, \eta^*).\end{aligned}$$

Using the property $(\mathcal{M}_2)$, we obtain

$$\Lambda(\eta^*, \eta^*, 0, 3\eta^*, \eta^*) = \eta^* \, \Lambda(1, 1, 0, 3, 1) = \lambda\eta^*,$$

where

$$\lambda = \Lambda(1, 1, 0, 3, 1) \in \ [0, 1).$$

Hence,

$$\eta^* \leq \lambda\eta^*.$$

Since $\lambda < 1$, the above inequality is impossible whenever $\eta^* > 0$. Therefore, the assumption $\theta^* < \eta^*$ cannot hold, and consequently

$$\eta^* \leq \theta^*.$$

Invoking the monotonicity property of Λ

$$\begin{aligned}\eta^* &\leq \Lambda(\theta^*, \eta^*, 0, \theta^* + 2\eta^*, \eta^*)\\ &\leq \Lambda(\theta^*, \theta^*, 0, 3\theta^*, \theta^*).\end{aligned}$$

Again by property $(\mathcal{M}_2)$,

$$\Lambda(\theta^*, \theta^*, 0, 3\theta^*, \theta^*) = \theta^* \, \Lambda(1, 1, 0, 3, 1) = \lambda\theta^*.$$

Therefore,

$$\eta^* \leq \lambda\theta^*,$$

which proves the lemma. □

Theorem 3.2.2. Let (U, S) be a complete SMS, let $\Lambda \in \mathcal{M}$, and let $\Gamma : U \to U$ be a self-mapping. Assume that whenever they $\eta^*, \theta^* \in U$ are distinct and

$$S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) > 0,$$

$\Rightarrow$

$$\begin{aligned} S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \Lambda\Big(& S(\eta^*, \eta^*, \theta^*),\ S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), \\ & S(\Gamma\eta^*, \Gamma\eta^*, \theta^*),\ S(\Gamma\theta^*, \Gamma\theta^*, \eta^*), \\ & S(\Gamma\theta^*, \Gamma\theta^*, \theta^*)\Big). \end{aligned}$$

Then Γ possesses a unique FP.

Proof. With $\eta_0^* \in U$ as an arbitrary initial value, we define the Picard sequence through:

$$\eta_{n+1}^* = \Gamma\eta_n^*, \qquad n = 0, 1, 2, \ldots$$

If

$$\eta_1^* = \Gamma\eta_1^*,$$

The result follows immediately. Now assume that it η_1^* is not a FP of Γ, i.e., $\Gamma\eta_1^* \neq \eta_1^*$.

$$\eta_1^* \neq \eta_2^*,$$

where

$$\eta_2^* = \Gamma\eta_1^*.$$

Since

$$S(\eta_1^*, \eta_1^*, \eta_2^*) = S(\Gamma\eta_0^*, \Gamma\eta_0^*, \Gamma\eta_1^*) > 0,$$

condition (3.2.1) can be applied to the pair (η_0^*, η_1^*). Therefore,

$$\begin{aligned} S(\eta_1^*, \eta_1^*, \eta_2^*) \leq \Lambda\Big(& S(\eta_0^*, \eta_0^*, \eta_1^*),\ S(\eta_1^*, \eta_1^*, \eta_0^*), \\ & S(\eta_1^*, \eta_1^*, \eta_1^*),\ S(\eta_2^*, \eta_2^*, \eta_0^*), \\ & S(\eta_2^*, \eta_2^*, \eta_1^*)\Big). \end{aligned}$$

Since

$$S(\eta_1^*, \eta_1^*, \eta_1^*) = 0$$

and the $S-$metric is symmetric, the previous inequality reduces to

$$\begin{aligned} S(\eta_1^*, \eta_1^*, \eta_2^*) \leq \Lambda\Big(& S(\eta_0^*, \eta_0^*, \eta_1^*),\ S(\eta_0^*, \eta_0^*, \eta_1^*), \\ & 0,\ S(\eta_0^*, \eta_0^*, \eta_2^*),\ S(\eta_1^*, \eta_1^*, \eta_2^*)\Big). \end{aligned}$$

Now, by axiom (S3),

$$S(\eta_0^*, \eta_0^*, \eta_2^*) \leq 2\, S(\eta_0^*, \eta_0^*, \eta_1^*) + S(\eta_1^*, \eta_1^*, \eta_2^*),$$

which implies

$$\begin{aligned} S(\eta_1^*, \eta_1^*, \eta_2^*) \leq \Lambda\Big(& S(\eta_0^*, \eta_0^*, \eta_1^*),\ S(\eta_0^*, \eta_0^*, \eta_1^*), \\ & 0,\ 2S(\eta_0^*, \eta_0^*, \eta_1^*) + S(\eta_1^*, \eta_1^*, \eta_2^*), \\ & S(\eta_1^*, \eta_1^*, \eta_2^*)\Big). \end{aligned}$$

Lemma 3.2.1 therefore yields

$$S(\eta_1^*, \eta_1^*, \eta_2^*) \leq \lambda\, S(\eta_0^*, \eta_0^*, \eta_1^*),$$

where

$$\lambda = \Lambda(1, 1, 0, 3, 1) \in \ [0, 1).$$

Repeating the same argument for successive iterates gives

$$S(\eta_n^*, \eta_n^*, \eta_{n+1}^*) \leq \lambda\, S(\eta_{n-1}^*, \eta_{n-1}^*, \eta_n^*), \qquad n = 1, 2, 3....$$

Consequently, by induction,

$$S(\eta_n^*, \eta_n^*, \eta_{n+1}^*) \leq \lambda^n\, S(\eta_0^*, \eta_0^*, \eta_1^*), \qquad n = 0, 1, 2, ...$$

If, for some index n, one has

$$\eta^*_{n+1} = \eta^*_n,$$

then

$$\eta^*_n = \Gamma\eta^*_n,$$

And therefore, the desired FP has already been obtained. Hence, in what follows, we assume that

$$\eta^*_n \neq \eta^*_{n+1}, \qquad \text{for every } n = 0, 1, 2, ...$$

We next verify that the sequence $\{\eta^*_n\}$ fulfills the Cauchy criterion. Let $m, n \in \mathbb{N}$ with $m \geq 1$. Repeated application of the triangle-type axiom (S3) yields

$$\begin{aligned} S(\eta^*_n, \eta^*_n, \eta^*_{n+m}) \leq\ & 2S(\eta^*_n, \eta^*_n, \eta^*_{n+1}) \\ & + 2S(\eta^*_{n+1}, \eta^*_{n+1}, \eta^*_{n+2}) \\ & + \cdots + 2S(\eta^*_{n+m-2}, \eta^*_{n+m-2}, \eta^*_{n+m-1}) \\ & + S(\eta^*_{n+m-1}, \eta^*_{n+m-1}, \eta^*_{n+m}). \end{aligned}$$

Since

$$S(\eta*_k, \eta*_k, \eta^*_{k+1}) \leq \lambda^k\, S(\eta^*_0, \eta^*_0, \eta^*_1),$$

for every $k \in \mathbb{N} \cup \{0\}$, we arrive at

$$\begin{aligned} S(\eta^*_n, \eta^*_n, \eta^*_{n+m}) &\leq 2 \sum_{k=n}^{n+m-1} \lambda^k\, S(\eta^*_0, \eta^*_0, \eta^*_1) \\ &= 2\lambda^n \left(1 + \lambda + \cdots + \lambda^{m-1}\right) S(\eta^*_0, \eta^*_0, \eta^*_1) \\ &= \frac{2\lambda^n(1-\lambda^m)}{1-\lambda}\, S(\eta^*_0, \eta^*_0, \eta^*_1). \end{aligned}$$

Because $0 \leq \lambda < 1$, the right-hand side vanishes in the limit as $n \to \infty$, independently of m. Consequently,

$$\lim_{n,m\to\infty} S(\eta^*_n, \eta^*_n, \eta^*_{n+m}) = 0,$$

thus the sequence $\{\eta_n^*\}$ is Cauchy in the space (U, S). By completeness of (U, S), there is a contained $\eta^* \in U$ such that

$$\lim_{n\to\infty} S(\eta_n^*, \eta_n^*, \eta^*) = 0.$$

We now show that this limit point is indeed a FP of Γ. To derive a contradiction, suppose that the limit point η^* is not a FP of Γ. Then,

$$\Gamma\eta^* \neq \eta^*,$$

and hence

$$S(\eta^*, \eta^*, \Gamma\eta^*) > 0.$$

For each $n = 0, 1, 2, ...$, the triangle-type axiom (S3) gives

$$\begin{aligned} S(\eta^*, \eta^*, \Gamma\eta^*) \leq 2S(\eta^*, \eta^*, \eta_{n+1}^*) \\ + S(\Gamma\eta^*, \Gamma\eta^*, \eta_{n+1}^*). \end{aligned}$$

Since

$$\eta_{n+1}^* = \Gamma\eta_n^*,$$

the contractive condition (3.2.1) yields

$$\begin{aligned} &S(\Gamma\eta^*, \Gamma\eta^*, \eta_{n+1}^*) \\ &= S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\eta_n^*) \\ &\leq \Lambda\Big(S(\eta^*, \eta^*, \eta_n^*),\ S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), \\ &\qquad S(\Gamma\eta^*, \Gamma\eta^*, \eta_n^*),\ S(\Gamma\eta_n^*, \Gamma\eta_n^*, \eta^*),\ S(\Gamma\eta_n^*, \Gamma\eta_n^*, \eta_n^*)\Big). \end{aligned}$$

Furthermore, by another application of axiom (S3),

$$S(\Gamma\eta^*, \Gamma\eta^*, \eta_n^*) \leq S(\eta^*, \eta^*, \Gamma\eta^*) + S(\eta_n^*, \eta_n^*, \eta^*),$$

while

$$\Gamma\eta_n^* = \eta_{n+1}^*.$$

Substituting these estimates into the preceding inequality gives

$$\begin{aligned} S(\eta^*, \eta^*, \Gamma\eta^*) \leq 2S(\eta^*, \eta^*, \eta^*_{n+1}) \\ + \Lambda\Big(S(\eta^*, \eta^*, \eta^*_n),\ S(\eta^*, \eta^*, \Gamma\eta^*), \\ S(\eta^*, \eta^*, \Gamma\eta^*) + S(\eta^*_n, \eta^*_n, \eta^*), \\ S(\eta^*_{n+1}, \eta^*_{n+1}, \eta^*),\ S(\eta^*_{n+1}, \eta^*_{n+1}, \eta^*_n)\Big). \end{aligned}$$

Upon passing to the limit as $n \to \infty$, and utilizing the continuity of Λ, we get

$$S(\eta^*_n, \eta^*_n, \eta^*) \longrightarrow 0,$$

it follows that

$$S(\eta^*, \eta^*, \Gamma\eta^*) \leq \Lambda\Big(0, S(\eta^*, \eta^*, \Gamma\eta^*), S(\eta^*, \eta^*, \Gamma\eta^*), 0, 0\Big).$$

Applying Lemma 3.2.1, we conclude that

$$S(\eta^*, \eta^*, \Gamma\eta^*) = 0,$$

which contradicts our assumption that

$$S(\eta^*, \eta^*, \Gamma\eta^*) > 0.$$

Therefore,

$$\Gamma\eta^* = \eta^*,$$

Consequently, it η^* is a FP of Γ. It remains to establish the uniqueness of the FP. Assume, toward a contradiction, that there exists another FP $\theta^* \in U$ of the mapping Γ satisfying

$$\theta^* \neq \eta^*.$$

Since both η^* and θ^* are fixed under Γ, we have

$$\Gamma\eta^* = \eta^*, \qquad \Gamma\theta^* = \theta^*.$$

The inequality $S(\eta^*, \eta^*, \theta^*) > 0$ therefore allows the application of the contractive condition (3.2.1). Hence,

$$\begin{aligned} S(\eta^*, \eta^*, \theta^*) &= S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \\ &\leq \Lambda\Big(S(\eta^*, \eta^*, \theta^*),\ S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), \\ &\qquad S(\Gamma\eta^*, \Gamma\eta^*, \theta^*),\ S(\Gamma\theta^*, \Gamma\theta^*, \eta^*),\ S(\Gamma\theta^*, \Gamma\theta^*, \theta^*)\Big). \end{aligned}$$

Using the identities

$$\Gamma\eta^* = \eta^*, \qquad \Gamma\theta^* = \theta^*,$$

together with

$$S(\eta^*, \eta^*, \eta^*) = 0, \qquad S(\theta^*, \theta^*, \theta^*) = 0,$$

the above estimate reduces to

$$S(\eta^*, \eta^*, \theta^*) \leq \Lambda\Big(S(\eta^*, \eta^*, \theta^*), 0, S(\eta^*, \eta^*, \theta^*), S(\theta^*, \theta^*, \eta^*), 0\Big).$$

Since the SM is symmetric,

$$S(\theta^*, \theta^*, \eta^*) = S(\eta^*, \eta^*, \theta^*),$$

and therefore

$$S(\eta^*, \eta^*, \theta^*) \leq \Lambda\Big(S(\eta^*, \eta^*, \theta^*), 0, S(\eta^*, \eta^*, \theta^*), S(\eta^*, \eta^*, \theta^*), 0\Big).$$

Invoking Lemma 3.2.1, we conclude that

$$S(\eta^*, \eta^*, \theta^*) = 0.$$

The defining property of an SM then yields

$$\eta^* = \theta^*,$$

which contradicts the assumption that the two FPs are distinct. Hence, no FP other than that η^* can exist. Accordingly, the self-mapping Γ admits exactly one

FP in U. □

Example 3.2.3. Consider the interval

$$U = [0, 1],$$

endowed with the SM

$$S(\eta^*, \theta^*, \vartheta^*) = |\eta^* - \theta^*| + |\eta^* + \theta^* - 2\vartheta^*|, \qquad \eta^*, \theta^*, \vartheta^* \in U.$$

Since completeness is assumed for the SMS (U, S), define the mappings

$$\Lambda : [0, \infty)^5 \longrightarrow [0, \infty)$$

and

$$\Gamma : U \longrightarrow U$$

by

$$\Lambda(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \alpha^*(\eta_1 + \eta_2 + \eta_5), \qquad \alpha^* \in \left[0, \frac{1}{3}\right),$$

and

$$\Gamma = \frac{\eta^*}{4\eta^* + 1},$$

respectively. It follows directly that the function Λ satisfies conditions $(\mathcal{M}_1)$–$(\mathcal{M}_3)$. Hence, all the conditions of the theorem are met, yielding the required conclusion.

$$\Lambda \in \mathcal{M}.$$

Now let $\eta^*, \theta^* \in U$ with $\eta^* \neq \theta^*$. Then

$$S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) = 2\,|\Gamma\eta^* - \Gamma\theta^*| > 0.$$

Furthermore,

$$\begin{aligned}
S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) &= 2\left|\frac{\eta^*}{1+4\eta^*} - \frac{\theta^*}{1+4\theta^*}\right| \\
&= 2\left|\frac{\eta^* - \theta^*}{(1+4\eta^*)(1+4\theta^*)}\right| \\
&\leq \frac{1}{4}\left(|\eta^* - \theta^*| + 8\frac{\eta^{*2}}{4\eta^*+1} + 8\frac{\theta^{*2}}{4\theta^*+1}\right) \\
&= \alpha^*\Big(S(\eta^*, \eta^*, \theta^*) + S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) + S(\Gamma\theta^*, \Gamma\theta^*, \theta^*)\Big),
\end{aligned}$$

where

$$\alpha^* = \frac{1}{4}.$$

The self-mapping Γ equivocally falls within the purview of Theorem 3.2.2, as it satisfies the contractive criterion stipulated in *(3.2.1)*. Consequently, an immediate application of the theorem ensures the existence and uniqueness of a FP Γ in U.

Corollary 3.2.4. The assertion of Theorem 3.2.2 remains true if the generalized contractive condition (3.2.1) is substituted by

$$S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \alpha^*\left(S(\Gamma\eta^*, \Gamma\eta^*, \theta^*) + S(\Gamma\theta^*, \Gamma\theta^*, \eta^*)\right), \qquad \alpha^* \in \left[0, \frac{1}{3}\right).$$

Proof. Define

$$\Lambda : [0,\infty)^5 \longrightarrow [0,\infty)$$

$$\Lambda(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \alpha^*(\eta_3 + \eta_4).$$

□

Remark 3.2.5. Viewed differently, Corollary 3.2.4 extends the traditional Chatterjea contraction theorem to the environment of SMSs, thereby broadening the FP result of Phaneendra and Swamy.

Corollary 3.2.6. The result of Theorem 3.2.2 remains applicable if condition (3.2.1) is replaced with

$$\begin{aligned}
S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \alpha^* S(\eta^*, \eta^*, \theta^*) &+ \beta^* S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \\
&+ \gamma^* S(\Gamma\theta^*, \Gamma\theta^*, \theta^*) + \delta^* S(\Gamma\theta^*, \Gamma\theta^*, \eta^*),
\end{aligned}$$

where

$$\alpha^* + \beta^* + 3\gamma^* + \delta^* \in \ [0, 1)$$

Proof. Consider the function

$$\Lambda : [0, \infty)^5 \longrightarrow [0, \infty)$$

by

$$\Lambda(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = \alpha^*\eta_1 + \beta^*\eta_2 + \delta^*\eta_5,$$

This mapping belongs to the family $\mathcal{M}$, and applying Theorem 3.2.2 immediately establishes the required result. □

Remark 3.2.7. Finally, Corollary 3.2.6 can be read as a Reich-type extension of the FP theorem to SMSs, generalizing the classical result of Reich.

The preceding corollaries demonstrate that numerous classical contraction principles arise naturally from suitable choices of $\mathcal{M}$-class functions. Although these contractive conditions often guarantee that a mapping admits a unique FP. This observation motivates the investigation of geometric fixed figures such as circles, ellipses, and elliptic discs. In the remainder of this chapter, we employ the $\mathcal{M}$-class framework to derive existence results for these geometric fixed sets in SMSs. For subsequent developments, we introduce the subclass

$$\mathcal{M}^* = \{\Lambda \in \mathcal{M} : \Lambda(0, 1, 1, 1, 1) \in \ [0, 1)\}.$$

The desired conclusion follows directly from the definition, namely, that

$$\mathcal{M}^* \subseteq \mathcal{M}.$$

Theorem 3.2.8. Let (U, S) be an SMS, let $\Lambda \in \mathcal{M}^*$, and let $\Gamma : U \to \ U$ be a self-mapping. Assume that, for every $\eta^* \in U$ satisfying $S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) > 0$, the inequality

$$S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \leq \Lambda\Big(S(\eta^*, \eta^*, \eta_0^*), S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*), S(\Gamma\eta^*, \Gamma\eta^*, \eta_0^*), S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta^*)\Big) \tag{3.2.2}$$

holds. Consequently, the circle

$$C\left(\eta_0^*, \kappa\right)$$

constitutes a fixed circle of the mapping (Λ), where

$$\kappa = \inf\left\{S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) : \Gamma\eta^* \neq \eta^*,\ \eta^* \in U\right\}.$$

Proof. Consider the circle $C\left(\eta_0^*, \kappa\right)$. We first verify that its center is fixed under the mapping Γ. Assume, on the contrary, that $\Gamma\eta_0^* \neq \eta_0^*$. Applying the contractive condition (3.2.2), together with the homogeneity and defining property of functions in $\mathcal{M}^*$, gives

$$\begin{aligned} S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*) \leq \Lambda\Big(&0, S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*), \\ &S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*), S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*), \\ &S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*)\Big) \\ = \Lambda&(0, 1, 1, 1, 1)\, S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*). \end{aligned}$$

Since $\Lambda(0, 1, 1, 1, 1) \in\ [0, 1)$, the above inequality implies

$$S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*) < S(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*),$$

which is impossible. Therefore,

$$\Gamma\eta_0^* = \eta_0^*.$$

Next, let $\eta^* \in C\left(\eta_0^*, \kappa\right)$ be arbitrary. Assume that $\Gamma\eta^* \neq \eta^*$. From the definition of the radius,

$$S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \geq \kappa$$

Using condition (3.2.2), together with the equality $\Gamma\eta_0^* = \eta_0^*$, we obtain

$$S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \leq \Lambda\Big(\kappa, S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), 0, 2S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) + \kappa, \kappa\Big).$$

Because

$$\kappa \leq S(\Gamma\eta^*, \Gamma\eta^*, \eta^*),$$

the monotonicity of Λ yields

$$S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \leq \Lambda\Big(S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), 0, 3S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), S(\Gamma\eta^*, \Gamma\eta^*, \eta^*)\Big).$$

Applying the positive homogeneity of Λ,

$$\begin{aligned} S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) &\leq \Lambda(1,1,0,3,1)\, S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \\ &= \lambda\, S(\Gamma\eta^*, \Gamma\eta^*, \eta^*), \end{aligned}$$

where

$$\lambda = \Lambda(1,1,0,3,1) \in [0,1).$$

Hence,

$$S(\Gamma\eta^*, \Gamma\eta^*, \eta^*) < S(\Gamma\eta^*, \Gamma\eta^*, \eta^*),$$

which is absurd. Therefore,

$$\Gamma\eta^* = \eta^*$$

for all $\eta^* \in C(\eta_0^*, \kappa)$. Consequently, the mapping Γ fixes every point of $C(\eta_0^*, \kappa)$, thereby establishing it $C(\eta_0^*, \kappa)$ as a fixed circle of Γ. □

Example 3.2.9. Consider the space $U = \mathbb{R}^3$, equipped with the SM

$$\begin{aligned} S(\eta^*, \theta^*, \vartheta^*) &= |\eta_1^* - \vartheta_1^*| + |\eta_1^* + \vartheta_1^* - 2\theta_1^*| \\ &\quad + |\eta_2^* - \vartheta_2^*| + |\eta_2^* + \vartheta_2^* - 2\theta_2^*| \\ &\quad + |\eta_3^* - \vartheta_3^*| + |\eta_3^* + \vartheta_3^* - 2\theta_3^*|, \end{aligned}$$

where

$$\eta^* = (\eta_1^*, \eta_2^*, \eta_3^*), \qquad \theta^* = (\theta_1^*, \theta_2^*, \theta_3^*), \qquad \vartheta^* = (\vartheta_1^*, \vartheta_2^*, \vartheta_3^*).$$

Let

$$\eta_0^* = (1, 2, 3)$$

and consider the circle centered at it η_0^* having radius 8. In the present SM, this circle is characterized by

$$|1 - \eta_1^*| + |2 - \eta_2^*| + |3 - \eta_3^*| = 4. \tag{3.2.3}$$

Define the self-map

$$\Gamma : U \to U$$

by

$$\Gamma(a^*, b^*, c^*) = \begin{cases} (a^*, b^*, c^*), & (a^*, b^*, c^*) \in C(\eta_0^*, 8), \\ (1, 0, 2), & \text{otherwise}. \end{cases}$$

It is readily verified that the mapping Γ satisfies all the assumptions of Theorem 3.2.8. Consequently, every point belonging to the circle $C(\eta_0^*, 8)$ remains fixed under Γ. Hence, $C(\eta_0^*, 8)$ is the unique fixed circle of the mapping; see Figure 3.1.

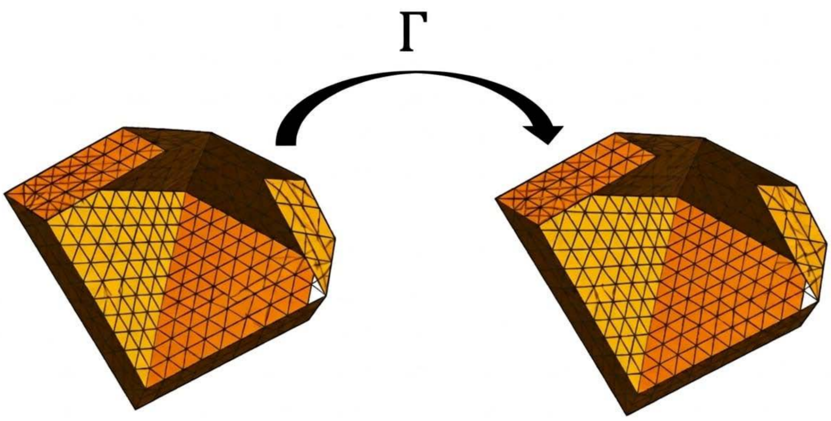

FIGURE 3.1: The fixed circle $C(\eta_0^*, 8)$ associated with the mapping Γ described in Example 3.2.9.

Building upon the seminal work of Joshi *et al.* [29], we formulate the notions of ellipses and elliptic discs in the context of SMSs and examine their intrinsic fixed geometric structures. The primary aim is to derive sufficient conditions ensuring that the FP set of a self-mapping possesses an ellipse or an elliptic disc through the utilization of a Caristi-type contractive framework.

Example 3.2.10. Let

$$U = \mathbb{R}^2$$

be equipped with the $S-$metric

$$S_1(\eta^*, \theta^*, \vartheta^*) = \sum_{\mathrm{i}=1}^{2} \left(|\eta_{\mathrm{i}}^* - \vartheta_{\mathrm{i}}^*| + |\eta_{\mathrm{i}}^* + \vartheta_{\mathrm{i}}^* - 2\theta_{\mathrm{i}}^*| \right).$$

Consider the focal points

$$\Phi_1^* = (2, 0), \qquad \Phi_2^* = (0, 2),$$

for which

$$S_1(\Phi_1^*, \Phi_1^*, \Phi_2^*) = 8.$$

For the parameter value 5, the corresponding ellipse is described by

$$E(\Phi_1^*, \Phi_2^*, 5) = \{\eta^* \in U : |2 - \eta_1^*| + |2 - \eta_2^*| + |\eta_1^*| + |\eta_2^*| = 5\}. \tag{3.2.4}$$

The above ellipse is centered at $(1, 1)$ the focal points $(2, 0)$ and $(0, 2)$. Its geometric representation is shown by the blue curve in Figure 3.2.

Theorem 3.2.11. Let

$$E(\Phi_1^*, \Phi_2^*, \tau)$$

be an ellipse in the SMS (U, S), and define

$$\varsigma(\eta^*) = S(\Phi_1^*, \Phi_1^*, \eta^*) + S(\Phi_2^*, \Phi_2^*, \eta^*), \qquad \eta^*, \Phi_1^*, \Phi_2^* \in U.$$

Suppose that the self-map

$$\Gamma : U \to U$$

satisfies the following conditions:

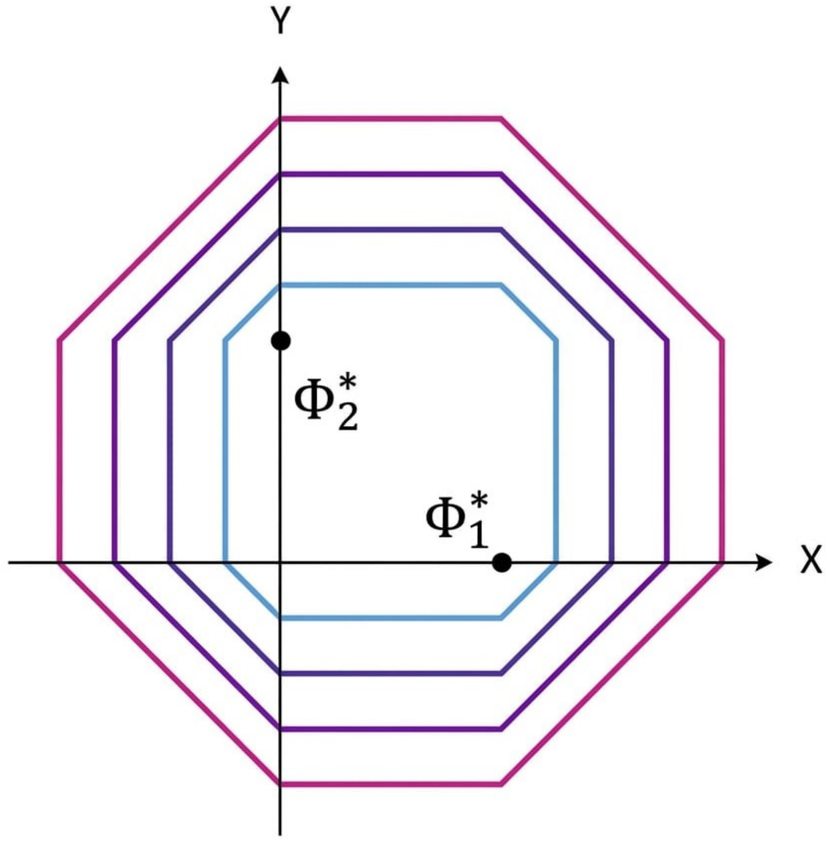

FIGURE 3.2: Illustration of the family of ellipses generated under the $S-$metric S_1, centered at $(1, 1)$ with foci at $(2, 0)$ and $(0, 2)$.

(E1): For every $\eta^* \in U$,

$$S(\eta^*, \eta^*, \Gamma\eta^*) \leq \varsigma(\eta^*) - \varsigma(\Gamma\eta^*).$$

(E2): For every $\eta^* \in U$,

$$S(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) \geq 2\tau.$$

(E3): Whenever

$$\eta^* \in E(\Phi_1^*, \Phi_2^*, \tau)$$

and

$$\theta^* \in U \setminus E(\Phi_1^*, \Phi_2^*, \tau),$$

one has

$$S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \sigma\, S(\eta^*, \eta^*, \theta^*), \qquad \sigma \in [0, 1).$$

Then

$$E(\Phi_1^*, \Phi_2^*, \tau)$$

constitutes the unique ellipse invariant under the mapping Γ.

Proof. Choose an arbitrary point

$$\eta^* \in E(\Phi_1^*, \Phi_2^*, \tau).$$

By the definition of the ellipse,

$$\varsigma(\eta^*) = 2\tau.$$

Using condition (E1), we obtain

$$\begin{aligned} S(\eta^*, \eta^*, \Gamma\eta^*) \leq\ & S(\Phi_1^*, \Phi_1^*, \eta^*) + S(\Phi_2^*, \Phi_2^*, \eta^*) \\ & - S(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) - S(\Phi_2^*, \Phi_2^*, \Gamma\eta^*). \end{aligned}$$

Now, using (E2), we obtain

$$S(\eta^*, \eta^*, \Gamma\eta^*) \leq 2\tau - 2\tau = 0.$$

Since every S-metric is non-negative,

$$S(\eta^*, \eta^*, \Gamma\eta^*) = 0,$$

which implies

$$\Gamma\eta^* = \eta^*.$$

Hence, every point of

$$E(\Phi_1^*, \Phi_2^*, \tau)$$

is fixed by the mapping Γ.

To verify uniqueness, suppose that there exists another fixed ellipse,

$$E(\Phi_1^{*'}, \Phi_2^{*'}, \tau').$$

Choose arbitrary points

$$u^* \in E(\Phi_1^*, \Phi_2^*, \tau)$$

and

$$v^* \in E(\Phi_1^{*'}, \Phi_2^{*'}, \tau').$$

Since both ellipses are invariant under Γ,

$$\begin{aligned} S(u^*, u^*, v^*) &= S(\Gamma u^*, \Gamma u^*, \Gamma v^*) \\ &\leq \sigma S(u^*, u^*, v^*), \end{aligned}$$

where $0 \leq \sigma < 1$.

This inequality is possible only if

$$S(u^*, u^*, v^*) = 0,$$

which implies

$$u^* = v^*,$$

contradicting the assumption that the two fixed ellipses are distinct.

Therefore, no fixed ellipse other than that

$$E(\Phi_1^*, \Phi_2^*, \tau)$$

can exist. Hence, it

$$E(\Phi_1^*, \Phi_2^*, \tau)$$

is the unique fixed ellipse of Γ. □

Theorem 3.2.12. The claim of Theorem 3.2.11 remains valid if assumptions (E1) and (E2) are replaced by the following conditions.

(E_1'):

$$S(\eta^*, \eta^*, \Gamma\eta^*) \leq \varsigma(\eta^*) + \varsigma(\Gamma\eta^*) - 4\tau.$$

(E_2'):

$$S(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) \leq 2\tau.$$

Theorem 3.2.13. Let

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

be an elliptic disc in the SMS (U, S), and let the auxiliary function

$$\varsigma(\eta^*) = S(\Phi_1^*, \Phi_1^*, \eta^*) + S(\Phi_2^*, \Phi_2^*, \eta^*)$$

be defined by *(3.2.6)*. Suppose that the self-map

$$\Gamma : U \to U$$

satisfies the following conditions.

(D1):

$$S(\eta^*, \eta^*, \Gamma\eta^*) \leq \varsigma(\eta^*) + \varsigma(\Gamma\eta^*) - 4\tau.$$

(D2):

$$S(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) \leq 2\tau, \qquad \forall\, \eta^* \in D(\Phi_1^*, \Phi_2^*, \tau).$$

Then every point of

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

is a fixed point of Γ. Consequently, it

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

is a fixed elliptic disc of the mapping.

Moreover, if the additional condition

(D3):

$$S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \sigma\, S(\eta^*, \eta^*, \theta^*),$$

holds for every

$$\eta^* \in D(\Phi_1^*, \Phi_2^*, \tau)$$

and every

$$\theta^* \in U \setminus D(\Phi_1^*, \Phi_2^*, \tau),$$

where

$$\sigma \in [0, 1),$$

then

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

constitutes the largest fixed elliptic disc corresponding to the semi-major axis of Γ.

Proof. The validity of the first assertion is established by an argument analogous to that employed in the above proof. Accordingly, every point belonging to

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

is fixed by the mapping Γ, showing that the elliptic disc is invariant under Γ. To establish the maximality of the fixed elliptic disc, Assume, for contradiction, that there exists another fixed elliptic disc.

$$D(\Phi_1^{*'}, \Phi_2^{*'}, \tau')$$

with

$$\tau' > \tau.$$

Choose arbitrary points

$$\eta^* \in D(\Phi_1^*, \Phi_2^*, \tau)$$

and

$$\theta^* \in D(\Phi_1^{*'}, \Phi_2^{*'}, \tau')$$

Since both elliptic discs are fixed by Γ, condition (D3) implies

$$\begin{aligned} S(\eta^*, \eta^*, \theta^*) &= S(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \\ &\leq \sigma\, S(\eta^*, \eta^*, \theta^*). \end{aligned}$$

Because $0 \leq \sigma < 1$, the above inequality can hold only if

$$S(\eta^*, \eta^*, \theta^*) = 0.$$

This contradicts the assumption that the second fixed elliptic disc possesses a strictly larger semi-major axis. Therefore, no fixed elliptic disc larger than

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

can exist. Hence,

$$D(\Phi_1^*, \Phi_2^*, \tau)$$

is the maximal elliptic disc fixed by the mapping and corresponding to the semi-major axis Γ. $\square$

Example 3.2.14. Let $U = \mathbb{R}^2$ be equipped with the S-metric

$$S(\eta^*, \theta^*, \vartheta^*) = \sum_{\mathrm{i}=1}^{2} \left(|e^{\eta_{\mathrm{i}}^*} - e^{\vartheta_{\mathrm{i}}^*}| + |e^{\eta_{\mathrm{i}}^*} + e^{\vartheta_{\mathrm{i}}^*} - 2e^{\theta_{\mathrm{i}}^*}| \right).$$

Consider the focal points

$$\Phi_1^* = (0, 0), \qquad \Phi_2^* = (\ln 2, 0).$$

The corresponding ellipse and elliptic disc are respectively described by

$$|1 - e^{\eta_1^*}| + 2|1 - e^{\eta_2^*}| + |2 - e^{\eta_1^*}| = 2, \tag{3.1}$$

and

$$|1 - e^{\eta_1^*}| + 2|1 - e^{\eta_2^*}| + |2 - e^{\eta_1^*}| \leq 2. \tag{3.2.10}$$

Define the self-map

$$\Gamma : U \to U$$

by

$$\Gamma(a^*, b^*) = \begin{cases} (a^*, b^*), & (a^*, b^*) \in D(\Phi_1^*, \Phi_2^*, \ln 2), \\ (a^* - \ln 15,\ b^* - \ln 15), & \text{otherwise.} \end{cases}$$

It is straightforward to verify that the mapping satisfies assumptions (D1), (D2), and (D3) of Theorem 3.2.13. Consequently, every point of the ellipse defined by (3.2.9) is fixed by Γ, while the elliptic disk determined by (3.2.10) forms the largest fixed elliptic disk associated with the mapping. The geometric configuration is illustrated in Figure 3.3.

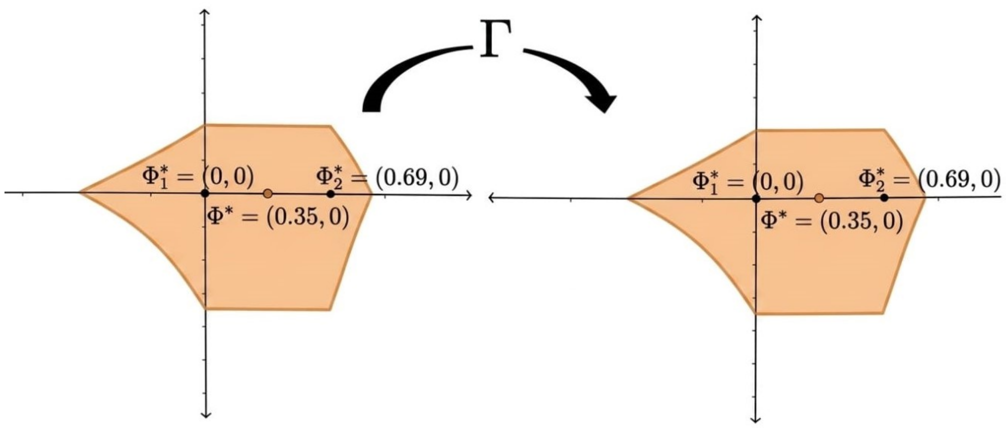

FIGURE 3.3: Geometric illustration of the mapping Γ: the boundary curve represents the fixed ellipse defined by (3.2.9), whereas the enclosed shaded region corresponds to the maximal fixed elliptic disc given by (3.2.10).

3.3 Application to a Satellite Web Coupling System

FP theory has become an indispensable tool in the analysis of nonlinear mathematical models arising in science and engineering. As an application of the framework reviewed in this chapter, we recall the satellite web coupling boundary value problem solved by Joshi, Tomar, and Abdeljawad [20], which establishes existence and uniqueness of solutions via the FP theory developed in the preceding sections. In this model, two cylindrical satellites are connected by a thin elastic membrane. Heat transfer through this membrane leads to the nonlinear boundary value problem

$$-\frac{d^2\eta^*}{dt^{*^2}} = \mu\,\eta^{*^4}, \qquad 0 < t^* < 1, \qquad \eta^*(0) = \eta^*(1) = 0, \tag{3.2}$$

where η^* denotes the temperature distribution at the point $t^* \in [0,1]$. The parameter

$$\mu = \frac{2a^* l^{*^2} K^{*^3}}{\epsilon\mathfrak{h}} > 0 \tag{3.3}$$

is a dimensionless constant, in which it K^* denotes the common absolute temperature of the satellites, l^* represents the inter-satellite distance, a^* is the radiation constant, ϵ denotes the thermal conductivity coefficient, and $\mathfrak{h}$ specifies the thickness of the connecting web. The Green function associated with (3.2) is

$$\mathcal{G}(t^*, \epsilon) = \begin{cases} t^*(1-\epsilon), & 0 < t^* < \epsilon, \\ \epsilon(1-t^*), & \epsilon < t^* < 1, \end{cases} \tag{3.4}$$

which transforms (3.2) into the equivalent integral equation

$$\eta^*(t^*) = 1 - \mu \int_0^1 \mathcal{G}(t^*, \epsilon)\eta^{*^4}(\epsilon)\, d\epsilon. \tag{3.5}$$

Let

$$\mathcal{U} = \mathcal{R}[0,1]$$

denote the family of all Riemann-integrable functions on $[0,1]$. Equip U with the $S-$metric

$$S(\eta^*,\theta^*,\vartheta^*) = \|\eta^*-\vartheta^*\|_\infty + \|\theta^*-\vartheta^*\|_\infty.$$

Then it (U,S) is a complete SMS. Throughout this application we employ the supremum norm

$$\|\eta^*\|_\infty = \sup_{t^*\in[0,1]} |\eta^*(t^*)|. \tag{3.6}$$

Theorem 3.3.1. Let $\Gamma : U \to U$ be a self-map in a complete SMS (U,S), satisfying

$$\|\eta^*(t)-\theta^*(t)\|_\infty > 0 \implies \|(\eta^{*^2}(\epsilon)+\theta^{*^2}(\epsilon))(\eta^*(\epsilon)+\theta^*(\epsilon))\|_\infty \le \frac{\kappa}{\mu},\ \kappa\in(0,8).$$

Consequently, the satellite web coupling boundary value problem (3.2) admits a unique solution.

Proof. Define a self-map $\Gamma : U \to U$ by

$$\Gamma\eta^*(t) = 1-\mu\int_0^1 \mathcal{G}(t^*,\epsilon)\eta^{*^4}(\epsilon)d\epsilon,\ \epsilon\in[0,1].$$

Evidently, a solution of (3.2) is a fixed point of the self-mapping Γ. However, $\|\Gamma\eta^*-\Gamma\theta^*\|_\infty > 0$, so $S(\Gamma\eta^*,\Gamma\eta^*,\Gamma\theta^*) > 0$. Now,

$$\begin{aligned}
S(\Gamma\eta^*,\Gamma\eta^*,\Gamma\theta^*) &= 2|\Gamma\eta^*-\Gamma(\theta^*)| \\
&= 2\left|1-\mu\int_0^1 \mathcal{G}(t^*,\epsilon)\eta^{*^4}(\epsilon)d\epsilon - 1 + \mu\int_0^1 \mathcal{G}(t^*,\epsilon)\theta^{*^4}(\epsilon)d\epsilon\right| \\
&= 2\mu\left|\int_0^1 (\theta^{*^4}(\epsilon)-\eta^{*^4}(\epsilon))\mathcal{G}(t^*,\epsilon)d\epsilon\right| \\
&= 2\mu\left|\int_0^1 (\theta^{*^2}(\epsilon)+\eta^{*^2}(\epsilon))(\theta^*(\epsilon)+\eta^*(\epsilon))(\theta^*(\epsilon)-\eta^*(\epsilon))\mathcal{G}(t^*,\epsilon)d\epsilon\right| \\
&= 2\mu\|\eta^*-\theta^*\|_\infty\|(\eta^{*^2}+\theta^{*^2})(\eta^*+\theta^*)\|_\infty\int_0^1 \mathcal{G}(t^*,\epsilon)d\epsilon \\
&\le 2\kappa\|\eta^*-\theta^*\|_\infty\left[\int_0^t \epsilon(1-t)d\epsilon + \int_t^1 t(1-\epsilon)d\epsilon\right] \\
&\le \frac{\kappa}{8}\|\eta^*-\theta^*\|_\infty \\
&= \alpha S(\eta^*,\eta^*,\theta^*),\ \alpha\in[0,1)\ (\text{since, } \kappa\in(0,8)).
\end{aligned}$$

Hence, all the hypotheses of Corollary 3.2.4 are satisfied. Consequently, Γ has a unique FP, and therefore the satellite web coupling problem (3.2) admits a unique solution. $\square$

Chapter 4

Fixed Point Results and their Geometry in S_b-Metric Spaces FPRG-SbMS

4.1 Introductory Results

Previous chapter developed a unified theory of fixed points and fixed figures—circles, discs, ellipses, and elliptic discs—in SMS built on the $\mathcal{M}-$class contractive framework [20]. We now extend this program to symmetric SbMS [14], where the triangle-type inequality is relaxed by a coefficient $b \geq 1$. This relaxation strictly enlarges the class of admissible spaces. Reproving the existence, uniqueness, and geometric fixed-figure results in this more general setting requires tracking the coefficient b through each estimate, most visibly in the Cauchy-sequence argument, where convergence now depends on the combined condition $b\lambda < 1$ rather than $\lambda < 1$ alone. We begin by confirming that the class of SbMS genuinely extends that of SMS before establishing the fixed-point and fixed-ellipse theorems that occupy the remainder of the chapter.

Definition 4.1.1. Let (U, S_b) be an SMS with $b \geq 1$. We denote by $\mathcal{M}$ the class of all continuous functions

$$\Lambda : [0, \infty)^5 \longrightarrow [0, \infty)$$

satisfying the following conditions [20]:

(M_1^*) : $\Lambda(\lambda x_1, \lambda x_2, \ldots, \lambda x_5) = \lambda\Lambda(x_1, x_2, \ldots, x_5)$ for all $(x_1, x_2, \ldots, x_5) \in [0, \infty)^5$ and $\lambda \geq 0$.

(M_2^*) : $\Lambda(x_1, x_2, \ldots, x_5) \leq \Lambda(y_1, y_2, \ldots, y_5)$ whenever $x_i \leq y_i$ for $i = 1, \ldots, 5$.

Furthermore, a function $\Lambda \in \mathcal{M}$ is said to satisfy (M_3^*) : for some fixed $b \geq 1$ such that

(M_3^*) $\Lambda(1, 1, 0, 3b, 1) \in [0, 1)$.

Theorem 4.1.2. Let (U, S_b) be a complete symmetric S_b-metric space with parameter $b \geq 1$ [14]. Consider a self-mapping $\Gamma : U \to U$ and a function $\Lambda \in \mathcal{M}$ [20]. Suppose that for all $\eta^*, \theta^* \in U$, whenever $S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) > 0$, the following inequality holds:

$$\begin{aligned} S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \Lambda\big(&S_b(\eta^*, \eta^*, \theta^*), S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*), \\ &S_b(\Gamma\eta^*, \Gamma\eta^*, \theta^*), S_b(\Gamma\theta^*, \Gamma\theta^*, \eta^*), S_b(\Gamma\theta^*, \Gamma\theta^*, \theta^*)\big). \end{aligned} \tag{4.1}$$

If $b\lambda < 1$ where $\lambda = \Lambda(1, 1, 0, 3, 1) \in [0, 1)$, then Γ possesses a unique fixed point.

Proof. Select an arbitrary point $\eta_0^* \in U$ and generate the iterative sequence

$$\eta_{n+1}^* = \Gamma\eta_n^*, \qquad n = 0, 1, 2, 3, \ldots$$

The resulting sequence is the corresponding Picard iterative sequence. If there exists $n \in \mathbb{N}$ such that

$$\eta_n^* = \eta_{n+1}^* = \Gamma\eta_n^*,$$

then η_n^* is a fixed point of Γ, and the desired conclusion follows immediately. Accordingly, we may assume that

$$\eta_n^* \neq \Gamma\eta_n^* \quad \forall\, n.$$

Contraction estimate. Consequently, $S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \Gamma\eta_1^*) > 0$. Setting $\eta_1^* = \Gamma\eta_0^*$ and $\eta_2^* = \Gamma\eta_1^*$, and applying (4.1) with $\eta^* = \eta_0^*$, $\theta^* = \eta_1^*$, together with the properties of the $M-$class function, we obtain the following:

$$\begin{aligned}
S_b(\eta_1^*, \eta_1^*, \eta_2^*) &= S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \Gamma\eta_1^*) \\
&\leq \Lambda\big[S_b(\eta_0^*, \eta_0^*, \eta_1^*),\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*),\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_1^*), \\
&\qquad\qquad S_b(\Gamma\eta_1^*, \Gamma\eta_1^*, \eta_0^*),\ S_b(\Gamma\eta_1^*, \Gamma\eta_1^*, \eta_1^*)\big] \\
&= \Lambda\big[S_b(\eta_0^*, \eta_0^*, \eta_1^*),\ S_b(\eta_1^*, \eta_1^*, \eta_0^*),\ S_b(\eta_1^*, \eta_1^*, \eta_1^*), \\
&\qquad S_b(\eta_2^*, \eta_2^*, \eta_0^*),\ S_b(\eta_2^*, \eta_2^*, \eta_1^*)\big].
\end{aligned}$$

Utilizing the symmetry property $S_b(z^*, z^*, \theta^*) = S_b(\theta^*, \theta^*, z^*)$,

$$S_b(\eta_1^*, \eta_1^*, \eta_2^*) \leq \Lambda\big[S_b(\eta_0^*, \eta_0^*, \eta_1^*),\ S_b(\eta_0^*, \eta_0^*, \eta_1^*),\ 0,\ S_b(\eta_2^*, \eta_2^*, \eta_0^*),\ S_b(\eta_1^*, \eta_1^*, \eta_2^*)\big].$$

Applying Lemma 3.2.10,

$$S_b(\eta_1^*, \eta_1^*, \eta_2^*) \leq \lambda\, S_b(\eta_0^*, \eta_0^*, \eta_1^*). \tag{4.2}$$

Similarly,

$$S_b(\eta_2^*, \eta_2^*, \eta_3^*) \leq \lambda\, S_b(\eta_1^*, \eta_1^*, \eta_2^*) \leq \lambda\big[\lambda\, S_b(\eta_0^*, \eta_0^*, \eta_1^*)\big] = \lambda^2\, S_b(\eta_0^*, \eta_0^*, \eta_1^*).$$

Proceeding inductively,

$$S_b(\eta_n^*, \eta_n^*, \eta_{n+1}^*) \leq \lambda^n\, S_b(\eta_0^*, \eta_0^*, \eta_1^*). \tag{4.3}$$

Now we demonstrate that $\{\eta_n^*\}$ is Cauchy for $\mathfrak{m} \geq 1$. Applying axiom (B3):

$$\begin{aligned} S_b(\eta_n^*, \eta_n^*, \eta_{n+m}^*) &\leq b\big[S_b(\eta_n^*, \eta_n^*, \eta_{n+1}^*) + S_b(\eta_n^*, \eta_n^*, \eta_{n+1}^*) + S_b(\eta_{n+m}^*, \eta_{n+m}^*, \eta_{n+1}^*)\big] \\ &= b\big[2\, S_b(\eta_n^*, \eta_n^*, \eta_{n+1}^*) + S_b(\eta_{n+1}^*, \eta_{n+1}^*, \eta_{n+m}^*)\big]. \end{aligned}$$

Repeated application yields:

$$\begin{aligned} S_b(\eta_n^*, \eta_n^*, \eta_{n+m}^*) \leq 2b\, S_b(\eta_n^*, \eta_n^*, \eta_{n+1}^*) & \\ + b\Big[b\big(S_b(\eta_{n+1}^*, \eta_{n+1}^*, \eta_{n+2}^*) + S_b(\eta_{n+1}^*, \eta_{n+1}^*, \eta_{n+2}^*) & \\ + S_b(\eta_{n+m}^*, \eta_{n+m}^*, \eta_{n+2}^*)\big)\Big]. & \end{aligned}$$

Continuing this expansion:

$$\begin{aligned} S_b(\eta_n^*, \eta_n^*, \eta_{n+m}^*) &\leq 2b\, S_b(\eta_n^*, \eta_n^*, \eta_{n+1}^*) + 2b^2\, S_b(\eta_{n+1}^*, \eta_{n+1}^*, \eta_{n+2}^*) \\ &\quad + \cdots + 2b^{\mathfrak{m}}\, S_b(\eta_{n+\mathfrak{m}-1}^*, \eta_{n+\mathfrak{m}-1}^*, \eta_{n+m}^*). \end{aligned}$$

Employing (4.3):

$$\begin{aligned} S_b(\eta_n^*, \eta_n^*, \eta_{n+m}^*) &\leq 2\, S_b(\eta_0^*, \eta_0^*, \eta_1^*)\big(b\lambda^n + b^2\lambda^{n+1} + \cdots + b^{\mathfrak{m}}\lambda^{n+\mathfrak{m}-1}\big) \\ &= 2b\, S_b(\eta_0^*, \eta_0^*, \eta_1^*)\big(\lambda^n + b\lambda^{n+1} + \cdots + b^{\mathfrak{m}-1}\lambda^{n+\mathfrak{m}-1}\big) \\ &= 2b\lambda^n\, S_b(\eta_0^*, \eta_0^*, \eta_1^*)\big(1 + b\lambda + \cdots + b^{\mathfrak{m}-1}\lambda^{\mathfrak{m}-1}\big). \end{aligned}$$

Since $b\lambda < 1$, the geometric series converges. Therefore

$$S_b(\eta_n^*, \eta_n^*, \eta_{n+m}^*) \to 0 \quad \text{as } n \to \infty.$$

Consequently, the sequence $\{\eta_n^*\}$ is Cauchy. Since (U, S_b) is complete, it contains $\eta^* \in U$ satisfying

$$\lim_{n\to\infty} S_b(\eta_n^*, \eta_n^*, \eta^*) = 0.$$

We now establish that $\Gamma\eta^* = \eta^*$. *Verification that* $\Gamma\eta^* = \eta^*$. Assume to the contrary that $\Gamma\eta^* \neq \eta^*$. Then $S_b(\eta^*, \eta^*, \Gamma\eta^*) > 0$. Applying (B3):

$$S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq b\big[\, 2\, S_b(\eta^*, \eta^*, \eta^*_{n+1}) + S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*_{n+1})\,\big].$$

Since $\eta^*_{n+1} = \eta^*_n$,

$$S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq b\big[\, 2\, S_b(\eta^*, \eta^*, \eta^*_{n+1}) + S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*_n)\,\big]. \tag{4.4}$$

Using (4.1):

$$\begin{aligned} S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*_n) \leq \Lambda\big[\, & S_b(\eta^*, \eta^*, \eta^*_n),\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*), \\ & S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\ S_b(\Gamma\eta^*_n, \Gamma\eta^*_n, \eta^*),\ S_b(\Gamma\eta^*_n, \Gamma\eta^*_n, \eta^*_n)\,\big]. \end{aligned}$$

Taking $n \to \infty$ and employing continuity of Λ, we obtain:

$$S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq \Lambda\big[0,\ S_b(\eta^*, \eta^*, \Gamma\eta^*),\ S_b(\eta^*, \eta^*, \Gamma\eta^*),\ 0,\ 0\big].$$

Thus $S_b(\eta^*, \eta^*, \Gamma\eta^*) = 0$. Hence $\Gamma\eta^* = \eta^*$. To establish uniqueness, let $\eta^{*'}$ be another fixed point of Γ. Then

$$\begin{aligned} S_b(\eta^*, \eta^*, \eta^{*'}) &= S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\eta^{*'}) \\ &\leq \Lambda\big[\, S_b(\eta^*, \eta^*, \eta^{*'}),\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^{*'}), \\ &\qquad S_b(\Gamma\eta^{*'}, \Gamma\eta^{*'}, \eta^*),\ S_b(\Gamma\eta^{*'}, \Gamma\eta^{*'}, \eta^{*'})\,\big] \\ &= \Lambda\big[\, S_b(\eta^*, \eta^*, \eta^{*'}),\ 0,\ S_b(\eta^*, \eta^*, \eta^{*'}),\ S_b(\eta^{*'}, \eta^{*'}, \eta^*),\ 0\big] \\ &= \Lambda\big[\, S_b(\eta^*, \eta^*, \eta^{*'}),\ 0,\ S_b(\eta^*, \eta^*, \eta^{*'}),\ S_b(\eta^*, \eta^*, \eta^{*'}),\ 0\big] \end{aligned}$$

Let $t^* = S_b(\eta^*, \eta^*, \eta^{*'})$. Then:

$$t^* \leq \Lambda\big[\, t^*,\ 0,\ t^*,\ t^*,\ 0\,\big] = t^* \cdot \Lambda(1, 0, 1, 1, 0).$$

Using the property of Λ:

$$\Rightarrow \quad S_b(\eta^*, \eta^*, \eta^{*'}) \leq t^* \, \Lambda(1, 0, 1, 1, 0) \leq t^* \lambda,$$

since $\lambda = \Lambda(1, 1, 0, 3, 1)$ Therefore,

$$t^* \leq \lambda t^* \; \Rightarrow \; t^* - \lambda t^* \leq 0 \; \Rightarrow \; t^*(1 - \lambda) \leq 0.$$

Since $t^* \geq 0$ and $\lambda \in [0, 1)$, we obtain $t^* = 0$, and thus $S_b(\eta^*, \eta^*, \eta^{*'}) = 0$. By property (B2), $\eta^* = \eta^{*'}$. Thus, the existence and uniqueness of a fixed point Γ in U are established. □

Example 4.1.3. Let $U = [0, 1]$ and define a mapping $S_b : U \times U \times U \to [0, \infty)$ by

$$S_b(\eta^*, \theta^*, \vartheta^*) = \big(|\eta^* - \theta^*| + |\eta^* + \theta^* - 2\vartheta^*|\big)^2, \quad \forall \, \eta^*, \theta^*, \vartheta^* \in U.$$

Then (U, S_b) is a complete SbMS with coefficient $b = 3$ [14]. Define $\Gamma : U \to U$ by $\Gamma\eta^* = \dfrac{\eta^*}{1 + 4\eta^*}$ and $\Lambda(x_1, x_2, x_3, x_4, x_5) = \frac{1}{4} \, x_1$. Clearly $\Lambda \in \mathcal{M}$ [20]. For $\eta^*, \theta^* \in U$,

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) = \big(2|\Gamma\eta^* - \Gamma\theta^*|\big)^2 = 4|\Gamma\eta^* - \Gamma\theta^*|^2.$$

Since

$$|\Gamma\eta^* - \Gamma\theta^*| = \left| \frac{\eta^* - \theta^*}{(1 + 4\eta^*)(1 + 4\theta^*)} \right| \leq |\eta^* - \theta^*|,$$

we obtain $S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq 4|\eta^* - \theta^*|^2 = S_b(\eta^*, \eta^*, \theta^*)$, so

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \tfrac{1}{4} \, S_b(\eta^*, \eta^*, \theta^*) = \Lambda\big(S_b(\eta^*, \eta^*, \theta^*), \ldots\big).$$

The contraction condition holds with $\lambda = \frac{1}{4}$ and $b\lambda = \frac{3}{4} < 1$. Solving $\Gamma\eta^* = \eta^*$:

$$\frac{\eta^*}{1 + 4\eta^*} = \eta^* \implies 4\eta^{*2} = 0 \implies \eta^* = 0.$$

Thus 0 is the unique fixed point of Γ.

Theorem 4.1.4. Let (U, S_b) be an S_b-metric space [14] and $\Lambda \in \mathcal{M}^*$ [20]. Let $\Gamma : U \to U$ be a self-map such that whenever $S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) > 0$, the following inequality holds:

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \leq \Lambda\big[S_b(\eta^*, \eta^*, \eta_0^*),\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*),\\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta_0^*),\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta^*)\big]. \tag{4.5}$$

Then the circle

$$C(\eta_0^*, \tau) = \{\eta^* \in U : S_b(\eta^*, \eta^*, \eta_0^*) = \tau\}, \quad \tau = \inf\{S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) : \Gamma\eta^* \neq \eta^*\},$$

is a fixed circle of Γ.

Proof. Let $C(\eta_0^*, \tau)$ be centered at η_0^* and suppose $\Gamma\eta_0^* \neq \eta_0^*$. Applying (4.5) with $\eta^* = \eta_0^*$,

$$\begin{aligned} S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*) &\leq \Lambda\big[0,\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*),\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*),\\ &\qquad S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*),\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*)\big]\\ &= S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*) \cdot \Lambda(0,1,1,1,1)\\ &< S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*), \end{aligned}$$

a contradiction; hence $\Gamma\eta_0^* = \eta_0^*$. Now let $\eta^* \in C(\eta_0^*, \tau)$ and assume $\Gamma\eta^* \neq \eta^*$. Thus, $S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) > 0 \Rightarrow S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \geq \tau$. Using (4.5), we have

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \leq \Lambda\big(S_b(\eta^*, \eta^*, \eta_0^*), S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\\ S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta_0^*), S_b(\Gamma\eta^*, \Gamma\eta^*, \eta_0^*), S_b(\Gamma\eta_0^*, \Gamma\eta_0^*, \eta^*)\big).$$

Since $S_b(\eta_0^*, \eta_0^*, \eta_0^*) = 0$ (as $\Gamma\eta_0^* = \eta_0^*$) and $S_b(\eta^*, \eta^*, \eta_0^*) = \tau$, we obtain

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) \leq \Lambda\big[\,\tau,\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\ 0,\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta_0^*),\ \tau\,\big].$$

Now, applying the S_b-triangle inequality to the fourth term $S_b(\Gamma\eta^*, \Gamma\eta^*, \eta_0^*)$:

$$\begin{aligned} S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) &\leq \Lambda\Big(\tau,\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\ 0,\ b\big(2S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) + \tau\big),\ \tau\Big)\\ &= \Lambda\Big(\tau,\ S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*),\ 0,\ 2b\, S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*) + b\tau,\ \tau\Big) \end{aligned}$$

$$\leq S_b(\Gamma\eta^*, \Gamma\eta^*, \eta^*)\,\Lambda(1, 1, 0, 3b, 1).$$

Since $\Lambda(1, 1, 0, 3b, 1) < 1$, we obtain a contradiction unless $\Gamma\eta^* = \eta^*$. Hence $C(\eta_0^*, \tau)$ is a fixed circle of Γ. □

Example 4.1.5. Let $U = \mathbb{R}^3$. Define a function

$$S_b : U \times U \times U \longrightarrow [0, \infty)$$

by

$$\begin{aligned} S_b(\eta^*, \theta^*, \vartheta^*) = \Big(&|\eta_1^* - \vartheta_1^*| + |\eta_1^* + \vartheta_1^* - 2\theta_1^*| \\ &+ |\eta_2^* - \vartheta_2^*| + |\eta_2^* + \vartheta_2^* - 2\theta_2^*| \\ &+ |\eta_3^* - \vartheta_3^*| + |\eta_3^* + \vartheta_3^* - 2\theta_3^*|\Big)^2, \end{aligned}$$

where

$$\eta^* = (\eta_1^*, \eta_2^*, \eta_3^*), \qquad \theta^* = (\theta_1^*, \theta_2^*, \theta_3^*),$$

and

$$\vartheta^* = (\vartheta_1^*, \vartheta_2^*, \vartheta_3^*) \in U.$$

Clearly,

$$S_b(\eta^*, \theta^*, \vartheta^*) \geq 0,$$

and

$$S_b(\eta^*, \theta^*, \vartheta^*) = 0 \quad \Longleftrightarrow \quad \eta^* = \theta^* = \vartheta^*.$$

Furthermore, for any $a^* \in U$, by using the triangle inequality, we obtain

$$\begin{aligned} &|\eta_i^* - \vartheta_i^*| + |\eta_i^* + \vartheta_i^* - 2\theta_i^*| \\ &\qquad \leq 2|\eta_i^* - a_i^*| + 2|\theta_i^* - a_i^*| + 2|\vartheta_i^* - a_i^*|, \qquad i = 1, 2, 3. \end{aligned}$$

Hence, by employing the inequality

$$(a+b+c)^2 \leq 3(a^2+b^2+c^2),$$

we have

$$S_b(\eta^*,\theta^*,\vartheta^*) \leq 3\Big[S_b(\eta^*,\eta^*,a^*)+S_b(\theta^*,\theta^*,a^*)+S_b(\vartheta^*,\vartheta^*,a^*)\Big].$$

Therefore, (U, S_b) is an S_b-metric space with coefficient $b = 3$.

Moreover,

$$S_b(\eta^*,\eta^*,\theta^*) = \left(2\sum_{i=1}^{3}|\eta_i^*-\theta_i^*|\right)^2 = S_b(\theta^*,\theta^*,\eta^*),$$

so (U, S_b) is symmetric.

Now, let

$$\eta_0^* = (1,2,3) \in U$$

and consider the circle

$$C(\eta_0^*,16) = \left\{\eta^* \in U : S_b(\eta_0^*,\eta_0^*,\eta^*) = 16\right\}.$$

Since

$$\begin{aligned} S_b(\eta_0^*,\eta_0^*,\eta^*) &= \Big(2|1-\eta_1^*|+2|2-\eta_2^*|+2|3-\eta_3^*|\Big)^2 \\ &= 4\Big(|1-\eta_1^*|+|2-\eta_2^*|+|3-\eta_3^*|\Big)^2, \end{aligned}$$

the condition

$$S_b(\eta_0^*,\eta_0^*,\eta^*) = 16$$

is equivalent to

$$|1-\eta_1^*|+|2-\eta_2^*|+|3-\eta_3^*| = 2.$$

Thus,

$$C(\eta_0^*,16) = \left\{\eta^* \in U : |1-\eta_1^*|+|2-\eta_2^*|+|3-\eta_3^*| = 2\right\}.$$

Define the self-mapping $T : U \to U$ by

$$T(\eta^*) = \eta_0^* + \frac{|1-\eta_1^*| + |2-\eta_2^*| + |3-\eta_3^*|}{2} \left(\eta^* - \eta_0^*\right).$$

Let $\eta^* \in C(\eta_0^*, 16)$. Then

$$|1-\eta_1^*| + |2-\eta_2^*| + |3-\eta_3^*| = 2.$$

Therefore,

$$\begin{aligned} T(\eta^*) &= \eta_0^* + \frac{2}{2}\left(\eta^* - \eta_0^*\right) \\ &= \eta_0^* + \eta^* - \eta_0^* \\ &= \eta^*. \end{aligned}$$

Hence,

$$T(\eta^*) = \eta^*, \qquad \forall\, \eta^* \in C(\eta_0^*, 16).$$

Consequently, every point of $C(\eta_0^*, 16)$ is a fp of T. Therefore,

$$\boxed{C(\eta_0^*, 16) \subseteq \mathrm{Fix}(T)}$$

and hence $C(\eta_0^*, 16)$ is a fixed circle of T in the symmetric S_b-metric space (U, S_b) with $b = 3$.

Notice that

$$T(\eta_0^*) = \eta_0^*,$$

so the center η_0^* is also a fixed point of T. Thus,

$$\mathrm{Fix}(T) = C(\eta_0^*, 16) \cup \{\eta_0^*\}.$$

Theorem 4.1.6. Let (U, S_b) be a complete symmetric S_b-metric space with $b \geq 1$ [14], $\Phi_1^*, \Phi_2^* \in U$, and $\tau \in [0, \infty)$. Define the elliptic disc

$$D = D(\Phi_1^*, \Phi_2^*, \tau)$$

and

$$\varsigma(\eta^*) = S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*), \quad \forall\, \eta^* \in U.$$

Suppose that a mapping $\Gamma : U \to U$ satisfies:

$$(E_1 D): \quad S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq \varsigma(\eta^*) + \varsigma(\Gamma\eta^*) - 4\tau, \quad \forall\, \eta^* \in U,$$

$$(E_2 D): \quad \varsigma(\Gamma\eta^*) \leq 2b\tau, \quad \forall\, \eta^* \in D,$$

$(E_3 D)$: there exists $\psi \in [0, 1)$ such that

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \frac{\psi}{b} S_b(\eta^*, \eta^*, \theta^*), \quad \forall\, \eta^* \in D,\ \forall\, \theta^* \in U \setminus D.$$

Then:

(i) D is a fixed elliptic disc of Γ.

(ii) D has the maximum possible semi-major axis among all fixed elliptic discs determined by the same foci.

Proof. Let $\eta^* \in D$. Then

$$\varsigma(\eta^*) \leq 2\tau \leq 2b\tau$$

since $b \geq 1$. Using condition $(E_1 D)$:

$$\begin{aligned} S_b(\eta^*, \eta^*, \Gamma\eta^*) &\leq \varsigma(\eta^*) + \varsigma(\Gamma\eta^*) - 4b\tau \\ &\leq 2b\tau + 2b\tau - 4b\tau \\ &= 0. \end{aligned}$$

Hence, $S_b(\eta^*, \eta^*, \Gamma\eta^*) = 0$, and by the identity axiom of an S_b-metric space, we have $\Gamma\eta^* = \eta^*$. Assume there exists another fixed elliptic disc.

$$D' = D'(\Phi_1^*, \Phi_2^*, \tau_1), \quad \tau_1 > \tau.$$

Choose $\eta^* \in D$ and $\theta^* \in D' \setminus D$. Since both discs are fixed, we have the following:

$$\Gamma\eta^* = \eta^*, \quad \Gamma\theta^* = \theta^*.$$

Applying condition (E_3D):

$$\begin{aligned} S_b(\eta^*, \eta^*, \theta^*) &= S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \\ &\leq \frac{\psi}{b} S_b(\eta^*, \eta^*, \theta^*). \end{aligned}$$

Thus,

$$\left(1 - \frac{\psi}{b}\right) S_b(\eta^*, \eta^*, \theta^*) \leq 0.$$

since $b \geq 1$ and $0 \leq \psi < 1$, it follows that:

$$1 - \frac{\psi}{b} > 0.$$

Hence, $S_b(\eta^*, \eta^*, \theta^*) = 0$, which implies $\eta^* = \theta^*$. This is a contradiction because $\theta^* \notin D$. Therefore, no larger fixed elliptic disc exists and D has the maximum possible semi-major axis. $\square$

Example 4.1.7. Let $U = \mathbb{R}$ and define the symmetric S_b-MS with $b = 2$ by:

$$S_b(\eta^*, \theta^*, \vartheta^*) = (|\eta^* - \vartheta^*| + |\theta^* - \vartheta^*|)^2.$$

Take $\Phi_1^* = -1$, $\Phi_2^* = 1$, and $\tau = 2$. Then:

$$\varsigma(\eta^*) = S_b(-1, -1, \eta^*) + S_b(1, 1, \eta^*).$$

Hence, the elliptic disc is defined as

$$D = \{\eta^* \in \mathbb{R} : \varsigma(\eta^*) \leq 4\}.$$

Define the mapping $\Gamma : U \to U$ by

$$\Gamma(\eta^*) = \eta^* + \max\{0, \varsigma(\eta^*) - 4\}. \tag{4.6}$$

We now show that every point of D is fixed by Γ.

Case 1. Let $\eta^* \in D$. Since

$$D = \{\eta^* \in \mathbb{R} : \varsigma(\eta^*) \leq 4\},$$

we have

$$\varsigma(\eta^*) \leq 4.$$

Therefore,

$$\varsigma(\eta^*) - 4 \leq 0,$$

and hence

$$\max\{0, \varsigma(\eta^*) - 4\} = 0.$$

Consequently,

$$\begin{aligned}\Gamma(\eta^*) &= \eta^* + \max\{0, \varsigma(\eta^*) - 4\}\\ &= \eta^* + 0\\ &= \eta^*.\end{aligned}$$

Thus,

$$\Gamma(\eta^*) = \eta^*, \qquad \forall\, \eta^* \in D.$$

Case 2. Let $\eta^* \notin D$. Then

$$\varsigma(\eta^*) > 4,$$

and therefore

$$\varsigma(\eta^*) - 4 > 0.$$

Hence,

$$\max\{0, \varsigma(\eta^*) - 4\} = \varsigma(\eta^*) - 4 > 0.$$

It follows that

$$\begin{aligned}\Gamma(\eta^*) &= \eta^* + \varsigma(\eta^*) - 4 \\ &\neq \eta^*.\end{aligned}$$

Therefore,

$$\Gamma(\eta^*) \neq \eta^*, \qquad \forall\, \eta^* \notin D.$$

Hence,

$$\text{Fix}(\Gamma) = D,$$

and consequently every point of D is a FP of Γ. Thus, D is a fixed elliptic disc of Γ. Hence, all hypotheses of the theorem are satisfied. Consequently:

(i) Every point of D is a FP of Γ.

(ii) D is a fixed elliptic disc.

(iii) D has the maximum possible semi-major axis.

Theorem 4.1.8 (Fixed Ellipse Theorem)**.** Let (U, S_b) be a complete SbMS with coefficient $b \geq 1$ [14], and let $\Phi_1^*, \Phi_2^* \in U$ be two distinct points (foci). Define the ellipse

$$E(\Phi_1^*, \Phi_2^*, \tau) = \Big\{\eta^* \in U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) = 2\tau\Big\}, \quad \tau > 0.$$

Let $\Gamma : U \to U$ be a self-map satisfying:

(A_1^*) : $\forall\, \eta^* \in E(\Phi_1^*, \Phi_2^*, \tau)$ with $\Gamma\eta^* \neq \eta^*$,

$$S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq \frac{1}{2}\Big(S_b(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S_b(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) - 2\tau\Big).$$

(A_2^*) : $\forall\, \eta^* \in E(\Phi_1^*, \Phi_2^*, \tau)$,

$$S_b(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S_b(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) \leq 2\tau.$$

(A_3^*) : $\forall\, \eta^* \in E(\Phi_1^*, \Phi_2^*, \tau)$ and $\theta^* \in U \setminus E(\Phi_1^*, \Phi_2^*, \tau)$,

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \psi\, S_b(\eta^*, \eta^*, \theta^*), \quad 0 \leq \psi < 1.$$

Then $E(\Phi_1^*, \Phi_2^*, \tau)$ is a unique fixed ellipse of Γ, i.e.,

$$\Gamma\eta^* = \eta^* \quad \forall\, \eta^* \in E(\Phi_1^*, \Phi_2^*, \tau),$$

and no other ellipse of the same type can be fixed by Γ.

Proof. Let $\eta^* \in E(\Phi_1^*, \Phi_2^*, \tau)$. Suppose $\Gamma\eta^* \neq \eta^*$. From (A_1^*),

$$S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq \frac{1}{2}\Big(S_b(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S_b(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) - 2\tau\Big).$$

Using (A_2^*),

$$S_b(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S_b(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) - 2\tau \leq 0.$$

Hence $S_b(\eta^*, \eta^*, \Gamma\eta^*) \leq 0$. Since $S_b \geq 0$, we obtain

$$S_b(\eta^*, \eta^*, \Gamma\eta^*) = 0 \;\Rightarrow\; \Gamma\eta^* = \eta^*.$$

Thus, every $\eta^* \in E(\Phi_1^*, \Phi_2^*, \tau)$ is a FP. Now assume there exist two distinct fixed ellipses E and E'. Take $\eta^* \in E$ and $\theta^* \in E'$. Then

$$\Gamma\eta^* = \eta^*, \quad \Gamma\theta^* = \theta^*.$$

Using (A_3^*),

$$S_b(\eta^*, \eta^*, \theta^*) = S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) \leq \psi\, S_b(\eta^*, \eta^*, \theta^*).$$

Thus $(1-\psi)\, S_b(\eta^*, \eta^*, \theta^*) \leq 0$. Since $0 \leq \psi < 1$, we obtain

$$S_b(\eta^*, \eta^*, \theta^*) = 0, \quad \text{hence} \quad \eta^* = \theta^*.$$

Therefore $E = E'$. This theorem introduces a contraction outside the ellipse and a boundary control condition ensuring both existence and uniqueness of a fixed geometrical structure. □

Example 4.1.9. Let

$$\Phi_1^* = (0,0), \qquad \Phi_2^* = (\ln 2, 0), \qquad p^* = (\ln 3, 0),$$

and consider the finite set

$$U = \{\Phi_1^*, \Phi_2^*, p^*\} \subset \mathbb{R}^2.$$

For

$$\eta^* = (\eta_1^*, \eta_2^*), \qquad \theta^* = (\theta_1^*, \theta_2^*), \qquad \vartheta^* = (\vartheta_1^*, \vartheta_2^*) \in U,$$

define

$$\begin{aligned} S_b(\eta^*, \theta^*, \vartheta^*) = \Big(&|e^{\eta_1^*} - e^{\vartheta_1^*}| + |e^{\eta_2^*} - e^{\vartheta_2^*}| \\ &+ |e^{\theta_1^*} - e^{\vartheta_1^*}| + |e^{\theta_2^*} - e^{\vartheta_2^*}| \Big)^2. \end{aligned} \tag{4.7}$$

For convenience, define

$$d(\eta^*, \vartheta^*) = |e^{\eta_1^*} - e^{\vartheta_1^*}| + |e^{\eta_2^*} - e^{\vartheta_2^*}|.$$

Then

$$S_b(\eta^*, \theta^*, \vartheta^*) = \left(d(\eta^*, \vartheta^*) + d(\theta^*, \vartheta^*) \right)^2.$$

Clearly,

$$S_b(\eta^*, \theta^*, \vartheta^*) \geq 0,$$

and

$$S_b(\eta^*, \theta^*, \vartheta^*) = 0 \iff \eta^* = \theta^* = \vartheta^*.$$

Moreover, for any $a^* \in U$, the triangle inequality for d gives

$$d(\eta^*, \vartheta^*) + d(\theta^*, \vartheta^*) \leq d(\eta^*, a^*) + d(\theta^*, a^*) + 2d(\vartheta^*, a^*).$$

Using

$$(x + y + 2z)^2 \leq 6(x^2 + y^2 + z^2) \leq 8(x^2 + y^2 + z^2),$$

we obtain

$$\begin{aligned} S_b(\eta^*, \theta^*, \vartheta^*) &\leq 8\Big(d(\eta^*, a^*)^2 + d(\theta^*, a^*)^2 + d(\vartheta^*, a^*)^2\Big) \\ &= 2\Big[S_b(\eta^*, \eta^*, a^*) + S_b(\theta^*, \theta^*, a^*) + S_b(\vartheta^*, \vartheta^*, a^*)\Big]. \end{aligned}$$

Thus, (U, S_b) is an S_b-MS with coefficient $b = 2$.

Furthermore,

$$S_b(\eta^*, \eta^*, \theta^*) = 4d(\eta^*, \theta^*)^2 = S_b(\theta^*, \theta^*, \eta^*),$$

so (U, S_b) is symmetric. Since U is finite, (U, S_b) is complete.

Now take the two distinct foci

$$\Phi_1^* = (0, 0), \qquad \Phi_2^* = (\ln 2, 0),$$

and let

$$\tau = 10.$$

Define

$$E(\Phi_1^*, \Phi_2^*, 10) = \{\eta^* \in U : S_b(\Phi_1^*, \Phi_1^*, \eta^*) + S_b(\Phi_2^*, \Phi_2^*, \eta^*) = 20\}.$$

For $p^* = (\ln 3, 0)$, we have

$$S_b(\Phi_1^*, \Phi_1^*, p^*) = (2|1-3|)^2 = 16,$$
$$S_b(\Phi_2^*, \Phi_2^*, p^*) = (2|2-3|)^2 = 4.$$

Therefore,

$$S_b(\Phi_1^*, \Phi_1^*, p^*) + S_b(\Phi_2^*, \Phi_2^*, p^*) = 20.$$

On the other hand,

$$S_b(\Phi_1^*, \Phi_1^*, \Phi_1^*) + S_b(\Phi_2^*, \Phi_2^*, \Phi_1^*) = 0 + 4 = 4,$$

and similarly,

$$S_b(\Phi_1^*, \Phi_1^*, \Phi_2^*) + S_b(\Phi_2^*, \Phi_2^*, \Phi_2^*) = 4 + 0 = 4.$$

Hence,

$$E(\Phi_1^*, \Phi_2^*, 10) = \{p^*\}.$$

Also,

$$S_b(\Phi_1^*, \Phi_1^*, \Phi_2^*) = 4 < 20 = 2\tau,$$

so the required non-degeneracy condition is satisfied.

Define the mapping

$$\Gamma : U \longrightarrow U$$

by

$$\Gamma(\eta^*) = p^*, \qquad \forall\, \eta^* \in U. \tag{4.8}$$

We now verify the conditions of Theorem 4.1.8.

Verification of (A_1^*).

The only point of the ellipse is p^*, and

$$\Gamma(p^*) = p^*.$$

Thus, there is no

$$\eta^* \in E(\Phi_1^*, \Phi_2^*, 10)$$

for which

$$\Gamma\eta^* \neq \eta^*.$$

Therefore, condition (A_1^*) is satisfied vacuously.

Verification of (A_2^*).

For

$$\eta^* = p^* \in E(\Phi_1^*, \Phi_2^*, 10),$$

we have

$$\Gamma\eta^* = p^*.$$

Consequently,

$$\begin{aligned}
&S_b(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S_b(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) \\
&\qquad = S_b(\Phi_1^*, \Phi_1^*, p^*) + S_b(\Phi_2^*, \Phi_2^*, p^*) \\
&\qquad = 16 + 4 \\
&\qquad = 20 = 2\tau.
\end{aligned}$$

Hence,

$$S_b(\Phi_1^*, \Phi_1^*, \Gamma\eta^*) + S_b(\Phi_2^*, \Phi_2^*, \Gamma\eta^*) \leq 2\tau,$$

and condition (A_2^*) holds.

Verification of (A_3^*).

Let

$$\eta^* \in E(\Phi_1^*, \Phi_2^*, 10)$$

and

$$\theta^* \in U \setminus E(\Phi_1^*, \Phi_2^*, 10).$$

Then necessarily

$$\eta^* = p^*,$$

while

$$\theta^* \in \{\Phi_1^*, \Phi_2^*\}.$$

Since Γ is constant,

$$\Gamma\eta^* = \Gamma\theta^* = p^*.$$

Therefore,

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) = S_b(p^*, p^*, p^*) = 0.$$

Choose

$$\psi = 0.$$

Clearly,

$$0 \leq \psi < 1,$$

and

$$S_b(\Gamma\eta^*, \Gamma\eta^*, \Gamma\theta^*) = 0 \leq \psi S_b(\eta^*, \eta^*, \theta^*).$$

Hence, condition (A_3^*) is also satisfied.

Thus, all the hypotheses of Theorem 4.1.8 are satisfied. Consequently,

$$\boxed{E(\Phi_1^*, \Phi_2^*, 10) = \{(\ln 3, 0)\}}$$

is a unique fixed ellipse of Γ.

Indeed,

$$\mathrm{Fix}(\Gamma) = \{(\ln 3, 0)\} = E(\Phi_1^*, \Phi_2^*, 10).$$

4.2 Application to a Satellite Web Coupling Problem in Complete Symmetric S_b-Metric Spaces

Motivated by the applications of fixed point theory in engineering and applied sciences, we further extend the satellite web coupling problem [20] to the framework of complete symmetric S_b-metric spaces [14]. Consider the nonlinear two-point boundary value problem:

$$\begin{cases} -\eta''(t) = \phi(t, \eta(t)), \quad t \in [0, 1], \\ \eta(0) = \eta(1) = 0, \end{cases} \tag{4.15}$$

where $\phi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous nonlinear boundary response function.

The above boundary value problem can be transformed into the equivalent Hammerstein integral equation:

$$\eta(t) = \int_0^1 \mathcal{G}(t, s)\phi(s, \eta(s))\, ds, \tag{4.16}$$

where $\mathcal{G}(t, s)$ is the Green's function defined by:

$$\mathcal{G}(t, s) = \begin{cases} s(1 - t), & 0 \leq s \leq t \leq 1, \\ t(1 - s), & 0 \leq t \leq s \leq 1. \end{cases} \tag{4.17}$$

The Green's function satisfies $\mathcal{G}(t, s) \geq 0$ for all $(t, s) \in [0, 1]^2$, and

$$\sup_{t\in[0,1]} \int_0^1 \mathcal{G}(t, s)\, ds = \frac{1}{8}. \tag{4.18}$$

Let $U = C([0, 1], \mathbb{R})$ be the space of continuous real-valued functions on $[0, 1]$. We equip U with the following induced symmetric S_b-metric:

$$S_b(\eta, \theta, \vartheta) = \sup_{t\in[0,1]} \left(|\eta(t) - \vartheta(t)|^p + |\theta(t) - \vartheta(t)|^p\right), \tag{4.19}$$

where $p > 1$. By using the convexity inequality $(a + b)^p \le 2^{p-1}(a^p + b^p)$, the space (U, S_b) forms a complete symmetric S_b-metric space with parameter $b = 2^{p-1}$ [14].

Define the operator $T : U \to U$ by

$$(T\eta)(t) = \int_0^1 \mathcal{G}(t, s)\phi(s, \eta(s))\, ds. \tag{4.20}$$

The existence and uniqueness of the solution of the satellite web coupling problem are established by applying the FP result in complete symmetric S_b-metric spaces.

Theorem 4.2.1. Let (U, S_b) be the complete symmetric S_b-metric space defined above [14], and let $T : U \to U$ be the operator

$$(T\eta)(t) = \int_0^1 \mathcal{G}(t, s)\phi(s, \eta(s))\, ds.$$

Assume that the continuous function $\phi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|\phi(s, \eta) - \phi(s, \theta)|^p \le L|\eta - \theta|^p, \tag{4.21}$$

for all $\eta, \theta \in \mathbb{R}$ and $s \in [0, 1]$, where the positive constant L satisfies

$$L < \frac{8^p}{b}. \tag{4.22}$$

Then the operator T possesses a unique FP in U. Hence, the satellite web coupling boundary value problem (4.15) admits a unique solution belonging to U.

Proof. Let $\eta, \theta, \vartheta \in U$. Then

$$S_b(T\eta, T\theta, T\vartheta) = \sup_{t\in[0,1]} \left(|T\eta(t) - T\vartheta(t)|^p + |T\theta(t) - T\vartheta(t)|^p\right).$$

Using the definition of T, we have

$$|T\eta(t) - T\vartheta(t)|^p = \left|\int_0^1 \mathcal{G}(t, s)\left[\phi(s, \eta(s)) - \phi(s, \vartheta(s))\right] ds\right|^p.$$

Applying Hölder's inequality,

$$|T\eta(t) - T\vartheta(t)|^p \leq \left(\int_0^1 \mathcal{G}(t,s)\, ds\right)^{p-1} \int_0^1 \mathcal{G}(t,s)\big|\phi(s,\eta(s)) - \phi(s,\vartheta(s))\big|^p\, ds.$$

Since

$$\int_0^1 \mathcal{G}(t,s)\, ds \leq \frac{1}{8}$$

and

$$|\phi(s,\eta(s)) - \phi(s,\vartheta(s))|^p \leq L|\eta(s) - \vartheta(s)|^p,$$

we obtain

$$|T\eta(t) - T\vartheta(t)|^p \leq \left(\frac{1}{8}\right)^{p-1} L \int_0^1 \mathcal{G}(t,s)|\eta(s) - \vartheta(s)|^p\, ds.$$

Similarly,

$$|T\theta(t) - T\vartheta(t)|^p \leq \left(\frac{1}{8}\right)^{p-1} L \int_0^1 \mathcal{G}(t,s)|\theta(s) - \vartheta(s)|^p\, ds.$$

Adding the above inequalities, we get

$$S_b(T\eta, T\theta, T\vartheta) \leq \sup_{t\in[0,1]} \left[\left(\frac{1}{8}\right)^{p-1} L \int_0^1 \mathcal{G}(t,s)\big(|\eta(s) - \vartheta(s)|^p + |\theta(s) - \vartheta(s)|^p\big)\, ds\right].$$

Since

$$|\eta(s) - \vartheta(s)|^p + |\theta(s) - \vartheta(s)|^p \leq S_b(\eta,\theta,\vartheta)$$

for every $s \in [0,1]$, therefore

$$S_b(T\eta, T\theta, T\vartheta) \leq \left(\frac{1}{8}\right)^{p-1} L\, S_b(\eta,\theta,\vartheta) \sup_{t\in[0,1]} \int_0^1 \mathcal{G}(t,s)\, ds \leq \left(\frac{1}{8}\right)^{p} L\, S_b(\eta,\theta,\vartheta).$$

Hence,

$$S_b(T\eta, T\theta, T\vartheta) \leq \lambda_0\, S_b(\eta,\theta,\vartheta),$$

where

$$\lambda_0 = \frac{L}{8^p}.$$

Because

$$L < \frac{8^p}{b},$$

we have

$$\lambda_0 < \frac{1}{b},$$

or equivalently,

$$b\lambda_0 < 1.$$

Thus, the contraction condition of the FP theorem for complete symmetric S_b-metric spaces [14, 20] is satisfied. Hence, the operator T has a unique FP $\eta^* \in U$. Since every FP of T satisfies

$$\eta(t) = \int_0^1 \mathcal{G}(t, s)\phi(s, \eta(s))\, ds,$$

which is equivalent to the original boundary value problem (4.15), the satellite web coupling problem admits a unique solution in the complete symmetric S_b-metric space U. Thus, the proof is complete. $\square$

Chapter 5

Conclusion

This thesis has developed a unified framework for the study of fixed figures in S-metric and symmetric S_b-metric spaces by employing the versatile $\mathcal{M}-$class contractive functions. The research extends several fundamental ideas in generalized fixed point theory and provides new theoretical results concerning fixed points, fixed circles, fixed ellipses, and fixed elliptic discs. This chapter summarizes the principal achievements of the thesis and highlights several promising directions for future investigation.

One of the central contributions of this work is the systematic use of the $\mathcal{M}$-class contractive framework as a common platform for a wide variety of contraction principles. Originally introduced by Joshi and Tomar [19] and subsequently generalized by Joshi, Tomar, and Abdeljawad [20], the $\mathcal{M}-$class approach has proved to be an effective analytical tool for studying geometric fixed figures in generalized metric spaces.

within the framework of S-metric spaces, we have developed a broad collection of sufficient criteria guaranteeing the existence of fixed circles, fixed ellipses, and fixed elliptic discs. These results concerning non-identity mappings and the existence of multiple ellipses further enrich the developed theory. The theoretical developments obtained for S-metric spaces have subsequently been extended to symmetric

S_b-metric spaces, wherein the generalized triangle inequality contains the relaxation parameter $b \geq 1$. The presence of this parameter introduces additional analytical difficulties that require techniques beyond those employed in ordinary S-metric spaces. In particular, the convergence analysis depends fundamentally on the condition $b\lambda < 1$, that guarantees the Cauchy property of the associated Picard sequence. Likewise, the inequality

$$\Lambda(1, 1, 0, 3b, 1) < 1$$

is instrumental in proving the existence of fixed circles. Furthermore, a new half-boundary control theorem for fixed ellipses has been established in the symmetric S_b-metric setting, thereby extending the corresponding S-metric results to a more general and flexible class of generalized metric spaces.

Another significant outcome of this thesis is the characterization of continuity and discontinuity of self-mappings on fixed ellipses. A necessary and sufficient criterion has been established to characterize the points of a fixed ellipse at which a mapping is continuous. This result provides a concrete contribution toward the open question raised by Rhoades [8] concerning discontinuous contractive mappings. Beyond its theoretical significance, the criterion also has potential applications in neural network theory, where discontinuous activation functions [9] are frequently associated with improved storage capacity and enhanced computational performance.

The collection of open problems presented throughout this thesis demonstrates that the proposed $\mathcal{M}-$class fixed figure framework offers numerous opportunities for further development. Several important questions remain unresolved, including extensions involving JS-contractions, simulation functions, interpolative contractions, generalized geometric figures, and broader classes of generalized metric spaces. In particular, the integration of fuzzy contractive conditions with directed graphs [7] into the $\mathcal{M}-$class framework presents a promising direction for future investigation. It is hoped that the methods, techniques, and results developed in this dissertation will provide a useful foundation for future research on the geometry of fixed point sets and their applications in nonlinear analysis, applied mathematics, and related scientific disciplines.

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