CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Fixed Point Results in Generalized b-Metric Spaces

by

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Dedicated to my ***Parents*** *and* ***Teachers***

CERTIFICATE OF APPROVAL

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Abstract

In this thesis, we propose a new generalization of the Banach Contraction Principle, referred to as the (Γ, ν)-φ-Ω-contraction, which is influenced by the concept of φ-contraction in generalized b-metric spaces. Our main focus is to investigate the conditions for the existence and uniqueness of fixed points for mappings in a generalized b-metric spaces. Furthermore, we present several examples to illustrate the enhancements introduced by this approach, along with an application to iterated solutions of non-linear integral equations.

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Abbreviations

BCP	Banach contraction principle
b**-MS**	b-metric space
CS	Cauchy Sequence
FP	Fixed point
GMS	Generalized metric space
GbMS	Generalized b-metric space
MS	Metric space
UFP	Unique Fixed Point

Symbols

$\eth$	Distance function
$C[\flat_1, \flat_2]$	Space of all real valued continuous functions on $[\flat_1, \flat_2]$
l^p	A sequence space
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\in$	Belongs to
$\exists$	There exist
$s.t$	Such that
$\forall$	For all

Chapter 1

Introduction

1.1 Background

Mathematics is crucial subject of scientific knowledge. It is the science of logical reasoning with the large range of applications in all aspects of life. It is a systematic way to understand this world wisely using different mathematical techniques and it is also regarded as “the mother of sciences”. Since, it offers different tools for precision, problem solving and innovations, it is divided into different branches. One of the most important branch of mathematics is functional analysis.

Functional analysis investigates mathematical spaces equipped with limit operations, continuity and mappings, providing essential tools for analysing functions and operators. It also provides an extension for analysing their properties.

One of the main part in functional analysis is fixed point (FP) theory, which studies the existence and properties of points that remain invariant under specific self mappings. Specifically, for a function a $\mathcal{S} : \mathcal{W} \to \mathcal{W}$, a point $\zeta \in \mathcal{W}$ is called a FP if $\mathcal{S}(\zeta) = \zeta$. The FP theory has wast applications in various fields of sciences such as,optimization theory, mathematical economics and approximation theory. In last 5-7 decades FP theory is playing a vital role and rising rapidly in research area of mathematics .

Henry Poincare [1] provide platform to initiate the metric FP theory. His work laid basic aspects for dynamical systems and topology which deeply influenced FP theory. In (1886) topological FP ideas were introduced by Poincare. He used qualitative (topological) approaches by examining the existence of FPs in surfaces flow and dynamical systems.

This basic step leads toward topological FP theory. A French mathematician, Maurice Frchet [2] was the pioneer of metric space (MS) in 1906. Later, in 1912, Brouwer [3] clearly examined the FP problem and establish the FP theorem to lead the equation $\mathcal{S}(\zeta) = \zeta$.

Stefan Banach, in 1922 established the Banach Contraction Principle (BCP) [4] which was the very crucial result in FP theory due to which Stefan Banach has risen as cornerstone of FP theory.

This basic idea ensures the existence and uniqueness of FPs for contraction mappings and also provides effective method for finding them. Depending upon Banach's work, many researchers have expanded FP theory in main two directions.

- One is mapping condition such as weaker contractive conditions or alternative mapping properties.
- Second is considering generalized spaces.

The BCP has undergone significant extensions and generalizations by various researchers. Notably, Edelstein [5] introduced a globally contractive mapping, broadening the BCP's scope. Later, In 1965, Presic S.B [6] expanded the principle to operators on product spaces, further developing the concept. Similarly, in 1968 proceeding ahead Kannan [7] introduced kannan mapping by generalizing BCP and by changing contraction mapping to Kannan mapping and prove the FP results known as Banach-Kannan contraction principle.

The concept of MSs has undergone numerous generalizations through modifications of its axioms. In 1989, Bakhtin [8], pioneered the concept of b-metric space (b-MS), laying the groundwork for extending the BCP to such spaces and enabling novel applications. Later on, in 1993, Czerwik [9], also worked on b-MS, characterized by a relaxed triangle inequality known as the b-triangle inequality. Czerwik also worked on generalized b-MS (GbMS). In 2017, Kamran et al. [10] generalized the concept of b-MS, which is known as extended b-MS by further weakning the triangular inequality. In 2018, Mlaiki et al. [11] gave us novel type of extended b-MS , that is controlled MS and double controlled MS [12]. In 2016, Kamran et al. [13] introduce the notion of a $\mathcal{C}^*$-algebra-valued b-MS. Later on Shehzad et al. [14] introduced a novel concept of (P, ψ)-type almost contractive conditions for fuzzy mappings in the context of b-MSs, broadening the applicability of FP theory. Researchers have derived numerous FP results within the bMS framework. Partial MS was introduced by Matthews [15] in 1994 characterized by the property that the point's distance to itself may be non-zero. Nazam et al. [16, 17] have established various FP and common FP theorems in the settings of partial MSs.

Wardowski [18] proposed a significant generalization of the classical Banach contraction by developing the concept of F-contractions, and demonstrated a FP theorem for complete MS.
A notable generalization of MSs was introduced by Branciari [19] known as generalized metric space (GMS). Every MS qualifies as a GMS but the converse is not necessarily true. Hence a GMS may lack the properties required to be a MS.
In 2006 Mustafa et al. [20] extends the concept of a traditional MS to a GMS by altering or relaxing one or more of its standard axioms. Despite these modifications, these spaces retain enough structural properties to facilitate the analysis of notions such as continuity, convergence, and FP results. This broader framework offers greater versatility, making it applicable in various mathematical fields, including topology, functional analysis, and FP theory. The GMS framework provides greater adaptability compared to the traditional MS, particularly in dealing with non-linear problems and functions involving multiple variables. Certain mappings that do not meet contraction conditions in standard MSs may still satisfy them within the context of GMS.

Jleli et al. [21, 22] introduced θ-contraction mappings and established a corresponding FP theorem, offering a novel generalization of the BCP that is distinct from existing literature. Kari et al. [23] established novel FP theorems in rectangular MSs and generalized asymmetric MSs, utilizing altered forms of generalized θ-contraction mappings.

Hussain and Salimi [24] introduced α-GF-contractions in 2014, proving corresponding FP theorems. Meanwhile, Hussain et al. [25] derived new FP results for generalized λ-μ-G-F-contraction mappings in complete b-MSs.
In this thesis, a new notion of generalized $(\Gamma$-$\nu)$-φ-Ω-contraction is presented which generalizes φ-contraction in frame of generalized b-metric space (GbMS), and provide illustrative examples along with corollaries that follow from our findings.

The rest of thesis is arranged as:

- **Chapter-2** provides foundational concepts and results essential for later chapters.

- **Chapter-3** gives a detailed review of Kari et al.[26], where they used $(\Gamma$-$\nu)$-φ-Ω-contraction in GMS to prove some FP results. Illustrative examples are provided to clarify our findings, and applications are also presented to demonstrate the validity and relevance of the main results.
- **Chapter-4** introduces a new class of contraction mappings known as $(\Gamma$-$\nu)$-φ-Ω-contraction in setting of GbMS. Some important FP results are established by using this idea. To explain the obtained results some examples and applications are provided.

- **Chapter-5** includes the conclusion of the thesis.

Chapter 2

Basic Definitions

In this chapter, some basics definitions with examples are presented that will be used in the next chapters. The first section covers some basics of MS and examples. The second section covers some generalization of MS including b-MS, GMS and GbMS with some fundamental results and examples. The third section deals with the FP and related concepts of (Γ-ν)-GMSs. Moreover φ-contraction is introduced with suitable examples.

2.1 Metric Space

A MS is a set equipped with a distance function (called a metric) that defines how far apart elements of the set are from each other, satisfying specific properties like non-negativity, symmetry, and the triangle inequality. Geometrically, you can think of a MS as a generalization of familiar spaces like the Euclidean plane, where points have measurable distances between them.
In 1906, Frechet developed the idea of MS.

Definition 2.1.1.

“A MS is a pair $(\mathcal{W}, \eth)$, where $\mathcal{W}$ is a non-empty set and $\eth$ is a metric on $\mathcal{W}$(or distance function on $\mathcal{W}$), that is, a function define on $\mathcal{W} \times \mathcal{W}$ $s.t$ $\forall\ \zeta, \xi, \Psi \in \mathcal{W}$ we have:

(M1): $\eth$ is real-valued, finite and non-negative,

(M2): $\eth(\zeta,\xi)=0$ if and only if $\zeta=\xi$,

(M3): $\eth(\zeta,\xi)=\eth(\xi,\zeta)$, (Symmetry)

(M4): $\eth(\zeta,\Psi)\leq\eth(\zeta,\xi)+\eth(\xi,\Psi)$. (Triangular inequality)" [2]

Example 2.1.2. Consider $\mathcal{W}=\mathbb{R}^2$ then the mapping $\eth:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ defined as:

$$\eth(\zeta,\xi)=\sqrt{(\zeta_1-\xi_1)^2+(\zeta_2-\xi_2)^2}$$

$\forall\ \zeta=(\zeta_1,\zeta_2),\xi=(\xi_1,\xi_2)\in\mathcal{W}$, is metric on $\mathbb{R}^2$ and $(\mathbb{R}^2,\eth)$ is a MS.

Definition 2.1.3.

"A sequence $\{\zeta_n\}$ in a metric space $\mathcal{W}=(\mathcal{W},\eth)$ is said to converge or to be convergent if there is a $\zeta\in\mathcal{W}$ *s.t*

$$\lim_{n\to\infty}\eth(\zeta_n,\zeta)=0.$$

ζ is called limit of $\{\zeta_n\}$ and we write

$$\lim_{n\to\infty}\zeta_n=\zeta\quad\text{or}\quad\zeta_n\to\zeta.$$

We say that $\{\zeta_n\}$ converges to ζ or has the limit ζ. If $\{\zeta_n\}$ is not convergent, it is said to be divergent." [2]

Example 2.1.4. Consider the real number line $\mathbb{R}$ equipped with the metric defined by

$$\eth(\zeta,\xi)=|\zeta-\xi|.$$

Then, the sequence $\{\zeta_n\}=\frac{1}{n}$ in $\mathcal{W}$ is a convergent sequence.

Definition 2.1.5.

"A mapping $\mathcal{S}:\mathcal{W}\to\mathbb{X}$ of a metric space $\mathcal{W}=(\mathcal{W},\eth)$ to $\mathbb{X}=(\mathbb{X},\eth_1)$ is continuous at a point $\zeta\in\mathcal{W}$ if and only if

$$\zeta_n\to\zeta_0\qquad\text{implies}\qquad\mathcal{S}\zeta_n\to\mathcal{S}\zeta_0.\text{"}\ [2]$$

Definition 2.1.6.

"A sequence $\{\zeta_n\}$ in a metric space $\mathcal{W} = (\mathcal{W}, \eth)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ $s.t$,

$$\eth(\zeta_m, \zeta_n) < \epsilon \quad \text{for every} \quad m, n > N.\text{"}[2]$$

Definition 2.1.7.

"A space $\mathcal{W}$ is said to be complete if every Cauchy sequence (CS) in $\mathcal{W}$ converges (that is, has a limit which is an element of $\mathcal{W}$)."[2]

Example 2.1.8. With usual metric on $\mathbb{R}$ the closed interval $[0, 1]$ is complete. Also the real line and complex plane are complete metric spaces.

2.2 Some Generalizations of Metric Space

In 1989, Bakhtin [27] introduce the concept of b-MS.

Definition 2.2.1.

"Let $\mathcal{W}$ be a non-empty set and let $s \geq 1$ be a given real number.
A function $\eth_b$: $\mathcal{W} \times \mathcal{W} \to [0, \infty)$ is called a b-metric if $\forall\ \zeta, \xi, \Psi \in \mathcal{W}$ the following conditions are satisfied,

$$(b_1):\ \eth_b(\zeta, \xi) = 0 \Leftrightarrow \zeta = \xi,$$

$$(b_2):\ \eth_b(\zeta, \xi) = \eth_b(\xi, \zeta),$$

$$(b_3):\ \eth_b(\zeta, \Psi) \leq s[\eth_b(\zeta, \xi) + \eth_b(\xi, \Psi)].$$

The pair $(\mathcal{W}, \eth_b)$ is called a b-MS."

Definition 2.2.2. "Let $(\mathcal{W}, \eth_b)$ be a bMS. A sequence $\{\zeta_n\}$ in $\mathcal{W}$ is said to be:

(i): Cauchy if and only if $\eth_b(\zeta_n, \zeta_m) \to 0$ as $n, m \to \infty$, (ii): Convergent if and only if there exist $\zeta \in \mathcal{W}$ $s.t$ $\eth_b(\zeta_n, \zeta) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} \zeta_n = \zeta$,

(iii): The b-MS $(\mathcal{W}, \eth_b)$ is complete if every CS is convergent.”[28]

Example 2.2.3. Consider $\mathcal{W} = \mathbb{R}$, the mapping $\eth_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by;

$$\eth_b(\zeta, \xi) = |\zeta - \xi|^2$$

is a b-metric on $\mathbb{R}$ with $s = 2$.

Example 2.2.4. Let $(\mathcal{W}, \eth)$ be a MS and;

$$\eth_b(\zeta, \xi) = (\eth(\zeta, \xi))^p$$

$\forall\ \zeta, \xi \in \mathcal{W}$ and $\mathtt{p} > 1$ in $\mathbb{R}$. Then, $(\mathcal{W}, \eth_b)$ is b-MS with $s = 2^{p-1}$.

Example 2.2.5. Consider the space $l_{\mathtt{p}}$,

$$\left\{ \{\mathtt{u_n}\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |\mathtt{u_n}|^{\mathtt{p}} < \infty \right\}, \ (0 < \mathtt{p} < 1).$$

Define $\eth_b : l_{\mathtt{p}} \times l_{\mathtt{p}} \to \mathbb{R}^+$ as:

$$\eth_b(\zeta, \xi) = \left(\sum_{n=1}^{\infty} |\mathtt{u_n} - \mathtt{v_n}|^{\mathtt{p}} \right)^{\frac{1}{\mathtt{p}}},$$

where $\zeta = \{\mathtt{u_n}\}$, $\xi = \{\mathtt{v_n}\}$. Then $\eth_b$ is a b-MS with $s = 2^{\frac{1}{\mathtt{p}}}$.

Definition 2.2.6.

“Let $\mathcal{W}$ be a non-empty set and $\eth_g : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be a mapping $s.t$ $\forall\ \zeta, \xi \in \mathcal{W}$ and if $\forall$ distinct points $\Psi, \Phi \in \mathcal{W}$, each of them different from ζ and ξ

(i): $\eth_g(\zeta, \xi) = 0 \Leftrightarrow \zeta = \xi$,

(ii): $\eth_g(\zeta, \xi)$=$\eth_g(\xi, \zeta)$ $\forall\ \zeta, \xi \in \mathcal{W}$,

(iii): $\eth_g(\zeta, \xi) \leq \eth_g(\zeta, \Psi) + \eth_g(\Psi, \phi) + \eth_g(\phi, \xi)$, Then $(\mathcal{W}, \eth_g)$ is called a GMS."[21]

Definition 2.2.7.

"Let $(\mathcal{W}, \eth_g)$ be a GMS, and let $\{\zeta_m\}_{m\in\mathbb{N}}$ be a sequence in $\mathcal{W}$, with $\zeta \in \mathcal{W}$. Then, we can say that :

(i): sequence $\{\zeta_m\}_{m\in\mathbb{N}}$ is convergent to ζ if and only if $\eth_g(\zeta, \zeta_m) \to 0$ as $n \to \infty$.

(ii): sequence $\{\zeta_m\}_{m\in\mathbb{N}}$ is Cauchy if and only if $\eth_g(\zeta_m, \zeta_n) \to 0$ as $n, m \to \infty$.

(iii): $(\mathcal{W}, \eth_g)$ is complete if every CS $\{\zeta_m\}$ in $\mathcal{W}$ converges to some element ζ in $\mathcal{W}$."[21]

Example 2.2.8. Let $\mathcal{L} = \{0, 2\}$, $\mathcal{M} = \{1/\mathsf{n},\ \mathsf{n} \in \mathbb{N}\}$ and $\mathcal{W} = \mathcal{L} \cup \mathcal{M}$. Define $\eth_g : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ as follows:

$$\eth_g(\zeta, \xi) = \begin{cases} 0, & \zeta = \xi; \\ 1, & \zeta \neq \xi \text{ and } \zeta, \xi \subseteq \mathcal{L} \text{ or } \zeta, \xi \subseteq \mathcal{M}; \\ \xi, & \zeta \in \mathcal{L}, \xi \in \mathcal{M}; \\ \zeta, & \zeta \in \mathcal{M}, \xi \in \mathcal{L}. \end{cases}$$

Then $(\mathcal{W}, \eth_g)$ is a complete GMS.

Example 2.2.9. Suppose $\mathcal{W} = \mathbb{R}$ and $\Psi \neq 0 \in \mathbb{R}$. Define $\eth_g : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ by

$$\eth_g(\zeta, \xi) = \begin{cases} 0 & \text{if } \zeta = \xi, \\ 3\Psi & \text{if } \zeta \text{ and } \xi \text{ are in } \{1, 2\}, \zeta \neq \xi, \\ \Psi & \text{if } \zeta \text{ and } \xi \text{ cannot both in } \{1, 2\}, \zeta \neq \xi. \end{cases}$$

Then, $(\mathcal{W}, \eth_g)$ is a GMS.

In [29] Stefan Czerwik introduced the idea of GbMS.

Definition 2.2.10.

"Let $\mathcal{W}$ be a non-empty set and $s \geq 1$ be a given real number, and let $\eth_{gb}$: $\mathcal{W} \times \mathcal{W} \to [0, \infty)$ be a mapping $s.t$ $\forall$ $\zeta, \xi \in \mathcal{W}$ and all distinct points $\Psi, \Phi \in \mathcal{W}$, each distinct from ζ and ξ:

$(\eth_{gb_1}) : \eth_{gb}(\zeta,\xi) = 0 \Leftrightarrow \zeta = \xi,$

$(\eth_{gb_2}) : \eth_{gb}(\zeta,\xi) = \eth_{gb}(\xi,\zeta),$

$(\eth_{gb_3}) : \eth_{gb}(\zeta,\xi) \leq s[\eth_{gb}(\zeta,\Psi) + \eth_{gb}(\Psi,\phi) + \eth_{gb}(\phi,\xi)].$

Then $(\mathcal{W},\eth_{gb})$ is called a GbMS or rectangular b-MS."[29]

Example 2.2.11. Let $(\mathcal{W},\eth_g)$ be a GMS and consider a real number $\mathsf{p} \geq 1$. Let $\eth_{gb}(\zeta,\xi) = (\eth_g(\zeta,\xi))^{\mathsf{p}}$. Then, $(\mathcal{W},\eth_{gb})$ is a GbMS when $\mathsf{s} \geq 3^{\mathsf{p}-1}$, where s is a b-metric coefficient.

2.3 Banach Contraction Principle (BCP) and Some of it's Generalizations

Banach FP theorem also known as BCP was first presented in 1922 by Polish mathematician Stefan Banach. Under certain conditions, It guarantees the existence and uniqueness of FP results in complete MSs.

Definition 2.3.1. "Let $(\mathcal{W},\eth)$ be a metric space. Then a mapping $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ is called a contraction mapping on $\mathcal{W}$ if there exists a positive real number $\lambda \in [0,1)$ $s.t$

$$\eth(\mathcal{S}\zeta,\mathcal{S}\xi) \leq \lambda\, \eth(\zeta,\xi), \quad \forall\, \zeta,\xi \in \mathcal{W}.$$

Geometrically, this means that any points ζ and ξ have images that are closer together than those points ζ and ξ; more precisely, the ratio $\dfrac{\eth(\mathcal{S}\zeta,\mathcal{S}\xi)}{\eth(\zeta,\xi)}$ does not exceed a constant which is strictly less than 1."[2]

Example 2.3.2. Consider the MS $\mathcal{W} = \mathbb{R}$, with metric $\eth(\zeta,\xi) = |\zeta - \xi|$. Define a function $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ as follows:

$$\mathcal{S}(\zeta) = \frac{\zeta}{4} + 1, \quad \forall\, \zeta \in \mathcal{W}.$$

Then $\mathcal{S}$ is a contraction mapping with $\lambda = \dfrac{1}{4}$.

Definition 2.3.3.

"A fixed point of a mapping $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ of a set $\mathcal{W}$ into itself is an $\zeta \in \mathcal{W}$ which is mapped onto itself (is "kept fixed" by $\mathcal{S}$), that is,

$$\mathcal{S}(\zeta) = \zeta,$$

the image $\mathcal{S}\zeta$ coincides with ζ." [2]

In general a mapping may or may not have FPs, and a FP may or may not be unique.

Example 2.3.4. Let $\mathcal{W} = \mathbb{R}$. Define the mapping $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ by $\mathcal{S}(\zeta) = \zeta^2$. Then $\zeta = 0, 1$ are the FPs of $\mathcal{S}$. Figure (2.1) represents the graphical picture of this mapping.

Graphical representation:

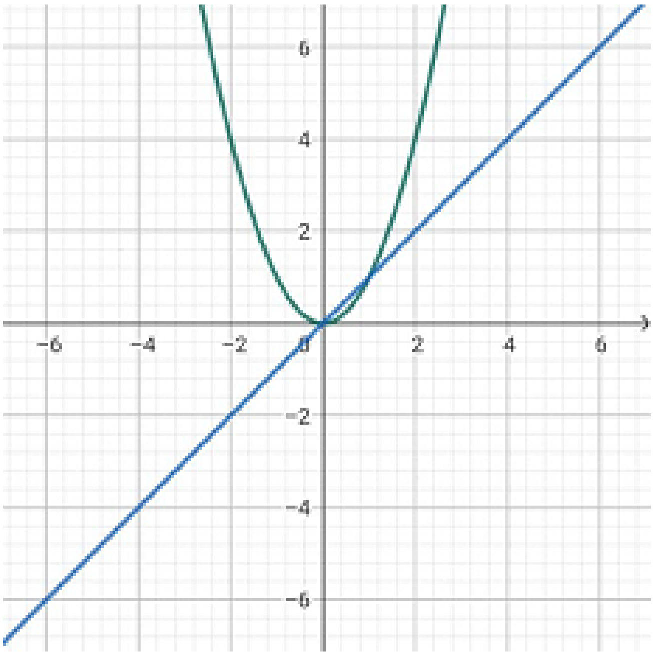

Figure 2.1: Mapping having more than one FPs

Example 2.3.5. Let $\mathcal{W} = \mathbb{R}$, define the mapping $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ by $\mathcal{S}(\zeta) = \zeta + 1$. Then $\mathcal{S}$ has no FP. Figure (2.2) represents the graphical picture of this mapping.

Graphical representation:

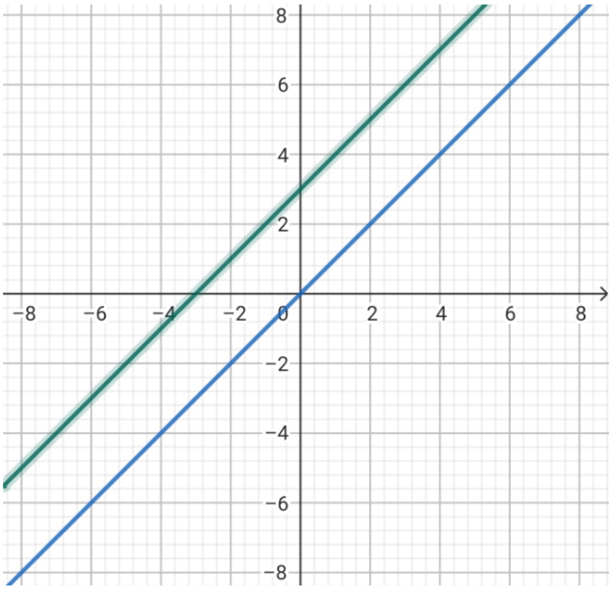

FIGURE 2.2: Mapping having no FPs

Example 2.3.6. Let $\mathcal{W} = \mathbb{R}$, define the mapping $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ by $\mathcal{S}(\zeta) = \dfrac{\zeta}{3} + 2$. Then $\mathcal{S} = 3$ is the UFP of $\mathcal{S}$. Figure (2.3) represents the graphical picture of this mapping.

Graphical representation:

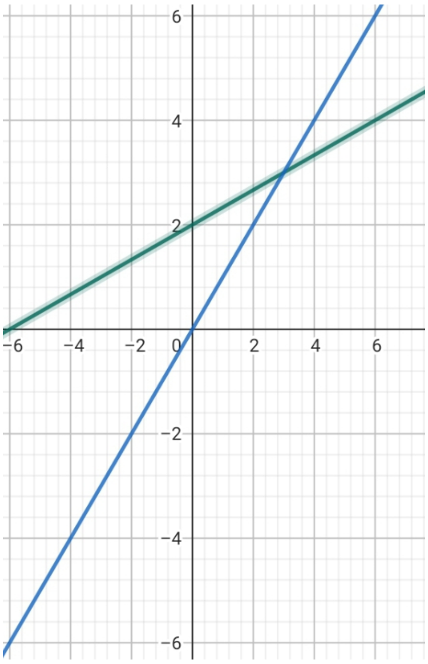

FIGURE 2.3: Mapping having unique FP (UFP)

Example 2.3.7. Let $\mathcal{W} = \mathbb{R}$, define the mapping $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ by considering the following trigonometric function

$$\mathcal{S}(\zeta) = \tan(\zeta).$$

This function has infinitely many FPs, one in each interval $\left(\frac{n\pi}{2}, \frac{(n+2)\pi}{2}\right)$ along the positive x-axis and one in each interval $\left(\frac{-(n+2)\pi}{2}, \frac{-n\pi}{2}\right)$ along the negative x-axis that are shown in Fig (2.4).

Graphical representation

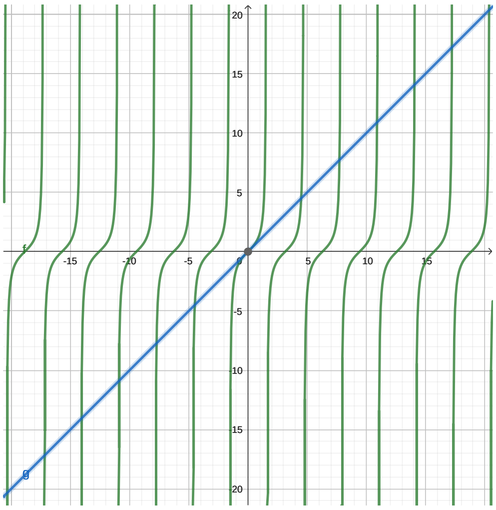

FIGURE 2.4: Mapping having infinitely many FP

Definition 2.3.8. "Let $\mathcal{S}$ be a self-mapping on $\mathcal{W}$ and let $\Gamma : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be a function. One says that $\mathcal{S}$ is an Γ-admissible mapping if

$$\forall \quad \zeta, \xi \in \mathcal{W}, \quad \Gamma(\zeta, \xi) \geq 1$$
$$\Rightarrow \Gamma(\mathcal{S}\zeta, \mathcal{S}\xi) \geq 1.\text{"}[25]$$

Definition 2.3.9. "Let $\mathcal{S}$ be a self-mapping on $\mathcal{W}$ and let $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be two functions. One says that $\mathcal{S}$ is an Γ-admissible mapping with respect to ν if

$$\forall \quad \zeta, \xi \in \mathcal{W}, \quad \Gamma(\zeta, \xi) \geq \nu(\zeta, \xi)$$
$$\Rightarrow \Gamma(\mathcal{S}\zeta, \mathcal{S}\xi) \geq \nu(\mathcal{S}\zeta, \mathcal{S}\xi).$$

Note that if we take $\nu(\zeta, \xi) = 1$, then this definition reduces to (2.3.8) . Also, if we take $\Gamma(\zeta, \xi) = 1$, then we say that $\mathcal{S}$ is an ν-sub-admissible mapping."[25]

Definition 2.3.10. “Let $\mathcal{S}$ be a self-mapping on $\mathcal{W}$ and let $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be two functions.
We say that $\mathcal{S}$ is an (Γ, ν)-admissible mapping if

$$\forall \ \ \zeta, \xi \in \mathcal{W}, \ \ \Gamma(\zeta, \xi) \geq 1 \Rightarrow \Gamma(\mathcal{S}\zeta, \mathcal{S}\xi) \geq 1,$$

and

$$\forall \ \ \zeta, \xi \in \mathcal{W}, \ \ \nu(\zeta, \xi) \leq 1 \Rightarrow \nu(\mathcal{S}\zeta, \mathcal{S}\xi) \leq 1.$$

One can easily see that an (Γ, ν)-admissible mapping is an Γ-admissible and ν-sub-admissible mapping simultaneously.”[30]

Definition 2.3.11. “ Let $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ and $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$.
We say that $\mathcal{S}$ is a triangular (Γ, ν)- admissible mapping if

$(\mathcal{S}_1) : \ \Gamma(\zeta, \xi) \geq 1 \Rightarrow \Gamma(\mathcal{S}\zeta, \mathcal{S}\xi) \geq 1$, for all $\zeta, \xi \in \mathcal{W}$

$(\mathcal{S}_2) : \ \nu(\zeta, \xi) \leq 1 \Rightarrow \nu(\mathcal{S}\zeta, \mathcal{S}\xi) \leq 1$, for all $\zeta, \xi \in \mathcal{W}$

$$(\mathcal{S}_3) : \begin{cases} \Gamma(\zeta, \xi) \geq 1 \\ \Gamma(\xi, \Psi) \geq 1 \\ \Rightarrow \Gamma(\zeta, \Psi) \geq 1 \ \ \forall \ \ \zeta, \xi, \Psi \in \mathcal{W} \end{cases}$$

$$(\mathcal{S}_4) : \begin{cases} \nu(\zeta, \xi) \leq 1 \\ \Gamma(\xi, \Psi) \leq 1 \\ \Rightarrow \nu(\zeta, \Psi) \leq 1 \ \ \forall \ \ \zeta, \xi, \Psi \in \mathcal{W}.\text{”} \end{cases} \qquad [30]$$

Definition 2.3.12. “ Let $(\mathcal{W}, \eth_g)$ be a GMS and let $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be two mappings. Then,

(i): $\mathcal{S}$ is a Γ-continuous mapping on $(\mathcal{W}, \eth_g)$ if for a given point $\zeta \in \mathcal{W}$ and sequence $\{\zeta_m\}$ in $\mathcal{W}$, $\zeta_m \to \zeta$ and $\Gamma(\zeta_m, \zeta_{m+1}) \geq 1 \ \ \forall \ \ m \in \mathbb{N}$, then $\mathcal{S}\zeta_m \to \mathcal{S}\zeta$.

(ii): $\mathcal{S}$ is a ν sub-continuous mapping on $(\mathcal{W}, \eth_g)$ if for a given point $\zeta \in \mathcal{W}$ and sequence $\{\zeta_m\}$ in $\mathcal{W}$, $\zeta_m \to \zeta$ and $\nu(\zeta_m, \zeta_{m+1}) \leq 1 \ \ \forall \ \ m \in \mathbb{N}$, then $\mathcal{S}\zeta_m \to \mathcal{S}\zeta$.

(iii): $\mathcal{S}$ is a (Γ, ν)-continuous mapping on $(\mathcal{W}, \eth_g)$ if for a given point $\zeta \in \mathcal{W}$ and sequence $\{\zeta_m\}$ in $\mathcal{W}$ $s.t$ $\zeta_m \to \zeta$ with $\Gamma(\zeta_m, \zeta_{m+1}) \geq 1$ and $\nu(\zeta_m, \zeta_{m+1}) \leq 1$ $\forall \ \ m \in \mathbb{N}$, we have $\mathcal{S}\zeta_m \to \mathcal{S}\zeta$."[30]

Example 2.3.13. Let $\mathcal{W} = [0, +\infty)$ be equipped with the usual metric $\eth_g(\zeta, \xi) = |\zeta - \xi|$.

Assume that $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ and $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be defined by

$$\mathcal{S}(\zeta) = \begin{cases} \frac{\zeta}{2}, & \text{if } \zeta \in [0,1], \\ \arcsin\zeta + \sinh\zeta + 2, & \text{if } \zeta \in (1, +\infty), \end{cases}$$

$$\Gamma(\zeta, \xi) = \begin{cases} \zeta^4 + \xi^2 + 1, & \text{if } \zeta, \xi \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\nu(\zeta, \xi) = \begin{cases} \frac{1}{1+\zeta^2+\xi^6}, & \text{if } \zeta, \xi \in [0,1], \\ 2, & \text{otherwise}. \end{cases}$$

Then $\mathcal{S}$ is (Γ, ν)-continuous (Γ-continuous and ν-sub-continuous) on $(\mathcal{W}, \eth_g)$.

Example 2.3.14. Let $\mathcal{W} = [0, +\infty)$ and $\eth_g(\zeta, \xi) = |\zeta - \xi|$.

Assume that $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ and $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be defined by

$$\mathcal{S}(\zeta) = \begin{cases} \zeta^5, & \text{if } \zeta \in [0,1], \\ \sin\pi\zeta + 2, & \text{if } \zeta \in (1, +\infty), \end{cases}$$

$$\Gamma(\zeta, \xi) = \begin{cases} \zeta^2 + \xi^2 + 1, & \text{if } \zeta, \xi \in [0,1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\nu(\zeta, \xi) = \zeta^2.$$

Then $\mathcal{S}$ is discontinuous mapping, but $\mathcal{S}$ is (Γ, ν)-continuous on $(\mathcal{W}, \eth_g)$.

Indeed, if $\zeta_n \to \zeta$ as $n \to \infty$ and $\Gamma(\zeta_n, \zeta_{n+1}) \geq \nu(\zeta_n, \zeta_{n+1})$, then $\zeta_n \in [0,1]$ and so $\lim\limits_{n\to\infty} \mathcal{S}\zeta_n = \lim\limits_{n\to\infty} \zeta_n^5 = \zeta^5 = \mathcal{S}\zeta$.

Definition 2.3.15. “Let $(\mathcal{W}, \eth_{gb})$ be a GbMS and let $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be two mappings. Then, the space $\mathcal{W}$ is said to be:

(i): Γ-complete if every CS in $\mathcal{W}$ with $\Gamma(\zeta_m, \zeta_{m+1}) \geq 1 \ \forall \ m \in \mathbb{N}$, converges in $\mathcal{W}$.

(ii): ν sub-complete, if every CS in $\mathcal{W}$ with $\nu(\zeta_m, \zeta_{m+1}) \leq 1 \ \forall \ m \in \mathbb{N}$, converges in $\mathcal{W}$.

(iii): (Γ, ν)-complete, if every CS in $\mathcal{W}$ with $\Gamma(\zeta_m, \zeta_{m+1}) \geq 1$ and $\nu(\zeta_m, \zeta_{m+1}) \leq 1 \ \forall \ m \in \mathbb{N}$, converges in $\mathcal{W}$.”[30]

Example 2.3.16. Let $\mathcal{W} = (0, \infty)$ and $\eth_{gb}(\zeta, \xi) = |\zeta - \xi|$ be a metric on $\mathcal{W}$. Let $\mathcal{B}^*$ be a closed subset of $\mathcal{W}$. Define $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ by

$$\Gamma(\zeta, \xi) = \begin{cases} \frac{(\zeta+\xi)}{2}, & \text{if } \zeta, \xi \in \mathcal{B}^*, \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu(\zeta, \xi) = 2\zeta\xi.$$

Clearly, $(\mathcal{W}, \eth_{gb})$ is not a complete MS, but $(\mathcal{W}, \eth_{gb})$ is an Γ-ν-complete MS. Indeed, if $\{\zeta_n\}$ is a CS in $\mathcal{W}$.

Definition 2.3.17. “ Let $(\mathcal{W}, \eth_{gb})$ be a GbMS and let $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be two mappings. Then, the space $\mathcal{W}$ is said to be:

(i): Γ-regular, if $\zeta_m \to \zeta$, $\Gamma(\zeta_m, \zeta_{m+1}) \geq 1 \ \forall \ m \in \mathbb{N}$, implies $\Gamma(\zeta_m, \zeta) \geq 1 \ \forall \ m \in \mathbb{N}$.

(ii): ν-sub-regular, if $\zeta_m \to \zeta$, $\nu(\zeta_m, \zeta_{m+1}) \leq 1 \ \forall \ m \in \mathbb{N}$, implies $\nu(\zeta_m, \zeta) \leq 1 \ \forall \ m \in \mathbb{N}$.

(iii): (Γ, ν)-regular, if $\zeta_m \to \zeta$, $\Gamma(\zeta_m, \zeta_{m+1}) \geq 1$ and $\nu(\zeta_m, \zeta_{m+1}) \leq 1 \ \forall \ m \in \mathbb{N}$, imply that $\Gamma(\zeta_m, \zeta) \geq 1$ and $\nu(\zeta_m, \zeta) \leq 1 \ \forall \ m \in \mathbb{N}$.”[30]

Example 2.3.18. Let $\mathcal{W} = \mathbb{R}$, and define $\eth_{gb}(\zeta, \xi) = |\zeta - \xi|^2$, and define the mapping

$$\Gamma(\zeta,\xi)=\begin{cases}2 & \text{if } |\zeta-\xi|\geq \frac{1}{n} \text{ for any } n\in\mathbb{N},\\ 0 & \text{if } \zeta=\xi,\\ 1 & \text{otherwise.}\end{cases}$$

$$\nu(\zeta,\xi)=\begin{cases}0 & \text{if } \zeta=\xi,\\ 1 & \text{if } |\zeta-\xi|\leq \frac{1}{n} \text{ for some } n\in\mathbb{N},\\ 2 & \text{otherwise.}\end{cases}$$

Then $(\mathcal{W},\eth_{gb})$ is a (Γ-ν)-regular GbMS.

Definition 2.3.19.

"Let $\Theta_{\mathcal{W}}$ denote the family of all functions $\varphi:(0,+\infty)\to(1,+\infty)$ that satisfy the following properties:

(φ_1): φ is increasing.

(φ_2): For each sequence $\{\zeta_m\}\subset(0,+\infty)$

$$\lim_{n\to 0}\zeta_m=0\Leftrightarrow \lim_{n\to\infty}\varphi(\zeta_m)=1.$$

($\varphi_{3\mathcal{W}}$): φ is continuous.

Throughout this thesis by $\Theta_{\mathcal{G}}$ we mean the family of all functions $\varphi:(0,+\infty)\to(1,+\infty)$ that satisfies φ_1, φ_2 and

($\varphi_{3\mathcal{G}}$) There exist $\mathtt{r}\in(0,1)$ and $l\in(0,\infty]$ $s.t$ $\lim_{l\to 0}\varphi(l)-\dfrac{1}{l^{\mathtt{r}}}=l$."[21]

Definition 2.3.20. "Let $(\mathcal{W},\eth_g)$ be a complete GMS and : $\mathcal{W}\to\mathcal{W}$ be a mapping. Then, $\mathcal{S}$ is said to be a φ-contraction if there exists a function $\varphi\in\Theta_G$ and a constant $\mathtt{d}\in(0,1)$ $s.t$ $\forall$ $\zeta,\xi\in\mathcal{W}$,

$$\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)>0\Rightarrow\varphi[\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)]\leq[\varphi(M(\zeta,\xi))]^{\mathtt{d}},$$

where

$$M(\zeta,\xi)=\max\{\eth_g(\zeta,\xi),\eth_g(\mathcal{S}\zeta,\xi),\eth_g(\zeta,\mathcal{S}\xi)\}.\text{"}[26]$$

Remark 2.1. The sets $\Theta_{\mathcal{G}}$ and $\Theta_{\mathcal{W}}$ are distinct.i.e, they contain different elements or have different properties.

Example 2.3.21. Define $\varphi : (0, +\infty) \to (1, +\infty)$ by

$$\varphi(t) = \begin{cases} \sqrt{t} + 1 & \text{if } t \in \left(0, \dfrac{1}{2}\right], \\ e^t & \text{if } t \in \left[\dfrac{1}{2}, +\infty\right). \end{cases}$$

Then $\varphi \in \Theta_{\mathcal{G}}$ but for any $t > 0$,

$$\lim_{t \to \frac{1}{2}^-} \varphi(t) = \sqrt{\frac{1}{2}} + 1$$
$$\lim_{t \to \frac{1}{2}^+} \varphi(t) = e^{\frac{1}{2}}.$$

Since $\sqrt{\dfrac{1}{2}} + 1 \neq e^{\frac{1}{2}}$, so, φ does not satisfy the condition $(\varphi_{3\mathcal{W}})$, then, $\varphi \notin \Theta_{\mathcal{G}}$.

Example 2.3.22. Define $\varphi : (0, +\infty) \to (1, +\infty)$ by

$$\varphi(t) = e^{\sqrt{te^t}}, t > 0.$$

Then, $\varphi \in \Theta_{\mathcal{W}}$, but for any $\mathtt{r} > 0$,

$$\lim_{t \to 0^+} \frac{\varphi(t) - 1}{t^{\mathtt{r}}} = \lim_{t \to 0^+} \frac{e^{\sqrt{te^t}} - 1}{t^{\mathtt{r}}} = \lim_{t \to 0^+} \frac{e^{\sqrt{te^t}}(e^t + te^t)}{2\mathtt{r}\sqrt{te^t} \times t^{\mathtt{r}-1}}.$$

$$\Rightarrow \lim_{t \to 0^+} \frac{\varphi(t) - 1}{t^{\mathtt{r}}} = \infty.$$

So, $\varphi \in \Theta_{\mathcal{G}}$.

Chapter 3

FP Results for (Γ, ν)-φ-Ω-contraction in Generalized MSs

3.1 Introduction

In this chapter, we extend the concept of φ-contraction in GMS by introducing the idea of a generalized (Γ, ν)-φ-Ω-contraction. This concept provides a more general framework for analysing contractions in GMS to demonstrate the applicability of our results. Some illustrative examples are also provided. Furthermore, we derive some useful corollaries from our main results, which can be used to acquire additional insights and extensions.

Definition 3.1.1. Consider the set σ of all functions Ω: $[0, +\infty)^5 \to [0, +\infty)$, satisfying the following:
for all $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 \in [0, +\infty)$ with $\ell_1\ \ell_2\ \ell_3\ \ell_4\ \ell_5 = 0,\ \ \exists\ \ \gamma \in]0, 1[$ such that $\Omega(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) = \gamma$.

Example 3.1.2. If $\Omega(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) = \min\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\} + \gamma$ where $\gamma \in]0, 1[$, then, $\Omega \in \sigma$.

Example 3.1.3.

If

$$\Omega(\ell_1,\ell_2,\ell_3,\ell_4,\ell_5)=\frac{\min\{\ell_1,\ell_2,\ell_3,\ell_4,\ell_5\}}{\max\{\ell_1,\ell_2,\ell_3,\ell_4,\ell_5+1\}+\gamma}$$

where $\gamma\in\,]0,1[$, then, $\Omega\in\sigma$.

Definition 3.1.4. Let $(\mathcal{W},\eth_g)$ be a (Γ,ν)-complete GMS, and suppose $\mathcal{S}$ be a self-mapping on $\mathcal{W}$, where $\Gamma,\nu:\mathcal{W}\times\mathcal{W}\to[0,+\infty)$ are two functions. Then $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction, if $\forall\ \zeta,\xi\in\mathcal{W}$ with ($\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1$) and $\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)>0$, we have

$$\varphi[\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi))]\leq[\varphi(M(\zeta,\xi))]^{\Omega(\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\zeta,\xi),\eth_g(\xi,\mathcal{S}\zeta),\eth_g(\mathcal{S}^2\zeta,\xi))},\tag{3.1}$$

where $\varphi\in\Theta_{\mathcal{W}}$, $\Omega\in\sigma$, and

$$\begin{aligned}M(\zeta,\xi)=\max\{&\eth_g(\zeta,\xi),\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\mathcal{S}\zeta,\xi),\\&\eth_g(\mathcal{S}^2\zeta,\xi),\eth_g(\mathcal{S}^2\zeta,\mathcal{S}\xi),\eth_g(\mathcal{S}^2\zeta,\mathcal{S}\zeta)\}.\end{aligned}\tag{3.2}$$

Example 3.1.5. Let $\mathcal{W}=[1,+\infty)$ and $\varphi(\jmath)=e^{\jmath}\ \forall\ \jmath\in\,]0,+\infty[$. Then, $\varphi\in\Theta_{\mathcal{W}}$.

Define $\eth_g:\mathcal{W}\times\mathcal{W}\to[0,+\infty)$ by

$$\eth_g(\zeta,\xi)=|\zeta-\xi|.$$

Then, $(\mathcal{W},\eth_g)$ is a complete GMS.

Define $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ by

$$\begin{aligned}&\mathcal{S}(\jmath)=\sqrt{\jmath}\ \ \forall\ \ \jmath\in[1,+\infty)\,,\\&\Gamma(\zeta,\xi)=1,\\&\nu(\zeta,\xi)=1,\end{aligned}$$

$$\Omega(\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5)=\frac{\min\{\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\}}{\max\{\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\}+1}\quad\forall\ \ \jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\in\mathbb{R}_+.$$

Then $\mathcal{S}$ is a (Γ,ν)-continuous and triangular (Γ,ν)-admissible mapping.

Case 1. $0\leq\zeta\leq\xi$

Now

$$\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)=(\sqrt{\xi}-\sqrt{\zeta}),$$

$$\begin{aligned}M(\eth_g(\zeta,\xi)) = \max\Bigg\{&\eth_g(\zeta,\xi),\eth_g(\zeta,\sqrt{\zeta}),\eth_g(\xi,\sqrt{\xi}),\eth_g(\xi,\sqrt{\zeta})\\ &\eth_g\left(\sqrt{\sqrt{\zeta}},\xi\right),\eth_g\left(\sqrt{\sqrt{\zeta}},\sqrt{\xi}\right),\eth_g\left(\sqrt{\sqrt{\zeta}},\sqrt{\zeta}\right)\Bigg\}.\end{aligned}$$

Since $\zeta\le\xi$, we get

$$\begin{aligned}M(\zeta,\xi) = \max\Big\{&(\xi-\zeta),(\zeta-\sqrt{\zeta}),(\xi-\sqrt{\xi}),(\xi-\sqrt{\zeta})\\ &\left(\xi-\sqrt{\sqrt{\zeta}}\right),\left(\sqrt{\xi}-\sqrt{\sqrt{\zeta}}\right),\left(\sqrt{\zeta}-\sqrt{\sqrt{\zeta}}\right)\Big\}.\end{aligned}$$

Thus,

$$M(\zeta,\xi)\ge\xi-\zeta,$$

$$\Rightarrow\varphi(M(\zeta,\zeta))\ge\varphi(\eth_g(\zeta,\xi))=e^{(\xi-\zeta)}.$$

Thus,

$$\begin{aligned}&\varphi(\eth_g(\zeta,\xi))^{\Omega(\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\zeta,\mathcal{S}\xi),\eth_g(\xi,\mathcal{S}\zeta),\eth_g(\mathcal{S}^2\zeta,\xi))}\\ &=e^{(\xi-\zeta)\Omega(\zeta-\sqrt{\zeta},\xi-\sqrt{\xi},\zeta-\sqrt{\xi},\xi-\sqrt{\zeta},\xi-\sqrt{\sqrt{\zeta}})}\\ &=e^{(\xi-\zeta)(\min\{\zeta-\sqrt{\zeta},\xi-\sqrt{\xi},\zeta-\sqrt{\xi},\xi-\sqrt{\zeta},\xi-\sqrt{\sqrt{\zeta}}\}/\max\{\zeta-\sqrt{\zeta},\xi-\sqrt{\xi},\zeta-\sqrt{\xi},\xi-\sqrt{\zeta},\xi-\sqrt{\sqrt{\zeta}}\}+1)}\\ &\le e^{(\xi-\zeta)}.\end{aligned}$$

Thus,

$$\varphi(\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi))\le\varphi(M(\eth_g(\zeta,\xi)))^{\Omega(\eth_g(\zeta,\xi)\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\xi,\mathcal{S}\zeta),\eth_g(\mathcal{S}^2\zeta,\xi)}.$$

Case 2. $\zeta>\xi>0$.

Similarly, it can be concluded that

$$\varphi(\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi))\le\varphi(M(\eth_g(\zeta,\xi)))^{\Omega(\eth_g(\zeta,\xi)\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\xi,\mathcal{S}\zeta),\eth_g(\mathcal{S}^2\zeta,\xi)}.$$

Hence, $\mathcal{S}$ is an (Γ,ν)-φ-Ω-contraction.

3.2 Main Results

We now present a novel concept of generalized φ-Ω-contraction in the framework of (Γ,ν)-GMS as follows.

Theorem 3.2.1. Suppose $(\mathcal{W},\eth_g)$ be a (Γ,ν)-complete GMS, and $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a mapping, which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

(iii) : $\exists\ \zeta_0\in\mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$.

(iv) : $\mathcal{S}$ is a (Γ,ν)-continuous.

Then $\mathcal{S}$ has a FP. Also, $\mathcal{S}$ has a UFP when $\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1\ \ \forall$ FPs $\zeta,\xi\in\mathcal{W}$.

Proof. Let $\zeta_0\in\mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$.
Define a sequence $\{\zeta_m\}$ by

$$\zeta_m=\mathcal{S}^m\zeta_0=\mathcal{S}\zeta_{m-1}.$$

Then by (iii)

$$\Gamma(\zeta_0,\zeta_1)=\Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1.$$

Since $\mathcal{S}$ is a triangular (Γ,ν) admissible mapping

$$\begin{aligned}\Rightarrow \Gamma(\mathcal{S}\zeta_0,\mathcal{S}\zeta_1)\geq 1 \text{ i.e}\\ \Gamma(\zeta_1,\zeta_2)\geq 1,\end{aligned}$$

similarly

$$\begin{aligned}&\nu(\zeta_0,\zeta_1)=\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1\\ &\Rightarrow \nu(\mathcal{S}\zeta_0,\mathcal{S}\zeta_1)\leq 1\\ &\Rightarrow \nu(\zeta_1,\zeta_2)\leq 1.\end{aligned}$$

Proceeding this way, we have $\Gamma(\zeta_{m-1},\zeta_m)\geq 1$ and $\nu(\zeta_{m-1},\zeta_m)\leq 1,\ \forall\ \ n\in\mathbb{N}$. By $\mathcal{S}_3$ and $\mathcal{S}_4$,

$$\Gamma(\zeta_m,\zeta_n)\geq 1 \text{ and } \nu(\zeta_m,\zeta_n)\leq 1,\quad \forall\ \ m,n\in\mathbb{N}, m\neq n. \tag{3.3}$$

Let $\zeta_n\neq\mathcal{S}\zeta_n$, i.e, $\eth_g(\zeta_{n-1},\zeta_n)>0\ \ \forall\ \ n\in\mathbb{N}$.

Also

$$\zeta_n\neq\zeta_m,\quad \forall\ \ m,n\in\mathbb{N}, m\neq n. \tag{3.4}$$

Indeed if $\zeta_n=\zeta_m$ for some $m=n+k>n$, then

$$\zeta_{n+1}=\mathcal{S}\zeta_n=\mathcal{S}\zeta_m=\zeta_{m+1}.$$

Express $\eth_{g_m}=\eth_g(\zeta_m,\zeta_{m+1})$, therefore $\varphi(\eth_{g_n})=\varphi(\eth_{g_m})$.

Then, (3.1) implies that

$$\begin{aligned}
&\varphi(\eth_g(\mathcal{S}\zeta_{m-1},\mathcal{S}\zeta_m))\\
&\leq[\varphi(M(\zeta_{m-1},\zeta_m))]^{\Omega(\eth_g(\zeta_{m-1},\zeta_m),\eth_g(\zeta_m,\zeta_{m+1}),\eth_g(\zeta_{m-1},\zeta_m),\eth_g(\zeta_m,\zeta_m),\eth_g(\zeta_{m+1},\zeta_m))}\\
&=\varphi(M(\zeta_{m-1},\zeta_m))]^{\Omega(\eth_{g_{m-1}},\,\eth_{g_{m-1}},\,\eth_{g_m},\,0,\,\eth_{g_{m+1}})},
\end{aligned} \tag{3.5}$$

here

$$\begin{aligned}
M(\zeta_{m-1},\zeta_m)=&\max\{\eth_g(\zeta_{m-1},\zeta_m),\eth_g(\zeta_{m-1},\zeta_m),\eth_g(\zeta_m,\zeta_{m+1})\\
&\eth_g(\zeta_m,\zeta_m),\eth_g(\zeta_{m+1},\zeta_m),\eth_g(\zeta_{m+1},\zeta_{m+1}),\eth_g(\zeta_{m+1},\zeta_m)\}\\
=&\max\{\eth_g(\zeta_{m-1},\zeta_m),\eth_g(\zeta_m,\zeta_{m+1})\}.
\end{aligned}$$

Since $\eth_{g_{m-1}}.\eth_{g_m}.0.\eth_{g_{m+1}}=0$ which implies that $\exists\ \ \gamma\in\,]0,1[\ \ s.t$

$$\Omega(\eth_{g_{m-1}},\eth_{g_{m-1}},\eth_{g_m},0,\eth_{g_{m+1}})=\gamma.$$

Thus, (3.5) becomes

$$\varphi(\eth_{g_n})\leq[\varphi(M(\zeta_{m-1},\zeta_m))]^{\gamma}. \tag{3.6}$$

Suppose that

$$M(\zeta_{m-1},\zeta_m)=\eth_g(\zeta_m,\zeta_{m+1}).$$

Then, by (3.6), we have

$$\varphi(\eth_{g_m}) \leq [\varphi(\eth_{g_m})]^{\gamma} < \varphi(\eth_{g_m}),$$

which is not possible, therefore

$$M(\zeta_{m-1}, \zeta_m) = \eth_g(\zeta_{m-1}, \zeta_m).$$

Since $\zeta_n = \zeta_m$

$$\Rightarrow \eth_{g_n} = \eth_{g_m} < \eth_{g_{m-1}}.$$

Continuing this process, we get $\eth_{g_n} = \eth_{g_m} < \eth_{g_{m-1}} < \eth_{g_{m-2}} < ... < \eth_{g_n}$, which is a contradiction. Thus, (3.4) holds.

Substituting $\zeta = \zeta_{n-1}$ and $\xi = \zeta_n$ in (3.1), $\forall \ \ n \in \mathbb{N}$.

$$\begin{aligned}
&\varphi(\eth_g(\mathcal{S}\zeta_{n-1}, \mathcal{S}\zeta_n) \\
&= \varphi(\eth_g(\zeta_n, \zeta_{n+1})) \\
&\leq [\varphi(M(\zeta_{n-1}, \zeta_n))]^{\Omega(\eth_g(\zeta_{n-1},\zeta_n),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_n,\zeta_n),\eth_g(\zeta_{n+1},\zeta_n))} \\
&= [\varphi(M(\zeta_{n-1}, \zeta_n))]^{\Omega(\eth_g(\zeta_{n-1},\zeta_n),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_{n+1}),0,\eth_g(\zeta_{n+1},\zeta_n))}.
\end{aligned}$$

Now by definition (3.1.1) $\exists \ \ \gamma \in (0, 1) \ \ s.t$

$$\Omega[\eth_g(\zeta_{n-1}, \zeta_n), \eth_g(\zeta_n, \zeta_{n+1}), \eth_g(\zeta_{n-1}, \zeta_{n+1}), 0, \eth_g(\zeta_{n+1}, \zeta_n)] = \gamma.$$

$$\begin{aligned}
\Rightarrow M(\zeta_{n-1}, \zeta_n) = \max\{&\eth_g(\zeta_{n-1}, \zeta_n), \eth_g(\zeta_{n-1}, \zeta_n), \eth_g(\zeta_n, \zeta_{n+1}), \\
&\eth_g(\zeta_{n+1}, \zeta_n), \eth_g(\zeta_{n+1}, \zeta_n), \eth_g(\zeta_{n+1}, \zeta_{n+1}), \eth_g(\zeta_{n+1}, \zeta_n)\} \\
= \max\{&\eth_g(\zeta_{n-1}, \zeta_n), \eth_g(\zeta_n, \zeta_{n+1})\}.
\end{aligned}$$

Suppose that

$$M(\zeta_{n-1}, \zeta_{n+1}) = \eth_g(\zeta_n, \zeta_{n+1}).$$

In that case,

$$\varphi(\eth_g(\zeta_n, \zeta_{n+1})) \leq [\varphi(\eth_g(\zeta_n, \zeta_{n+1})]^{\gamma} < \varphi(\eth_g(\zeta_n, \zeta_{n+1})),$$

which contradicts our assumption.

Therefore,

$$M(\zeta_{n-1}, \zeta_{n+1}) = \eth_g(\zeta_{n-1}, \zeta_n).$$

Hence

$$\varphi(\eth_g(\zeta_n, \zeta_{n+1})) \leq [\varphi(\eth_g(\zeta_{n-1}, \zeta_n)]^\gamma < \varphi(\eth_g(\zeta_{n-1}, \zeta_n)).$$

Since φ is continuous,

$$\Rightarrow \eth_g(\zeta_n, \zeta_{n+1}) < \eth_g(\zeta_{n-1}, \zeta_n). \tag{3.7}$$

Thus, $\eth_g(\zeta_n, \zeta_{n+1})_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers that decreases strictly.

Therefore, $\exists\ \lambda_1 \geq 0\ \ s.t$

$$\lim_{n\to\infty} \eth_g(\zeta_{n+1}, \zeta_n) = \lambda_1.$$

Now, we claim that $\lambda_1 = 0$. Assuming on contrary, suppose that λ_1 is greater than 0.

Given that the sequence $\eth_g(\zeta_n, \zeta_{n+1})$ is non-negative and strictly decreasing $\forall\ n$, we can conclude that

$$\eth_g(\zeta_n, \zeta_{n+1}) \geq \lambda_1\ \ \forall\ \ n \in \mathbb{N},\ \text{ that is}$$

$$\eth_g(\zeta_n, \zeta_{n+1}) \geq \lim_{n\to\infty} \eth_g(\zeta_{n+1}, \zeta_n)$$

$$\Rightarrow \varphi(\eth_g(\zeta_n, \zeta_{n+1})) \geq \lim_{n\to\infty} \varphi(\eth_g(\zeta_{n+1}, \zeta_n)).$$

Using property of φ

$$\varphi(\eth_g(\zeta_n, \zeta_{n+1})) > 1.$$

Consider $\varphi(\eth_g(\zeta_n, \zeta_{n+1}))$ and applying contraction condition

$$\varphi(\eth_g(\zeta_n, \zeta_{n+1})) \leq [\varphi(M(\zeta_{n-1}, \zeta_n)]^\gamma.$$

As

$$M(\zeta_{n-1}, \zeta_n) = \eth_g(\zeta_{n-1}, \zeta_n)$$

$$\Rightarrow \varphi(\eth_g(\zeta_n, \zeta_{n+1})) \leq [\varphi(\eth_g(\zeta_{n-1}, \zeta_n))]^{\gamma}. \tag{3.8}$$

Now consider

$$\begin{aligned}\varphi(\eth_g(\zeta_{n-1}, \zeta_n))]^{\gamma} \leq& \varphi(\eth_g(\zeta_{n-1}, \zeta_n))]\\ \leq& [\varphi(\eth_g(\zeta_{n-2}, \zeta_{n-1}))]^{\gamma}.\end{aligned}$$

$$\Rightarrow \varphi(\eth_g(\zeta_{n-1}, \zeta_n)) \leq [\varphi(\eth_g(\zeta_{n-2}, \zeta_{n-1}))]^{\gamma}.$$

Now (3.8) gives

$$\begin{aligned}\varphi(\eth_g(\zeta_n, \zeta_{n+1})) \leq& [\varphi(\eth_g(\zeta_{n-1}, \zeta_n))]^{\gamma}\\ \leq& \varphi(\eth_g(\zeta_{n-2}, \zeta_{n-1}))]^{\gamma^2}.\end{aligned}$$

Proceeding this way, it can be concluded that

$$\begin{aligned}1 <& \varphi(\eth_g(\zeta_n, \zeta_{n+1}))\\ \leq& \varphi(\eth_g(\zeta_0, \zeta_1))^{\gamma^n}.\end{aligned}$$

Taking $\lim n \to \infty$ in above inequality, we get

$$1 < \varphi(\lambda_1) \leq 1,$$

a contradiction. Therefore,

$$\lim_{n \to \infty} \eth_g(\zeta_n, \zeta_{n+1}) = 0. \tag{3.9}$$

Now substituting $\zeta = \zeta_{n-1}$ and $\xi = \zeta_{n+1}$ in (3.1), $\forall \ \ n \in \mathbb{N}$, we have

$$\begin{aligned}&\varphi(\eth_g(\zeta_n, \zeta_{n+2}))\\ \leq&[\varphi(M(\zeta_{n-1}, \zeta_{n+1}))]^{\Omega(\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_n),\eth_g(\zeta_{n+1},\zeta_{n+2}),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\zeta_{n+1},\zeta_{n+1}))}\\ =&[\varphi(M(\zeta_{n-1}, \zeta_n))]^{\Omega(\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_n),\eth_g(\zeta_{n+1},\zeta_{n+2}),\eth_g(\zeta_n,\zeta_{n+1}),0))},\end{aligned}$$

here

$$M(\zeta_{n-1},\zeta_{n+1}) = \max\{\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_n),\eth_g(\zeta_{n+1},\zeta_{n+2}),$$

$$\eth_g(\zeta_{n+1},\zeta_n),\eth_g(\zeta_{n+1},\zeta_{n+1}),\eth_g(\zeta_{n+1},\zeta_n),\eth_g(\zeta_{n+1},\zeta_n)\}.$$

Since

$$\eth_g(\zeta_{n+1},\zeta_{n+2}) \le \eth_g(\zeta_n,\zeta_{n+1}) \le \eth_g(\zeta_{n-1},\zeta_n),$$

$$\Rightarrow M(\zeta_{n-1},\zeta_{n+1}) = \max\{\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_n)\},$$

and $\exists\ \ \gamma \in (0,1)\ \ s.t$

$$\Omega(\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_n),\eth_g(\zeta_{n+1},\zeta_{n+2}),\eth_g(\zeta_n,\zeta_{n+1}),0) = \gamma.$$

Then,

$$\varphi(\eth_g(\zeta_n,\zeta_{n+2}) \le [\varphi(\max\{\eth_g(\zeta_{n-1},\zeta_{n+1}),\eth_g(\zeta_{n-1},\zeta_n)\})]^{\gamma}. \tag{3.10}$$

Take $\mathsf{a}^*_{\mathsf{n}} = \eth_g(\zeta_n,\zeta_{n+2})$ and $\mathsf{b}^*_{\mathsf{n}} = \eth_g(\zeta_n,\zeta_{n+1})$.

Thus, (3.10) can be written as:

$$\varphi(\mathsf{a}^*_{\mathsf{n}}) \le [\varphi(\max\{\mathsf{a}^*_{\mathsf{n-1}},\mathsf{b}^*_{\mathsf{n-1}}\})]^{\gamma},$$

using (φ_1)

$$\mathsf{a}^*_{\mathsf{n}} < \max\{\mathsf{a}^*_{\mathsf{n-1}},\mathsf{b}^*_{\mathsf{n-1}}\}.$$

By (3.7), we have

$$\mathsf{b}^*_{\mathsf{n}} \le \mathsf{b}^*_{\mathsf{n-1}} \le \max\{\mathsf{a}^*_{\mathsf{n-1}},\mathsf{b}^*_{\mathsf{n-1}}\},$$

$$\Rightarrow \max\{\mathsf{a}^*_{\mathsf{n}},\mathsf{b}^*_{\mathsf{n}}\} \le \max\{\mathsf{a}^*_{\mathsf{n-1}},\mathsf{b}^*_{\mathsf{n-1}}\},\quad \forall\ \ n \in \mathbb{N}.$$

Therefore, the sequence $\max\{\mathsf{a}^*_{\mathsf{n-1}},\mathsf{b}^*_{\mathsf{n-1}}\}_{n\in\mathbb{N}}$ forms a decreasing sequence of non-negative real numbers. Thus, $\exists\ \lambda_2 \ge 0\ \ s.t$

$$\lim_{n\to\infty} \max\{\mathsf{a}^*_{\mathsf{n}},\mathsf{b}^*_{\mathsf{n}}\} = \lambda_2.$$

We assume that $\lambda_2 > 0$. By (3.9) and

$$\lim_{n\to\infty}\sup \mathsf{b}_\mathsf{n}^* = \lim_{n\to\infty}\sup \eth_g(\zeta_{n-1},\zeta_n) = 0,$$

$$\lim_{n\to\infty}\sup \mathsf{a}_\mathsf{n}^* = \lim_{n\to\infty}\sup \max\{\mathsf{a}_\mathsf{n}^*,\mathsf{b}_\mathsf{n}^*\} = \lim_{n\to\infty}\max\{\mathsf{a}_\mathsf{n}^*,\mathsf{b}_\mathsf{n}^*\}.$$

Taking the $\lim\sup n \to \infty$ in (3.10), and using the properties of φ, we get

$$\begin{aligned}\varphi(\lim_{n\to\infty}\sup \mathsf{a}_\mathsf{n}^*) <& \varphi(\lim_{n\to\infty}\sup \max\{\mathsf{a}_{\mathsf{n}-1}^*,\mathsf{b}_{\mathsf{n}-1}^*\})\\ =&\varphi(\lim_{n\to\infty}\max\{\mathsf{a}_{\mathsf{n}-1}^*,\mathsf{b}_{\mathsf{n}-1}^*\}).\end{aligned}$$

Therefore,

$$\varphi(\lambda_2) < \varphi(\lambda_2),$$

which is a contradiction. Therefore,

$$\lim_{n\to\infty}\eth_g(\zeta_n,\zeta_{n+2}) = 0. \tag{3.11}$$

Now, we aim to show that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is a CS.

To argue by contradiction, assume that this is not the case. Specifically, suppose $\exists$ a positive real number $\epsilon > 0$ and two sequences of natural numbers, $\{n_{(k)}\}$ and $\{m_{(k)}\}$, $s.t$ $m_{(k)} > n_{(k)} > k$ and,

$$\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}) \geq \epsilon$$

and

$$\eth_g(\zeta_{m_{(k)}-1},\zeta_{n_{(k)}-1}) < \epsilon.$$

Now, using (3.7), (3.10) , (3.11) and the quadrilateral inequality

$$\epsilon \leq \eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}) \leq [\eth_g(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1}) + \eth_g(\zeta_{m_{(k)+1}},\zeta_{m_{(k)}-1}) + \eth_g(\zeta_{m_{(k)}-1},\zeta_{n_{(k)}})]$$

$$\leq [\eth_g(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1}) + \eth_g(\zeta_{m_{(k)}+1},\zeta_{m_{(k)}-1}) + \epsilon].$$

Taking $\lim\limits_{k\to\infty}$ in the above inequality, it gives

$$\begin{aligned}\lim_{k\to\infty}\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}) &\leq [\lim_{k\to\infty}(\eth_g(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1})+\eth_g(\zeta_{m_{(k)}+1},\zeta_{m_{(k)}-1})+\epsilon)]\\ &=(0+0+\epsilon)\\ \Rightarrow \lim_{k\to\infty}\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}) &\leq \epsilon.\end{aligned}$$

Again applying quadrilateral inequality,

$$\eth_g(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}) \leq [\eth_g(\zeta_{m_{(k)}+1},\zeta_{m_{(k)}-1})+\eth_g(\zeta_{m_{(k)}-1)},\zeta_{m_{(k)}})+\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}})].$$

When $k\to\infty$ the above inequality becomes

$$\begin{aligned}&\lim_{k\to\infty}\eth_g(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}})\\ \leq&[\lim_{k\to\infty}(\eth_g(\zeta_{m_{(k)}+1},\zeta_{m_{(k)}-1})+\eth_g(\zeta_{m_{(k)}-1)},\zeta_{m_{(k)}})+\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}))]\\ \leq&(0+0+\epsilon)\leq\epsilon.\end{aligned}$$

Similarly, we can get

$$\lim_{k\to\infty}\eth_g(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}+1})\leq\epsilon.$$

Using quadrilateral inequality once more,
one can obtain

$$\eth_g(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}})\leq\eth_g(\zeta_{m_{(k)}+2},\zeta_{m_{(k)}})+\eth_g(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1})+\eth_g(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}).$$

Letting $k\to\infty$ in the above inequality,

$$\begin{aligned}&\lim_{k\to\infty}\eth_g(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}})\\ \leq&[\lim_{k\to\infty}(\eth_g(\zeta_{m_{(k)}+2},\zeta_{m_{(k)}})+\eth_g(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1})+\eth_g(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}))]\\ \leq&(0+0+\epsilon)\leq\epsilon.\end{aligned}$$

Similarly

$$\lim_{k\to\infty}\eth_g(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}+1})\leq\epsilon.$$

From (3.1) and taking $\zeta = \zeta_{m_{(k)}}$ and $\xi = \zeta_{n_{(k)}}$

$$\begin{aligned} M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}) = \max\{&\eth_g(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}), \eth_g(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}), \eth_g(\zeta_{n_{(k)}}, \zeta_{n_{(k)}+1}), \\ &\eth_g(\zeta_{m_{(k)}+2}, \zeta_{m_{(k)}+1}), \eth_g(\zeta_{m_{(k)}+2}, \zeta_{n_{(k)}}), \\ &\eth_g(\zeta_{m_{(k)}+2}, \zeta_{n_{(k)}+1}), \eth_g(\zeta_{m_{(k)}+2}, \zeta_{m_{(k)}+1})\}. \end{aligned}$$

Therefore,

$$\lim_{k\to\infty} M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}) \le \max\{\epsilon, 0, 0, \epsilon, \epsilon, \epsilon, 0\} = \epsilon.$$

Using (3.1) with $\zeta = \zeta_{m_{(k)}}$ and $\xi = \zeta_{n_{(k)}}$, one can obtain

$$\begin{aligned} &\varphi[\eth_g(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1})] \\ \le&[\varphi(M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}))^{\Omega[\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_g(\zeta_{m_{(k)}},\mathcal{S}\zeta_{m_{(k)}}),\eth_g(\zeta_{n_{(k)}},\mathcal{S}\zeta_{n_{(k)}}),\eth_g(\mathcal{S}\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_g(\mathcal{S}^2\zeta_{m_{(k)}},\zeta_{n_{(k)}})} \\ =&[\varphi(M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}))]^{\Omega[\eth_g(\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_g(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1}),\eth_g(\zeta_{n_{(k)}},\zeta_{n_{(k)}+1}),\eth_g(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}),\eth_g(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}})]}. \end{aligned}$$

As Ω is a continuous function, therefore

$$\begin{aligned} &\lim_{k\to\infty} \Omega\Big[\eth_g\Big(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}\Big), \eth_g\Big(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}\Big), \eth_g\Big(\zeta_{n_{(k)}}, \zeta_{n_{(k)}+1}\Big), \\ &\eth_g\Big(\zeta_{m_{(k)}+1}, \zeta_{n_{(k)}}\Big), \eth_g\Big(\zeta_{m_{(k)}+2}, \zeta_{n_{(k)}}\Big)\Big] \\ =&\Omega\Big[\lim_{k\to\infty}\Big(\eth_g\Big(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}\Big), \eth_g\Big(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}\Big), \eth_g\Big(\zeta_{n_{(k)}}, \zeta_{n_{(k)}+1}\Big), \\ &\eth_g\Big(\zeta_{m_{(k)}+1}, \zeta_{n_{(k)}}\Big), \eth_g\Big(\zeta_{m_{(k)}+2}, \zeta_{n_{(k)}}\Big)\Big)\Big] \\ =& \Omega[\epsilon, 0, 0, \epsilon, \epsilon]. \end{aligned}$$

So, $\exists\ \ \gamma \in\ ]0, 1[\ \ s.t\ \ \Omega[\epsilon, 0, 0, \epsilon, \epsilon] = \gamma.$

Thus,

$$\lim_{k\to\infty} \varphi(\eth_g(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1})) \le [\lim_{k\to\infty} \varphi(M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}))]^{\gamma}$$

Using the continuity of φ in the above inequality, it becomes

$$\begin{gathered} \varphi\Big[\lim_{k\to\infty}\Big(\eth_g(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1}\Big)\Big)\Big] \le \Big[\varphi\Big(\lim_{k\to\infty} M\Big(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}\Big)\Big)\Big]^{\gamma} \\ \Rightarrow \varphi[\epsilon] \le [\varphi(\epsilon)]^{\gamma} \end{gathered}$$

Hence,

$$\varphi(\epsilon) \leq [\varphi(\epsilon)]^{\gamma} < \varphi(\epsilon),$$

which is a contradiction.

So,

$$\lim_{n,m\to\infty} \eth_g(\zeta_m,\zeta_n) = 0.$$

Hence, $\{\zeta_n\}$ is a CS in $\mathcal{W}$. By using completeness property of $(\mathcal{W},\eth_g)$, $\exists\ \Psi \in \mathcal{W}$

s.t

$$\lim_{n\to\infty} \eth_g(\zeta_n,\Psi) = 0.$$

Now, we will show that $\eth_g(\mathcal{S}\Psi,\Psi) = 0$.

Assume by contradiction

$$\eth_g(\mathcal{S}\Psi,\Psi) > 0.$$

Now by quadrilateral inequality we get,

$$\eth_g(\Psi,\mathcal{S}\Psi) \leq [\eth_g(\Psi,\zeta_n) + \eth_g(\zeta_n,\mathcal{S}\zeta_n) + \eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)]. \tag{3.12}$$

By letting $n\to\infty$ in inequality (3.12), we obtain

$$\begin{aligned}\lim_{n\to\infty}[\eth_g(\Psi,\mathcal{S}\Psi)] \leq & [\lim_{n\to\infty}(\eth_g(\Psi,\zeta_n) + \eth_g(\zeta_n,\mathcal{S}\zeta_n) + \eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi))] \\ = & [\lim_{n\to\infty}(\eth_g(\Psi,\zeta_n) + \eth_g(\zeta_n,\zeta_{n+1}) + \eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi))] \\ = & [\lim_{n\to\infty}(\eth_g(\Psi,\zeta_n) + \eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi))] \qquad (3.13) \\ = & [0 + \lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)] \\ \Rightarrow \eth_g(\Psi,\mathcal{S}\Psi) \leq & \lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi).\end{aligned}$$

Consider

$$\varphi(\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi),$$

and applying contraction condition

$$\begin{aligned}\varphi(\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi) \leq & \varphi[M(\zeta_n,\Psi)]^{\gamma} \\ \Rightarrow \lim_{n\to\infty}\varphi(\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi) \leq & \lim_{n\to\infty}\varphi[M(\zeta_n,\Psi)]^{\gamma}.\end{aligned}$$

As φ is continuous

$$\lim_{n\to\infty}\varphi(\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\leq\varphi[\lim_{n\to\infty}M(\zeta_n,\Psi)]^{\gamma}.$$

Here

$$\begin{aligned}M(\zeta_n,\Psi)=\max\{&\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\mathcal{S}\zeta_n),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}\zeta_n,\Psi),\\&\eth_g(\mathcal{S}^2\zeta_n,\mathcal{S}\Psi),\eth_g(\mathcal{S}^2\zeta_n,\Psi),\eth_g(\mathcal{S}^2\zeta_n,\mathcal{S}\zeta_n)\}\end{aligned}$$

$$\begin{aligned}=\max\{&\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_{n+1},\Psi),\\&\eth_g(\zeta_{n+2},\mathcal{S}\Psi),\eth_g(\zeta_{n+2},\Psi),\eth_g(\zeta_{n+2},\zeta_{n+1})\}.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}M(\zeta_n,\Psi)=\max\Big\{&\lim_{n\to\infty}[\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_{n+1},\Psi),\\&\eth_g(\zeta_{n+2},\mathcal{S}\Psi),\eth_g(\zeta_{n+2},\Psi),\eth_g(\zeta_{n+2},\zeta_{n+1})]\Big\}\end{aligned}$$

$$=\max\Big\{0,0,\eth_g(\Psi,\mathcal{S}\Psi),0,\eth_g(\Psi,\mathcal{S}\Psi),0,0\Big\}$$

$$=\eth_g(\Psi,\mathcal{S}\Psi).$$

Thus,

$$\lim_{n\to\infty}M(\zeta_n,\Psi)=\eth_g(\Psi,\mathcal{S}\Psi),$$

and $\exists\ \gamma\in]0,1[\ \ s.t$

$$\lim_{n\to\infty}\Omega(\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_n,\mathcal{S}\Psi),\eth_g(\zeta_{n+2},\Psi))=\gamma.$$

Hence

$$\begin{aligned}\varphi(\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)&\leq\varphi[M(\zeta_n,\Psi)]^{\gamma}\\&\leq\varphi[\eth_g(\Psi,\mathcal{S}\Psi)]^{\gamma}\\&<\eth_g(\Psi,\mathcal{S}\Psi).\end{aligned}$$

As φ is increasing, so

$$(\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\leq\eth_g(\Psi,\mathcal{S}\Psi).$$

Taking $n\to\infty$, the above inequality becomes

$$\lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\leq\eth_g(\Psi,\mathcal{S}\Psi). \tag{3.14}$$

Combining (3.13) and (3.14) ,we get

$$\eth_g(\Psi,\mathcal{S}\Psi)\leq\lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\leq\eth_g(\Psi,\mathcal{S}\Psi).$$

Therefore,

$$\lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)=\eth_g(\Psi,\mathcal{S}\Psi). \tag{3.15}$$

Given that $\zeta_n\to\Psi$ as $n\to\infty\ \ \forall\ \ n\in\mathbb{N}$ and since $\mathcal{S}$ is an (Γ,ν)-continuous, we conclude that $\lim\limits_{n\to\infty}\mathcal{S}\zeta_n=\mathcal{S}\Psi$. Then,

$$\begin{aligned}&\lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)=\eth_g(\mathcal{S}\Psi,\mathcal{S}\Psi)=0\\&\Rightarrow\eth_g(\mathcal{S}\Psi,\Psi)=0.\end{aligned}$$

So $\Psi=\mathcal{S}\Psi$.

For uniqueness, let $\Psi,\zeta\in\mathcal{W}$ are two FPs of $\mathcal{S}\ \ s.t\ \ \zeta\neq\Psi$.

Therefore,

$$\eth_g(\Psi,\zeta)=\eth_g(\mathcal{S}\Psi,\mathcal{S}\zeta)>0.$$

Applying (3.1) with $\zeta=\Psi$ and $\xi=\zeta$,

we have

$$\begin{aligned}\varphi(\eth_g(\Psi,\zeta))&=\varphi(\eth_g(\mathcal{S}\zeta,\mathcal{S}\Psi))\\&\leq[\varphi(M(\Psi,\zeta))]^{\Omega(\eth_g(\Psi,\zeta),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\zeta,\mathcal{S}\Psi),\eth_g(\mathcal{S}^2\Psi,\zeta))}\\&=[\varphi(M(\Psi,\zeta))]^{\Omega(\eth_g(\Psi,\zeta),\eth_g(\Psi,\Psi),\eth_g(\zeta,\zeta),\eth_g(\zeta,\Psi),\eth_g(\Psi,\zeta))}\\&=[\varphi(M(\Psi,\zeta))]^{\Omega(\eth_g(\Psi,\zeta),0,0,\eth_g(\zeta,\Psi),\eth_g(\Psi,\zeta))}\\&=[\varphi(M(\Psi,\zeta))]^{\gamma},\end{aligned}$$

here

$$\begin{aligned}M(\Psi,\zeta) &= \max\{\eth_g(\Psi,\zeta),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\mathcal{S}\Psi,\zeta),\\ &\quad \eth_g(\mathcal{S}^2\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^2\Psi,\zeta),\eth_g(\mathcal{S}^2\Psi,\mathcal{S}\zeta)\}\end{aligned}$$

$$\begin{aligned}&= \max\{\eth_g(\Psi,\zeta),\eth_g(\Psi,\Psi),\eth_g(\zeta,\zeta),\eth_g(\Psi,\zeta),\eth_g(\Psi,\Psi),\eth_g(\Psi,\zeta),\eth_g(\Psi,\zeta)\}\\ &= \eth_g(\Psi,\zeta).\end{aligned}$$

Therefore,

$$\varphi(\eth_g(\mathcal{S}\Psi,\mathcal{S}\zeta)) \leq [\varphi(\eth_g(\Psi,\zeta))]^\gamma < \varphi(\eth_g(\Psi,\zeta)),$$

which implies that

$$\eth_g(\Psi,\zeta) < \eth_g(\Psi,\zeta),$$

this contradiction implies $\zeta = \Psi$, thereby concluding the proof. □

Corollary 3.2.2. Let $(\mathcal{W},\eth_g)$ be a (Γ,ν)-complete GMS, and $\Gamma,\nu : \mathcal{W}\times\mathcal{W} \to [0,+\infty[$. Suppose $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ be a self-mapping which fulfils the following assertions:

(i) : $\varphi[\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)] \leq [\varphi(M(\zeta,\xi))]^{\mathsf{k}}, \mathsf{k} \in (0,1)$ and $\varphi \in \Theta_{\mathcal{G}}$,

(ii) : $\mathcal{S}$ is continuous.

Then, $\mathcal{S}$ has a UFP.

Proof. Define a function $\Omega(\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5) = \mathtt{k} \ \forall \ \jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5 \in \mathbb{R}_+$.
Clearly $\Omega \in \sigma$.
Taking

$$\Gamma(\zeta,\xi) = 1, \nu(\zeta,\xi) = 1.$$

Thus, $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction, and $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping. Hence following Theorem (3.2.1), $\mathcal{S}$ has a UFP ζ. □

It is an evident that if ζ is a FP of $\mathcal{S}$ then it's also a FP of $\mathcal{S}^n$ for every natural number n.

Jeong and Rhoades [31] first introduce the idea of property P that if the set of FPs of $\mathcal{S}$ coincides with the set of FPs of $\mathcal{S}^n \ \forall \ n \in \mathbb{N}$, i.e.
$Fix(\mathcal{S}) = Fix(\mathcal{S}^n)$, then, a mapping $\mathcal{S}$ is said to have property P (or have no periodic points).

Theorem 3.2.3. Consider $(\mathcal{W}, \eth_g)$ be a (Γ,ν)-complete GMS and let $\Gamma,\nu : \mathcal{W}\times\mathcal{W} \to [0,+\infty)$ be two functions. $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ be a mapping which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

(iii) : $\Gamma(\Psi,\mathcal{S}\Psi) \geq 1$ and $\nu(\Psi,\mathcal{S}\Psi) \leq 1, \ \forall \ \Psi \in Fix(\mathcal{S})$,

Then, $\mathcal{S}$ has the property p, that is
$\mathcal{S}^n\zeta = \mathcal{S}\zeta$.

Proof. Let for some fixed $n > 1$, let $\Psi \in Fix(\mathcal{S}^n)$. As $\Gamma(\Psi,\mathcal{S}\Psi) \geq 1$ and $\nu(\Psi,\mathcal{S}\Psi) \leq 1$ and $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping, then

$$\Gamma(\mathcal{S}\Psi,\mathcal{S}^2\Psi) \geq 1 \text{ and } \nu(\mathcal{S}\Psi,\mathcal{S}^2\Psi) \leq 1.$$

Similarly, it can be written as:

$$\Gamma(\mathcal{S}^n\Psi,\mathcal{S}^{n+1}\Psi) \geq 1 \text{ and } \nu(\mathcal{S}^n\Psi,\mathcal{S}^{n+1}\Psi) \leq 1 \ \forall \ n \in \mathbb{N}.$$

By $(\mathcal{S}_3)$ and $(\mathcal{S}_4)$

$$\Gamma(\mathcal{S}^m\Psi,\mathcal{S}^n\Psi) \geq 1 \text{ and } \nu(\mathcal{S}^m\Psi,\mathcal{S}^n\Psi) \leq 1, \ \forall \ m,n \in \mathbb{N}, n \neq m.$$

Consider that $\Psi \notin Fix(\mathcal{S})$, i.e , $\eth_g(\Psi,\mathcal{S}\Psi) > 0$.
Also we can write $\eth_g(\Psi,\mathcal{S}\Psi)$ as

$$\eth_g(\Psi,\mathcal{S}\Psi) = \eth_g(\mathcal{S}^n\Psi,\mathcal{S}\Psi) = \eth_g(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi), \tag{3.16}$$

consider $\eth_g(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)$ and using (3.1)

$$\begin{aligned}
&\varphi(\eth_g(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)\\
&\leq [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\Omega(\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}\mathcal{S}^{n-1}\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\Psi,\mathcal{S}^2\mathcal{S}^{n-1}\Psi))}\\
&= [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\Omega(\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}^n\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^n\Psi,\Psi),\eth_g(\Psi,\mathcal{S}^{n+1}\Psi))}\\
&= [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\Omega(\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}^n\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^n\Psi,\Psi),0)}.
\end{aligned}$$

So, $\exists\ \gamma \in (0,1)\ \ s.t$

$$\Omega(\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}^n\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^n\Psi,\Psi),0)=\gamma.$$

Since φ is continuous and an increasing function, therefore, (3.16) gives

$$\varphi(\eth_g(\Psi,\mathcal{S}\Psi))=\varphi(\eth_g(\mathcal{S}^n\Psi,\mathcal{S}\Psi))=\varphi(\eth_g(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)).$$

$$\text{Also } \varphi(\eth_g(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi))\leq[\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\gamma}$$

$$\Rightarrow \varphi(\eth_g(\Psi,\mathcal{S}\Psi)\leq[\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\gamma}. \tag{3.17}$$

Now to find $M(\mathcal{S}^{n-1}\Psi,\Psi))$,
we proceed as follows:

$$\begin{aligned}
&M(\Psi,\mathcal{S}^{n-1}\Psi)\\
&=\max\{\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}\mathcal{S}^{n-1}\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^2\mathcal{S}^{n-1}\Psi,\Psi),\\
&\qquad\eth_g(\mathcal{S}^2\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^2\mathcal{S}^{n-1}\Psi,\mathcal{S}^{n-1}\Psi)\}
\end{aligned}$$

$$\begin{aligned}
&=\max\{\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}^n\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}\mathcal{S}^n\Psi,\Psi),\\
&\qquad\eth_g(\mathcal{S}\mathcal{S}^n\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}\mathcal{S}^n\Psi,\mathcal{S}^{n-1}\Psi)\}\\
&=\max\{\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}^{n-1}\Psi,\Psi),\eth_g(\mathcal{S}\Psi,\Psi),\\
&\qquad\eth_g(\mathcal{S}\Psi,\mathcal{S}\Psi),\eth_g(\mathcal{S}\Psi,\mathcal{S}^{n-1}\Psi)\}.
\end{aligned}$$

As $n\to\infty$ and $\eth_g(\mathcal{S}^{n-1}\Psi,\mathcal{S}^n\Psi)\to 0$

$$\Rightarrow \lim_{n\to\infty} M(\Psi,\mathcal{S}^{n-1}\Psi)=\eth_g(\Psi,\mathcal{S}\Psi).$$

Therefore (3.17) becomes

$$\varphi(\eth_g(\Psi,\mathcal{S}\Psi) \leq [\varphi(\eth_g(\Psi,\mathcal{S}\Psi))]^{\gamma} < \varphi(\eth_g(\Psi,\mathcal{S}\Psi)),$$

which is a contradiction. Hence,

$$\eth_g(\Psi,\mathcal{S}\Psi) = 0,$$

which implies that $Fix(\mathcal{S}^n) = Fix(\mathcal{S})$. Thus $\mathcal{S}$ has a property P.

Considering the following assertions, we prove that if $\mathcal{S}$ is not necessarily continuous then Theorem (3.2.1) still holds. □

Theorem 3.2.4. Let $(\mathcal{W},\eth_g)$ be a (Γ,ν)-complete GMS and let Γ,ν are two functions $s.t$ $\Gamma,\nu : \mathcal{W}\times\mathcal{W} \to [0,+\infty)$.

Suppose $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self mapping which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

(iii) : $\exists\ \zeta_0 \in \mathcal{W}$ $s.t$ $\Gamma(\zeta_0,\mathcal{S}\zeta_0) \geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0) \leq 1$.

(iv) : $(\mathcal{W},\eth_g)$ is (Γ,ν)-regular,

then, $\mathcal{S}$ has a FP. Moreover, $\mathcal{S}$ has a UFP with $\Gamma(\Psi,\zeta) \geq 1$ and $\nu(\Psi,\zeta) \leq 1$ $\forall\ \Psi,\zeta \in Fix(\mathcal{S})$.

Proof. Let $\zeta_0 \in \mathcal{W}$ $s.t$ $\Gamma(\zeta_0,\mathcal{S}\zeta_0) \geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0) \leq 1$.

Following the proof of Theorem (3.2.1), it can be concluded that

$$\Gamma(\zeta_n,\mathcal{S}\zeta_{n+1}) \geq 1 \text{ and } \nu(\zeta_n,\mathcal{S}\zeta_{n+1}) \leq 1, \text{ and}$$
$$\zeta_n \to \Psi \text{ as } n\to\infty,$$

where $\zeta_{n+1} = \mathcal{S}\zeta_n$. From (iv), $\Gamma(\zeta_{n+1},\Psi) \geq 1$ and $\nu(\zeta_{n+1},\Psi) \leq 1$ hold $\forall\ n \in \mathbb{N}$.

Suppose that $\mathcal{S}\Psi = \zeta_{0+1} = \mathcal{S}\zeta_{n_0}$ for some $n_0 \in \mathbb{N}$. By Theorem (3.2.1), it is known that the elements of the sequence $\{\zeta_n\}$ are different. So, $\mathcal{S}\Psi \neq \mathcal{S}\zeta_n$, that is $\eth_g(\mathcal{S}\Psi, \mathcal{S}\zeta_n) > 0 \ \ \forall \ \ n > n_0$. Consequently, one can apply (3.1) to ζ_n and Ψ $\forall \ \ n > n_0$ to obtain

$$\begin{aligned}\varphi(\eth_g(\mathcal{S}\zeta_n, \mathcal{S}\Psi)) \leq & [\varphi(M(\zeta_n, \Psi))]^{\Omega(\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\mathcal{S}\zeta_n),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_n,\mathcal{S}\Psi),\eth_g(\mathcal{S}^2\zeta_n,\Psi))} \\ = & [\varphi(M(\zeta_n, \Psi))]^{\Omega(\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_n,\mathcal{S}\Psi),\eth_g(\zeta_{n+2},\Psi))},\end{aligned}$$

which implies that

$$\varphi(\eth_g(\mathcal{S}\zeta_n, \mathcal{S}\Psi)) \leq [\varphi(M(\zeta_n, \Psi))]^{\Omega(\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_n,\mathcal{S}\Psi),\eth_g(\zeta_{n+2},\Psi))}. \tag{3.18}$$

Now

$$\begin{aligned}\lim_{n\to\infty} M(\zeta_n, \Psi) = & \max\Big\{ \lim_{n\to\infty}[\eth_g(\zeta_n, \Psi), \eth_g(\zeta_n, \zeta_{n+1}), \eth_g(\Psi, \mathcal{S}\Psi), \eth_g(\zeta_{n+1}, \Psi), \\ & \eth_g(\zeta_{n+2}, \mathcal{S}\Psi), \eth_g(\zeta_{n+2}, \Psi), \eth_g(\zeta_{n+2}, \zeta_{n+1})]\Big\}.\end{aligned}$$

As it is described earlier that $\zeta_n \to \Psi$ as $n \to \infty$.

Then, the above equality becomes

$$\begin{aligned}&= \max\Big\{\eth_g(\Psi, \Psi), \eth_g(\Psi, \Psi), \eth_g(\Psi, \mathcal{S}\Psi), \eth_g(\Psi, \Psi), \\ &\lim_{n\to\infty} \eth_g(\zeta_{n+2}, \mathcal{S}\Psi), \eth_g(\Psi, \Psi), \eth_g(\Psi, \Psi)\Big\} \\ &= \max\Big\{0, 0, \eth_g(\Psi, \mathcal{S}\Psi), 0, \lim_{n\to\infty} \eth_g(\zeta_{n+2}, \mathcal{S}\Psi), 0, 0\Big\}.\end{aligned}$$

Since

$$0 \leq \eth_g(\zeta_{n+2}, \mathcal{S}\Psi) \leq \eth_g(\zeta_{n+2}, \zeta_n) + \eth_g(\zeta_n, \Psi) + \eth_g(\zeta_n, \mathcal{S}\Psi).$$

Taking $\lim n \to \infty$

$$\begin{aligned}\lim_{n\to\infty} \eth_g(\zeta_{n+2}, \mathcal{S}\Psi) \leq & \lim_{n\to\infty}[\eth_g(\zeta_{n+2}, \zeta_n) + \eth_g(\zeta_n, \Psi) + \eth_g(\zeta_n, \mathcal{S}\Psi)] \\ \leq & \eth_g(\Psi, \Psi) + \eth_g(\Psi, \Psi) + \eth_g(\Psi, \mathcal{S}\Psi) \qquad (3.19) \\ \Rightarrow \lim_{n\to\infty} \eth_g(\zeta_{n+2}, \mathcal{S}\Psi) \leq & \eth_g(\Psi, \mathcal{S}\Psi).\end{aligned}$$

Thus,

$$\lim_{n\to\infty} M(\zeta_n, \Psi) \leq \eth_g(\Psi, \mathcal{S}\Psi).$$

Also

$$\lim_{n\to\infty}\Omega(\eth_g(\zeta_n,\Psi),\eth_g(\zeta_n,\zeta_{n+1}),\eth_g(\Psi,\mathcal{S}\Psi),\eth_g(\zeta_n,\mathcal{S}\Psi),\eth_g(\zeta_{n+2},\Psi))=\gamma.$$

If $\eth_g(\Psi,\mathcal{S}\Psi)>0$, subsequently by (3.19) and certainly φ and Ω are continuous, (3.18) becomes

$$\varphi(\lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi))\leq[\varphi(\eth_g(\Psi,\mathcal{S}\Psi))]^{\gamma}<[\varphi(\eth_g(\Psi,\mathcal{S}\Psi))].$$

Using (3.15), we get

$$\varphi(\eth_g(\Psi,\mathcal{S}\Psi))<\varphi[\eth_g(\Psi,\mathcal{S}\Psi)],$$

which is a contradiction. Hence, $\eth_g(\Psi,\mathcal{S}\Psi)=0$, that is, Ψ is a FP of $\mathcal{S}$, which implies that $\Psi=\mathcal{S}\Psi$. One can prove the uniqueness proof following the Theorem (3.2.1). □

Definition 3.2.5. Let $(\mathcal{W},\eth_g)$ be a $(\Gamma-\nu)$-GMS, and let $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self-mapping. Suppose that $\Gamma,\nu:\mathcal{W}\times\mathcal{W}\to[0,+\infty[$ are two functions. Then $\mathcal{S}$ is a (Γ,ν)-$\varphi_{\mathcal{G}}$-Ω-contraction, if $\forall\ \zeta,\xi\in\mathcal{W}$ with $\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1$ and $\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)>0$,we have

$$\varphi(\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi))\leq[\varphi(M(\zeta,\xi))]^{\Omega(\eth_g(\zeta,\xi),\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\mathcal{S}\zeta,\xi),\eth_g(\mathcal{S}^2\zeta,\xi))}, \tag{3.20}$$

where $\varphi\in\Theta_G$, $\Omega\in\sigma$ and

$$M(\zeta,\xi)=\max\{\eth_g(\zeta,\xi),\eth_g(\zeta,\mathcal{S}\zeta),\eth_g(\xi,\mathcal{S}\xi),\eth_g(\mathcal{S}\zeta,\xi),\eth_g(\mathcal{S}^2\zeta,\xi),\eth_g(\mathcal{S}^2\zeta,\mathcal{S}\xi),\eth_g(\mathcal{S}^2\zeta,\mathcal{S}\zeta)\}.$$

Theorem 3.2.6. Let $(\mathcal{W},\eth_g)$ be a (Γ,ν)-complete GMS, and let Γ,ν are two functions $s.t$ $\Gamma,\nu:\mathcal{W}\times\mathcal{W}\to[0,+\infty)$. Suppose $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self-mapping which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

(iii) : $\exists\ \zeta_0\in\mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$.

(iv) : $\mathcal{S}$ is a (Γ,ν)-continuous.

Then, $\mathcal{S}$ has a FP. Also, $\mathcal{S}$ has a UFP when $\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1\ \forall\ \zeta,\xi\in\mathcal{W}$.

Proof. Let $\zeta_0\in\mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$. Following the proof of Theorem (3.2.1),
it can be concluded that

$$\Gamma(\zeta_n,\zeta_{n+1})\geq 1 \text{ and } \nu(\zeta_n,\zeta_{n+1})\leq 1,$$

and

$$\lim_{n\to\infty}\eth_g(\zeta_n,\zeta_{n+1})=0\ ,$$

$$\lim_{n\to\infty}\eth_g(\zeta_n,\zeta_{n+2})=0.$$

By property (φ_3), $\exists\ \ \mathtt{r}\in\,]0,1[$ and $l\in(0,+\infty]\ \ s.t$

$$\lim_{n\to\infty}\frac{(\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1)}{\eth_g(\zeta_n,\zeta_{n+1})^{r}}=l.$$

Suppose that $l<\infty$. So, $\exists\ \ n_1\in\mathbb{N}\ \ s.t$

$$\left|\frac{\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1}{\eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}}-l\right|<\frac{l}{2},\quad\forall\ \ n\geq n_1. \tag{3.21}$$

$$\Rightarrow -\frac{l}{2}<\frac{\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1}{\eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}}-l<\frac{l}{2}$$

$$\Rightarrow (-\frac{l}{2}+l)\ \eth_g(\zeta_n,\zeta_{n+1})^{r}<\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1<(\frac{l}{2}+l)\ \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \frac{l}{2}\ \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}<\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1<\frac{3l}{2}\ \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}<\frac{2}{l}(\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1)<3\eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}<\mathcal{A}^*(\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1),\ \text{where } \mathcal{A}^*=\frac{2}{l}$$

$$\Rightarrow n\eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}<\mathcal{A}^*.n(\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1).$$

Now let $l=\infty$. Suppose $\mathcal{R}>0$ be an arbitrary non-negative number.
Then, $\exists\ \ n_2\in\mathbb{N}\ \ s.t$

$$\left|\frac{\varphi(\eth_g(x_n,x_{n+1}))-1}{(\eth_g(x_n,x_{n+1}))^r}\right|>\mathcal{R}. \tag{3.22}$$

Since $\eth_g(\zeta_n,\zeta_{n+1})>0$ and also $\varphi(\eth_g(\zeta_n,\zeta_{n+1}))$ is positive, so absolute value is unnecessary and one can write it as

$$\frac{\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1}{(\eth_g(\zeta_n,\zeta_{n+1}))^{\mathtt{r}}}>\mathcal{R}$$

$$\Rightarrow\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1>\mathcal{R}\ \ \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow\ \ \eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}<\frac{1}{\mathcal{R}}\ \ \varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1$$

$$\Rightarrow n[\eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}]<n.\mathcal{A}^*[\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1],\ \ \forall\ \ n\geq n_2,\ \text{where}\ \mathcal{A}^*=\frac{1}{\mathcal{R}}.$$

Therefore, in all cases, $\exists\ \ \mathcal{A}^*>0$ and $\mathsf{c}\in\mathbb{N}(\mathsf{c}=\max(n_1,n_2))\ \ s.t$

$$n[\eth_g(\zeta_n,\zeta_{n+1})^{\mathtt{r}}]<\mathcal{A}^*.n[\varphi(\eth_g(\zeta_n,\zeta_{n+1}))-1],\ \ \forall\ \ n\geq n_{\mathsf{c}}. \tag{3.23}$$

Consider $\varphi(\eth_g(\zeta_n,\zeta_{n+1}))=\varphi(\eth_g(\mathcal{S}\zeta_{n-1},\mathcal{S}\zeta_n))$.
By using definition (3.2.5)

$$\varphi(\eth_g(\zeta_n,\zeta_{n+1}))\leq M(\zeta_{n-1},\zeta_n))^{\mathtt{r}}. \tag{3.24}$$

As $M(\zeta_{n-1},\zeta_n)=\eth_g(\zeta_{n-1},\zeta_n)$, therefore (3.24) becomes

$$\varphi(\eth_g(\zeta_n,\zeta_{n+1}))<\varphi(\eth_g(\zeta_n,\zeta_{n+1}))\leq\eth_g(\zeta_{n-1},\zeta_n)^{\mathtt{r}}.$$

Using this in (3.23)

$$n[\eth_g(\zeta_n,\zeta_{n+1})]^{\mathtt{r}}<\mathcal{A}^*.n[\varphi(\eth_g(\zeta_n,\zeta_{n+1}))]-\mathcal{A}^*.n<\mathcal{A}^*.n[\eth_g(\zeta_{n-1},\zeta_n)]^{\mathtt{r}}-\mathcal{A}^*.n. \tag{3.25}$$

Now applying contraction condition on $\varphi(\eth_g(\zeta_{n-1},\zeta_n))$

$$\begin{aligned}\varphi(\eth_g(\zeta_{n-1},\zeta_n)) \leq& \varphi(M(\zeta_{n-2},\zeta_{n-1}))^{\mathtt{r}}\\ =&\varphi(\eth_g(\zeta_{n-2},\zeta_{n-1}))^{\mathtt{r}}.\end{aligned}$$

Therefore (3.25) becomes

$$\begin{aligned}n[\eth_g(\zeta_n,\zeta_{n+1})]^{\mathtt{r}} < (\mathcal{A}^*.n[\eth_g(\zeta_{n-1},\zeta_n)]^{\mathtt{r}} - \mathcal{A}^*.n) =& \mathcal{A}^*.n[\eth_g(\zeta_{n-1},\zeta_n)^r - 1]\\ <& \mathcal{A}^*.n[\varphi(\eth_g(\zeta_{n-2},\zeta_{n-1}))^{r^2} - 1].\end{aligned}$$

Continuing in the same way, we obtain

$$n[\eth_g(\zeta_n,\zeta_{n+1})]^{\mathtt{r}} < n.\mathcal{A}^*[\varphi(\eth_g(\zeta_n,\zeta_{n+1})) - 1] < \ldots < \mathcal{A}^*.n[\varphi(\eth_g(\zeta_0,\mathtt{u}_1)))^{\mathtt{r}^n} - 1].$$

Taking $n\to\infty$ in the above inequality to get

$$\begin{aligned}&\lim_{n\to\infty} n[\eth_g(\zeta_n,\zeta_{n+1})]^{\mathtt{r}} = 0\\ &\Rightarrow n[\eth_g(\zeta_n,\zeta_{n+1})]^{\mathtt{r}} < \epsilon,\ \forall\ \epsilon > 0.\end{aligned}$$

Choosing $\epsilon = 1$ and also $\exists\ \ n_3 \in \mathbb{N}\ \ s.t$

$$\eth_g(\zeta_n,\zeta_{n+1}) \leq \frac{1}{n^{1/\mathtt{r}}},\quad \forall\ \ n \geq n_3.$$

Now consider $\eth_g(\zeta_n,\zeta_{n+2})$ and by using condition (φ_3), $\exists\ \ \mathtt{r} \in (0,1)$ and $\mathtt{h} \in\ ]0,+\infty[\ \ s.t$

$$\lim_{n\to\infty}\frac{\varphi(\eth_g(\zeta_n,\zeta_{n+2})) - 1}{(\eth_g(\zeta_n,\zeta_{n+2}))^{\mathtt{r}}} = \mathtt{h}.$$

Now suppose that $\mathtt{h} < \infty$ and $\exists\ \ n_4 \in \mathbb{N}$, then Following the same procedure, one can get

$$n(\eth_g(\zeta_n,\zeta_{n+2}))^{\mathtt{r}} < \mathcal{J}^*.n(\varphi(\eth_g(\zeta_n,\zeta_{n+2})) - 1),\quad \forall\ \ n \geq n_4.$$

Now let $\mathtt{h} = \infty$ and let $\exists$ an arbitrary non-negative number $\mathcal{Q}^* > 0$ and also $\exists\ \ n_5 \in \mathbb{N}$, then one can get

$$n[\eth_g(\zeta_n,\zeta_{n+2})]^{\mathtt{r}} < n.\mathcal{J}^*[\varphi(\eth_g(\zeta_n,\zeta_{n+2})) - 1],\quad \forall\ \ n \geq n_5,\ \text{where } \mathcal{J}^* = 1/\mathcal{Q}^*.$$

Following the same procedure as in the equation (3.23), we get

$$n[\eth_g(\zeta_n,\zeta_{n+2})^{\mathtt{r}}] < n.\mathcal{J}^*[\varphi(\eth_g(\zeta_n,\zeta_{n+2}))-1] < ... < n.\mathcal{J}^*[\varphi(\eth_g(\zeta_0,\zeta_2)))^{\mathtt{r}^n}-1].$$

As $n \to \infty$

$$\lim_{n\to\infty} n[\eth_g(\zeta_n,\zeta_{n+2})^{\mathtt{r}}] = 0$$

$$\Rightarrow n[\eth_g(\zeta_n,\zeta_{n+2})^{\mathtt{r}}] < 1\ .$$

Then, $\exists \;\; n_6 \in \mathbb{N} \;\; s.t$

$$\eth_g(\zeta_n,\zeta_{n+2}) \le \frac{1}{n^{1/\mathtt{r}}}, \quad \forall \;\; n \ge n_6.$$

If $\mathtt{m} > \mathtt{n}$, then

$$\begin{aligned}
\eth_g(\zeta_n,\zeta_m) \le& [\eth_g(\zeta_n,\zeta_{n+1}) + \eth_g(\zeta_{n+1},\zeta_m)] \\
\le& \eth_g(\zeta_n,\zeta_{n+1}) + \eth_g(\zeta_{n+1},\zeta_{n+2}) + \eth_g(\zeta_{n+2},\zeta_m) \\
\le& \eth_g(\zeta_n,\zeta_{n+1}) + ... + \eth_g(\zeta_{m-1},\zeta_m) \\
\Rightarrow \eth_g(\zeta_n,\zeta_m) \le& [\eth_g(\zeta_n,\zeta_{n+1}) + \eth_g(\zeta_{n+1},\zeta_{n+2}) + ... + \eth_g(\zeta_{m-1},\zeta_m)],
\end{aligned}$$

By combining the result, it can be concluded that

$$\eth_g(\zeta_n,\zeta_m) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Therefore,

$$\eth_g(\zeta_n,\zeta_m) \le \sum_{i=n}^{\infty} \frac{1}{i^{1/\mathtt{r}}}. \tag{3.26}$$

Since $\mathtt{r} \in (0,1)$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/\mathtt{r}}}$ is a p-series with $p = \frac{1}{\mathtt{r}} > 1$, which converges. So, in (3.26) taking the limit $n, m \to \infty$, we get

$$\lim_{n\to\infty} \eth_g(\zeta_n,\zeta_m) = 0.$$

Hence,

it is clear that $\{\zeta_n\}_{n\in\mathbb{N}}$ is a CS. As $\mathcal{W}$ is complete, then $\exists\ \ \Psi\in\mathcal{W}\ \ s.t$

$$\lim_{n\to\infty}\eth_g(\zeta_n,\Psi)=0.$$

As $\mathcal{S}$ is (Γ,ν)-continuous, therefore

$$\lim_{n\to\infty}\eth_g(\mathcal{S}\zeta_n,\mathcal{S}\Psi)=0,$$

$$\Rightarrow \Psi=\lim_{n\to\infty}\zeta_{n+1}=\lim_{n\to\infty}\mathcal{S}\zeta_n=\mathcal{S}\Psi.$$

This clearly shows that Ψ is a FP of $\mathcal{S}$. □

Corollary 3.2.7. Let $(\mathcal{W},\eth_g)$ be a (Γ,ν)-complete GMS and also suppose that Γ,ν are two functions $\ s.t\ \ \Gamma,\nu: C\times C\to[0,+\infty[$. Let $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self-mapping which fulfils the following assertions:

(i) : $\varphi[\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)]\leq[\varphi(M(\zeta,\xi))]^{\mathsf{r}}, \mathsf{r}\in(0,1)\ \varphi\in\Theta_{\mathcal{G}}$.

(ii) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(iii) : $\exists\ \zeta_0\in\mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$.

(iv) : $\mathcal{S}$ is a (Γ,ν)-continuous.

Then, $\mathcal{S}$ has a FP.

Also, $\mathcal{S}$ has a UFP when

$$\Gamma(\zeta,\xi)\geq 1 \text{ and } \nu(\zeta,\xi)\leq 1\ \forall\ \zeta,\xi\in\mathcal{W}.$$

Proof. Define a function $\Omega:\mathbb{R}_+^5\to\mathbb{R}_+$ by

$$\Omega(e_1,e_2,e_3,e_4,e_5)=r\ \forall\ e_1,e_2,e_3,e_4,e_5\in\mathbb{R}_+$$

. Clearly $\Omega\in\sigma$ and $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction. Following the proof of Theorem (3.2.6), $\mathcal{S}$ has a UFP $\zeta\in\mathcal{W}$. □

Example 3.2.8. Let $\mathcal{W} = (0, +\infty)$ and $a \in (0, 1)$.

Define $\eth_g : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ by

$$\eth_g(\zeta, \xi) = (|\zeta - \xi|).$$

Then, $(\mathcal{W}, \eth_g)$ is a complete GMS.

Define $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ by

$$\mathcal{S}(\jmath) = a\sqrt{\jmath} \;\; \forall \;\; \jmath \in (0, +\infty),$$

and $\varphi(\jmath) = e^{\jmath}$

$$\Gamma(\zeta, \xi) = \frac{\max\{\zeta, \xi\} + a}{\min\{\zeta, \xi\} + a}, \;\; \forall \;\; \zeta, \xi \in \mathbb{R}_+,$$

$$\nu(\zeta, \xi) = \frac{\min\{\zeta, \xi\} + a}{\max\{\zeta, \xi\} + a}, \;\; \forall \;\; \zeta, \xi \in \mathbb{R}_+,$$

$$\Omega(\jmath_1, \jmath_2, \jmath_3, \jmath_4, \jmath_5) = \sqrt{a} \;\; \forall \;\; \jmath_1, \jmath_2, \jmath_3, \jmath_4, \jmath_5 \in \mathbb{R}_+.$$

Then, $\mathcal{S}$ is an (Γ, ν)-continuous and triangular (Γ, ν)-admissible mapping.

Case 1 : $0 \leq \zeta \leq \xi$

$\eth(\mathcal{S}\zeta, \mathcal{S}\xi) = (a\sqrt{\xi} - a\sqrt{\zeta}),$

$$M(\eth(\zeta, \xi)) = \max\Big\{\eth(\zeta, \xi), \eth(\zeta, a\sqrt{\zeta}), \eth(\xi, a\sqrt{\xi}), \eth(\xi, a\sqrt{\zeta}),$$
$$\eth\left(a^2\sqrt{\sqrt{\zeta}}, \xi\right), \eth\left(a^2\sqrt{\sqrt{\zeta}}, a\sqrt{\xi}\right), \eth\left(a^2\sqrt{\sqrt{\zeta}}, a\sqrt{\zeta}\right)\Big\}.$$

Since $\zeta \leq \xi$ and $a \in (0, 1)$, we get

$$M(\zeta, \xi) = \max\Big\{(\xi - \zeta), (\zeta - a\sqrt{\zeta}), (\xi - a\sqrt{\xi}), (\xi - a\sqrt{\zeta}),$$
$$\left(\xi - a^2\sqrt{\sqrt{\zeta}}\right), \left(a\sqrt{\xi} - a^2\sqrt{\sqrt{\zeta}}\right), \left(a\sqrt{\zeta} - a^2\sqrt{\sqrt{\zeta}}\right)\Big\}.$$

Thus,

$$M(\zeta, \xi) \geq \xi - \zeta \geq a(\xi - \zeta).$$

In contrast,

$$a(\xi-\zeta)=\sqrt{a}\sqrt{a}(\xi-\zeta),$$

$$\Rightarrow \varphi(M(\zeta,\xi))\geq \varphi(\eth(\zeta,\xi))=e^{(\xi-\zeta)}.$$

Thus,

$$\varphi(\eth(\zeta,\xi))^{\Omega(\eth(\zeta,\mathcal{S}\zeta),\eth(\xi,\mathcal{S}\xi),\eth(\zeta,\mathcal{S}\xi),\eth(\xi,\mathcal{S}\zeta),\eth(\mathcal{S}^2\zeta,\xi))}=e^{\sqrt{a}(\xi-\zeta)}=e^{\sqrt{a}(\sqrt{\xi}-\sqrt{\zeta})(\sqrt{\xi}+\sqrt{\zeta})},$$

$$\varphi(\eth(\mathcal{S}\zeta,\mathcal{S}\xi))=e^{a(\sqrt{\xi}-\sqrt{\zeta})}.$$

As $\zeta,\xi\in\,]0,\infty[$

$$e^{a(\sqrt{\xi}-\sqrt{\zeta})}\leq e^{\sqrt{a}(\sqrt{\xi}-\sqrt{\zeta})(\sqrt{\xi}+\sqrt{\zeta})}.$$

Thus,

$$\varphi(\eth(\mathcal{S}\zeta,\mathcal{S}\xi))\leq \varphi(\eth(\zeta,\xi))^{\Omega(\eth(\zeta,\mathcal{S}\zeta),\eth(\xi,\mathcal{S}\xi),\eth(\zeta,\mathcal{S}\xi),\eth(\xi,\mathcal{S}\zeta),\eth(\mathcal{S}^2\zeta,\xi))}.$$

Case : 2 $\zeta>\xi>0$

$\eth(\mathcal{S}\zeta,\mathcal{S}\xi)=(a\sqrt{\zeta}-a\sqrt{\xi}),$

$$M(\eth(\zeta,\xi))=\max\Big\{\eth(\zeta,\xi),\eth(\zeta,a\sqrt{\zeta}),\eth(\xi,a\sqrt{\xi}),\eth(\xi,a\sqrt{\zeta}),$$
$$\eth\left(a^2\sqrt{\sqrt{\zeta}},\xi\right),\eth\left(a^2\sqrt{\sqrt{\zeta}},a\sqrt{\xi}\right),\eth\left(a^2\sqrt{\sqrt{\zeta}},a\sqrt{\zeta}\right)\Big\}.$$

As $\zeta>\xi$ and $a\in\,]0,1[$,

$$M(\zeta,\xi)=\max\Big\{(\zeta-\xi),(\zeta-a\sqrt{\zeta}),(\xi-a\sqrt{\xi}),(|\xi-a\sqrt{\zeta}|),$$
$$\left(|\xi-a^2\sqrt{\sqrt{\zeta}}|\right),\left(|a\sqrt{\xi}-a^2\sqrt{\sqrt{\zeta}}|\right),\left(a\sqrt{\zeta}-a^2\sqrt{\sqrt{\zeta}}\right)\Big\}.$$

Thus,

$$M(\zeta,\xi)\geq\xi-\zeta\geq a(\zeta-\xi).$$

Since,

$$a(\xi-\zeta)=\sqrt{a}\sqrt{a}(\zeta-\xi).$$

$$\Rightarrow \varphi(M(\zeta,\xi))\geq \varphi(\eth(\zeta,\xi))=e^{(\zeta-\xi)}.$$

Thus,

$$\begin{aligned}&\varphi(\eth(\zeta,\xi))^{\Omega(\eth(\zeta,\mathcal{S}\zeta),\eth(\xi,\mathcal{S}\xi),\eth(\zeta,\mathcal{S}\xi),\eth(\xi,\mathcal{S}\zeta),\eth(\mathcal{S}^2\zeta,\xi))}\\ =&e^{\sqrt{a}(\zeta-\xi)}\\ =&e^{\sqrt{a}(\sqrt{\zeta}-\sqrt{\xi})(\sqrt{\zeta}+\sqrt{\xi})},\end{aligned}$$

$$\varphi(\eth(\mathcal{S}\zeta,\mathcal{S}\xi))=e^{a(\sqrt{\zeta}-\sqrt{\xi})}.$$

As $\zeta,\xi\in\,]0,\infty[$

$$e^{a(\sqrt{\zeta}-\sqrt{\xi})}\leq e^{\sqrt{a}(\sqrt{\zeta}-\sqrt{\xi})(\sqrt{\zeta}+\sqrt{\xi})}.$$

Thus, $\varphi(\eth(\mathcal{S}\zeta,\mathcal{S}\xi))\leq\varphi(\eth(\zeta,\xi))^{\Omega(\eth(\zeta,\mathcal{S}\zeta),\eth(\xi,\mathcal{S}\xi),\eth(\zeta,\mathcal{S}\xi),\eth(\xi,\mathcal{S}\zeta),\eth(\mathcal{S}^2\zeta,\xi))}$, where $\varphi\in\Theta_{\mathcal{W}}\cap\Theta_{\mathcal{G}}$. Hence, conditions (3.1) and (3.20) are satisfied. Therefore, $\mathcal{S}$ has a UFP $\Psi=\frac{1}{9}$.

3.3 Applications to Non-linear Integral Equations

This section establishes the existence and uniqueness of solutions for integral equations of Fredholm type by applying Theorem (3.2.1) and (3.2.6).

$$\text{Let } \zeta(\mathsf{t})=\mathsf{v}^*\int_{\flat_1}^{\flat_2}\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))d\mathsf{i}, \tag{3.27}$$

where $\flat_1,\flat_2\in\mathbb{R}$, $\zeta\in\mathcal{C}([\flat_1,\flat_2],\mathbb{R})$, and $\mathsf{g}:[\flat_1,\flat_2]^2\times\mathbb{R}\to\mathbb{R}$ are continuous functions and v^* represents a constant influenced by parameters $\flat_1$ and $\flat_2$.

Theorem 3.3.1. Suppose the function $\mathsf{g}:[\flat_1,\flat_2]^2\times\mathbb{R}\to\mathbb{R}$ is $s.t$

$$|\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))-\mathsf{g}(\mathsf{t},\mathsf{i},\xi(\mathsf{i}))|\leq|\zeta(\mathsf{t})-\xi(\mathsf{t})|\;\forall\;\mathsf{t},\mathsf{i}\in\mathbb{R} \tag{3.28}$$

and $\zeta,\xi\in\mathcal{C}([\flat_1,\flat_2],\mathbb{R})$.

Then, $\zeta\in\mathcal{C}([\flat_1,\flat_2],\mathbb{R})$ is the unique solution of (3.27) with $|\mathsf{v}^*|\leq\frac{\flat_1}{\flat_2}$.

Proof. Let $\mathcal{W} = \mathcal{C}([\flat_1, \flat_2], \mathbb{R})$ and $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ defined by

$$\mathcal{S}(\zeta)(t) = v^* \int_{\flat_1}^{\flat_2} g(t, i, \zeta(i) di. \ \forall \ \zeta \in \mathcal{W}, \tag{3.29}$$

and $\varphi(t) = e^t$.

Define $\eth_g : \mathcal{W} \times \mathcal{W} \to [0, +\infty[$ by

$$\eth_g(\zeta, \xi) = \left(\max_{t \in [\flat_1, \flat_2]} |\zeta(t) - \xi(t)| \right).$$

Then, $(\mathcal{W}, \eth_g)$ is a complete GMS.

Suppose that $\zeta, \xi \in \mathcal{W}$ and $t, i \in [\flat_1, \flat_2]$.

Then

$$\begin{aligned} |\mathcal{S}\zeta(t) - \mathcal{S}\xi(t)| &= |v^*| \left(\left| \int_{\flat_1}^{\flat_2} g(t, i, \zeta(i)) di - \int_{\flat_1}^{\flat_2} g(t, i, \xi(i)) di \right| \right) \\ &= |v^*| \left| \int_{\flat_1}^{\flat_2} g(t, i, \zeta(i)) di - g(t, i, \xi(i)) di \right| \\ &\leq |v^*| \int_{\flat_1}^{\flat_2} |g(t, i, \zeta(i)) - g(t, i, \xi(i))| di. \end{aligned}$$

By (3.28)

$$\begin{aligned} |\mathcal{S}\zeta(t) - \mathcal{S}\xi(t)| &\leq |v^*| \int_{\flat_1}^{\flat_2} (|\zeta(i) - \xi(i)|) di \\ &= |v^*| \left[\int_{\flat_1}^{\flat_2} (|\zeta(i)| - |\xi(i)|) di \right]. \end{aligned} \tag{3.30}$$

From (3.30)

$$\begin{aligned} \max_{t \in [\flat_1, \flat_2]} (|\mathcal{S}\zeta(t) - \mathcal{S}\xi(t)|) &= \max_{t \in [\flat_1, \flat_2]} |v^*| \int_{\flat_1}^{\flat_2} |g(t, i, \zeta(i)) - g(t, i, \xi(i)) di| \\ &\leq \max_{t \in [\flat_1, \flat_2]} |v^*| \int_{\flat_1}^{\flat_2} (|\zeta(i)| - |\xi(i)|) di \\ &\leq |v^*| \int_{\flat_1}^{\flat_2} \left(\max_{i \in [\flat_1, \flat_2]} |\zeta(i)| - |\xi(i)| \right) di. \end{aligned} \tag{3.31}$$

As $\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi) > 0$ and also $\eth_g(\zeta,\xi) > 0$ for any $\zeta \neq \xi$, then from (3.31)

$$e^{[\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)]} = e^{\left[|\mathsf{v}^*| \max\limits_{\mathsf{t}\in[\flat_1,\flat_2]} \int_{\flat_1}^{\flat_2} |\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i})) - \mathsf{g}(\mathsf{t},\mathsf{i},\xi(\mathsf{i}))d\mathsf{i}|\right]}.$$

As $|\mathsf{v}^*| \leq \frac{\flat_1}{\flat_2}$, the above equation becomes

$$e^{[\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)]} \leq e^{\left[\frac{\flat_1}{\flat_2}\int_{\flat_1}^{\flat_2}\left(\left(\max\limits_{\mathsf{i}\in[\flat_1,\flat_2]} |\zeta(\mathsf{i})| - |\xi(\mathsf{i})|\right)d\mathsf{i}\right)\right]}.$$

Thus,

$$\varphi(\eth_g(\mathcal{S}\zeta,\mathcal{S}\xi)) \leq [\varphi(M(\zeta,\xi))]^{\Omega},$$

$\forall\ \zeta,\xi \in \mathcal{W}$ and $\Omega(\mathsf{t}) = \left|\frac{\flat_1}{\flat_2}\right|$. Then, $\mathcal{S}$ satisfies (3.1) and (3.20). Thus $\mathcal{S}$ has a UFP which is a solution of (3.31).

□

Chapter 4

FP Results for (Γ, ν)-φ-Ω-Contraction in Generalized bMSs

4.1 Introduction

In this chapter, a new category of contraction mapping known as (Γ, ν)-φ-Ω-contraction in GbMS has been introduced. To generalize the notion of a (Γ, ν)-φ-Ω-contraction, these mappings utilize two auxiliary functions φ and Ω and parameters (Γ, ν) . The introduction of these two functions and parameters gives us a more flexible approach to establish the existence and uniqueness of FPs.

Throughout this chapter we will consider the distance $\eth_{gb}$ continuous.

Definition 4.1.1. Let $(\mathcal{W}, \eth_{gb})$ be a (Γ, ν)-complete GbMS, and let $\mathcal{S}$ be a self-mapping on $\mathcal{W}$, where $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ are two functions. Then $\mathcal{S}$ is a (Γ, ν)-φ-Ω-contraction, if $\forall\ \zeta, \xi \in \mathcal{W}$ with $(\Gamma(\zeta, \xi) \geq 1$ and $\nu(\zeta, \xi) \leq 1)$ and $\eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi) > 0$, we have

$$\varphi[s^3 \eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi))] \leq [\varphi(M(\zeta, \xi))]^{\Omega(\eth_{gb}(\zeta, \mathcal{S}\zeta), \eth_{gb}(\xi, \mathcal{S}\xi), \eth_{gb}(\zeta, \xi), \eth_{gb}(\xi, \mathcal{S}\zeta), \eth_{gb}(\mathcal{S}^2\zeta, \xi))}, \quad (4.1)$$

where $\varphi \in \Theta_{\mathcal{W}}$, $\Omega \in \sigma$, and

$$\begin{aligned} M(\zeta,\xi) = \max\{&\eth_{gb}(\zeta,\xi), \eth_{gb}(\zeta,\mathcal{S}\zeta), \eth_{gb}(\xi,\mathcal{S}\xi), \eth_{gb}(\mathcal{S}\zeta,\xi), \\ &\eth_{gb}(\mathcal{S}^2\zeta,\xi), \eth_{gb}(\mathcal{S}^2\zeta,\mathcal{S}\xi), \eth_{gb}(\mathcal{S}^2\zeta,\mathcal{S}\zeta)\}. \end{aligned} \tag{4.2}$$

Example 4.1.2. Let $\mathcal{W} = (0,+\infty)$ and $\varphi(\jmath) = e^{\jmath}$ $\forall$ $\jmath \in \,]0,+\infty[\,.$
Then, $\varphi \in \Theta_{\mathcal{W}}$.
Define $\eth_{gb} : \mathcal{W} \times \mathcal{W} \to [0,+\infty)$ by

$$\eth_{gb}(\zeta,\xi) = (\xi - \zeta)^2.$$

Then, $(\mathcal{W}, \eth_{gb})$ is a complete GbMS with $s = 3$.

Define $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ by

$$\begin{aligned} &\mathcal{S}(\jmath) = \frac{\sqrt{\jmath}}{2} \ \ \forall \ \ \jmath \in (0,+\infty)\,, \\ &\Gamma(\zeta,\xi) = 1, \\ &\nu(\zeta,\xi) = 1, \end{aligned}$$

$$\Omega(\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5) = \frac{\min\{\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\}}{\max\{\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\}+1} \ \ \forall \ \ \jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5 \in \mathbb{R}_+.$$

Then $\mathcal{S}$ is a (Γ,ν)-continuous and triangular (Γ,ν)-admissible mapping.
$\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi) = \Big(\frac{\sqrt{\xi}}{2} - \frac{\sqrt{\zeta}}{2}\Big)^2 = \frac{(\sqrt{\xi}-\sqrt{\zeta})^2}{4}$,

$$\begin{aligned} &M(\eth_{gb}(\zeta,\xi)) \\ &= \max\left\{\eth_{gb}(\zeta,\xi), \eth_{gb}\Big(\zeta,\frac{\sqrt{\zeta}}{2}\Big), \eth_{gb}\Big(\xi,\frac{\sqrt{\xi}}{2}\Big), \eth_{gb}\Big(\xi,\frac{\sqrt{\zeta}}{2}\Big)\right. \\ &\left.\eth_{gb}\left(\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}},\xi\right), \eth_{gb}\left(\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}},\frac{\sqrt{\xi}}{2}\right), \eth_{gb}\left(\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}},\frac{\sqrt{\zeta}}{2}\right)\right\}. \end{aligned}$$

$$\begin{aligned} M(\zeta,\xi) =\max&\left\{\Big(\xi-\zeta\Big)^2, \Big(\zeta-\frac{\sqrt{\zeta}}{2}\Big)^2, \Big(\xi-\frac{\sqrt{\xi}}{2}\Big)^2, (\xi-\frac{\sqrt{\zeta}}{2})^2\right. \\ &\left.\left(\xi-\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}}\right)^2, \left(\frac{\sqrt{\xi}}{2}-\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}}\right)^2, \left(\frac{\sqrt{\zeta}}{2}-\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}}\right)^2\right\}. \end{aligned}$$

Thus,

$$M(\zeta, \xi) \geq (\xi - \zeta)^2,$$

$$\Rightarrow \varphi(M(\zeta, \xi)) \geq \varphi(\eth_{gb}(\zeta, \xi)) = e^{(\xi-\zeta)^2}.$$

Thus,

$$\begin{aligned}
&\varphi(\eth_{gb}(\zeta, \xi))^{\Omega(\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\xi,\mathcal{S}\xi),\eth_{gb}(\zeta,\mathcal{S}\xi),\eth_{gb}(\xi,\mathcal{S}\zeta),\eth_{gb}(\mathcal{S}^2\zeta,\xi))}\\
=&e^{(\xi-\zeta)^2\Omega\left(\zeta-\frac{\sqrt{\zeta}}{2}\right)^2,\left(\xi-\frac{\sqrt{\xi}}{2}\right)^2,\left(\zeta-\frac{\sqrt{\xi}}{2}\right)^2,\left(\xi-\frac{\sqrt{\zeta}}{2}\right)^2,\left(\xi-\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}}\right)^2}\\
=&e^{(\xi-\zeta)^2\left(\left\{\frac{\min\left(\zeta-\frac{\sqrt{\zeta}}{2}\right)^2,\left(\xi-\frac{\sqrt{\xi}}{2}\right)^2,\left(\zeta-\frac{\sqrt{\xi}}{2}\right)^2,\left(\xi-\frac{\sqrt{\zeta}}{2}\right)^2,\left(\xi-\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}}\right)^2}{\max\left(\zeta-\frac{\sqrt{\zeta}}{2}\right)^2,\left(\xi-\frac{\sqrt{\xi}}{2}\right)^2,\left(\zeta-\frac{\sqrt{\xi}}{2}\right)^2,\left(\xi-\frac{\sqrt{\zeta}}{2}\right)^2,\left(\xi-\frac{\sqrt{\sqrt{\zeta}}}{2\sqrt{2}}\right)^2}\right\}+1\right)}\\
\leq&e^{(\xi-\zeta)^2}.
\end{aligned}$$

Now

$$\varphi(\eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi)) = e^{\frac{(\sqrt{\xi}-\sqrt{\zeta})^2}{4}}.$$

As

$$\xi - \zeta = (\sqrt{\xi} - \sqrt{\zeta})(\sqrt{\xi} + \sqrt{\zeta}).$$

Solving for square root difference

$$|\sqrt{\xi} - \sqrt{\zeta}| = \frac{|\xi - \zeta|}{\sqrt{\xi} + \sqrt{\zeta}},$$

since

$$\sqrt{\xi} > 0, \sqrt{\zeta} > 0$$

$$\Rightarrow \sqrt{\xi} + \sqrt{\zeta} > 0,$$

$\Rightarrow \sqrt{\xi} + \sqrt{\zeta}$ is strictly larger than 1. Hence

$$\begin{aligned}
|\sqrt{\xi} - \sqrt{\zeta}| &< |\xi - \zeta|\\
\Rightarrow (\sqrt{\xi} - \sqrt{\zeta})^2 &< (\xi - \zeta)^2.
\end{aligned}$$

Since $\frac{1}{4} < 1$

$$\Rightarrow \frac{1}{4}(\sqrt{\xi} - \sqrt{\zeta})^2 < \frac{1}{4}(\xi - \zeta)^2 < (\xi - \zeta)^2,$$

with $s=3$, it can be concluded that

$$\begin{aligned}\frac{1}{4}s^3\Big(\zeta+\xi-2\sqrt{\zeta\xi}\Big) &\leq(\zeta^2+\xi^2-2\zeta\xi)\\ \Rightarrow e^{\frac{1}{4}s^3\Big(\zeta+\xi-2\sqrt{\zeta\xi}\Big)} &\leq e^{(\zeta^2+\xi^2-2\zeta\xi)}.\end{aligned}$$

Thus,

$$\varphi(s^3\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi))\leq\varphi(M(\eth_{gb}(\zeta,\xi)))^{\Omega(\eth_{gb}(\zeta,\xi)\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\xi,\mathcal{S}\xi),\eth_{gb}(\xi,\mathcal{S}\zeta),\eth_{gb}(\mathcal{S}^2\zeta,\xi)}.$$

Hence, $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

4.2 Main Results

Theorem 4.2.1. Let $(\mathcal{W},\eth_{gb})$ be a (Γ,ν)-complete GbMS, and $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a mapping, that fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping .

(ii) : $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

(iii) : $\exists\ \zeta_0\in\mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$.

(iv) : $\mathcal{S}$ is a (Γ,ν)-continuous.

Then $\mathcal{S}$ has a FP. Also, $\mathcal{S}$ has a UFP when $\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1\ \ \forall$ FPs $\zeta,\xi\in\mathcal{W}$.

Proof. Let $\zeta_0\in\mathcal{W}\ \ s.t$

$$\Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1 \text{ and } \nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$$

. Define a sequence $\{\zeta_m\}$ by $\zeta_m=\mathcal{S}^m\zeta_0=\mathcal{S}\zeta_{m-1}$. Then by (iii)

$$\Gamma(\zeta_0,\zeta_1)=\Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1,$$

since $\mathcal{S}$ is a triangular (Γ,ν) admissible mapping, therefore

$$\begin{aligned}&\Gamma(\mathcal{S}\zeta_0,\mathcal{S}\zeta_1)\geq 1 \text{ i.e}\\&\Gamma(\zeta_1,\zeta_2)\geq 1,\end{aligned}$$

similarly

$$\begin{aligned}&\nu(\zeta_0,\zeta_1)=\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1\\&\Rightarrow \nu(\mathcal{S}\zeta_0,\mathcal{S}\zeta_1)\leq 1\\&\Rightarrow \nu(\zeta_1,\zeta_2)\leq 1.\end{aligned}$$

Proceeding this way, we have

$$\Gamma(\zeta_{m-1},\zeta_m)\geq 1 \text{ and } \nu(\zeta_{m-1},\zeta_m)\leq 1,\ \forall\ n\in\mathbb{N}.$$

By $\mathcal{S}_3$ and $\mathcal{S}_4$,

$$\Gamma(\zeta_m,\zeta_n)\geq 1 \text{ and } \nu(\zeta_m,\zeta_n)\leq 1,\quad\forall\quad m,n\in\mathbb{N}, m\neq n. \tag{4.3}$$

Let $\zeta_n\neq\mathcal{S}\zeta_n$, i.e, $\eth_{gb}(\zeta_{n-1},\zeta_n)>0\quad\forall\quad n\in\mathbb{N}$.
Also

$$\zeta_n\neq\zeta_m,\quad\forall\quad m,n\in\mathbb{N}, m\neq n. \tag{4.4}$$

Indeed if $\zeta_n=\zeta_m$ for some $m=n+k>n$, then

$$\zeta_{n+1}=\mathcal{S}\zeta_n=\mathcal{S}\zeta_m=\zeta_{m+1}.$$

Express $\eth_{gb_m}=\eth_{gb}(\zeta_m,\zeta_{m+1})$,

$\Rightarrow \varphi(\eth_{gb_n})=\varphi(\eth_{gb_m})$. Then, (4.1) implies that

$$\begin{aligned}&\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_{m-1},\mathcal{S}\zeta_m))\\&\leq[\varphi(M(\zeta_{m-1},\zeta_m))]^{\Omega(\eth_{gb}(\zeta_{m-1},\zeta_m),\eth_{gb}(\zeta_m,\zeta_{m+1}),\eth_{gb}(\zeta_{m-1},\zeta_m),\eth_{gb}(\zeta_m,\zeta_m),\eth_{gb}(\zeta_{m+1},\zeta_m))} \quad (4.5)\\&=\varphi(M(\zeta_{m-1},\zeta_m))]^{\Omega(\eth_{gb_{m-1}},\,\eth_{gb_{m-1}},\,\eth_{gb_m},\,0,\,\eth_{gb_{m+1}})},\end{aligned}$$

here

$$\begin{aligned}M(\zeta_{m-1},\zeta_m) =&\max\{\eth_{gb}(\zeta_{m-1},\zeta_m),\eth_{gb}(\zeta_{m-1},\zeta_m),\eth_{gb}(\zeta_m,\zeta_{m+1})\\ &\eth_{gb}(\zeta_m,\zeta_m),\eth_{gb}(\zeta_{m+1},\zeta_m),\eth_{gb}(\zeta_{m+1},\zeta_{m+1}),\eth_{gb}(\zeta_{m+1},\zeta_m)\}\\ =&\max\{\eth_{gb}(\zeta_{m-1},\zeta_m),\eth_{gb}(\zeta_m,\zeta_{m+1})\}.\end{aligned}$$

Since $\eth_{gb_{m-1}}.\eth_{gb_m}.0.\eth_{gb_{m+1}} = 0$

which implies that $\exists \ \ \gamma \in]0,1[\ \ s.t$

$$\Omega(\eth_{gb_{m-1}},\eth_{gb_{m-1}},\eth_{gb_m},0,\eth_{gb_{m+1}}) = \gamma.$$

Thus, (4.5) becomes

$$\varphi(s^3\eth_{gb_n}) \leq [\varphi(M(\zeta_{m-1},\zeta_m))]^{\gamma}. \tag{4.6}$$

Suppose that

$$M(\zeta_{m-1},\zeta_m) = \eth_{gb}(\zeta_m,\zeta_{m+1}).$$

Then, by (4.6), we have

$$\varphi(s^3\eth_{gb_m}) \leq [\varphi(\eth_{gb_m})]^{\gamma} < \varphi(\eth_{gb_m}),$$

which is not possible, therefore

$$M(\zeta_{m-1},\zeta_m) = \eth_{gb}(\zeta_{m-1},\zeta_m).$$

Since $\zeta_n = \zeta_m$

$$\Rightarrow \eth_{gb_n} = \eth_{gb_m} < \eth_{gb_{m-1}}.$$

Continuing this process, we get $\eth_{gb_n} = \eth_{gb_m} < \eth_{gb_{m-1}} < \eth_{gb_{m-2}} < ... < \eth_{gb_n}$, which is a contradiction.

Thus, (4.4) holds.

Substituting $\zeta = \zeta_{n-1}$ and $\xi = \zeta_n$ in (4.1), $\forall \ \ n \in \mathbb{N}$.

$$\begin{aligned}&\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_{n-1},\mathcal{S}\zeta_n)\\ &=\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1}))\end{aligned}$$

$$\leq [\varphi(M(\zeta_{n-1},\zeta_n))]^{\Omega(\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_n,\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_n))}$$
$$= [\varphi(M(\zeta_{n-1},\zeta_n))]^{\Omega(\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),0,\eth_{gb}(\zeta_{n+1},\zeta_n))}.$$

Now by definition (3.1.1) $\exists\ \gamma \in (0,1)\ \ s.t$

$$\Omega[\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),0,\eth_{gb}(\zeta_{n+1},\zeta_n)] = \gamma.$$

Now

$$\begin{aligned} M(\zeta_{n-1},\zeta_n) &= \max\{\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_n,\zeta_{n+1}),\\ &\quad \eth_{gb}(\zeta_{n+1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_{n+1}),\eth_{gb}(\zeta_{n+1},\zeta_n)\}\\ &= \max\{\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_n,\zeta_{n+1})\}. \end{aligned}$$

Suppose that

$$M(\zeta_{n-1},\zeta_{n+1}) = \eth_{gb}(\zeta_n,\zeta_{n+1}).$$

In that case,

$$\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) \leq [\varphi(\eth_{gb}(\zeta_n,\zeta_{n+1})]^{\gamma} < \varphi(\eth_{gb}(\zeta_n,\zeta_{n+1})).$$

which contradicts our assumption.

Therefore,

$$M(\zeta_{n-1},\zeta_{n+1}) = \eth_{gb}(\zeta_{n-1},\zeta_n).$$

Hence

$$\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) \leq [\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n)]^{\gamma} < \varphi(\eth_{gb}(\zeta_{n-1},\zeta_n)).$$

Since φ is continuous,

$$\Rightarrow \eth_{gb}(\zeta_n,\zeta_{n+1}) < \eth_{gb}(\zeta_{n-1},\zeta_n). \tag{4.7}$$

Thus, $\eth_g(\zeta_n,\zeta_{n+1})_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers that decreases strictly. Therefore, $\exists\ \lambda_1 \geq 0\ \ s.t$

$$\lim_{n\to\infty}\eth_{gb}(\zeta_{n+1},\zeta_n) = \lambda_1.$$

Now, we claim that $\lambda_1 = 0$. Assuming on contrary, suppose that λ_1 is greater than 0.

Given that the sequence $\eth_{gb}(\zeta_n,\zeta_{n+1})$ is non-negative and strictly decreasing $\forall\ n$, we can conclude that

$\eth_{gb}(\zeta_n,\zeta_{n+1}) \geq \lambda_1\ \ \forall\ \ n\in\mathbb{N}$, that is

$$\begin{aligned}\eth_{gb}(\zeta_n,\zeta_{n+1}) &\geq \lim_{n\to\infty}\eth_{gb}(\zeta_{n+1},\zeta_n)\\ \Rightarrow \varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) &\geq \lim_{n\to\infty}\varphi(\eth_{gb}(\zeta_{n+1},\zeta_n)).\end{aligned}$$

Using property of φ

$$\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) > 1.$$

Consider $\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1}))$ and applying contraction condition

$$\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) \leq [\varphi(M(\zeta_{n-1},\zeta_n)]^\gamma.$$

As $M(\zeta_{n-1},\zeta_n) = \eth_{gb}(\zeta_{n-1},\zeta_n)$

$$\Rightarrow \varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) \leq [\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n))]^\gamma. \tag{4.8}$$

Now consider

$$\begin{aligned}\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n))]^\gamma \leq& \varphi(\eth_{gb}(\zeta_{n-1},\zeta_n))]\\ \leq& \varphi(s^3\eth_{gb}(\zeta_{n-1},\zeta_n)) \leq [\varphi(\eth_{gb}(\zeta_{n-2},\zeta_{n-1}))]^\gamma.\end{aligned}$$

Hence

$$\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n)) \leq [\varphi(\eth_{gb}(\zeta_{n-2},\zeta_{n-1}))]^\gamma.$$

Now (4.8) gives

$$\begin{aligned}\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) \leq& [\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n))]^\gamma\\ \leq& \varphi(\eth_{gb}(\zeta_{n-2},\zeta_{n-1}))]^{\gamma^2}.\end{aligned}$$

Proceeding this way, it can be concluded that

$$1 < \varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1})) \leq \varphi(\eth_{gb}(\zeta_0,\zeta_1))^{\gamma^n}.$$

Taking $\lim n \to \infty$ in above inequality, we get

$$1 < \varphi(s^3\lambda_1) \leq 1,$$

a contradiction. Therefore,

$$\lim_{n\to\infty} \eth_{gb}(\zeta_n,\zeta_{n+1}) = 0. \tag{4.9}$$

Now substituting $\zeta = \zeta_{n-1}$ and $\xi = \zeta_{n+1}$ in (4.1), $\forall \;\; n \in \mathbb{N}$, we have

$$\begin{aligned}
&\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+2}))\\
\leq&[\varphi(M(\zeta_{n-1},\zeta_{n+1}))]^{\Omega(\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_{n+2}),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\zeta_{n+1},\zeta_{n+1}))}\\
=&[\varphi(M(\zeta_{n-1},\zeta_n))]^{\Omega(\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_{n+2}),\eth_{gb}(\zeta_n,\zeta_{n+1}),0))},
\end{aligned}$$

here

$$\begin{aligned}
&M(\zeta_{n-1},\zeta_{n+1}) = \max\{\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_{n+2}),\\
&\eth_{gb}(\zeta_{n+1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_{n+1}),\eth_{gb}(\zeta_{n+1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_n)\}.
\end{aligned}$$

Since

$$\eth_{gb}(\zeta_{n+1},\zeta_{n+2}) \leq \eth_{gb}(\zeta_n,\zeta_{n+1}) \leq \eth_{gb}(\zeta_{n-1},\zeta_n),$$

$$\Rightarrow M(\zeta_{n-1},\zeta_{n+1}) = \max\{\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_n)\},$$

and $\exists \;\; \gamma \in (0,1) \;\; s.t$

$$\Omega(\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_n),\eth_{gb}(\zeta_{n+1},\zeta_{n+2}),\eth_{gb}(\zeta_n,\zeta_{n+1}),0) = \gamma.$$

Then,

$$\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+2}) \leq [\varphi(\max\{\eth_{gb}(\zeta_{n-1},\zeta_{n+1}),\eth_{gb}(\zeta_{n-1},\zeta_n)\})]^{\gamma}. \tag{4.10}$$

Take $\mathsf{a}^*_\mathsf{n} = \eth_{gb}(\zeta_n, \zeta_{n+2})$ and $\mathsf{b}^*_\mathsf{n} = \eth_{gb}(\zeta_n, \zeta_{n+1})$.

Thus, (4.10) can be written as:

$$\varphi(s^3\mathsf{a}^*_\mathsf{n}) \leq [\varphi(\max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\})]^\gamma$$

$$\Rightarrow \varphi(\mathsf{a}^*_\mathsf{n}) < \varphi(s^3\mathsf{a}^*_\mathsf{n}) \leq [\varphi(\max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\})]^\gamma,$$

using (φ_1)

$$\mathsf{a}^*_\mathsf{n} < \max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\}.$$

By (4.7), we have

$$\mathsf{b}^*_\mathsf{n} \leq \mathsf{b}^*_{\mathsf{n}-1} \leq \max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\},$$

which implies that

$$\max\{\mathsf{a}^*_\mathsf{n}, \mathsf{b}^*_\mathsf{n}\} \leq \max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\}, \quad \forall \;\; n \in \mathbb{N}.$$

Therefore, the sequence $\max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\}_{n\in\mathbb{N}}$ forms a decreasing sequence of non-negative real numbers.

Thus, $\exists \;\; \lambda_2 \geq 0 \;\; s.t$

$$\lim_{n\to\infty} \max\{\mathsf{a}^*_\mathsf{n}, \mathsf{b}^*_\mathsf{n}\} = \lambda_2.$$

We assume that $\lambda_2 > 0$. By (4.9) and

$$\lim_{n\to\infty} \sup \mathsf{b}^*_\mathsf{n} = \lim_{n\to\infty} \sup \eth_{gb}(\zeta_{n-1}, \zeta_n) = 0,$$

$$\begin{aligned}\lim_{n\to\infty} \sup \mathsf{a}^*_\mathsf{n} &= \lim_{n\to\infty} \sup \max\{\mathsf{a}^*_\mathsf{n}, \mathsf{b}^*_\mathsf{n}\} \\ &= \lim_{n\to\infty} \max\{\mathsf{a}^*_\mathsf{n}, \mathsf{b}^*_\mathsf{n}\}.\end{aligned}$$

Taking the $\lim\sup n \to \infty$ in (4.10), and using the properties of φ, we get

$$\begin{aligned}\varphi(s^3 \lim_{n\to\infty} \sup \mathsf{a}^*_\mathsf{n}) <&\varphi(\lim_{n\to\infty} \sup \max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\}) \\ =&\varphi(\lim_{n\to\infty} \max\{\mathsf{a}^*_{\mathsf{n}-1}, \mathsf{b}^*_{\mathsf{n}-1}\}) \\ \Rightarrow \varphi(s^3\lambda_2) <&\varphi(\lambda_2),\end{aligned}$$

which is a contraction.

Therefore,

$$\lim_{n\to\infty} \eth_{gb}(\zeta_n, \zeta_{n+2}) = 0. \tag{4.11}$$

Now, we aim to show that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is a CS.

To argue by contradiction, assume that this is not the case.

Specifically, suppose $\exists$ a non-negative real number $\epsilon > 0$ and two sequences of natural numbers, $\{n_{(k)}\}$ and $\{m_{(k)}\}$, $s.t\ \ m_{(k)} > n_{(k)} > k$ and,

$$\eth_{gb}(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}) \geq \epsilon$$

and

$$\eth_{gb}(\zeta_{m_{(k)}-1}, \zeta_{n_{(k)}-1}) < \epsilon.$$

Now, using (4.7), (4.10) , (4.11) and the quadrilateral inequality

$$\begin{aligned}
\epsilon \leq & \eth_{gb}(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}) \\
\leq & s[\eth_{gb}(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}) + \eth_{gb}(\zeta_{m_{(k)+1}}, \zeta_{m_{(k)}-1}) + \eth_{gb}(\zeta_{m_{(k)}-1}, \zeta_{n_{(k)}})] \\
\leq & s[\eth_{gb}(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}) + \eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{m_{(k)}-1}) + \epsilon].
\end{aligned}$$

Taking $\lim\limits_{k\to\infty}$ in the above inequality, it gives

$$\begin{aligned}
& \lim_{k\to\infty} \eth_{gb}(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}) \\
\leq & s[\lim_{k\to\infty} (\eth_{gb}(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}) + \eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{m_{(k)}-1}) + \epsilon)] \\
= & s(0 + 0 + \epsilon)
\end{aligned}$$

$$\Rightarrow \lim_{k\to\infty} \eth_{gb}(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}) \leq s\epsilon.$$

Again applying quadrilateral inequality,

$$\eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{n_{(k)}}) \leq s[\eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{m_{(k)}-1}) + \eth_{gb}(\zeta_{m_{(k)}-1}), \zeta_{m_{(k)}}) + \eth_{gb}(\zeta_{m_{(k)}}, \zeta_{n_{(k)}})].$$

When $k\to\infty$ the above inequality becomes

$$\begin{aligned}\lim_{k\to\infty}\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}) \leq & s[\lim_{k\to\infty}(\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{m_{(k)}-1})+\eth_{gb}(\zeta_{m_{(k)}-1)},\zeta_{m_{(k)}})\\ &+\eth_{gb}(\zeta_{m_{(k)}},\zeta_{n_{(k)}}))]\\ \leq & s(0+0+s\epsilon)\\ \Rightarrow \lim_{k\to\infty}\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}) \leq & s^2\epsilon.\end{aligned}$$

Similarly, we can get

$$\lim_{k\to\infty}\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}+1})\leq s^2\epsilon.$$

Using quadrilateral inequality once more, one can obtain

$$\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}})\leq \eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{m_{(k)}})+\eth_{gb}(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1})+\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}).$$

Letting $k\to\infty$ in the above inequality,

$$\begin{aligned}&\lim_{k\to\infty}\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}})\\ \leq & s[\lim_{k\to\infty}(\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{m_{(k)}})+\eth_{gb}(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1})+\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}))]\\ \leq & s(0+0+s^2\epsilon)\\ \leq & s^3\epsilon.\end{aligned}$$

Similarly

$$\lim_{k\to\infty}\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}+1})\leq s^3\epsilon.$$

From (4.1) and taking $\zeta=\zeta_{m_{(k)}}$ and $\xi=\zeta_{n_{(k)}}$

$$\begin{aligned}&M(\zeta_{m_{(k)}},\zeta_{n_{(k)}})\\ &=\max\{\eth_{gb}(\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_{gb}(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1}),\eth_{gb}(\zeta_{n_{(k)}},\zeta_{n_{(k)}+1}),\\ &\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{m_{(k)}+1}),\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}}),\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}+1}),\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{m_{(k)}+1})\}.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}M(\zeta_{m_{(k)}},\zeta_{n_{(k)}})\leq\max\{\epsilon,0,0,s^3\epsilon,s^3\epsilon,s^3\epsilon,0\}=s^3\epsilon.$$

Using (4.1) with $\zeta = \zeta_{m_{(k)}}$ and $\xi = \zeta_{n_{(k)}}$, one can obtain

$$\begin{aligned}
&\varphi[s\eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1})]\\
\leq&\varphi[s^3\eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1})]\\
\leq&[\varphi(M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}))]^{\Omega[\eth_{gb}(\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_{gb}(\zeta_{m_{(k)}},\mathcal{S}\zeta_{m_{(k)}}),\eth_{gb}(\zeta_{n_{(k)}},\mathcal{S}\zeta_{n_{(k)}}),\eth_{gb}(\mathcal{S}\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_{gb}(\mathcal{S}^2\zeta_{m_{(k)}},\zeta_{n_{(k)}})}\\
=&[\varphi(M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}))]^{\Omega[\eth_{gb}(\zeta_{m_{(k)}},\zeta_{n_{(k)}}),\eth_{gb}(\zeta_{m_{(k)}},\zeta_{m_{(k)}+1}),\eth_{gb}(\zeta_{n_{(k)}},\zeta_{n_{(k)}+1}),\eth_{gb}(\zeta_{m_{(k)}+1},\zeta_{n_{(k)}}),\eth_{gb}(\zeta_{m_{(k)}+2},\zeta_{n_{(k)}})]}.
\end{aligned}$$

As Ω is a continuous function, therefore

$$\begin{aligned}
&\lim_{k\to\infty} \Omega\Big[\eth_{gb}\Big(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}\Big), \eth_{gb}\Big(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}\Big), \eth_{gb}\Big(\zeta_{n_{(k)}}, \zeta_{n_{(k)}+1}\Big),\\
&\eth_{gb}\Big(\zeta_{m_{(k)}+1}, \zeta_{n_{(k)}}\Big), \eth_{gb}\Big(\zeta_{m_{(k)}+2}, \zeta_{n_{(k)}}\Big)\Big]
\end{aligned}$$

$$\begin{aligned}
=&\Omega\Big[\lim_{k\to\infty}\Big(\eth_{gb}\Big(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}\Big), \eth_{gb}\Big(\zeta_{m_{(k)}}, \zeta_{m_{(k)}+1}\Big), \eth_{gb}\Big(\zeta_{n_{(k)}}, \zeta_{n_{(k)}+1}\Big),\\
&\eth_{gb}\Big(\zeta_{m_{(k)}+1}, \zeta_{n_{(k)}}\Big), \eth_{gb}\Big(\zeta_{m_{(k)}+2}, \zeta_{n_{(k)}}\Big)\Big)\Big]
\end{aligned}$$

$$= \Omega[s\epsilon, 0, 0, s^2\epsilon, s^3\epsilon].$$

So, $\exists \;\; \gamma \in\,]0, 1[\;\; s.t \;\; \Omega[s\epsilon, 0, 0, s^2\epsilon, s^3\epsilon] = \gamma$.

Thus,

$$\lim_{k\to\infty} \varphi(s\eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1})) \leq [\lim_{k\to\infty} \varphi(M(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}))]^{\gamma}.$$

Using the continuity of φ in the above inequality, it becomes

$$\varphi\Big[s \lim_{k\to\infty}\Big(\eth_{gb}(\zeta_{m_{(k)}+1}, \zeta_{n(k)+1}\Big)\Big)\Big] \leq \Big[\varphi\Big(\lim_{k\to\infty} M\Big(\zeta_{m_{(k)}}, \zeta_{n_{(k)}}\Big)\Big)\Big]^{\gamma}$$

$$\Rightarrow \varphi[s(s^2\epsilon)] \leq [\varphi(s^3\epsilon)]^{\gamma}.$$

Hence,

$$\varphi(s^3\epsilon) \leq [\varphi(s^3\epsilon)]^{\gamma} < \varphi(s^3\epsilon),$$

which is a contradiction.

So,

$$\lim_{n,m\to\infty} \eth_{gb}(\zeta_m, \zeta_n) = 0.$$

Hence, $\{\zeta_n\}$ is a CS in $\mathcal{W}$. By using completeness property of $(\mathcal{W}, \eth_{gb})$, $\exists\ \Psi \in \mathcal{W}$ $s.t$

$$\lim_{n\to\infty} \eth_{gb}(\zeta_n, \Psi) = 0.$$

Now, we will show that

$$\eth_{gb}(\mathcal{S}\Psi, \Psi) = 0.$$

Suppose by contradiction

$$\eth_{gb}(\mathcal{S}\Psi, \Psi) > 0.$$

Now by quadrilateral inequality we get,

$$\eth_{gb}(\Psi, \mathcal{S}\Psi) \leq s[\eth_{gb}(\Psi, \zeta_n) + \eth_{gb}(\zeta_n, \mathcal{S}\zeta_n) + \eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi)]. \tag{4.12}$$

By letting $n \to \infty$ in inequality (4.12), we obtain

$$\begin{aligned}\lim_{n\to\infty}[\eth_{gb}(\Psi, \mathcal{S}\Psi)] \leq & s[\lim_{n\to\infty}(\eth_{gb}(\Psi, \zeta_n) + \eth_{gb}(\zeta_n, \mathcal{S}\zeta_n) + \eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi))] \\ =& s[\lim_{n\to\infty}(\eth_{gb}(\Psi, \zeta_n) + \eth_{gb}(\zeta_n, \zeta_{n+1}) + \eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi))] \\ =& s[\lim_{n\to\infty}(\eth_{gb}(\Psi, \zeta_n) + \eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi))] \\ =& s[0 + \lim_{n\to\infty}\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi)]\end{aligned}$$

$$\begin{aligned}&\Rightarrow \eth_{gb}(\Psi, \mathcal{S}\Psi) \leq \lim_{n\to\infty} s\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi) \leq \lim_{n\to\infty} s^3\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi) \\ &\Rightarrow \eth_{gb}(\Psi, \mathcal{S}\Psi) \leq \lim_{n\to\infty} s^3\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi).\end{aligned} \tag{4.13}$$

Consider

$$\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi),$$

and applying contraction condition

$$\begin{aligned}\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi) \leq& \varphi[M(\zeta_n, \Psi)]^\gamma \\ \Rightarrow \lim_{n\to\infty}\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi) \leq& \lim_{n\to\infty}\varphi[M(\zeta_n, \Psi)]^\gamma.\end{aligned}$$

As φ is continuous

$$\lim_{n\to\infty}\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n, \mathcal{S}\Psi) \leq \varphi[\lim_{n\to\infty} M(\zeta_n, \Psi)]^\gamma.$$

Here

$$M(\zeta_n,\Psi)=\max\{\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\zeta_n),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}\zeta_n,\Psi),$$
$$\eth_{gb}(\mathcal{S}^2\zeta_n,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^2\zeta_n,\Psi),\eth_{gb}(\mathcal{S}^2\zeta_n,\mathcal{S}\zeta_n)\}$$

$$=\max\{\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+1},\Psi),$$
$$\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi),\eth_{gb}(\zeta_{n+2},\zeta_{n+1})\}.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}M(\zeta_n,\Psi)&=\max\Big\{\lim_{n\to\infty}[\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+1},\Psi),\\&\qquad\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi),\eth_{gb}(\zeta_{n+2},\zeta_{n+1})]\Big\}\\&=\max\Big\{0,0,\eth_{gb}(\Psi,\mathcal{S}\Psi),0,\eth_{gb}(\Psi,\mathcal{S}\Psi),0,0\Big\}\\&=\eth_{gb}(\Psi,\mathcal{S}\Psi).\end{aligned}$$

Thus,

$$\lim_{n\to\infty}M(\zeta_n,\Psi)=\eth_{gb}(\Psi,\mathcal{S}\Psi),$$

and $\exists\ \gamma\in\,]0,1[\ \ s.t$

$$\lim_{n\to\infty}\Omega(\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi))=\gamma.$$

Hence

$$\begin{aligned}\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi)&\le\varphi[M(\zeta_n,\Psi)]^{\gamma}\\&\le\varphi[\eth_{gb}(\Psi,\mathcal{S}\Psi)]^{\gamma}<\eth_{gb}(\Psi,\mathcal{S}\Psi).\end{aligned}$$

As φ is increasing, so

$$(s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\le\eth_{gb}(\Psi,\mathcal{S}\Psi).$$

Taking $n\to\infty$, the above inequality becomes

$$\lim_{n\to\infty}s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\le\eth_{gb}(\Psi,\mathcal{S}\Psi).\tag{4.14}$$

Combining (4.13) and (4.14) ,we get

$$\begin{aligned}\eth_{gb}(\Psi,\mathcal{S}\Psi) \leq \lim_{n\to\infty} s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\\ \leq&\eth_{gb}(\Psi,\mathcal{S}\Psi).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi) = \eth_{gb}(\Psi,\mathcal{S}\Psi). \tag{4.15}$$

Given that $\zeta_n \to \Psi$ as $n \to \infty\ \ \forall\ \ n \in \mathbb{N}$ and since $\mathcal{S}$ is an (Γ,ν)-continuous, we conclude that

$$\lim_{n\to\infty} \mathcal{S}\zeta_n = \mathcal{S}\Psi$$

. Then,

$$\begin{aligned}&\lim_{n\to\infty} s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi)\\ &=s^3\eth_{gb}(\mathcal{S}\Psi,\mathcal{S}\Psi)\\ &=0\\ \Rightarrow \eth_{gb}(\mathcal{S}\Psi,\Psi)\ &=0.\end{aligned}$$

So

$\Psi = \mathcal{S}\Psi$. For uniqueness, now, suppose that $\Psi,\zeta \in \mathcal{W}$ are two FPs of $\mathcal{S}$ $s.t$ $\zeta \neq \Psi$. Therefore,

$$\eth_{gb}(\Psi,\zeta) = \eth_{gb}(\mathcal{S}\Psi,\mathcal{S}\zeta) > 0.$$

Applying (4.1) with $\zeta = \Psi$ and $\xi = \zeta$, we have

$$\begin{aligned}&\varphi(\eth_{gb}(\Psi,\zeta))\\ &= \varphi(\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\Psi))\\ &\leq \varphi(s^3\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\Psi))\\ &\leq [\varphi(M(\Psi,\zeta))]^{\Omega(\eth_{gb}(\Psi,\zeta),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\zeta,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^2\Psi,\zeta))}\\ &= [\varphi(M(\Psi,\zeta))]^{\Omega(\eth_{gb}(\Psi,\zeta),\eth_{gb}(\Psi,\Psi),\eth_{gb}(\zeta,\zeta),\eth_{gb}(\zeta,\Psi),\eth_{gb}(\Psi,\zeta))}\\ &= [\varphi(M(\Psi,\zeta))]^{\Omega(\eth_{gb}(\Psi,\zeta),0,0,\eth_{gb}(\zeta,\Psi),\eth_{gb}(\Psi,\zeta))}\\ &= [\varphi(M(\Psi,\zeta))]^{\gamma},\end{aligned}$$

here

$$\begin{aligned}M(\Psi,\zeta) &= \max\{\eth_{gb}(\Psi,\zeta),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\mathcal{S}\Psi,\zeta),\\ &\quad \eth_{gb}(\mathcal{S}^2\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^2\Psi,\zeta),\eth_{gb}(\mathcal{S}^2\Psi,\mathcal{S}\zeta)\}\\ &= \max\{\eth_{gb}(\Psi,\zeta),\eth_{gb}(\Psi,\Psi),\eth_{gb}(\zeta,\zeta),\eth_{gb}(\Psi,\zeta),\eth_{gb}(\Psi,\Psi),\eth_{gb}(\Psi,\zeta),\eth_{gb}(\Psi,\zeta)\}\\ &= \eth_{gb}(\Psi,\zeta).\end{aligned}$$

Therefore,

$$\varphi(\eth_{gb}(\mathcal{S}\Psi,\mathcal{S}\zeta)) \leq [\varphi(\eth_{gb}(\Psi,\zeta))]^{\gamma} < \varphi(\eth_{gb}(\Psi,\zeta)),$$

which implies that

$$\eth_{gb}(\Psi,\zeta) < \eth_{gb}(\Psi,\zeta),$$

this contradiction implies $\zeta = \Psi$, thereby concluding the proof. □

Corollary 4.2.2. Let $(\mathcal{W},\eth_{gb})$ be a (Γ,ν)-complete GbMS, and $\Gamma,\nu : \mathcal{W}\times\mathcal{W} \to [0,+\infty)$. Suppose $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ be a self-mapping which fulfils the following assertions:

(i) : $\varphi[s^3\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi)] \leq [\varphi(M(\zeta,\xi))]^{\mathsf{k}}, \mathsf{k} \in (0,1)$ and $\varphi \in \Theta_{\mathcal{G}}$,

(ii) : $\mathcal{S}$ is continuous.

Then, $\mathcal{S}$ has a UFP.

Proof. Define a function $\Omega(\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5) = \mathtt{k}\ \ \forall\ \ \jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5 \in \mathbb{R}_+$.
Clearly $\Omega \in \sigma$.
Taking

$$\Gamma(\zeta,\xi) = 1, \nu(\zeta,\xi) = 1.$$

Thus, $\mathcal{S}$ is a (Γ,ν)-φ-Ω- contraction, and $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping. Hence following Theorem (4.2.1), $\mathcal{S}$ has a UFP ζ. □

It is an evident that if ζ is a FP of $\mathcal{S}$ then it's also a FP of $\mathcal{S}^n$ for every natural number n. Then, a mapping $\mathcal{S}$ is said to have property P (or have no periodic points).

Theorem 4.2.3. Consider $(\mathcal{W}, \eth_{gb})$ be a (Γ, ν)-complete GbMS and let $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$ be two functions. $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ be a mapping which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ, ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ, ν)-φ-Ω-contraction.

(iii) : $\Gamma(\Psi, \mathcal{S}\Psi) \geq 1$ and $\nu(\Psi, \mathcal{S}\Psi) \leq 1, \ \forall \ \ \Psi \in Fix(\mathcal{S})$,

Then, $\mathcal{S}$ has the property p, that is
$\mathcal{S}^n\zeta = \mathcal{S}\zeta$.

Proof. Let for some fixed $n > 1$, let $\Psi \in Fix(\mathcal{S}^n)$. As

$$\Gamma(\Psi, \mathcal{S}\Psi) \geq 1$$

and

$$\nu(\Psi, \mathcal{S}\Psi) \leq 1$$

, and $\mathcal{S}$ is a triangular (Γ, ν)-admissible mapping, then

$$\Gamma(\mathcal{S}\Psi, \mathcal{S}^2\Psi) \geq 1 \text{ and } \nu(\mathcal{S}\Psi, \mathcal{S}^2\Psi) \leq 1.$$

Similarly, it can be written as:

$$\Gamma(\mathcal{S}^n\Psi, \mathcal{S}^{n+1}\Psi) \geq 1 \text{ and } \nu(\mathcal{S}^n\Psi, \mathcal{S}^{n+1}\Psi) \leq 1 \ \ \forall \ \ n \in \mathbb{N}.$$

By $(\mathcal{S}_3)$ and $(\mathcal{S}_4)$

$$\Gamma(\mathcal{S}^m\Psi, \mathcal{S}^n\Psi) \geq 1 \text{ and } \nu(\mathcal{S}^m\Psi, \mathcal{S}^n\Psi) \leq 1, \ \ \forall \ \ m, n \in \mathbb{N}, n \neq m.$$

Suppose that $\Psi \notin Fix(\mathcal{S})$, i.e , $\eth_{gb}(\Psi, \mathcal{S}\Psi) > 0$.
Also we can write $\eth_{gb}(\Psi, \mathcal{S}\Psi)$ as

$$\eth_{gb}(\Psi, \mathcal{S}\Psi) = \eth_{gb}(\mathcal{S}^n\Psi, \mathcal{S}\Psi) = \eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi, \mathcal{S}\Psi), \tag{4.16}$$

consider $\eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)$ and using (4.1)

$$\begin{aligned}
&\varphi(s^3\eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)\\
&\leq [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\Omega(\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\mathcal{S}\mathcal{S}^{n-1}\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\Psi,\mathcal{S}^2\mathcal{S}^{n-1}\Psi))}\\
&= [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\Omega(\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\mathcal{S}^{n}\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^{n}\Psi,\Psi),\eth_{gb}(\Psi,\mathcal{S}^{n+1}\Psi))}\\
&= [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\Omega(\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\mathcal{S}^{n}\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^{n}\Psi,\Psi),0)}.
\end{aligned}$$

So, $\exists \ \ \gamma \in (0,1) \ \ s.t$

$$\Omega(\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\mathcal{S}^{n}\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^{n}\Psi,\Psi),0)=\gamma.$$

Since φ is continuous and an increasing function, therefore, (4.16) gives

$$\varphi(\eth_{gb}(\Psi,\mathcal{S}\Psi))=\varphi(\eth_{gb}(\mathcal{S}^{n}\Psi,\mathcal{S}\Psi))=\varphi(\eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)).$$

Also

$$\begin{aligned}
\varphi(\eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi)) <&\varphi(s^3\eth_{gb}(\mathcal{S}\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi))\\
\leq&[\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\gamma}\\
\Rightarrow \varphi(\eth_{gb}(\Psi,\mathcal{S}\Psi) \leq [\varphi(M(\mathcal{S}^{n-1}\Psi,\Psi))]^{\gamma}. \qquad (4.17)
\end{aligned}$$

Now to find $M(\mathcal{S}^{n-1}\Psi,\Psi))$,

we proceed as follows:

$$\begin{aligned}
&M(\Psi,\mathcal{S}^{n-1}\Psi)\\
&=\max\{\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\mathcal{S}\mathcal{S}^{n-1}\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\\
&\qquad \eth_{gb}(\mathcal{S}^2\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^2\mathcal{S}^{n-1}\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^2\mathcal{S}^{n-1}\Psi,\mathcal{S}^{n-1}\Psi)\}\\
&\\
&=\max\{\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\mathcal{S}^{n}\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\\
&\qquad \eth_{gb}(\mathcal{S}\mathcal{S}^{n}\Psi,\Psi),\eth_{gb}(\mathcal{S}\mathcal{S}^{n}\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}\mathcal{S}^{n}\Psi,\mathcal{S}^{n-1}\Psi)\}\\
&\\
&=\max\{\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^{n-1}\Psi,\Psi),\\
&\qquad \eth_{gb}(\mathcal{S}\Psi,\Psi),\eth_{gb}(\mathcal{S}\Psi,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}\Psi,\mathcal{S}^{n-1}\Psi)\}.
\end{aligned}$$

As $n \to \infty$ and $\eth_{gb}(\mathcal{S}^{n-1}\Psi, \mathcal{S}^n\Psi) \to 0$

$$\Rightarrow \lim_{n\to\infty} M(\Psi, \mathcal{S}^{n-1}\Psi) = \eth_{gb}(\Psi, \mathcal{S}\Psi).$$

Therefore (4.17) becomes

$$\varphi(\eth_{gb}(\Psi, \mathcal{S}\Psi) \leq [\varphi(\eth_{gb}(\Psi, \mathcal{S}\Psi))]^{\gamma} < \varphi(\eth_{gb}(\Psi, \mathcal{S}\Psi)),$$

which is a contradiction. Hence,

$$\eth_{gb}(\Psi, \mathcal{S}\Psi) = 0,$$

which implies that $Fix(\mathcal{S}^n) = Fix(\mathcal{S})$. Thus $\mathcal{S}$ has a property P.

Considering the following assertions, we prove that if $\mathcal{S}$ is not necessarily continuous then theorem (4.2.1) still holds. □

Theorem 4.2.4. Let $(\mathcal{W}, \eth_{gb})$ be a (Γ, ν)-complete GbMS and let Γ, ν are two functions $s.t$ $\Gamma, \nu : \mathcal{W} \times \mathcal{W} \to [0, +\infty)$.

Suppose $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ be a self mapping which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ, ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ, ν)-φ-Ω-contraction.

(iii) : $\exists\ \zeta_0 \in \mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0, \mathcal{S}\zeta_0) \geq 1$ and $\nu(\zeta_0, \mathcal{S}\zeta_0) \leq 1$.

(iv) : $(\mathcal{W}, \eth_{gb})$ is (Γ, ν)-regular,

then, $\mathcal{S}$ has a FP. Moreover, $\mathcal{S}$ has a UFP with $\Gamma(\Psi, \zeta) \geq 1$ and $\nu(\Psi, \zeta) \leq 1$ $\forall\ \ \Psi, \zeta \in Fix(\mathcal{S})$.

Proof. Let $\zeta_0 \in \mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0, \mathcal{S}\zeta_0) \geq 1$ and $\nu(\zeta_0, \mathcal{S}\zeta_0) \leq 1$.

Following the proof of Theorem (4.2.1), it can be concluded that

$$\Gamma(\zeta_n, \mathcal{S}\zeta_{n+1}) \geq 1 \text{ and } \nu(\zeta_n, \mathcal{S}\zeta_{n+1}) \leq 1, \text{ and}$$
$$\zeta_n \to \Psi \text{ as } n \to \infty,$$

where $\zeta_{n+1}=\mathcal{S}\zeta_n$. From (iv), $\Gamma(\zeta_{n+1},\Psi)\geq 1$ and $\nu(\zeta_{n+1},\Psi)\leq 1$ hold $\forall\ n\in\mathbb{N}$. Suppose that $\mathcal{S}\Psi=\zeta_{0+1}=\mathcal{S}\zeta_{n_0}$ for some $n_0\in\mathbb{N}$. By Theorem (4.2.1), it is known that the elements of the sequence $\{\zeta_n\}$ are different. So, $\mathcal{S}\Psi\neq\mathcal{S}\zeta_n$, that is $\eth_{gb}(\mathcal{S}\Psi,\mathcal{S}\zeta_n)>0\ \ \forall\ \ n>n_0$.

Consequently, one can apply (4.1) to ζ_n and $\Psi\ \ \forall\ \ n>n_0$ to obtain

$$\begin{aligned}\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi))\leq&[\varphi(M(\zeta_n,\Psi))]^{\Omega(\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\zeta_n),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\Psi),\eth_{gb}(\mathcal{S}^2\zeta_n,\Psi))}\\ =&[\varphi(M(\zeta_n,\Psi))]^{\Omega(\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi))},\end{aligned}$$

which implies that

$$\varphi(s^3\eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi))\leq[\varphi(M(\zeta_n,\Psi))]^{\Omega(\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi))}. \tag{4.18}$$

Now

$$\begin{aligned}\lim_{n\to\infty}M(\zeta_n,\Psi)=\max\Big\{&\lim_{n\to\infty}[\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+1},\Psi),\\ &\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi),\eth_{gb}(\zeta_{n+2},\zeta_{n+1})]\Big\}.\end{aligned}$$

As it is described earlier that $\zeta_n\to\Psi$ as $n\to\infty$.

Then, the above equality becomes

$$\begin{aligned}=\max\Big\{&\eth_{gb}(\Psi,\Psi),\eth_{gb}(\Psi,\Psi),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\Psi,\Psi),\\ &\lim_{n\to\infty}\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi),\eth_{gb}(\Psi,\Psi),\eth_{gb}(\Psi,\Psi)\Big\}\\ =\max\Big\{&0,0,\eth_{gb}(\Psi,\mathcal{S}\Psi),0,\lim_{n\to\infty}\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi),0,0\Big\}.\end{aligned}$$

Since

$$0\leq\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi)\leq\eth_{gb}(\zeta_{n+2},\zeta_n)+\eth_{gb}(\zeta_n,\Psi)+\eth_{gb}(\zeta_n,\mathcal{S}\Psi).$$

Taking $\lim n\to\infty$

$$\begin{aligned}\lim_{n\to\infty}\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi)\leq&\lim_{n\to\infty}[\eth_{gb}(\zeta_{n+2},\zeta_n)+\eth_{gb}(\zeta_n,\Psi)+\eth_{gb}(\zeta_n,\mathcal{S}\Psi)]\\ \leq&\eth_{gb}(\Psi,\Psi)+\eth_{gb}(\Psi,\Psi)+\eth_{gb}(\Psi,\mathcal{S}\Psi) \qquad (4.19)\\ \lim_{n\to\infty}\eth_{gb}(\zeta_{n+2},\mathcal{S}\Psi)\leq&\eth_{gb}(\Psi,\mathcal{S}\Psi).\end{aligned}$$

Thus,

$$\lim_{n\to\infty} M(\zeta_n,\Psi) \leq \eth_{gb}(\Psi,\mathcal{S}\Psi).$$

Also

$$\lim_{n\to\infty} \Omega(\eth_{gb}(\zeta_n,\Psi),\eth_{gb}(\zeta_n,\zeta_{n+1}),\eth_{gb}(\Psi,\mathcal{S}\Psi),\eth_{gb}(\zeta_n,\mathcal{S}\Psi),\eth_{gb}(\zeta_{n+2},\Psi)) = \gamma.$$

If $\eth_{gb}(\Psi,\mathcal{S}\Psi) > 0$, subsequently by (4.19) and certainly φ and Ω are continuous, (4.18) becomes

$$\varphi(s^3 \lim_{n\to\infty} \eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi)) \leq [\varphi(\eth_{gb}(\Psi,\mathcal{S}\Psi))]^{\gamma} < [\varphi(\eth_{gb}(\Psi,\mathcal{S}\Psi))].$$

Using (4.15), we get

$$\varphi(\eth_{gb}(\Psi,\mathcal{S}\Psi)) < \varphi[\eth_{gb}(\Psi,\mathcal{S}\Psi)],$$

which is a contradiction.

Hence, $\eth_{gb}(\Psi,\mathcal{S}\Psi) = 0$, that is, Ψ is a FP of $\mathcal{S}$, which implies that $\Psi = \mathcal{S}\Psi$. One can prove the uniqueness of FP following the Theorem (4.2.1). □

Definition 4.2.5. Let $(\mathcal{W},\eth_{gb})$ be a (Γ,ν)-GbMS, and let $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self-mapping. Suppose that $\Gamma,\nu:\mathcal{W}\times\mathcal{W}\to[0,+\infty[$ are two functions. Then $\mathcal{S}$ is a (Γ,ν)-$\varphi_{\mathcal{G}}$-Ω-contraction, if $\forall\ \zeta,\xi\in\mathcal{W}$ with $\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1$ and $\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi) > 0$,we have

$$\varphi(s^3\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi)) \leq [\varphi(M(\zeta,\xi))]^{\Omega(\eth_{gb}(\zeta,\xi),\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\xi,\mathcal{S}\xi),\eth_{gb}(\mathcal{S}\zeta,\xi),\eth_{gb}(\mathcal{S}^2\zeta,\xi))}, \tag{4.20}$$

where $\varphi\in\Theta_{\mathcal{G}}$, $\Omega\in\sigma$ and

$$M(\zeta,\xi) = \max\{\eth_{gb}(\zeta,\xi),\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\xi,\mathcal{S}\xi),\eth_{gb}(\mathcal{S}\zeta,\xi),$$

$$\eth_{gb}(\mathcal{S}^2\zeta,\xi),\eth_{gb}(\mathcal{S}^2\zeta,\mathcal{S}\xi),\eth_{gb}(\mathcal{S}^2\zeta,\mathcal{S}\zeta)\}.$$

Theorem 4.2.6. Let $(\mathcal{W},\eth_{gb})$ be a (Γ,ν)-complete GbMS, and let Γ,ν are two functions $s.t$ $\Gamma,\nu:\mathcal{W}\times\mathcal{W}\to[0,+\infty)$. Suppose $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self-mapping which fulfils the following assertions:

(i) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(ii) : $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.

(iii) : $\exists\ \ \zeta_0 \in \mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0, \mathcal{S}\zeta_0) \geq 1$ and $\nu(\zeta_0, \mathcal{S}\zeta_0) \leq 1$.

(iv) : $\mathcal{S}$ is a (Γ,ν)-continuous.

Then, $\mathcal{S}$ has a FP. Also, $\mathcal{S}$ has a UFP when $\Gamma(\zeta,\xi) \geq 1$ and $\nu(\zeta,\xi) \leq 1\ \forall\ \zeta,\xi \in \mathcal{W}$.

Proof. Let $\zeta_0 \in \mathcal{W}\ \ s.t\ \ \Gamma(\zeta_0, \mathcal{S}\zeta_0) \geq 1$ and $\nu(\zeta_0, \mathcal{S}\zeta_0) \leq 1$.
Following the proof of Theorem (4.2.1),
it can be concluded that

$$\Gamma(\zeta_n, \zeta_{n+1}) \geq 1 \text{ and } \nu(\zeta_n, \zeta_{n+1}) \leq 1,$$

and

$$\lim_{n\to\infty} \eth_{gb}(\zeta_n, \zeta_{n+1}) = 0\ ,\ \lim_{n\to\infty} \eth_{gb}(\zeta_n, \zeta_{n+2}) = 0.$$

By property (φ_3), $\exists\ \ \mathtt{r} \in\,]0,1[$ and $l \in (0,+\infty]\ \ s.t$

$$\lim_{n\to\infty} \frac{(\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1)}{\eth_{gb}(\zeta_n, \zeta_{n+1})^r} = l.$$

Suppose that $l < \infty$. So, $\exists\ \ n_1 \in \mathbb{N}\ \ s.t$

$$\left| \frac{\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1}{\eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}} - l \right| < \frac{l}{2},\quad \forall\ \ n \geq n_1. \tag{4.21}$$

$$\Rightarrow -\frac{l}{2} < \frac{\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1}{\eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}} - l < \frac{l}{2}$$

$$\Rightarrow (-\frac{l}{2} + l)\ \eth_{gb}(\zeta_n, \zeta_{n+1})^r < \varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1 < (\frac{l}{2} + l)\ \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \frac{l}{2}\ \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}} < \varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1 < \frac{3l}{2}\ \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}} < \frac{2}{l}(\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1) < 3\eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}} < \mathcal{A}^*(\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1),$$

where $\mathcal{A}^* = \dfrac{2}{l}$.

$$\Rightarrow n\eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}} < \mathcal{A}^*.n(\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1).$$

Now suppose that $l = \infty$ and suppose $\mathcal{R} > 0$ be an arbitrary non-negative number. Then, $\exists \ \ n_2 \in \mathbb{N} \ \ s.t$

$$\left| \frac{\varphi(\eth_{gb}(x_n, x_{n+1})) - 1}{(\eth_{gb}(x_n, x_{n+1}))^r} \right| > \mathcal{R}. \tag{4.22}$$

Since $\eth_{gb}(\zeta_n, \zeta_{n+1}) > 0$ and also $\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1}))$ is non-negative, so absolute value is unnecessary and one can write it as

$$\frac{\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1}{(\eth_{gb}(\zeta_n, \zeta_{n+1}))^{\mathtt{r}}} > \mathcal{R}$$

$$\Rightarrow \varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1 > \mathcal{R} \ \ \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}$$

$$\Rightarrow \ \ \eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}} < \frac{1}{\mathcal{R}} \ \ \varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1$$

$$\Rightarrow n[\eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}] < n.\mathcal{A}^*[\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1], \ \ \forall \ \ n \geq n_2,$$

where $\mathcal{A}^* = \dfrac{1}{\mathcal{R}}$.
Therefore, in all cases, $\exists \ \ \mathcal{A}^* > 0$ and $\mathtt{c} \in \mathbb{N}(\mathtt{c} = \max(n_1, n_2)) \ \ s.t$

$$n[\eth_{gb}(\zeta_n, \zeta_{n+1})^{\mathtt{r}}] < \mathcal{A}^*.n[\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) - 1], \ \ \forall \ \ n \geq n_{\mathtt{c}}. \tag{4.23}$$

Consider

$$\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) = \varphi(\eth_g(\mathcal{S}\zeta_{n-1}, \mathcal{S}\zeta_n)).$$

By using definition (4.2.5)

$$\varphi(\eth_{gb}(\zeta_n, \zeta_{n+1})) < \varphi(s^3\eth_{gb}(\zeta_n, \zeta_{n+1})) \leq M(\zeta_{n-1}, \zeta_n))^{\mathtt{r}}. \tag{4.24}$$

As $M(\zeta_{n-1},\zeta_n)=\eth_{gb}(\zeta_{n-1},\zeta_n)$,

therefore (4.24) becomes

$$\varphi(\eth_{gb}(\zeta_n,\zeta_{n+1}))<\varphi(s^3\eth_{gb}(\zeta_n,\zeta_{n+1}))\leq\eth_{gb}(\zeta_{n-1},\zeta_n)^{\mathtt{r}}.$$

Using this in (4.23)

$$\begin{aligned}&n[\eth_{gb}(\zeta_n,\zeta_{n+1})]^{\mathtt{r}}\\ <&\mathcal{A}^*.n[\varphi(\eth_{gb}(\zeta_n,\zeta_{n+1}))]-\mathcal{A}^*.n\\ <&\mathcal{A}^*.n[\eth_{gb}(\zeta_{n-1},\zeta_n)]^{\mathtt{r}}-\mathcal{A}^*.n.\end{aligned}\tag{4.25}$$

Now applying contraction condition on $\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n))$

$$\begin{aligned}&\varphi(\eth_{gb}(\zeta_{n-1},\zeta_n))\\ <&\varphi(s^3\eth_{gb}(\zeta_{n-1},\zeta_n))\\ \leq&\varphi(M(\zeta_{n-2},\zeta_{n-1}))^{\mathtt{r}}\\ =&\varphi(\eth_{gb}(\zeta_{n-2},\zeta_{n-1}))^{\mathtt{r}}.\end{aligned}$$

Therefore (4.25) becomes

$$\begin{aligned}&n[\eth_{gb}(\zeta_n,\zeta_{n+1})]^{\mathtt{r}}\\ <&(\mathcal{A}^*.n[\eth_{gb}(\zeta_{n-1},\zeta_n)]^{\mathtt{r}}-\mathcal{A}^*.n)\\ =&\mathcal{A}^*.n[\eth_{gb}(\zeta_{n-1},\zeta_n)^{r}-1]\\ <&\mathcal{A}^*.n[\varphi(\eth_{gb}(\zeta_{n-2},\zeta_{n-1}))^{r^2}-1].\end{aligned}$$

Continuing in the same way, we obtain

$$n[\eth_{gb}(\zeta_n,\zeta_{n+1})]^{\mathtt{r}}<n.\mathcal{A}^*[\varphi(\eth_{gb}(\zeta_n,\zeta_{n+1}))-1]<\ldots<\mathcal{A}^*.n[\varphi(\eth_{gb}(\zeta_0,\mathsf{u}_1)))^{\mathtt{r}^n}-1].$$

Taking $n\to\infty$ in the above inequality to get

$$\begin{aligned}&\lim_{n\to\infty}n[\eth_{gb}(\zeta_n,\zeta_{n+1})]^{\mathtt{r}}=0\\ &\Rightarrow n[\eth_{gb}(\zeta_n,\zeta_{n+1})]^{\mathtt{r}}<\epsilon,\ \forall\ \epsilon>0.\end{aligned}$$

Choosing $\epsilon = 1$ and also $\exists\ \ n_3 \in \mathbb{N}\ \ s.t$

$$\eth_{gb}(\zeta_n, \zeta_{n+1}) \leq \frac{1}{n^{1/\mathtt{r}}},\ \ \forall\ \ n \geq n_3.$$

Now consider

$$\eth_{gb}(\zeta_n, \zeta_{n+2})$$

, and by using condition (φ_3), $\exists\ \ \mathtt{r} \in (0,1)$ and $\mathtt{h} \in]0, +\infty[\ \ s.t$

$$\lim_{n\to\infty} \frac{\varphi(\eth_{gb}(\zeta_n, \zeta_{n+2})) - 1}{(\eth_{gb}(\zeta_n, \zeta_{n+2}))^{\mathtt{r}}} = \mathtt{h}.$$

Now suppose that $\mathtt{h} < \infty$ and $\exists\ \ n_4 \in \mathbb{N}$, then following the same procedure, one can get

$$n(\eth_{gb}(\zeta_n, \zeta_{n+2}))^{\mathtt{r}} < \mathcal{J}^*.n(\varphi(\eth_{gb}(\zeta_n, \zeta_{n+2})) - 1),\ \ \forall\ \ n \geq n_4.$$

Now let $\mathtt{h} = \infty$ and suppose $\exists$ an arbitrary non-negative number $\mathcal{Q}^* > 0$ and also $\exists\ \ n_5 \in \mathbb{N}$, then one can get

$$n[\eth_{gb}(\zeta_n, \zeta_{n+2})]^{\mathtt{r}} < n.\mathcal{J}^*[\varphi(\eth_{gb}(\zeta_n, \zeta_{n+2})) - 1],\ \ \forall\ \ n \geq n_5,$$

where $\mathcal{J}^* = 1/\mathcal{Q}^*$.

Following the same procedure as in the equation (4.23), we get

$$n[\eth_{gb}(\zeta_n, \zeta_{n+2})^{\mathtt{r}}] < n.\mathcal{J}^*[\varphi(\eth_{gb}(x_n, x_{n+2})) - 1] < ... < n.\mathcal{J}^*[\varphi(\eth_{gb}(\zeta_0, \mathsf{u}_2)))^{\mathtt{r}^n} - 1].$$

As $n \to \infty$

$$\lim_{n\to\infty} n[\eth_{gb}(\zeta_n, \zeta_{n+2})^{\mathtt{r}}] = 0$$
$$\Rightarrow n[\eth_{gb}(\zeta_n, \zeta_{n+2})^{\mathtt{r}}] < 1\ .$$

Then, $\exists\ \ n_6 \in \mathbb{N}\ \ s.t$

$$\eth_{gb}(\zeta_n, \zeta_{n+2}) \leq \frac{1}{n^{1/\mathtt{r}}},\ \ \forall\ \ n \geq n_6.$$

If $\mathtt{m} > \mathtt{n}$, then

$$\begin{aligned}\eth_{gb}(\zeta_n,\zeta_m) \leq & s[\eth_{gb}(\zeta_n,\zeta_{n+1}) + \eth_{gb}(\zeta_{n+1},\zeta_m)] \\ \leq & s\eth_{gb}(\zeta_n,\zeta_{n+1}) + s^2\eth_{gb}(\zeta_{n+1},\zeta_{n+2}) + s^2\eth_{gb}(\zeta_{n+2},\zeta_m) \\ \leq & s\eth_{gb}(\zeta_n,\zeta_{n+1}) + ... + s^n\eth_{gb}(\zeta_{m-1},\zeta_m) \\ \Rightarrow \eth_{gb}(\zeta_n,\zeta_m) \leq & s^{\mathtt{k}}[\eth_{gb}(\zeta_n,\zeta_{n+1}) + \eth_{gb}(\zeta_{n+1},\zeta_{n+2}) + ... + \eth_{gb}(\zeta_{m-1},\zeta_m)],\end{aligned}$$

where $\mathtt{k}$ is the number of times the triangular inequality is applied and $\mathtt{k} \in \mathbb{N}$.

By combining the result, it can be concluded that

$$\eth_{gb}(\zeta_n,\zeta_m) \leq s^k \sum_{i=n}^{m-1} \frac{1}{i^{1/r}}.$$

Therefore,

$$\eth_{gb}(\zeta_n,\zeta_m) \leq s^{\mathtt{k}} \sum_{i=n}^{\infty} \frac{1}{i^{1/\mathtt{r}}}. \tag{4.26}$$

Since $\mathtt{r} \in (0,1)$, the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/\mathtt{r}}}$ is a p-series with $p = \frac{1}{\mathtt{r}} > 1$, which converges. Therefore $s^{\mathtt{k}} < \infty$.

So, in (4.26) taking the limit $n, m \to \infty$, we get

$$\lim_{n\to\infty} \eth_{gb}(\zeta_n,\zeta_m) = 0.$$

Hence, it is clear that $\{\zeta_n\}_{n\in\mathbb{N}}$ is a CS.

As $\mathcal{W}$ is complete, then $\exists \quad \Psi \in \mathcal{W} \quad s.t$

$$\lim_{n\to\infty} \eth_{gb}(\zeta_n,\Psi) = 0.$$

As $\mathcal{S}$ is (Γ,ν)-continuous, therefore

$$\lim_{n\to\infty} \eth_{gb}(\mathcal{S}\zeta_n,\mathcal{S}\Psi) = 0,$$

$$\Rightarrow \Psi = \lim_{n\to\infty} \zeta_{n+1} = \lim_{n\to\infty} \mathcal{S}\zeta_n = \mathcal{S}\Psi.$$

This clearly shows that Ψ is a FP of $\mathcal{S}$. □

Corollary 4.2.7. Let $(\mathcal{W},\eth_{gb})$ be a (Γ,ν)-complete GbMS and also suppose that Γ,ν are two functions $s.t$ $\Gamma,\nu:\mathcal{W}\times\mathcal{W}\to[0,+\infty[$. Let $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ be a self-mapping which fulfils the following assertions:

(i) : $\varphi[\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi)]\leq[\varphi(M(\zeta,\xi))]^{\mathsf{r}},\mathsf{r}\in\,]0,1[\ \varphi\in\Theta_{\mathcal{G}}$.

(ii) : $\mathcal{S}$ is a triangular (Γ,ν)-admissible mapping.

(iii) $\exists\ \zeta_0\in\mathcal{W}$ $s.t$ $\Gamma(\zeta_0,\mathcal{S}\zeta_0)\geq 1$ and $\nu(\zeta_0,\mathcal{S}\zeta_0)\leq 1$.

(iv) $\mathcal{S}$ is a (Γ,ν)-continuous.

Then, $\mathcal{S}$ has a FP. Also, $\mathcal{S}$ has a UFP when $\Gamma(\zeta,\xi)\geq 1$ and $\nu(\zeta,\xi)\leq 1\ \forall\ \zeta,\xi\in\mathcal{W}$.

Proof. Define a function $\Omega:\mathbb{R}_+^5\to\mathbb{R}_+$ by

$$\Omega(e_1,e_2,e_3,e_4,e_5)=r\ \forall\ e_1,e_2,e_3,e_4,e_5\in\mathbb{R}_+$$

. Clearly $\Omega\in\sigma$ and $\mathcal{S}$ is a (Γ,ν)-φ-Ω-contraction.
Following the proof of Theorem (4.2.6), $\mathcal{S}$ has a UFP $\zeta\in\mathcal{W}$. □

Example 4.2.8. Suppose $\mathcal{W}=[1,+\infty)$, define $\eth_{gb}:\mathcal{W}\times\mathcal{W}\to[0,+\infty)$ by

$$\eth_{gb}(\zeta,\xi)=(\xi-\zeta)^2.$$

Then, $(\mathcal{W},\eth_{gb})$ is a complete GbMS with $s=3$.
Define $\mathcal{S}:\mathcal{W}\to\mathcal{W}$ by

$$\begin{aligned}\mathcal{S}(\jmath)=&2a\sqrt{\jmath}\ \ \forall\ \ \jmath\in[1,+\infty),a\in(0,1)\text{ and}\\ \varphi(\jmath)=&e^{\jmath}\end{aligned}$$

$$\Gamma(\zeta,\xi)=\frac{\max\{\zeta,\xi\}+a}{\min\{\zeta,\xi\}+a},\quad\forall\ \ \zeta,\xi\in\mathbb{R}_+,$$

$$\nu(\zeta,\xi)=\frac{\min\{\zeta,\xi\}+a}{\max\{\zeta,\xi\}+a},\quad\forall\ \ \zeta,\xi\in\mathbb{R}_+,$$

$$\Omega(\jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5)=\sqrt{a}\ \ \forall\ \ \jmath_1,\jmath_2,\jmath_3,\jmath_4,\jmath_5\in\mathbb{R}_+.$$

Then, $\mathcal{S}$ is a (Γ,ν)-continuous and triangular (Γ,ν)-admissible mapping.

Now to prove $\mathcal{S}$ is (Γ,ν)-φ-Ω-contraction

$$\begin{aligned}
\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi) =&(2a\sqrt{\xi}-2a\sqrt{\zeta})^2,\\
=&4a^2(\sqrt{\xi}-\sqrt{\zeta})^2,\\
M(\eth_{gb}(\zeta,\xi)) =& \max\Big\{\eth_{gb}(\zeta,\xi),\eth_{gb}(\zeta,2a\sqrt{\zeta}),\eth_{gb}(\xi,2a\sqrt{\xi}),\eth_{gb}(\xi,2a\sqrt{\zeta}),\\
&\eth_{gb}\left(4a^2\sqrt{\sqrt{\zeta}},\xi\right),\eth_{gb}\left(4a^2\sqrt{\sqrt{\zeta}},2a\sqrt{\xi}\right),\eth_{gb}\left(4a^2\sqrt{\sqrt{\zeta}},2a\sqrt{\zeta}\right)\Big\}\\
=& \max\Big\{(\xi-\zeta)^2,(\zeta-2a\sqrt{\zeta})^2,(\xi-2a\sqrt{\xi})^2,(\xi-2a\sqrt{\zeta})^2,\\
&\left(\xi-4a^2\sqrt{\sqrt{\zeta}}\right)^2,\left(2a\sqrt{\xi}-4a^2\sqrt{\sqrt{\zeta}}\right)^2,\left(2a\sqrt{\zeta}-4a^2\sqrt{\sqrt{\zeta}}\right)^2\Big\}.
\end{aligned}$$

Thus,

$$M(\zeta,\xi)\geq(\xi-\zeta)^2\geq a(\xi-\zeta)^2. \tag{4.27}$$

Since,

$$a(\xi-\zeta)^2=\sqrt{a}\sqrt{a}(\xi-\zeta)^2.$$

Since φ is increasing, from (4.27) we obtain

$$\varphi(M(\zeta,\xi))\geq\varphi(\eth_{gb}(\zeta,\xi))=e^{(\xi-\zeta)^2}.$$

Thus,

$$\begin{aligned}
&\varphi(\eth_{gb}(\zeta,\xi))^{\Omega(\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\xi,\mathcal{S}\xi),\eth_{gb}(\zeta,\mathcal{S}\xi),\eth_{gb}(\xi,\mathcal{S}\zeta),\eth_{gb}(\mathcal{S}^2\zeta,\xi))}\\
=&e^{\sqrt{a}(\xi-\zeta)^2}\\
=&e^{\sqrt{a}(\sqrt{\xi}-\sqrt{\zeta})^2(\sqrt{\xi}+\sqrt{\zeta})^2}.
\end{aligned}$$

Now $\varphi(\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi))=e^{4a^2(\sqrt{\xi}-\sqrt{\zeta})^2}$. Since $\zeta,\xi\in[1,+\infty)$ and,

$$\begin{aligned}
&\sqrt{\zeta}\geq 1,\ \sqrt{\xi}\geq 1\\
&\Rightarrow\sqrt{\zeta}+\sqrt{\xi}\geq 2\\
&\Rightarrow(\sqrt{\zeta}+\sqrt{\xi})^2\geq 4.
\end{aligned}$$

As

$$\begin{aligned}(\xi-\zeta)^2 =&\left(\sqrt{\xi}-\sqrt{\zeta}\right)^2\left(\sqrt{\xi}+\sqrt{\zeta}\right)^2\\ \geq&4\left(\sqrt{\xi}-\sqrt{\zeta}\right)^2\\ \Rightarrow 4\left(\sqrt{\xi}-\sqrt{\zeta}\right)^2 \leq&(\xi-\zeta)^2.\end{aligned} \tag{4.28}$$

As

$$\begin{aligned}\eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi) =&4a^2(\sqrt{\xi}-\sqrt{\zeta})^2\\ \leq&\mathsf{a}^2(\xi-\zeta)^2\\ =&\mathsf{a}^{\frac{3}{2}}\sqrt{a}(\xi-\zeta)^2\\ \leq&\sqrt{a}(\xi-\zeta)^2.\end{aligned}$$

As $\zeta, \xi \in [1, +\infty)$, with $s = 3 \;\; \exists \;\; a \in (0, 1) \;\; s.t \;\; a = \dfrac{1}{10}$, it can be concluded that

$$s^3a^2(\zeta+\xi-2\sqrt{\zeta\xi}) \leq\sqrt{a}(\zeta^2+\xi^2-2\zeta\xi)$$

$$\Rightarrow e^{s^3a^2(\zeta+\xi-2\sqrt{\zeta\xi})} \leq e^{\sqrt{a}(\zeta^2+\xi^2-2\zeta\xi)}$$

Hence,
$\varphi(s^3\eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi)) \leq \varphi(\eth_{gb}(\zeta, \xi))^{\Omega(\eth_{gb}(\zeta,\mathcal{S}\zeta),\eth_{gb}(\xi,\mathcal{S}\xi),\eth_{gb}(\zeta,\mathcal{S}\xi),\eth_{gb}(\xi,\mathcal{S}\zeta),\eth_{gb}(\mathcal{S}^2\zeta,\xi))}$,
where $\varphi \in \Theta_{\mathcal{W}} \cap \Theta_{\mathcal{G}}$. Hence, contraction assertions (4.1) and (4.20) are satisfied, and $\Psi = 1$ is the UFP of $\mathcal{S}$.

4.3 Applications to Non-linear Integral Equations

This section establishes the existence and uniqueness of solutions for integral equations of Fredholm type by applying Theorem (4.2.1) and (4.2.6).

$$\text{Let } \zeta(\mathsf{t}) = \mathsf{v}^*\int_{\flat_1}^{\flat_2} \mathsf{g}(\mathsf{t}, \mathsf{i}, \zeta(\mathsf{i}))d\mathsf{i}, \tag{4.29}$$

where $\flat_1, \flat_2 \in \mathbb{R}$, $\zeta \in \mathcal{C}([\flat_1, \flat_2], \mathbb{R})$, and $\mathsf{g} : [\flat_1, \flat_2]^2 \times \mathbb{R} \to \mathbb{R}$ are continuous functions and v^* represents a constant influenced by parameters $\flat_1$ and $\flat_2$.

Theorem 4.3.1. Suppose the function g is $s.t$

$$s^3|\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i})) - \mathsf{g}(\mathsf{t},\mathsf{i},\xi(\mathsf{i}))|^2 \leq |\zeta(\mathsf{t}) - \xi(\mathsf{t})|^2 \ \forall\ \mathsf{t},\mathsf{i} \in \mathbb{R} \tag{4.30}$$

and $\zeta, \xi \in \mathcal{C}([\flat_1, \flat_2], \mathbb{R})$.

Then, $\zeta \in \mathcal{C}([\flat_1, \flat_2], \mathbb{R})$ is the unique solution of (4.29) with $|\mathsf{v}^*| \leq \dfrac{\flat_1}{\flat_2}$.

Proof. Let $\mathcal{W} = \mathcal{C}([\flat_1, \flat_2], \mathbb{R})$ and $\mathcal{S} : \mathcal{W} \to \mathcal{W}$ defined by

$$\mathcal{S}(\zeta)(\mathsf{t}) = \mathsf{v}^* \int_{\flat_1}^{\flat_2} \mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))d\mathsf{i}.\ \forall \zeta \in \mathcal{W}, \tag{4.31}$$

and $\varphi(\mathsf{t}) = e^{\mathsf{t}}$.

Define $\eth_{gb} : \mathcal{W} \times \mathcal{W} \to [0, +\infty[$ by by

$$\eth_{gb}(\zeta, \xi) = \left(\max_{\mathsf{t} \in [\flat_1, \flat_2]} |\zeta(\mathsf{t}) - \xi(\mathsf{t})| \right)^2.$$

Then, $(\mathcal{W}, \eth_{gb})$ is a complete GbMS. Suppose that $\zeta, \xi \in \mathcal{W}$ and $\mathsf{t}, \mathsf{i} \in [\flat_1, \flat_2]$.

Then

$$\begin{aligned} s^3|\mathcal{S}\zeta(\mathsf{t}) - \mathcal{S}\xi(\mathsf{t})|^2 &= s^3|\mathsf{v}^*|^2 \left(\left| \int_{\flat_1}^{\flat_2} \mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))d\mathsf{i} - \int_{\flat_1}^{\flat_2} \mathsf{g}(\mathsf{t},\mathsf{i},\xi(\mathsf{i}))d\mathsf{i} \right| \right)^2 \\ &= s^3|\mathsf{v}^*|^2 \left| \int_{\flat_1}^{\flat_2} \mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))d\mathsf{i} - \mathsf{g}(\mathsf{t},\mathsf{i},\xi(\mathtt{r}))d\Psi \right|^2 \\ &\leq s^3|\mathsf{v}^*|^2 \int_{\flat_1}^{\flat_2} |\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i})) - \mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))|^2 d\mathsf{i} \end{aligned}$$

By (4.30)

$$\begin{aligned} s^3|\mathcal{S}\zeta(\mathsf{t}) - \mathcal{S}\xi(\mathsf{t})|^2 &\leq |\mathsf{v}^*|^2 \int_{\flat_1}^{\flat_2} (|\zeta(\mathsf{i}) - \xi(\mathsf{i})|)^2 d\mathsf{i} \\ &= |\mathsf{v}^*|^2 \left[\int_{\flat_1}^{\flat_2} ((|\zeta(\mathsf{i})| - |\xi(\mathsf{i})|))^2 d\mathsf{i} \right]. \end{aligned} \tag{4.32}$$

From (4.32)

$$\begin{aligned} s^3 \max_{\mathsf{t}\in[\flat_1,\flat_2]} (|\mathcal{S}\zeta(\mathsf{t}) - \mathcal{S}\xi(\mathsf{t})|)^2 &= s^3 \max_{\mathsf{t}\in[\flat_1,\flat_2]} \ |\mathsf{v}^*|^2 \int_{\flat_1}^{\flat_2} |\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i})) - \mathsf{g}(\mathsf{t},\mathsf{i},\xi(\mathsf{i}))d\mathsf{i}|^2 \\ &\leq \max_{\mathsf{t}\in[\flat_1,\flat_2]} \ |\mathsf{v}^*|^2 \int_{\flat_1}^{\flat_2} ((|\zeta(\mathsf{i})| - |\xi(\mathsf{i})|))^2 d\mathsf{i} \\ &\leq |\mathsf{v}^*|^2 \int_{\flat_1}^{\flat_2} \left(\left(\max_{\mathsf{i}\in[\flat_1,\flat_2]} \ |\zeta(\mathsf{i})| - |\xi(\mathsf{i})| \right)^2 d\mathsf{i} \right). \end{aligned} \tag{4.33}$$

As $\eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi) > 0$ and also $\eth_{gb}(\zeta, \xi) > 0$ for any $\zeta \neq \xi$, then from (4.33)

$$e^{[s^3\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi)]} = e^{\left[s^3|\mathsf{v}^*|^2 \ \max_{\mathsf{t}\in[\flat_1,\flat_2]} \int_{\flat_1}^{\flat_2} |\mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i})) - \mathsf{g}(\mathsf{t},\mathsf{i},\zeta(\mathsf{i}))d\mathsf{i}|^2\right]}$$

As $|\mathsf{v}^*| \leq \dfrac{\flat_1}{\flat_2}$, the above equation becomes

$$e^{[s^3\eth_{gb}(\mathcal{S}\zeta,\mathcal{S}\xi)]} \leq e^{\left[\frac{{\flat_1}^2}{{\flat_2}^2} \int_{\flat_1}^{\flat_2} \left(\left(\max_{\mathsf{i}\in[\flat_1,\flat_2]} \ |\zeta(\mathsf{i})| - |\xi(\mathsf{i})| \right) d\mathsf{i} \right)^2 \right]}.$$

Thus,

$$\varphi(s^3\eth_{gb}(\mathcal{S}\zeta, \mathcal{S}\xi)) \leq [\varphi(M(\zeta, \xi))]^{\Omega},$$

$\forall \ \ \zeta, \xi \in \mathcal{W}$ and $\Omega(\mathsf{t}) = \left|\dfrac{\flat_1}{\flat_2}\right|$. Then, $\mathcal{S}$ satisfies (4.1) and (4.20). Thus $\mathcal{S}$ has a UFP which is a solution of (4.33). $\square$

Chapter 5

Conclusion

- This thesis reviews the work of Kari et al.[26] on "FP Results for (Γ, ν)-φ-Ω-Contraction in Generalized MSs".

- A new class of (Γ,ν)-φ-Ω-contraction, has been introduced within the framework of GbMSs, and demonstrate their continuity. Corresponding FP theorems are established. Some examples are also provided to illustrate (Γ,ν)-φ-Ω-contraction in GbMSs.

- Motivated by the above work, the existence and uniqueness of FP has been proved by using (Γ,ν)-φ-Ω-contraction in GbMSs.

- Finally we present an application through which we investigate the existence of solutions for a Fredholm-type integral equation.

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