CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Time Relaxation Effects in a Thermo-Bioconvective Porous Cavity

by

Noor ul Absar

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

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I dedicate my thesis to my beloved family, friends and specially to my

*father **(Azhar Mehmood)**,*

*mother **(Shehnaz Akhter)**,*

and

*brother **(Faheem Azhar)**.*

CERTIFICATE OF APPROVAL

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Abstract

The study investigates the heat and mass transfer phenomena within a square porous cavity that is filled with oxytactic microorganisms, utilizing the Cattaneo-Christov heat flux model. The flow and heat transfer process in the porous medium are analyzed through the Darcy model. The governing equations, which incorporate mass, momentum, and energy conservation laws, are solved numerically using the finite element method. Results are presented through streamlines, isotherms, concentration profiles, and tables, illustrating the impact of various parameters such as the Rayleigh number Ra, bioconvection Rayleigh number Rb, Lewis number Le, and Péclet number Pe on the fluid dynamics, heat transfer, and mass transfer behavior. The study emphasizes the effects of these parameters on the Nusselt and Sherwood numbers at the vertical cavity walls. To validate the computed results a table on the Nusselt and Sherwood number is presented which compares the computed solutions with the available literature. A strong agreement is found between the computed numerical values and the one present in the literature. This shows the accuracy of the implementation of the finite element code for the presented model problem. The results are computed for varying values of the physical parameters and are discussed in detail. Overall, the research provides valuable insights into the complex interactions of heat, fluid flow, and microorganism concentration within a porous cavity, focusing on the effects of key dimensionless numbers. These findings are essential for optimizing processes involving bioconvection and heat transfer in porous media, particularly in biological or ecological systems where microorganisms play a critical role in the dynamics.

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Abbreviations

CCHFM	Cattaneo–Christov Heat Flux Model
DF	Darcy-Forchheimer
GM	Gyrotactic Microorganisms
GFEM	Galerkin Finite Element Method
LTNECs	Local Thermal Non-equilibrium Conditions
MHD	Magnetohydrodynamics
TPD	Thermophoretic Particle Deposition
THNF	Trihybrid Nanofluid
TB	Thermo-Bioconvection

Symbols

ρ	Density
μ	Dynamical viscosity
ξ_j	Shape function
η_j	Interpolation function
Ra	Rayleigh number
Rb	Bioconvection Rayleigh number
$\tilde{w}$	Weight function
Le	Lewis number
Ω	Computational domain
$\tilde{U}$	Test function
$\bar{W}$	Infinite dimensional test space
Pe	Peclet number
R_t	Relaxation Parameter
Pr	Prandtl number
K^*	Block stiffness vector
X^*	Block solution vector
Q^*	Block boundary vector
u_i	Nodal variable
$\mathbf{v}$	Fluid Filtration Velocity Vector
T	Fluid Temperature
α_m	Effective Thermal Diffusivity
K	Permeability of the Porous Medium
n	Number of Motile Microorganisms
p	Pressure

β	Volume Expansion Coefficient
j	Flux of Microorganisms
b	Chemotaxis Constant
W_C	Maximum Cell Swimming Speed
Nu	Nusselt number
Sh	Sherwood number

Chapter 1

Literature Review

The study of convective heat transfer in porous media has grown exponentially due to its relevance in a wide array of engineering applications, such as enhancing heat exchangers, improving the efficiency of geothermal energy systems, and optimizing thermal management in various industrial processes. Researchers are particularly focused on understanding the complex fluid dynamics within these media, which involve the interaction between the fluid phase and the solid phase that make up the porous structure. This has profound implications in energy systems, where heat transfer efficiency is critical. Additionally, the ability of porous media to retain and transport fluids has proven useful in environmental applications such as soil remediation, groundwater contamination control, and carbon sequestration, where precise control of heat and fluid movement is vital. The versatility of porous media is also harnessed in the development of advanced thermal storage systems, where its ability to store and release heat over extended periods plays a crucial role in reducing energy consumption. Furthermore, with the rise of sustainability concerns, the use of porous media in energy storage and conversion devices such as thermal energy storage units and advanced catalytic reactors has become an area of intense research. These devices often rely on the efficient transfer of heat and the dynamic interactions within porous structures to achieve enhanced performance. As these technologies continue to evolve, understanding the underlying heat transfer mechanisms in porous media is becoming increasingly important for designing efficient, low-carbon solutions across various sectors. In summary, the study of heat transfer in porous media is at the intersection of various critical fields, ranging from energy efficiency to environmental protection, making it a key area for future research and technological innovation. Thermal equilibrium and non-equilibrium models were utilized by Badruddin et al. [1] to examine the behavior of heat transfer in a square porous annulus. The results of the study showed that the bottom wall had a higher Nusselt number than the top wall. Badruddin et al. [2] used the finite element method to solve differential equations in order to analyze the cylinder solid-porous interface in conjugate heat and mass transfer. The temperature along the solid-porous interface rose as the cylinder height increased, according to their results. Nonlinear heat transfer with radiation for magnetic field non-Newtonian Casson flow field in porous annulus was examined and studied by Jalili et al. [3]. Their findings showed that concentration is directly impacted by temperature and velocity, but indirectly by the magnetic field.

Bioconvection is the process by which large fluid forms, such as falling plumes, naturally form. Its causes are complex interactions between various phenomena at various physical scales. The directed movement of self-moving microbes, which have a density greater than that of water, is what propels the bioconvection process. The relationship between microorganisms that consume oxygen and Zhang et al. [4] studied thermo-bioconvection nanofluid flow in a Darcy-Forchheimer medium on a porous Riga plate. Hillesdon and Pedley [5] investigated oxytactic bacteria suspensions using the linear theory of bioconvection. Mandal et al. [6] used entropy analysis to investigate the stratification of nanofluid flow containing gyrotactic microorganisms along an inclined stretching cylinder.

Arafa et al. [7] investigated the hydrothermal bioconvective THNF (trihybrid nanofluid) flow over a stretched rotating disk with heat generation. Thermo-bioconvection is a vital technique for researching fluid motion in hot springs that are home to motile thermophiles or heat-loving microorganisms. Kuznetsov [8] used oxytactic bacteria in a shallow-depth permeable layer to study thermo-bioconvection. Subsequently, he [9] investigated the mechanism of the co-operation of gyrotactic microbes and nanoparticles. The effects of heat transmission and MHD on GM (gyrotactic microorganisms) floating in a nanofluid flow across a sheet were examined by Kashkooli et al. [10]. The objective of producing nanofluid composites is to enhance a single nanoparticle's rheological or thermal conductivity in order to improve its characteristics. Single nanoparticles make up nanofluids, which improve the rheological characteristics and heat-absorbing capacity of different fluids. Similar to improved rheological properties, a composite of nanofluids is notable due to liquid-state thermal energy transfer. Rajeswari and De. [11] examined the multi-stratified effects on stagnation point nanofluid flow using gyrotactic microorganisms over a porous substrate. Gomathi and De. [12] examined the effects of hall currents and ion slip on Casson Williamson nanofluid flow with porous media. Nisha and De. [13] looked into the effects of ion slip and Hall currents on Sisko nanofluid flow with porous medium and chemical reaction using a statistical approach. Abbas et al. [14] examined the effects of Stefan blowing on TPD (thermophoretic particle deposition) in Sisko nanofluid flow over a Riga plate with Dufour and Soret effects. A hybrid nanofluid, is a kind of nanofluid created by mixing two different kinds of nanoparticles into fluid. The materials, sizes, and shapes of the nanoparticles that are used can vary. The hybrid nanofluid has better qualities than either the base fluid or the individual NPs. Applications for hybrid nanofluids are numerous and include energy storage, environmental engineering, biomedical engineering, and heat transfer.

Heat transfer, which is fueled by temperature differences between or within an object, is the foundation of many industrial and technological applications, including power generation, cooling atomic reactors, and the production of energy in general. The fundamental law for analyzing heat transfer has been Fourier's Law [15], but due to its limitations, Cattaneo [16] introduced a relaxation time term to address the paradox of instantaneous heat conduction. The Oldroyd upper-convected derivative was added by Christov [17] to further improve this theory, leading to the creation of the Cattaneo–Christov heat flux theory [17].

In order to fully understand complex heat transfer scenarios, computational methods are essential. Current studies provide evidence for this trend. Mohanty et al. [18] looked into the analysis of heat and irreversibility for unsteady HNF (hybrid nanofluid) flow over a spinning sphere using the Cattaneo-Christov heat flux model and the interfacial nanolayer mechanism. Mohanty et al. [19] investigated an infinite porous disk with Cattaneo-Christov heat flux to ascertain the impact of

Marangoni convection and interfacial nanolayer on Darcy-Forchheimer HNF. Tri-hybrid nanofluids are blends of three distinct types of nanofluids. The purpose of tri-hybrid nanofluids is to improve heat conductivity. The substantial thermal energy generation in tangent hyperbolic material based on trihybrid nanomaterials was investigated by Nazir et al. [20]. Abbas et al. [21] investigated the effects of TPD (thermophoretic particle deposition) in a THNF (trihybrid nanofluid) containing LTNECs. Ahmed et al. investigated the time-dependent squeezing flow of THNF with variable characteristics between parallel rotating plates under varied electric and magnetic fields [22]. Abbas et al. [23] examined the combined effects of electrophoresis and thermophoresis in a trihybrid nanofluid Darcy Forchheimer flow on a Riga plate. Ahmed et al. [24] looked into the Cattaneo-Christov double-diffusion on Sutterby THNF flow with Oxytactic microorganisms. Convection movement in permeable media is essential to many industrial and environmental systems, such as heat exchanger design and geothermal energy systems. The non-Darcian porous medium is a modified version of the standard Darcian model that takes inertia and boundary topographies. Darcy's law, as it is widely known, is valid in a small area with low permeability and slow motion. Rasool et al. [25] study the numerical treatment of HNF flow in a Darcy-Forchheimer medium using radiative mass and heat transfer laws. Lund et al. [26] look into several solutions of the HNF Darcy-Forchheimer flow with viscous dissipation effect. Shafiq et al. [27] investigate the comparative analysis of the hyperbolic flow approaching a cylindrical surface in the Darcy-Forchheimer tangent using an artificial neural network. The simulation of Boger fluid chemical reactive flow over a sheet with heat generation and LTNECs (local thermal non-equilibrium conditions) was examined by Abbas et al. [28]. Mohanty et al. [29] investigated the numerical analysis of the significance of interfacial nanolayer thickness on DF (Darcy-Forchheimer) Casson HNF flow using a moving needle and Cattaneo-Christov flux model. The analysis of irreversibility for Darcy–Forchheimer Casson HNF flow resulting from a rotating disk with heat radiation and CCHF (Cattaneo–Christov heat flux) was examined by Mohanty et al. [30]. Fluid mechanics states that complex systems experience a variety of chemical reactions within the fluid. These reactions can be classified as mixed, heterogeneous, or homogeneous, and they all have a big influence on flow patterns.

1.1 Thesis Contribution

The study thoroughly investigates heat and mass transfer within a square porous cavity that is filled with oxytactic microorganisms, analyzing the intricate interactions between fluid flow, heat transfer, and microorganism dynamics under varying conditions. By employing the Cattaneo-Christov heat flux model to capture non-Fourier heat conduction, alongside the Darcy model and Boussinesq approximation to describe fluid flow and thermal buoyancy, the governing equations are solved numerically using the finite element method (FEM) with FreeFem++ software.

Key dimensionless numbers, such as the Rayleigh number (Ra), bioconvection Rayleigh number (Rb), Lewis number (Le), and Péclet number (Pe), are carefully analyzed for their impact on flow patterns, heat transfer efficiency, and mass transport.

The research further explores the effects of time relaxation, which arises from the interplay of heat transfer, microbial motility, and diffusion processes, and how this influences flow stability, pattern formation, and transient behavior in the system. The study presents detailed results, including Nusselt and Sherwood numbers, which shed light on the efficiency of heat and mass transfer at the cavity walls.

The study's findings have critical implications for optimizing bioconvection processes, particularly in biological systems where microorganisms play a significant role in ecological dynamics.

Additionally, the insights gained from these simulations provide valuable guidance for the design of bio-reactors and other systems requiring efficient heat and mass transfer in porous media. This research advances the understanding of microbial behavior and bioconvection phenomena, contributing to the development of more efficient and sustainable systems for biological and ecological applications.

1.2 Thesis Attributes

A brief outline of the thesis's contents is provided below.

In **Chapter 2**, the fundamental laws governing a system playing a crucial role in solving the governing model of Partial Differential Equations (PDEs) by providing the necessary framework for simplifying complex physical processes are explained. The key non-dimensionalized parameters, such as the Peclet number, Rayleigh number, Bioconvection Rayleigh number, Lewis number, Relaxation time, and Prandtl number, are introduced to capture the balance between different physical forces and provide insight into the system's dynamic behavior.

Each of these parameters offers a simplified description of the system, allowing for an easier identification of dominant effects (e.g., convection, diffusion, or heat transfer) and guiding the solution of the system's PDEs. The cavity's boundary conditions, including thermal and concentration variations along the surfaces, are clearly defined, establishing the framework for the analysis. Initially, the physical model is presented in its dimensional form, accounting for the temperature gradients, microorganism concentration, and the porosity of the medium. To facilitate a more straightforward and scalable approach, the model is then non-dimensionalized by introducing key non-dimensional parameters, such as the Rayleigh number, Prandtl number, and Peclet number, among others, which help to simplify the governing equations.

Chapter 3 investigates the proposed problem, thermo-bioconvection in a square porous cavity filled with oxygen microorganisms, considering the intricate interactions between heat, fluid flow, and microbial behavior. The cavity's boundary conditions, including thermal and concentration variations along the surfaces, are clearly defined, establishing the framework for the analysis.

Initially, the physical model is presented in its dimensional form, accounting for the temperature gradients, microorganism concentration, and the porosity of the medium. To facilitate a more straightforward and scalable approach, the model is then non-dimensionalized by introducing key non-dimensional parameters, such as the Rayleigh number, Prandtl number, and Peclet number, among others, which help to simplify the governing equations. This non-dimensionalization significantly reduces the complexity of the analysis, making it easier to identify the dominant factors that influence the flow behavior.

Chapter 4 investigates the time relaxation effects in a thermo-bioconvective porous cavity flow filled with oxytactic microorganisms, utilizing the Galerkin Finite Element Method (FEM) via weighted residual in FreeFEM++ software. The process starts with the dimensional equations, which are then non-dimensionalized using scaling transformations to simplify the problem by reducing the number of physical parameters. After the equations are transformed into their dimensionless form, they are integrated over the flow domain, transitioning into the weak formulation required for numerical resolution. FreeFEM++ is used to solve the resulting dimensionless equations using Galerkin finite element discretization. It utilizes stream functions, isotherms, oxygen isoconcentrations, and microorganisms isoconcentrations to comprehensively examine the flow dynamics within a square cavity, focusing on the Time Relaxation Effects in Thermo-Bioconvective Porous Cavity Flow. The stream functions are used to derive the streamlines, which represent the flow patterns in the cavity, with their behavior offering valuable insights into the overall flow structure.

Chapter 5 contains the conclusion of this work.

The work's references are enumerated in Bibliography.

Chapter 2

Basic Terminologies and Governing Laws

In this chapter, all the definitions, governing laws and dimensionless parameters related to the governing model are discussed, providing a thorough understanding of the key concepts involved.

2.1 Foundational Concepts

2.1.1 Fluid

"A substance that cannot keep its own shape but instead adopts that of its container is referred to as a fluid." [31]

2.1.2 Fluid Mechanics

"The field of study known as fluid mechanics examines the behaviour of fluids(liquids or gases) both at rest and in motion."[31]

2.1.3 Fluid Dynamics

"The area of mathematics and physics that deals with describing and understanding how liquids and gases move." [31]

2.1.4 Fluid Statics

"The area of fluid mechanics known as fluid statics is responsible for studying incompressible fluids at rest." [31]

2.1.5 Pressure

"The continuous physical force exerted on the unit area of surface is said to be pressure. It is expressed by P and mathematically, it can be written as

$$P = \frac{F}{A},$$

where F and A denote the applied physical force and area of the surface."[32]

2.1.6 Density

"Density is defined as mass per unit volume. It is represented by Greek letter ρ and mathematically, it is defined as

$$\rho = \frac{m}{V},$$

where m and V are the mass and volume of the substance, respectively."[32]

2.1.7 Viscosity

"The resistance of a fluid to a change in shape or movement of neighboring components relative to one another is known as its viscosity. Mathematically,

$$\mu = \frac{\tau}{\frac{\partial u}{\partial y}},$$

where μ is viscosity coefficient, τ is shear stress and $\frac{\partial u}{\partial y}$ represents the velocity gradient."[31]

2.1.8 Kinematic Viscosity

"It is described as the relationship between the fluid's dynamic viscosity and density. It is represented by the symbol ν referred to as nu. Mathematically,

$$\nu = \frac{\mu}{\rho}.\text{"}[31]$$

2.1.9 Thermal Conductivity

"The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity which may be a function of a number of variables."[33]

2.1.10 Thermal Diffusivity

"The rate at which heat diffuses by conducting through a material depends on the thermal diffusivity and can be defined as,

$$\alpha = \frac{k}{\rho C_p},$$

where α is the thermal diffusivity, k is the thermal conductivity, ρ is the density and C_p is the specifc heat at constant pressure."[34]

2.2 Types of Fluid

2.2.1 Ideal Fluid

"A fluid, which is incompressible and has zero viscosity, is known as an ideal fluid. Ideal fluid is only an imaginary fluid as all the fluids, which exist, have some viscosity." [33]

2.2.2 Real Fluid

"A fluid is considered to be real if it has viscosity. All of the fluids are actual fluids in real life." [33]

2.2.3 Newtonian Fluid

"A Newtonian fluid is defined as one with constant viscosity, with zero shear rate at zero shear stress." [33]

2.2.4 Non-Newtonian Fluid

"A non-Newtonian fluid is a fluid that does not follow Newton law of viscosity, i.e., constant viscosity independent of stress.

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)^m, \quad m \neq 0$$

$$\tau_{xy} = \mu\left(\frac{du}{dy}\right)^m ." [33]$$

2.2.5 Magnetohydrodynamics

"The interaction of magnetic fields and fluid flow is the subject of magnetohydrodynamics (MHD). The fluids under consideration must be both electrically conducting and non-magnetic, which restricts us to liquids, hot ionic gases, and strong electrolyte." [35]

2.3 Types of Flow

2.3.1 Rotational Flow

"Rotational flow is that type of flow in which the fluid particles while flowing along stream-lines, also rotate about their own axis." [31]

2.3.2 Irrotational Flow

"Irrotational flow is that type of flow in which the fluid particles, while flowing along stream lines, do not rotate about their own axis, then this type of flow is called irrotational flow."[31]

2.3.3 Compressible Flow

"Compressible flow is that type of flow in the the density of the fluid changes from point to point, or in other words the density (ρ) is not constant for the fluid. Mathematically,

$$\rho \neq k,$$

where k stands constant." [31]

2.3.4 Incompressible Flow

"Incompressible flow is that type of flow in which the density is constant for the fluid. Liquids are generally incompressible, whereas gases are compressibles. Mathemtically,

$$\rho = k,$$

where k stands constant."[31]

2.3.5 Steady Flow

"The flow is referred to as steady flow if the flow properties, such as depth of flow, velocity of flow, and rate of flow at any location in open channel flow, do not fluctuate with regard to time. Mathematically,

$$\frac{\partial Q}{\partial t} = 0,$$

where Q is any fluid property." [31]

2.3.6 Unsteady Flow

"Unsteady flow is defined as flow in an open channel that changes with respect to time at any location in terms of velocity, depth, or rate. Mathematically,

$$\frac{\partial Q}{\partial t} \neq 0,$$

where Q is any fluid property." [31]

2.3.7 Internal Flow

"Flows completely bounded by a solid surfaces are called internal or duct flows." [36]

2.3.8 External Flow

"Flows over bodies immersed in an unbounded fluid are said to be an external flows." [36]

2.4 Modes of Heat Transfer

2.4.1 Heat Transfer

"Heat transfer is a branch of engineering that deals with the transfer of thermal energy from one point to another within a medium or from one medium to another due to the occurrence of a temperature difference."[33]

2.4.2 Conduction

"The transfer of heat within a medium due to a diffusion process is called conduction." [33]

2.4.3 Convection

"Convection heat transfer is usually defined as energy transport effected by the motion of a fluid.

This convection heat transfer between two dissimilar media is governed by Newton's law of cooling."[33]

2.4.4 Thermal Radiation

"Thermal Radiation is defined as radiant (electromagnetic) energy emitted by a medium and is solely to the temperature of the medium."[33]

2.4.5 Porous Medium

"A material containing the pores in it is called porous material or a porous medium. Pores are usually filled with fluid, i.e., liquid or gases.

A porous medium is often considered by its porosity. Many natural materials such as soil, rocks (e.g., aquifers, petroleum, zeolites), biological tissues (e.g., wood, bones, cork) and hand made substances such as ceramics and cements can be characterized as porous media." [32]

2.5 Governing Laws of Fluid Dynamics

2.5.1 Continuity Equation

"The principle of conservation of mass can be stated as the time rate of change of mass is fixed volume is equal to the net rate of flow of mass across the surface.

Mathematically, it can be written as

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0.\text{"}[33]$$

2.5.2 Momentum Equation

"The momentum equation states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newtons Third Law of action and reaction governs the internal forces.

Mathematically, it can be written as

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla.[\rho \mathbf{u}(\mathbf{u})] = -\nabla T + \rho g.\text{"}[33]$$

2.5.3 Energy Equation

"The law of conservation of energy states that the time rate of change of the total energy is equal to the sum of the rate of work done by the applied forces and change of heat content per unit time.

$$\frac{\partial \rho}{\partial t} + \nabla.\rho \mathbf{u} = -\nabla \mathbf{q} + Q + \phi,$$

where ϕ is the dissipation function." [33]

2.6 Dimensionless Parameters

2.6.1 Nusselt Number (Nu)

"It is the relationship between the convective to the conductive heat transfer through the boundary of the surface. Mathematically, it is defined as

$$\mathrm{Nu} = \frac{h \cdot L}{k}, \tag{2.1}$$

where Nu represents the Nusselt number, h stands for convective heat transfer, L stands for characteristic length and k stands for thermal conductivity."[37]

2.6.2 Sherwood Number (Sh)

"It is a non-dimensional quantity which describes the ratio of the mass transport by convection to the transfer of mass by diffusion.

Mathematically,

$$\mathrm{Sh} = \frac{h_m L}{D}, \tag{2.2}$$

where Sh is the Sherwood number, h_m represents convective mass transfer coefficient, L is the characteristic length and D is the mass diffusivity."[36]

2.6.3 Peclet Number (Pe)

"The Peclet number (Pe) is a dimensionless parameter defined as the product of the Reynolds and Prandtl numbers and characterizes the relative importance of advection and diffusion in heat transport.

Mathematically,

$$Pe = Re.Pr = \frac{u \cdot L}{D}, \tag{2.3}$$

where u denotes flow velocity, L represents characteristic length, and D is the mass diffusivity." [38]

2.6.4 Darcy Number (D_a)

"The Darcy number D_a represents the effect of the permeability of medium according to its cross sectional area

$$D_a = \frac{\kappa}{H^2}, \tag{2.4}$$

where κ shows the permeability of porus medium and H is the length of prescribed geometry. It was first introduced by Henry Darcy. It is transformed by the non-dimensionalizing the differential form of Darcy's law."[39]

2.6.5 Lewis Number (Le)

"The Lewis number L_e is a dimensionless quantity that compares the rate of heat transfer to the rate of mass transfer in a fluid. In simpler terms, it tells you how effectively a fluid can transport heat compared to how effectively it can transport mass.

$$\mathrm{Le} = \frac{\alpha}{D}, \tag{2.5}$$

where α represents the thermal diffusivity and D represents mass diffusivity." [39]

2.6.6 Prandtl Number (Pr)

"The ratio of kinematic diffusivity to heat the diffusivity is said to be Prandtl number. It is denoted by Pr. Mathematically, it can be written as

$$\mathrm{Pr} = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} = \frac{\nu}{\alpha}, \tag{2.6}$$

where ν is the kinematic viscosity and α is the thermal diffusivity". [39]

2.7 Galerkin FEM Method

The Galerkin Finite Element Method (FEM) is a powerful numerical technique for solving differential equations. It combines the idea of weighted residuals with the finite element approach. Some steps are given below to illustrate the galerkin method.

Step 1: (Problem Definition) Start with the governing differential equation you want to solve. That equation represents the physical problem you are modeling. Then define the conditions at the boundaries of your domain. These conditions specify the behavior of the solution at the edges of the problem space.

Step 2: (Weak Formulation) Multiply the differential equation by a weight function (also called a test function) and integrate over the problem domain. Apply integration by parts to reduce the order of the highest derivative in the equation.

This step is crucial for the Galerkin method. The result is the weak form of the differential equation. It involves lower-order derivatives, making it easier to work with.

Step 3: (Discretization) Divide the problem domain into smaller, simpler elements (e.g., line segments in 1D, triangles in 2D). Choose basis functions (also called shape functions) to represent the solution within each element. These functions are typically polynomials. Express the approximate solution as a linear combination of the basis functions, with unknown coefficients.

Step 4: (System of Equations) The orthogonality condition results in a system of linear equations, often expressed in matrix form. Solve the system of equations to find the unknown coefficients of the basis functions.

2.7.1 Example: 1D Heat Equation

To demonstrate the Galerkin method, consider the following one dimensional heat equation.

$$\frac{d^2u}{dx^2} + f(x) = 0, \quad 0 < x < 1$$

with the following boundary conditions

$$u(0) = 0, \quad u(1) = 0.$$

1. Weak Formulation: Multiply by a weight function $v(x)$, integrate, and apply integration by parts:

   $$\int \frac{d^2u}{dx^2} v \, dx = -\int \frac{du}{dx}\frac{dv}{dx} \, dx + \left[\frac{du}{dx} v\right]_0^1.$$

   Using the boundary conditions, the second term vanishes. The weak form becomes

   $$-\int \frac{du}{dx}\frac{dv}{dx} \, dx + \int f(x) v \, dx = 0.$$

2. Discretization: Divide the domain $[0, 1]$ into elements. Let's use linear basis functions $\phi_i(x)$. The approximate solution is expressed as

$$u(x) \approx \sum u_i \phi_i(x).$$

3. Galerkin Method: Choose weight functions $v(x) = \phi_j(x)$. Substitute the approximate solution into the weak form:

$$-\int \left(\sum u_i \frac{d\phi_i}{dx} \right) \frac{d\phi_j}{dx}\, dx + \int f(x)\phi_j(x)\, dx = 0.$$

4. System of Equations: This leads to a system of linear equations:

$$[K]\{u\} = \{F\},$$

   where, $K_{ij} = -\int \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\, dx$ is the Stiffness matrix, $F_j = \int f(x)\phi_j(x)\, dx$ is the Force vector, and $\{u\}$ is the vector of unknown coefficients u_i.

5. Solve: Solve the system of equations to find the coefficients u_i.

6. Solution: The approximate solution is obtained by $u(x) \approx \sum u_i \phi_i(x)$.

Chapter 3

Thermo-Bioconvection in Porous Cavity by Oxytactic Microorganisms

The primary goal of this chapter is to build upon the foundational work of Kuznetsov (2006) [8] by extending it to the context of thermo-bioconvection within a square porous cavity that is saturated with oxytactic microorganisms. The study makes certain simplifying assumptions: the porous matrix is inert in terms of microbial interactions, meaning it does not absorb or otherwise alter the concentration of the bacteria.

Furthermore, it is assumed that the pore sizes within the matrix are significantly larger than the microorganisms themselves, ensuring that the microorganisms' oxytactic behavior—defined as the movement or aggregation of bacteria in response to a temperature or concentration gradient—remains unaffected by the presence of the porous medium. This allows the analysis to focus solely on the interactions between the temperature gradient, the microorganisms' behavior, and the resulting convective flows. The work presented in this chapter is expected to contribute novel and original insights to the literature concerning convection in porous media that are biologically active, particularly where the dynamics of oxytaxis and heat transfer are involved.

It is anticipated that these results will deepen our understanding of such systems, which have implications for biological processes in porous environments, including those relevant in natural systems and industrial applications.

The findings presented here can offer new perspectives on the role of microorganism behavior in thermo-convective flows, thereby extending the theoretical and practical understanding of these complex systems.

3.1 Mathematical Modeling

Consider a two-dimensional porous square cavity that is filled with oxytactic microorganisms and has walls of length H. Presumably, the enclosure's left hot and right cooled walls are kept at constant temperatures T_H and T_C, respectively, with $T_H > T_C$, and the walls at the top and bottom are adiabatic. In opposition to the y-axis is the direction in which the gravity vector, **g**, acts.

The model that is being presented here is based on a continuum model that Hillesdon and Pedley [5] developed for a suspension of oxytactic microorganisms. An energy equation and a buoyancy term in the momentum equation resulting from the temperature variation across the cavity complete this model.

The suspension is taken to be diluted and the Boussinesq approximation is applied. The governing equations are expressed as in the following form:

Continuity equation

$$\nabla.\mathbf{v} = 0. \tag{3.1}$$

Momentum (Darcy) equation

$$\frac{\mu}{K} v = -\nabla p + \left[\gamma \Delta\rho n - \rho_f \beta \left(T - T_C\right)\right] g. \tag{3.2}$$

Time relaxed energy equation

$$v.\nabla T = \alpha_m \nabla^2 T. \tag{3.3}$$

Oxygen conservation equation

$$v.\nabla C = D_C \nabla^2 C - \delta_n. \tag{3.4}$$

Cell conservation equation

$$\nabla.\mathbf{j} = 0, \tag{3.5}$$

where

$$\mathbf{j} = n\mathbf{v} + n\tilde{\mathbf{v}} - D_n \nabla_n. \tag{3.6}$$

Here v is the fluid filtration velocity vector; T is the fluid temperature; α_m is the effective thermal diffusivity of the porous medium; K is the permeability of the porous medium; n is the number density of motile microorganisms; p is the excess pressure (above hydro-static); β is the volume expansion coefficient of water at constant pressure; j is the flux of microorganisms due to macroscopic motion of fluid; directional swimming of microorganisms up the oxygen gradients and a diffusive process that models all random motions of microorganisms, respectively; μ is the dynamic viscosity of the suspension (the suspension includes fluid plus microorganisms, μ assumed to be constant and approximately the same as that of water); ρ_f is the density of the fluid; $\Delta\rho = \rho_{cell} - \rho_f$ is the density difference between cells and fluid; γ is the average volume of a microorganism; D_C is the diffusivity of oxygen; D_n is the diffusivity of microorganisms; ∇^2 is the Laplacian operator; $-\delta n$ describes the consumption of oxygen by the microorganisms; $\Delta C = C_{min}$ where C_0 is the free-surface oxygen concentration; and C_{min} is the minimum oxygen concentration that microorganisms need in order to be active. The terms on the right-hand side of Eq. (3.6) represent the flux of microorganisms due to macroscopic motion of the fluid, the directional swimming of microorganisms up the oxygen gradients, and a diffusive process that models all random motions of microorganisms, respectively. The average directional swimming velocity of a microorganism $\tilde{v}$ in Eq. (3.6) can be approximate as

$$\tilde{\mathbf{v}} = bW_C \nabla C / \Delta C, \tag{3.7}$$

where b is the chemotaxis constant [m] and Wc is the maximum cell swimming speed [m/s] (the product bWc is assumed to be constant). Eq. (3.1)- (3.5) can be written in Cartesian co-ordinates x and y as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.8}$$

$$\frac{\mu}{K} u = -\frac{\partial p}{\partial x}, \tag{3.9}$$

$$\frac{\mu}{K} v = -\frac{\partial p}{\partial y} - [\gamma \Delta \rho n - \rho_f \beta (T - T_C)] g, \tag{3.10}$$

or, if we eliminate the pressure gradients,

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{gK}{\mu}\left(\rho_f \beta \frac{\partial T}{\partial x} - \gamma \Delta \rho \frac{\partial n}{\partial x}\right), \tag{3.11}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{3.12}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_C \left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - \delta_n, \tag{3.13}$$

$$\frac{\partial}{\partial x}\left[un + \tilde{u}n - D_n \frac{\partial n}{\partial x}\right] + \frac{\partial}{\partial y}\left[vn + \tilde{v}n - D_n \frac{\partial n}{\partial y}\right] = 0, \tag{3.14}$$

where u and v are the velocity components along the $x-$ and $y-$axes, respectively, and $\tilde{u}$ and $\tilde{v}$ are provided by $\tilde{u} = \left(\frac{bW_c}{\Delta C}\right)\frac{\partial C}{\partial x}$ and $\tilde{v} = \left(\frac{bW_c}{\Delta C}\right)\frac{\partial C}{\partial y}$. Equations (3.8)-(3.14) can be written in dimensionless form using the following variable

$$X = \frac{x}{H}, \quad Y = \frac{y}{H}, \quad \theta = \frac{(T - T_C)}{\Delta T}, \quad \phi = \frac{(C - C_{min})}{\Delta C}, \quad N = \frac{n}{n_0}, \quad p = \frac{\alpha_m^2 P}{H^2}, \tag{3.15}$$

where n_0 is the average number density of the microorganisms (number density of the microorganisms in a well-stirred suspension) and we take velocities u and v, which are defined as $u = (\alpha_m/H)U$ and $v = (\alpha_m/H)V$. As a result, we get the following dimensionless partial differential equations for the model problem under consideration:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{3.16}$$

$$\frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} = -Ra\left[\frac{\partial \theta}{\partial X} - Rb\frac{\partial N}{\partial X}\right], \tag{3.17}$$

$$U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}, \tag{3.18}$$

$$U\frac{\partial \phi}{\partial X}+V\frac{\partial \phi}{\partial Y}=\frac{1}{Le}\left[\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\right]-\frac{\sigma}{Le}N, \tag{3.19}$$

$$\frac{\partial}{\partial X}(UN)+\frac{\partial}{\partial Y}(VN)+Pe\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\right)+\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\right)\right)=\frac{1}{Le\chi}\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right], \tag{3.20}$$

where the parameters Ra is the traditional Rayleigh number for a porous medium, Le is the Lewis number, Rb is the bio-convection Rayleigh number, Pe is the Péclet number, which can be regarded as a ratio of two characteristic velocities, one due to oxytactic swimming and the other due to random, diffusive swimming (Hillesdon and Pedley 1996) and σ is a constant, which characterizes the ratio of the rate of oxygen consumption to the rate of oxygen diffusion. These parameters are defined as

$$R_a=\frac{gK\beta\Delta TH}{\nu\alpha_m},\ R_b=\frac{\gamma\Delta\rho n_0}{\rho_f\beta\Delta T},\ \sigma=\frac{\delta n_0H^2}{D_c\Delta C},\ L_e=\frac{\alpha_m}{D_C},\ \chi=\frac{D_C}{D_n},\ P_e=\frac{bW_C}{\alpha_m}. \tag{3.21}$$

The following dimensionless boundary conditions for Eqs. (3.16)- (3.20) can be presented as

$$\begin{aligned}
&\Psi=0 \quad \text{at all walls}\\
&\theta=1,\quad \phi=1,\quad N=1 \quad \text{at} \quad X=0,\\
&\theta=0,\quad \phi=1,\quad N=1 \quad \text{at} \quad X=1,\\
&\frac{\partial\theta}{\partial Y}=0,\quad \phi=1,\quad PeN\frac{\partial\phi}{\partial Y}-\frac{\partial N}{\partial Y}=0 \quad \text{at} \quad Y=0,\\
&\frac{\partial\theta}{\partial Y}=0,\quad \frac{\partial\phi}{\partial Y}=0,\quad \frac{\partial N}{\partial Y}=0 \quad \text{at} \quad Y=1.
\end{aligned}$$

The definitions for the local Nusselt and Sherwood numbers (Nu_Y, Sh_Y) on the vertical walls and average Nusselt and Sherwood numbers (Nu, Sh) are given by

$$Nu_Y=-\left(\frac{\partial\theta}{\partial X}\right)_{X=0,1},\quad Sh_Y=-\left(\frac{\partial\phi}{\partial X}\right)_{X=0,1}, \tag{3.22}$$

and

$$Nu = \int_0^1 Nu_Y dY, \quad Sh = \int_0^1 Sh_Y dY, \tag{3.23}$$

respectively.

3.2 Weak Formulation

In order to transform differential equations into integral form, weak formulation is a variational method that multiplies the dependent variable by a suitable test function and then integrates the result over the whole computational domain. The weak formulation of the strong form of the governing PDEs from Eqs. (3.16) to (3.20) is written below:

The weak form for u-component of momentum equation (3.16) as follows:

$$\frac{\partial U}{\partial t} + A_1 U = -\frac{\partial P}{\partial X},$$

Multiplying by test function $\tilde{U}$ first and then integrate over computational domain,

$$A_1 U\tilde{U} - q\frac{\partial U}{\partial X}\tilde{U} = -\frac{\partial P}{\partial X}\tilde{U}, \tag{3.24}$$

as

$$\frac{\partial}{\partial X}(U\tilde{U}) = U\frac{\partial \tilde{U}}{\partial X} + \frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}(U\tilde{U}) - U\frac{\partial \tilde{U}}{\partial X} = \frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow -\left[\frac{\partial}{\partial X}(U\tilde{U}) - U\frac{\partial \tilde{U}}{\partial X}\right] = -\frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow -\frac{\partial}{\partial X}(U\tilde{U}) + U\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial U}{\partial X}\tilde{U}$$

$$\frac{\partial}{\partial X}(P\tilde{U}) = P\frac{\partial \tilde{U}}{\partial X} + \frac{\partial P}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}(P\tilde{U}) - P\frac{\partial \tilde{U}}{\partial X} = \frac{\partial P}{\partial X}\tilde{U}$$

$$\Rightarrow -\left[\frac{\partial}{\partial X}(P\tilde{U}) - P\frac{\partial \tilde{U}}{\partial X}\right] = -\frac{\partial P}{\partial X}\tilde{U}$$

$$\Rightarrow -\frac{\partial}{\partial X}(P\tilde{U}) + P\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial P}{\partial X}\tilde{U}.$$

So, equation (3.24) becomes

$$A_1 U\tilde{U} - q\frac{\partial}{\partial X}(U\tilde{U}) + qU\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial}{\partial X}(P\tilde{U}) + P\frac{\partial \tilde{U}}{\partial X},$$

$$\begin{aligned} A_1\int\limits_{\Omega^n} U\tilde{U}d\Omega - q\int\limits_{\Omega^n}\frac{\partial}{\partial X}(U\tilde{U})d\Omega + q\int\limits_{\Omega^n} U\frac{\partial \tilde{U}}{\partial X}d\Omega = \\ -\int\limits_{\Omega^n}\frac{\partial}{\partial X}(P\tilde{U})d\Omega + P\int\limits_{\Omega^n}\frac{\partial \tilde{U}}{\partial X}d\Omega, \end{aligned} \tag{3.25}$$

$$\begin{aligned} A_1\int\limits_{\Omega^n} U\tilde{U}d\Omega - \cancelto{0}{q\int\limits_{\Omega^n}\frac{\partial}{\partial X}(U\tilde{U})d\Omega} + q\int\limits_{\Omega^n} U\frac{\partial \tilde{U}}{\partial X}d\Omega = \\ -\cancelto{0}{\int\limits_{\Omega^n}\frac{\partial}{\partial X}(P\tilde{U})d\Omega} + P\int\limits_{\Omega^n}\frac{\partial \tilde{U}}{\partial X}d\Omega, \end{aligned}$$

$$A_1\int\limits_{\Omega^n} U\tilde{U}d\Omega + q\int\limits_{\Omega^n} U\frac{\partial \tilde{U}}{\partial X}d\Omega = P\int\limits_{\Omega^n}\frac{\partial \tilde{U}}{\partial X}d\Omega.$$

Taking Domain as current time step value, we have

$$A_1\int\limits_{\Omega^{n+1}} U^{n+1}\tilde{U}d\Omega + q^{n+1}\int\limits_{\Omega^{n+1}} U^{n+1}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega = P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega,$$

which is weak form of u-component of momentum equation.

Similarly, we obtain the weak form for v-component of momentum equation (3.16) as follows:

$$A_2 V = -\frac{\partial P}{\partial Y} - A_3\Big[A_4 N - \theta\Big].$$

Multiplying by test function $\tilde{V}$ first and then integrate over computational domain,

$$\begin{aligned} A_2 V\tilde{V} - q\frac{\partial V}{\partial Y}\tilde{V} = -\frac{\partial P}{\partial Y}\tilde{V} - A_3\Big[A_4 N - \theta\Big]\tilde{V}, \\ \text{as } \frac{\partial}{\partial Y}(V\tilde{V}) = V\frac{\partial \tilde{V}}{\partial Y} + \frac{\partial V}{\partial Y}\tilde{V}, \end{aligned} \tag{3.26}$$

$$\Rightarrow \frac{\partial}{\partial Y}(V\tilde{V}) - V\frac{\partial \tilde{V}}{\partial Y} = \frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow -\left[\frac{\partial}{\partial Y}(V\tilde{V}) - V\frac{\partial \tilde{V}}{\partial Y}\right] = -\frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow -\frac{\partial}{\partial Y}(V\tilde{V}) + V\frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial V}{\partial Y}\tilde{V}$$

$$\frac{\partial}{\partial Y}(P\tilde{V}) = P\frac{\partial \tilde{V}}{\partial Y} + \frac{\partial P}{\partial Y}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial Y}(P\tilde{V}) - P\frac{\partial \tilde{V}}{\partial Y} = \frac{\partial P}{\partial Y}\tilde{V}$$

$$\Rightarrow -\left[\frac{\partial}{\partial Y}(P\tilde{V}) - P\frac{\partial \tilde{V}}{\partial Y}\right] = -\frac{\partial P}{\partial Y}\tilde{V}$$

$$\Rightarrow -\frac{\partial}{\partial Y}(P\tilde{V}) + P\frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial P}{\partial Y}\tilde{V}.$$

So, equation (3.26) becomes:

$$A_2 V\tilde{V} - q\frac{\partial}{\partial Y}(V\tilde{V}) + qV\frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial}{\partial Y}(P\tilde{V}) + P\frac{\partial \tilde{V}}{\partial Y} - A_3\left[A_4 N - \theta\right]\tilde{V},$$

$$\begin{aligned} A_2 \int_{\Omega^n} V\tilde{V}d\Omega - q\int_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega + q\int_{\Omega^n} V\frac{\partial \tilde{V}}{\partial X}d\Omega = \\ -\int_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})d\Omega + P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega - A_3A_4\int_{\Omega^n} N\tilde{V}d\Omega + A_3\int_{\Omega^n}\theta\tilde{V}d\Omega, \end{aligned} \tag{3.27}$$

$$\begin{aligned} A_2 \int_{\Omega^n} V\tilde{V}d\Omega - \cancelto{0}{q\int_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega} + q\int_{\Omega^n} V\frac{\partial \tilde{V}}{\partial X}d\Omega = \\ -\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})d\Omega} + P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega - A_3A_4\int_{\Omega^n} N\tilde{V}d\Omega + A_3\int_{\Omega^n}\theta\tilde{V}d\Omega, \end{aligned}$$

$$\begin{aligned} A_2 \int_{\Omega^n} V\tilde{V}d\Omega + q\int_{\Omega^n} V\frac{\partial \tilde{V}}{\partial X}d\Omega = \\ P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega - A_3A_4\int_{\Omega^n} N\tilde{V}d\Omega + A_3\int_{\Omega^n}\theta\tilde{V}d\Omega. \end{aligned}$$

Taking domain as current time step value, we have

$$A_2 \int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V} d\Omega + q^{n+1} \int\limits_{\Omega^{n+1}} V^{n+1} \frac{\partial \tilde{V}}{\partial X^{n+1}} d\Omega =$$

$$P^{n+1} \int\limits_{\Omega^{n+1}} \frac{\partial \tilde{V}}{\partial Y^{n+1}} d\Omega - A_3 A_4 \int\limits_{\Omega^{n+1}} N^{n+1}\tilde{V} d\Omega + A_3 \int\limits_{\Omega^{n+1}} \theta^{n+1}\tilde{V} d\Omega,$$

which is weak form of v-component of momentum equation. The energy equation (3.17) in the same way becomes:

$$U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}$$

Multiplying by test function $\tilde{\theta}$ first and then integrate over computational domain,

$$\left[\left(U\frac{\partial \theta}{\partial X}\right) + \left(V\frac{\partial \theta}{\partial Y}\right)\right]\tilde{\theta} = \frac{\partial^2 \theta}{\partial X^2}\tilde{\theta} + \frac{\partial^2 \theta}{\partial Y^2}\tilde{\theta}, \tag{3.28}$$

$$\int\limits_{\Omega^n} \left[\left(U\frac{\partial \theta}{\partial X}\right) + \left(V\frac{\partial \theta}{\partial Y}\right)\right]\tilde{\theta} d\Omega = \int\limits_{\Omega^n} \frac{\partial^2 \theta}{\partial X^2}\tilde{\theta} d\Omega + \int\limits_{\Omega^n} \frac{\partial^2 \theta}{\partial Y^2}\tilde{\theta} d\Omega,$$

$$\frac{1}{\delta t}\left(\theta^n \ o \ X^n\right)\tilde{\theta} d\Omega = \int\limits_{\Omega^n} \frac{\partial^2 \theta}{\partial X^2}\tilde{\theta} d\Omega + \int\limits_{\Omega^n} \frac{\partial^2 \theta}{\partial Y^2}\tilde{\theta} d\Omega.$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega} \psi \Delta\phi \ d\Omega = -\int\limits_{\Omega} \nabla\phi\nabla\psi \ d\Omega + \int\limits_{\Omega} \psi(\nabla\phi n) \ d\Gamma.$$

Here, $\psi = \tilde{\theta}$, $\Delta\phi = \frac{\partial^2 \theta}{\partial X^2}$, $\Delta\phi = \frac{\partial^2 \theta}{\partial Y^2}$, $\nabla\phi = \frac{\partial \theta}{\partial X}$, $\nabla\phi = \frac{\partial \theta}{\partial Y}$,
$\nabla\psi = \frac{\partial \tilde{\theta}}{\partial X}$, $\nabla\psi = \frac{\partial \tilde{\theta}}{\partial Y}$,
as $\quad \nabla\phi n = \frac{\partial \phi}{\partial n} = n_x\frac{\partial \phi}{\partial X} + n_y\frac{\partial \phi}{\partial Y} \quad (\phi = \theta).$
So,

$$\int\limits_{\Omega^n} \left[\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2}\right]\tilde{\theta} d\Omega = \Bigg[-\int\limits_{\Omega^n} \frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial X} d\Omega$$

$$+ \oint\limits_{\Gamma} \tilde{\theta}\left(n_x\frac{\partial \theta}{\partial X}\right) d\Gamma\Bigg] + \Bigg[-\int\limits_{\Omega^n} \frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial Y} d\Omega + \oint\limits_{\Gamma} \tilde{\theta}\left(n_y\frac{\partial \theta}{\partial Y}\right) d\Gamma\Bigg].$$

Now (3.28) becomes

$$\frac{1}{\delta t}\left(\theta^n \circ X^n\right)\tilde{\theta}\, d\Omega = -\int\limits_{\Omega^n} \frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial X}\, d\Omega + \oint\limits_{\Gamma} \tilde{\theta}\left(n_x \frac{\partial \theta}{\partial X}\right) d\Gamma$$
$$-\int\limits_{\Omega^n} \frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial Y}\, d\Omega + \oint\limits_{\Gamma} \tilde{\theta}\left(n_y \frac{\partial \theta}{\partial Y}\right) d\Gamma.$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\Big(\theta^n \ o \ X^n\Big)\tilde{\theta} d\Omega = -\int\limits_{\Omega^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{\theta}}{\partial X^{n+1}} d\Omega + \oint\limits_{\Gamma} \tilde{\theta}\Big(n_x \frac{\partial \theta^{n+1}}{\partial X^{n+1}}\Big) d\Gamma$$
$$-\int\limits_{\Omega^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{\theta}}{\partial Y^{n+1}} d\Omega + \oint\limits_{\Gamma} \tilde{\theta}\Big(n_y \frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\Big) d\Gamma,$$

which is weak form of energy equation.

The oxygen conservation equation (3.18) in the same way becomes:

$$U\frac{\partial \phi}{\partial X}+V\frac{\partial \phi}{\partial Y} = A_6\Big(\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\Big)-A_7 N.$$

Multiplying by test function $\tilde{\phi}$ first and then integrate over computational domain,

$$\Big[\Big(U\frac{\partial \phi}{\partial X}\Big)+\Big(V\frac{\partial \phi}{\partial Y}\Big)\Big]\tilde{\phi} = A_6\Big(\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\Big)\tilde{\phi}-A_7 N\tilde{\phi},$$

$$\int\limits_{\Omega^n}\Big[\Big(U\frac{\partial \phi}{\partial X}\Big)+\Big(V\frac{\partial \phi}{\partial Y}\Big)\Big]\tilde{\phi} d\Omega = A_6\int\limits_{\Omega^n}\Big(\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\Big)\tilde{\phi} d\Omega-A_7\int\limits_{\Omega^n} N\tilde{\phi} d\Omega. \tag{3.29}$$

So (3.29), becomes

$$-\frac{1}{\delta t}\Big(\phi^n \ o \ X^n\Big)\tilde{\phi} d\Omega = A_6\int\limits_{\Omega^n}\Big(\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\Big)\tilde{\phi} d\Omega-A_7\int\limits_{\Omega^n} N\tilde{\phi} d\Omega. \tag{3.30}$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega} \psi\Delta\phi \ d\Omega = -\int\limits_{\Omega} \nabla\phi\nabla\psi \ d\Omega + \int\limits_{\Omega} \psi(\nabla\phi n) \ d\Gamma.$$

Here, $\psi = \tilde{\phi}$, $\Delta\phi = \frac{\partial^2 \phi}{\partial X^2}$, $\Delta\phi = \frac{\partial^2 \phi}{\partial Y^2}$, $\nabla\phi = \frac{\partial \phi}{\partial X}$, $\nabla\phi = \frac{\partial \phi}{\partial Y}$,

$$\nabla\psi = \frac{\partial\tilde{\phi}}{\partial X}, \ \nabla\psi = \frac{\partial\tilde{\phi}}{\partial Y}.$$

As $$\nabla\phi n = \frac{\partial\phi}{\partial n} = n_x\frac{\partial\phi}{\partial X} + n_y\frac{\partial\phi}{\partial Y} \qquad (\phi = \phi).$$

So,

$$\int_{\Omega^n}\left[\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2}\right]\tilde{\phi}d\Omega$$
$$= \left[-\int_{\Omega^n}\frac{\partial\phi}{\partial X}\frac{\partial\tilde{\phi}}{\partial X}d\Omega + \oint_{\Gamma}\tilde{\phi}\left(n_x\frac{\partial\phi}{\partial X}\right)d\Gamma - \int_{\Omega^n}\frac{\partial\phi}{\partial Y}\frac{\partial\tilde{\phi}}{\partial Y}d\Omega + \oint_{\Gamma}\tilde{\phi}\left(n_y\frac{\partial\phi}{\partial Y}\right)d\Gamma\right],$$

Now (3.30), becomes:

$$-\frac{1}{\delta t}\left(\phi^n \ o \ X^n\right)\tilde{\phi}d\Omega = A_6\left[\int_{\Omega^n}\frac{\partial\phi}{\partial X}\frac{\partial\tilde{\phi}}{\partial X}d\Omega + \oint_{\Gamma}\tilde{\phi}\left(n_x\frac{\partial\phi}{\partial X}\right)d\Gamma - \int_{\Omega^n}\frac{\partial\phi}{\partial Y}\frac{\partial\tilde{\phi}}{\partial Y}d\Omega\right.$$
$$\left.+\oint_{\Gamma}\tilde{\phi}\left(n_y\frac{\partial\phi}{\partial Y}\right)d\Gamma\right] - A_7\int_{\Omega^n}N\tilde{\phi}d\Omega.$$

Taking domain as current time step value, we have

$$-\frac{1}{\delta t}\left(\phi^n \ o \ X^n\right)\tilde{\phi}d\Omega = A_6\left[-\int_{\Omega^{n+1}}\frac{\partial\phi^{n+1}}{\partial X^{n+1}}\frac{\partial\tilde{\phi}}{\partial X^{n+1}}d\Omega + \oint_{\Gamma}\tilde{\phi}\left(n_x\frac{\partial\phi^{n+1}}{\partial X^{n+1}}\right)d\Gamma\right.$$
$$\left.-\int_{\Omega^{n+1}}\frac{\partial\phi^{n+1}}{\partial Y^{n+1}}\frac{\partial\tilde{\phi}}{\partial Y^{n+1}}d\Omega + \oint_{\Gamma}\tilde{\phi}\left(n_y\frac{\partial\phi^{n+1}}{\partial Y^{n+1}}\right)d\Gamma\right] - A_7\int_{\Omega^{n+1}}N^{n+1}\tilde{\phi}d\Omega,$$

which is weak form of oxygen conservation equation.

The cell conservation equation (3.19) in the same way becomes:

$$\frac{\partial}{\partial Y}(VN)+A_8\left(\frac{\partial}{\partial X}\left(N\frac{\partial\phi}{\partial X}\right)+\frac{\partial}{\partial Y}\left(N\frac{\partial\phi}{\partial Y}\right)\right) = A_9\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right],$$

Multiplying by test function $\tilde{N}$ first and then integrate over computational domain,

$$\left(U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N} + \left(V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N} + A_8\left(N\frac{\partial^2\phi}{\partial X^2} + N\frac{\partial^2\phi}{\partial Y^2}\right)\tilde{N}$$
$$= A_9\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}, \tag{3.31}$$

as $$\frac{\partial}{\partial X}\left(N\frac{\partial\phi}{\partial X}\tilde{N}\right) = N\frac{\partial\phi}{\partial X}\frac{\partial}{\partial X}(\tilde{N}) + N\frac{\partial}{\partial X}\left(\frac{\partial\phi}{\partial X}\right)\tilde{N} + \frac{\partial}{\partial X}(N)\frac{\partial\phi}{\partial X}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) = N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} + N\frac{\partial^2 \phi}{\partial X^2}\tilde{N} + \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} = N\frac{\partial^2 \phi}{\partial X^2}\tilde{N}$$

$$\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) = N\frac{\partial \phi}{\partial Y}\frac{\partial}{\partial Y}(\tilde{N}) + N\frac{\partial}{\partial Y}\left(\frac{\partial \phi}{\partial Y}\right)\tilde{N} + \frac{\partial}{\partial Y}(N)\frac{\partial \phi}{\partial Y}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) = N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} + N\frac{\partial^2 \phi}{\partial Y^2}\tilde{N} + \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N} = N\frac{\partial^2 \phi}{\partial Y^2}\tilde{N}.$$

So (3.31), becomes

$$\frac{1}{2}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N} + \frac{1}{2}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N}$$
$$+ A_8\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} + \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right)\right.$$
$$\left. - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right) = A_9\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N},$$

$$\frac{1}{2}\int\limits_{\Omega^n}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega + \frac{1}{2}\int\limits_{\Omega^n}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y}\right.$$
$$\left. + N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega + A_8\int\limits_{\Omega^n}\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} + \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right)\right.$$
$$\left. - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega = A_9\int\limits_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega,$$

$$\frac{1}{2}\int\limits_{\Omega^n}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N}\,d\Omega$$
$$+ \frac{1}{2}\int\limits_{\Omega^n}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N}\,d\Omega$$
$$+ A_8\int\limits_{\Omega^n}\left[\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}\right.$$
$$\left. + \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right]d\Omega$$
$$= A_9\int\limits_{\Omega^n}\left(\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right)\tilde{N}\,d\Omega.$$

$$\frac{1}{2}\int\limits_{\Omega^n}\left(U\frac{\partial N}{\partial X}+U\frac{\partial N}{\partial X}+N\frac{\partial U}{\partial X}+N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega+\frac{1}{2}\int\limits_{\Omega^n}\left(V\frac{\partial N}{\partial Y}+V\frac{\partial N}{\partial Y}+N\frac{\partial V}{\partial Y}\right.$$

$$\left.+N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega+A_8\int\limits_{\Omega^n}\left(\cancelto{0}{\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right)}-N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X}-\frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}\right.$$

$$\left.+\cancelto{0}{\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right)}-N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}-\frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega=A_9\int\limits_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega,$$

$$\frac{1}{2}\int\limits_{\Omega^n}\left(U\frac{\partial N}{\partial X}+U\frac{\partial N}{\partial X}+N\frac{\partial U}{\partial X}+N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega+\frac{1}{2}\int\limits_{\Omega^n}\left(V\frac{\partial N}{\partial Y}+V\frac{\partial N}{\partial Y}+N\frac{\partial V}{\partial Y}\right.$$

$$\left.+N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega+A_8\int\limits_{\Omega^n}\left(-N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X}-\frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}-N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}-\frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega$$

$$=A_9\int\limits_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega. \tag{3.32}$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega}\psi\Delta\phi\ d\Omega=-\int\limits_{\Omega}\nabla\phi\nabla\psi\ d\Omega+\int\limits_{\Omega}\psi(\nabla\phi n)\ d\Gamma,$$

here $\psi=\tilde{N},\ \Delta\phi=\frac{\partial^2 N}{\partial X^2},\ \Delta\phi=\frac{\partial^2 N}{\partial Y^2},\ \nabla\phi=\frac{\partial N}{\partial X},\ \nabla\phi=\frac{\partial N}{\partial Y},$
$\nabla\psi=\frac{\partial \tilde{N}}{\partial X},\ \nabla\psi=\frac{\partial \tilde{N}}{\partial Y}$

as $\nabla\phi$

n= $\frac{\partial \phi}{\partial n}=n_x\frac{\partial N}{\partial X}+n_y\frac{\partial N}{\partial Y}.\qquad (\phi=N)$

So

$$\int\limits_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega$$

$$=\left[-\int\limits_{\Omega^n}\frac{\partial N}{\partial X}\frac{\partial \tilde{N}}{\partial X}d\Omega+\oint\limits_{\Gamma}\tilde{N}\left(n_x\frac{\partial N}{\partial X}\right)d\Gamma-\int\limits_{\Omega^n}\frac{\partial N}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}d\Omega+\oint\limits_{\Gamma}\tilde{N}\left(n_y\frac{\partial N}{\partial Y}\right)d\Gamma\right].$$

Now (3.32), becomes

$$\frac{1}{2}\int_{\Omega^n}\left(U\frac{\partial N}{\partial X}+U\frac{\partial N}{\partial X}+N\frac{\partial U}{\partial X}+N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega+\frac{1}{2}\int_{\Omega^n}\left(V\frac{\partial N}{\partial Y}+V\frac{\partial N}{\partial Y}+N\frac{\partial V}{\partial Y}\right.$$

$$\left.+N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega+A_8\int_{\Omega^n}\left(-N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X}-\frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}-N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}-\frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega$$

$$=A_9\left(\left[-\int_{\Omega^n}\frac{\partial N}{\partial X}\frac{\partial \tilde{N}}{\partial X}d\Omega+\oint_{\Gamma}\tilde{N}\left(n_x\frac{\partial N}{\partial X}\right)d\Gamma-\int_{\Omega^n}\frac{\partial N}{\partial Y}\right.\right.$$

$$\left.\left.\frac{\partial \tilde{N}}{\partial Y}d\Omega+\oint_{\Gamma}\tilde{N}\left(n_y\frac{\partial N}{\partial Y}\right)d\Gamma\right]\right),$$

which is weak form of cell conservation equation.

3.3 Results and Discussion

This section contains the results obtained for the model presented in the previous section. The computed results are validated by computing the average values of the Nusselt number and the average Sherwood number for the case discussed in [40]. The results for the calculated average Nusselt number are shown in Table 3.1

Ra	*Rb*	*Le*	*Pe*	$Nu\|_{X=1}$	
				Sheremet and Pop [40]	*Present*
10	10	1	0.1	1.0775	1.04261
			1	1.0723	1.04566
				$Sh\|_{X=1}$	
				Sheremet and Pop [40]	*Present*
10	10	1	0.1	0.3897	0.389745
			1	0.3807	0.388705

TABLE 3.1: Comparison of the average Nusselt and Sherwood numbers.

Chapter 4

Time Relaxation Effects in a Thermo-Bioconvective Porous Cavity

In the context of analyzing the time relaxation effects in a thermo-bioconvective flow model within a porous cavity filled with oxytactic microorganisms, the Cattaneo Christov heat flux model plays a crucial role in modifying the classical Fourier law by incorporating a finite relaxation time, τ, that accounts for the delayed response of heat flux to temperature gradients. Unlike classical Fourier's law, which assumes instantaneous thermal equilibrium, the Cattaneo-Christov model introduces a time relaxation term in the heat flux equation:

$$\tau \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k \nabla T, \tag{4.1}$$

where $\mathbf{q}$ is the heat flux vector, k is the thermal conductivity, and τ is the relaxation time. This modification acknowledges the fact that heat transfer cannot occur instantaneously and instead propagates with a finite speed, a more realistic assumption in many systems, especially in porous media and biological systems like the one being modeled. The time relaxation effect influences the microbe behavior, as microorganisms exhibit chemotactic or thermotactic responses to thermal gradients.

In this case, the microorganisms' movement would not immediately align with temperature changes but would instead follow a delayed response, potentially leading to damped oscillations in the temperature and microorganism concentration profiles. These changes significantly impact the system's bio-convection patterns, with more accurate predictions of heat flow and microbe behavior under transient thermal conditions.

4.1 Mathematical Modeling

Consider a two-dimensional porous square cavity that is filled with oxytactic microorganisms and has walls of length H. Presumably, the enclosure's left hot and right cooled walls are kept at constant temperatures T_H and T_C, respectively, with $T_H > T_C$, and the Walls at the top and bottom are adiabatic. In opposition to the y-axis is the direction in which the gravity vector, **g**, acts.

The model that is being presented here is based on a continuum model that Hillesdon and Pedley (1996) developed for a suspension of oxytactic microorganisms. An energy equation and a buoyancy term in the momentum equation resulting from the temperature variation across the cavity complete this model. The suspension is taken to be diluted and the Boussinesq approximation is applied. Because bioconvection has a low flow velocity, the inertia terms are disregarded (Pedley and Hillesdon) [5]. The governing equations are expressed as in the following form:

Continuity equation

$$\nabla.\mathbf{v} = 0. \tag{4.2}$$

Momentum (Darcy) equation

$$\frac{\partial \mathbf{v}}{\partial t^*} + \frac{\mu}{K}\mathbf{v} = -\nabla p + \left[\gamma \Delta \rho n - \rho_f \beta \left(T - T_C\right)\right] g. \tag{4.3}$$

Time relaxed energy equation

$$\frac{\partial T}{\partial t^*} + \mathbf{v}.\nabla T = \alpha_m \nabla^2 T - \nabla \cdot \boldsymbol{q}. \tag{4.4}$$

Oxygen conservation equation

$$\frac{\partial C}{\partial t^*} + \mathbf{v}.\nabla C = D_C \nabla^2 C - \delta_n. \tag{4.5}$$

Cell conservation equation

$$\nabla.\mathbf{j} = 0, \tag{4.6}$$

where

$$\mathbf{j} = n\mathbf{v} + n\tilde{\mathbf{v}} - D_n \nabla_n. \tag{4.7}$$

Here v is the fluid filtration velocity vector; T is the fluid temperature; t^* is the physical time; α_m is the effective thermal diffusivity of the porous medium; K is the permeability of the porous medium; n is the number density of motile microorganisms; p is the excess pressure (above hydro-static); β is the volume expansion coefficient of water at constant pressure; j is the flux of microorganisms due to macroscopic motion of fluid; directional swimming of microorganisms up the oxygen gradients and a diffusive process that models all random motions of microorganisms, respectively; μ is the dynamic viscosity of the suspension (the suspension includes fluid plus microorganisms, μ assumed to be constant and approximately the same as that of water); ρ_f is the density of the fluid; $\Delta\rho = \rho_{cell} - \rho_f$ is the density difference between cells and fluid; γ is the average volume of a microorganism; D_C is the diffusivity of oxygen; D_n is the diffusivity of microorganisms; ∇^2 is the Laplacian operator; $-\delta n$ describes the consumption of oxygen by the microorganisms; $\Delta C = C_{min}$ where C_0 is the free-surface oxygen concentration; and C_{min} is the minimum oxygen concentration that microorganisms need in order to be active. The terms on the right-hand side of Eq. (4.7) represent the flux of microorganisms due to macroscopic motion of the fluid, the directional swimming of microorganisms up the oxygen gradients, and a diffusive process that models all random motions of microorganisms, respectively. The average directional swimming velocity of a microorganism $\tilde{v}$ in Eq. (4.7) can be approximate as (Hillesdon and Pedley 1996).

$$\tilde{\mathbf{v}} = bW_C \nabla C / \Delta C, \tag{4.8}$$

where b is the chemotaxis constant [m] and Wc is the maximum cell swimming speed [m/s] (the product bWc is assumed to be constant).

Eq. (4.1)- (4.6) can be written in Cartesian co-ordinates x and y as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.9}$$

$$\frac{\partial u}{\partial t^*} + \frac{\mu}{K}u = -\frac{\partial p}{\partial x}, \tag{4.10}$$

$$\frac{\partial v}{\partial t^*} + \frac{\mu}{K}v = -\frac{\partial p}{\partial y} - [\gamma \Delta \rho n - \rho_f \beta (T - T_C)]g, \tag{4.11}$$

$$\begin{aligned}\frac{\partial T}{\partial t^*} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \lambda_2\Bigg[\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\frac{\partial T}{\partial x} \\ + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\frac{\partial T}{\partial y} + u^2\frac{\partial^2 T}{\partial x^2} + v^2\frac{\partial^2 T}{\partial y^2} + 2uv\frac{\partial^2 T}{\partial x \partial y}\Bigg],\end{aligned} \tag{4.12}$$

$$\frac{\partial C}{\partial t^*} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_C\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - \delta_n, \tag{4.13}$$

$$\frac{\partial n}{\partial t^*} + \frac{\partial}{\partial x}\left[un + \tilde{u}n - D_n\frac{\partial n}{\partial x}\right] + \frac{\partial}{\partial y}\left[vn + \tilde{v}n - D_n\frac{\partial n}{\partial y}\right] = 0, \tag{4.14}$$

where u and v are the velocity components along the $x-$ and $y-$axes, respectively, and $\tilde{u}$ and $\tilde{v}$ are provided by $\tilde{u} = \left(\frac{bW_c}{\Delta C}\right)\frac{\partial C}{\partial x}$ and $\tilde{v} = \left(\frac{bW_c}{\Delta C}\right)\frac{\partial C}{\partial y}$. Equations (4.9)-(4.14) can be written in dimensionless form using the following variable

$$X = \frac{x}{H}, Y = \frac{y}{H}, \theta = \frac{(T - T_C)}{\Delta T}, \phi = \frac{(C - C_{min})}{\Delta C}, N = \frac{n}{n_0}, t^* = \frac{H^2 t}{\alpha_m}, p = \frac{\alpha_m^2 P}{H^2}, \tag{4.15}$$

where n_0 is the average number density of the microorganisms (number density of the microorganisms in a well-stirred suspension) and we take velocities u and v, which is defined as $u = (\alpha_m/H)U$ and $v = (\alpha_m/H)V$. As a result, we get the following dimensionless partial differential equations for the model problem under consideration:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{4.16}$$

$$\frac{\partial U}{\partial t}+\frac{Pr}{Da}U=-\frac{\partial P}{\partial X}, \tag{4.17}$$

$$\frac{\partial V}{\partial t}+\frac{Pr}{Da}V=-\frac{\partial P}{\partial Y}-\frac{PrRa}{Da}\left[R_bN-\theta\right], \tag{4.18}$$

$$\frac{\partial \theta}{\partial t}+U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{\partial^2 \theta}{\partial X^2}+\frac{\partial^2 \theta}{\partial Y^2}+\frac{R_t}{Pr}\left[U\frac{\partial U}{\partial X}\frac{\partial \theta}{\partial X}+V\frac{\partial U}{\partial Y}\frac{\partial \theta}{\partial X}+U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y}\right.$$
$$\left.+V\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial Y}+U^2\frac{\partial^2 \theta}{\partial X^2}+V^2\frac{\partial^2 \theta}{\partial Y^2}+2UV\frac{\partial^2 \theta}{\partial X\partial Y}\right], \tag{4.19}$$

$$\frac{\partial \phi}{\partial t}+U\frac{\partial \phi}{\partial X}+V\frac{\partial \phi}{\partial Y}=\frac{1}{Le}\left[\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\right]-\frac{\sigma}{Le}N, \tag{4.20}$$

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial X}(UN)+\frac{\partial}{\partial Y}(VN)+Pe\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\right)+\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\right)\right)$$
$$=\frac{1}{Le\chi}\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right], \tag{4.21}$$

where the parameters Ra is the traditional Rayleigh number for a porous medium, Le is the Lewis number, Rb is the bio-convection Rayleigh number, Pe is the Péclet number, which can be regarded as a ratio of two characteristic velocities, one due to oxytactic swimming and the other due to random, diffusive swimming (Hillesdon and Pedley 1996) and σ is a constant, which characterizes the ratio of the rate of oxygen consumption to the rate of oxygen diffusion, Pr is the prandtl number, R_t is relaxation time and D_a is darcy parameter. These parameters are defined as

$$R_a=\frac{gK\beta\Delta TH}{\nu\alpha_m},\quad R_b=\frac{\gamma\Delta\rho n_0}{\rho_f\beta\Delta T},\quad \sigma=\frac{\delta n_0H^2}{D_c\Delta C},\quad L_e=\frac{\alpha_m}{D_C},$$
$$\chi=\frac{D_C}{D_n},\quad P_e=\frac{bW_C}{\alpha_m},\quad P_r=\frac{\mu}{\alpha_m},\quad R_t=\frac{\lambda_2\mu}{H^2},\quad D_a=\frac{K}{H^2}. \tag{4.22}$$

The following dimensionless boundary conditions for Eqs. (4.16)- (4.1) can be presented as

$$
\begin{aligned}
&\Psi = 0 \quad \text{at all walls}\\
&\theta = 1, \quad \phi = 1, \quad N = 1 \quad \text{at} \quad X = 0,\\
&\theta = 0, \quad \phi = 1, \quad N = 1 \quad \text{at} \quad X = 1,\\
&\frac{\partial \theta}{\partial Y} = 0, \quad \phi = 1, \quad PeN\frac{\partial \phi}{\partial Y} - \frac{\partial N}{\partial Y} = 0 \quad \text{at} \quad Y = 0,\\
&\frac{\partial \theta}{\partial Y} = 0, \quad \frac{\partial \phi}{\partial Y} = 0, \quad \frac{\partial N}{\partial Y} = 0 \quad \text{at} \quad Y = 1.
\end{aligned}
$$

The definitions for the local Nusselt and Sherwood numbers (Nu_Y, Sh_Y) on the vertical walls and average Nusselt and Sherwood numbers (Nu, Sh) are given by

$$
Nu_Y = -\left(\frac{\partial \theta}{\partial X}\right)_{X=0,1}, \quad Sh_Y = -\left(\frac{\partial \phi}{\partial X}\right)_{X=0,1}, \tag{4.23}
$$

and

$$
Nu = \int_0^1 Nu_Y dY, \quad Sh = \int_0^1 Sh_Y dY, \tag{4.24}
$$

respectively.

4.2 Weak Formulation of the Model Problem

4.2.1 Variational Formulation

Converting the governing equations into integral equations is the goal of variational formulation. The PDEs are multiplied by test functions of the same space to convert the equations from the strong form to the weak form after being integrated over the entire domain. In the end, we used the set of approximated trial functions that are only valid over a portion of the domain to obtain an approximated solution.

4.2.1.1 Strong Form of Governing Equations

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{4.25}$$

$$\frac{\partial U}{\partial t}+A_1U=-\frac{\partial P}{\partial X}, \tag{4.26}$$

$$\frac{\partial V}{\partial t}+A_2V=-\frac{\partial P}{\partial Y}-A_3\left[A_4N-\theta\right], \tag{4.27}$$

$$\frac{\partial \theta}{\partial t}+U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+A_5\Bigg[U\frac{\partial U}{\partial X}\frac{\partial \theta}{\partial X}+V\frac{\partial U}{\partial Y}\frac{\partial \theta}{\partial X}+U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y} + V\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial Y}+U^2\frac{\partial^2\theta}{\partial X^2}+V^2\frac{\partial^2\theta}{\partial Y^2}+2UV\frac{\partial^2\theta}{\partial X\partial Y}\Bigg], \tag{4.28}$$

$$\frac{\partial \phi}{\partial t}+U\frac{\partial \phi}{\partial X}+V\frac{\partial \phi}{\partial Y}=A_6\left[\frac{\partial^2\phi}{\partial X^2}+\frac{\partial^2\phi}{\partial Y^2}\right]-A_7N, \tag{4.29}$$

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial X}(UN)+\frac{\partial}{\partial Y}(VN)+A_8\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\right)+\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\right)\right)=A_9\left[\frac{\partial^2N}{\partial X^2}+\frac{\partial^2N}{\partial Y^2}\right]. \tag{4.30}$$

In the above set of equations, the values of $A_1,\ A_2,...A_9$ are expressed as:

$$A_1=A_2=\frac{Pr}{Da},\qquad A_3=\frac{PrRa}{Da},\qquad A_4=R_b,\qquad A_5=\frac{R_t}{Pr},$$
$$A_6=\frac{1}{Le},\qquad A_7=\frac{\sigma}{Le},\qquad A_8=Pe,\qquad A_9=\frac{1}{Le\chi}.$$

4.3 Weak Formulation

In order to transform differential equations into integral form, weak formulation is a variational method that multiplies the dependent variables by a suitable test function and then integrates the result over the whole computational domain. In this instance, the continuously varying infinite dimensional space Ω defines the solution spaces U, V, θ, ϕ, N, and P; in practice, the solution cannot be reached in such a vast space. The main objective is to identify some suitable spaces to obtain functions with finite parameters or properties. We first need to define a few special functions, which we will call test functions for the residuals, so that we can use the weak formulation to find an approximate solution. Assume that W and Q represent the infinite-dimensional test spaces where $W = [H_1(\Omega),\ H_1(\Omega),\ H_1(\Omega)]$ and $Q = L_2(\Omega)$. The corresponding test functions, $\tilde{U}$, $\tilde{V}$, $\tilde{\theta}$, $\tilde{\phi}$, $\tilde{N}$ and q should be such that $\tilde{U}$, $\tilde{V}$, $\tilde{\theta}$, $\tilde{\phi}$, $\tilde{N} \in W$ and $q \in Q$. The test functions $\tilde{U}$, $\tilde{V}$, $\tilde{\theta}$, $\tilde{\phi}$, $\tilde{N} \in W$ multiplies the momentum, energy, oxygen conservation, cell conservation components and continuity equation by $q \in Q$ in the variational formulation. The weak formulation of the strong form of governing PDEs from Eqs. (4.25) to (4.30) is written below: The weak form for u-component of momentum equation (4.26) as follows:

$$\frac{\partial U}{\partial t} + A_1 U = -\frac{\partial P}{\partial X},$$

Multiplying by test function $\tilde{U}$ first and then integrate over computational domain,

$$\frac{\partial U}{\partial t}\tilde{U} + A_1 U\tilde{U} - q\frac{\partial U}{\partial X}\tilde{U} = -\frac{\partial P}{\partial X}\tilde{U} \tag{4.31}$$

$$\text{As } \frac{\partial}{\partial X}(U\tilde{U}) = U\frac{\partial \tilde{U}}{\partial X} + \frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}(U\tilde{U}) - U\frac{\partial \tilde{U}}{\partial X} = \frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow -\left[\frac{\partial}{\partial X}(U\tilde{U}) - U\frac{\partial \tilde{U}}{\partial X}\right] = -\frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow -\frac{\partial}{\partial X}(U\tilde{U}) + U\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial U}{\partial X}\tilde{U}$$

$$\frac{\partial}{\partial X}(P\tilde{U}) = P\frac{\partial \tilde{U}}{\partial X} + \frac{\partial P}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}(P\tilde{U}) - P\frac{\partial \tilde{U}}{\partial X} = \frac{\partial P}{\partial X}\tilde{U}$$

$$\Rightarrow -\left[\frac{\partial}{\partial X}(P\tilde{U}) - P\frac{\partial \tilde{U}}{\partial X}\right] = -\frac{\partial P}{\partial X}\tilde{U}$$

$$\Rightarrow -\frac{\partial}{\partial X}(P\tilde{U}) + P\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial P}{\partial X}\tilde{U}.$$

So, equation (4.31) becomes:

$$\frac{\partial U}{\partial t}\tilde{U} + A_1 U\tilde{U} - q\frac{\partial}{\partial X}(U\tilde{U}) + qU\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial}{\partial X}(P\tilde{U}) + P\frac{\partial \tilde{U}}{\partial X},$$

$$\int_{\Omega^n} \frac{\partial U}{\partial t}\tilde{U} d\Omega + A_1 \int_{\Omega^n} U\tilde{U} d\Omega - q \int_{\Omega^n} \frac{\partial}{\partial X}(U\tilde{U}) d\Omega + q \int_{\Omega^n} U \frac{\partial \tilde{U}}{\partial X} d\Omega = - \int_{\Omega^n} \frac{\partial}{\partial X}(P\tilde{U}) d\Omega + P \int_{\Omega^n} \frac{\partial \tilde{U}}{\partial X} d\Omega, \tag{4.32}$$

as $$\int_{\Omega^n} \frac{\partial U}{\partial t}\tilde{U} d\Omega = \int_{\Omega^n} \frac{(U^{n+1} - U^n)}{\delta t}\tilde{U} d\Omega = \frac{1}{\delta t}\int_{\Omega^n} U^{n+1}\tilde{U} d\Omega - \frac{1}{\delta t}\int_{\Omega^n} U^n\tilde{U} d\Omega,$$

$$\begin{aligned} &\frac{1}{\delta t}\int_{\Omega^n} U^{n+1}\tilde{U}\, d\Omega - \frac{1}{\delta t}\int_{\Omega^n} U^n\tilde{U}\, d\Omega + A_1 \int_{\Omega^n} U\tilde{U}\, d\Omega \\ &- q \int_{\Omega^n} \frac{\partial}{\partial X}(U\tilde{U})\, d\Omega + q \int_{\Omega^n} U \frac{\partial \tilde{U}}{\partial X}\, d\Omega \\ &= - \int_{\Omega^n} \frac{\partial}{\partial X}(P\tilde{U})\, d\Omega + P \int_{\Omega^n} \frac{\partial \tilde{U}}{\partial X}\, d\Omega. \end{aligned}$$

$$\begin{aligned} &\frac{1}{\delta t}\int_{\Omega^n} U^{n+1}\tilde{U}\, d\Omega - \frac{1}{\delta t}\int_{\Omega^n} U^n\tilde{U}\, d\Omega + A_1 \int_{\Omega^n} U\tilde{U}\, d\Omega \\ &- \cancelto{0}{q \int_{\Omega^n} \frac{\partial}{\partial X}(U\tilde{U})\, d\Omega} + q \int_{\Omega^n} U \frac{\partial \tilde{U}}{\partial X}\, d\Omega \\ &= - \cancelto{0}{\int_{\Omega^n} \frac{\partial}{\partial X}(P\tilde{U})\, d\Omega} + P \int_{\Omega^n} \frac{\partial \tilde{U}}{\partial X}\, d\Omega. \end{aligned}$$

$$\frac{1}{\delta t}\int_{\Omega^n} U^{n+1}\tilde{U} d\Omega - \frac{1}{\delta t}\int_{\Omega^n} U^n\tilde{U} d\Omega + A_1 \int_{\Omega^n} U\tilde{U} d\Omega + q \int_{\Omega^n} U \frac{\partial \tilde{U}}{\partial X} d\Omega = P \int_{\Omega^n} \frac{\partial \tilde{U}}{\partial X} d\Omega.$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} U^{n+1}\tilde{U} d\Omega - \frac{1}{\delta t}\int_{\Omega^{n+1}} U^n\tilde{U} d\Omega + A_1 \int_{\Omega^{n+1}} U^{n+1}\tilde{U} d\Omega + q^{n+1} \int_{\Omega^{n+1}} U^{n+1} \frac{\partial \tilde{U}}{\partial X^{n+1}} d\Omega = P^{n+1} \int_{\Omega^{n+1}} \frac{\partial \tilde{U}}{\partial X^{n+1}} d\Omega,$$

which is weak form of u-component of momentum equation.

Similarly, we obtain the weak form for v-component of momentum equation (4.26) as follows:

$$\frac{\partial V}{\partial t}+A_2V=-\frac{\partial P}{\partial Y}-A_3\Big[A_4N-\theta\Big].$$

Multiplying by test function $\tilde{V}$ first and then integrate over computational domain,

$$\frac{\partial V}{\partial t}\tilde{V}+A_2V\tilde{V}-q\frac{\partial V}{\partial Y}\tilde{V}=-\frac{\partial P}{\partial Y}\tilde{V}-A_3\Big[A_4N-\theta\Big]\tilde{V}, \tag{4.33}$$

$$\begin{aligned}
\text{as}\quad \frac{\partial}{\partial Y}(V\tilde{V}) &= V\frac{\partial\tilde{V}}{\partial Y}+\frac{\partial V}{\partial Y}\tilde{V}\\
\Rightarrow \frac{\partial}{\partial Y}(V\tilde{V})-V\frac{\partial\tilde{V}}{\partial Y} &= \frac{\partial V}{\partial Y}\tilde{V}\\
\Rightarrow -\Big[\frac{\partial}{\partial Y}(V\tilde{V})-V\frac{\partial\tilde{V}}{\partial Y}\Big] &= -\frac{\partial V}{\partial Y}\tilde{V}\\
\Rightarrow -\frac{\partial}{\partial Y}(V\tilde{V})+V\frac{\partial\tilde{V}}{\partial Y} &= -\frac{\partial V}{\partial Y}\tilde{V}\\
\frac{\partial}{\partial Y}(P\tilde{V}) &= P\frac{\partial\tilde{V}}{\partial Y}+\frac{\partial P}{\partial Y}\tilde{V}\\
\Rightarrow \frac{\partial}{\partial Y}(P\tilde{V})-P\frac{\partial\tilde{V}}{\partial Y} &= \frac{\partial P}{\partial Y}\tilde{V}\\
\Rightarrow -\Big[\frac{\partial}{\partial Y}(P\tilde{V})-P\frac{\partial\tilde{V}}{\partial Y}\Big] &= -\frac{\partial P}{\partial Y}\tilde{V}\\
\Rightarrow -\frac{\partial}{\partial Y}(P\tilde{V})+P\frac{\partial\tilde{V}}{\partial Y} &= -\frac{\partial P}{\partial Y}\tilde{V}.
\end{aligned}$$

So, equation (4.33) becomes:

$$\frac{\partial V}{\partial t}\tilde{V}+A_2V\tilde{V}-q\frac{\partial}{\partial Y}(V\tilde{V})+qV\frac{\partial\tilde{V}}{\partial Y}=-\frac{\partial}{\partial Y}(P\tilde{V})+P\frac{\partial\tilde{V}}{\partial Y}-A_3\Big[A_4N-\theta\Big]\tilde{V},$$

$$\begin{aligned}
&\int_{\Omega^n}\frac{\partial V}{\partial t}\tilde{V}d\Omega+A_2\int_{\Omega^n}V\tilde{V}d\Omega-q\int_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega+q\int_{\Omega^n}V\frac{\partial\tilde{V}}{\partial X}d\Omega=\\
&\quad-\int_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})d\Omega+P\int_{\Omega^n}\frac{\partial\tilde{V}}{\partial Y}d\Omega-A_3A_4\int_{\Omega^n}N\tilde{V}d\Omega+A_3\int_{\Omega^n}\theta\tilde{V}d\Omega.
\end{aligned} \tag{4.34}$$

$$\text{As}\int_{\Omega^n}\frac{\partial V}{\partial t}\tilde{V}d\Omega=\int_{\Omega^n}\frac{(V^{n+1}-V^n)}{\delta t}\tilde{V}d\Omega=\frac{1}{\delta t}\int_{\Omega^n}V^{n+1}\tilde{V}d\Omega-\frac{1}{\delta t}\int_{\Omega^n}V^n\tilde{V}d\Omega,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^n} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^n} V^{n}\tilde{V}d\Omega + A_2\int\limits_{\Omega^n} V\tilde{V}d\Omega - q\int\limits_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega + q\int\limits_{\Omega^n} V\frac{\partial \tilde{V}}{\partial X}d\Omega =$$

$$-\int\limits_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})d\Omega + P\int\limits_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega - A_3A_4\int\limits_{\Omega^n} N\tilde{V}d\Omega + A_3\int\limits_{\Omega^n}\theta\tilde{V}d\Omega,$$

$$\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^n} V^{n+1}\tilde{V}\,d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^n} V^{n}\tilde{V}\,d\Omega + A_2\int\limits_{\Omega^n} V\tilde{V}\,d\Omega \\
&\quad - \cancelto{0}{q\int\limits_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})\,d\Omega} \quad + q\int\limits_{\Omega^n} V\frac{\partial \tilde{V}}{\partial X}\,d\Omega \\
&= -\cancelto{0}{\int\limits_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})\,d\Omega} \quad + P\int\limits_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}\,d\Omega \\
&\quad - A_3A_4\int\limits_{\Omega^n} N\tilde{V}\,d\Omega + A_3\int\limits_{\Omega^n}\theta\tilde{V}\,d\Omega.
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^n} V^{n+1}\tilde{V}\,d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^n} V^{n}\tilde{V}\,d\Omega + A_2\int\limits_{\Omega^n} V\tilde{V}\,d\Omega \\
&\quad + q\int\limits_{\Omega^n} V\frac{\partial \tilde{V}}{\partial X}\,d\Omega \\
&= P\int\limits_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}\,d\Omega - A_3A_4\int\limits_{\Omega^n} N\tilde{V}\,d\Omega + A_3\int\limits_{\Omega^n}\theta\tilde{V}\,d\Omega.
\end{aligned}$$

Taking Domain as current time step value, we have

$$\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V}\,d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V^{n}\tilde{V}\,d\Omega + A_2\int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V}\,d\Omega \\
&\quad + q^{n+1}\int\limits_{\Omega^{n+1}} V^{n+1}\frac{\partial \tilde{V}}{\partial X^{n+1}}\,d\Omega \\
&= P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}}\,d\Omega - A_3A_4\int\limits_{\Omega^{n+1}} N^{n+1}\tilde{V}\,d\Omega + A_3\int\limits_{\Omega^{n+1}}\theta^{n+1}\tilde{V}\,d\Omega.
\end{aligned}$$

which is weak form of v-component of momentum equation.

The energy equation (4.28) in the same way becomes:

$$\frac{\partial\theta}{\partial t}+U\frac{\partial\theta}{\partial X}+V\frac{\partial\theta}{\partial Y}=\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}+A_5\Bigg[U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}+V\frac{\partial U}{\partial Y}\frac{\partial\theta}{\partial X}$$
$$+U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}+V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}+U^2\frac{\partial^2\theta}{\partial X^2}+V^2\frac{\partial^2\theta}{\partial Y^2}+2UV\frac{\partial^2\theta}{\partial X\partial Y}\Bigg].$$

Multiplying by test function $\tilde{\theta}$ first and then integrate over computational domain,

$$\frac{\partial\theta}{\partial t}\tilde{\theta}+\left[\left(U\frac{\partial\theta}{\partial X}\right)+\left(V\frac{\partial\theta}{\partial Y}\right)\right]\tilde{\theta}=\frac{\partial^2\theta}{\partial X^2}\tilde{\theta}+\frac{\partial^2\theta}{\partial Y^2}\tilde{\theta}+A_5\Bigg[U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}+V\frac{\partial U}{\partial Y}\frac{\partial\theta}{\partial X}$$
$$+U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}+V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}+U^2\frac{\partial^2\theta}{\partial X^2}+V^2\frac{\partial^2\theta}{\partial Y^2}+2UV\frac{\partial^2\theta}{\partial X\partial Y}\Bigg]\tilde{\theta}, \tag{4.35}$$

$$as \qquad \frac{\partial}{\partial X}\left(U^2\frac{\partial\theta}{\partial X}\tilde{\theta}\right)=U^2\frac{\partial\theta}{\partial X}\frac{\partial}{\partial X}(\tilde{\theta})+U^2\frac{\partial}{\partial X}\left(\frac{\partial\theta}{\partial X}\right)\tilde{\theta}+\frac{\partial}{\partial X}(U^2)\frac{\partial\theta}{\partial X}\tilde{\theta}$$

$$\Rightarrow\frac{\partial}{\partial X}\left(U^2\frac{\partial\theta}{\partial X}\tilde{\theta}\right)=U^2\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X}+U^2\frac{\partial^2\theta}{\partial X^2}\tilde{\theta}+2U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\tilde{\theta}$$

$$\Rightarrow\frac{\partial}{\partial X}\left(U^2\frac{\partial\theta}{\partial X}\tilde{\theta}\right)-U^2\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X}-2U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\tilde{\theta}=U^2\frac{\partial^2\theta}{\partial X^2}\tilde{\theta}$$

$$\frac{\partial}{\partial Y}\left(V^2\frac{\partial\theta}{\partial Y}\tilde{\theta}\right)=V^2\frac{\partial\theta}{\partial Y}\frac{\partial}{\partial Y}(\tilde{\theta})+V^2\frac{\partial}{\partial Y}\left(\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}+\frac{\partial}{\partial Y}(V^2)\frac{\partial\theta}{\partial Y}\tilde{\theta}$$

$$\Rightarrow\frac{\partial}{\partial Y}\left(V^2\frac{\partial\theta}{\partial Y}\tilde{\theta}\right)=V^2\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y}+V^2\frac{\partial^2\theta}{\partial Y^2}\tilde{\theta}+2V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\tilde{\theta}$$

$$\Rightarrow\frac{\partial}{\partial Y}\left(V^2\frac{\partial\theta}{\partial Y}\tilde{\theta}\right)-V^2\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y}-2V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\tilde{\theta}=V^2\frac{\partial^2\theta}{\partial Y^2}\tilde{\theta}$$

$$2UV\frac{\partial^2\theta}{\partial X\partial Y}\tilde{\theta}=\frac{\partial}{\partial X}\left(UV\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}+\frac{\partial}{\partial Y}\left(UV\frac{\partial\theta}{\partial X}\right)\tilde{\theta}$$

$$\Rightarrow\frac{\partial}{\partial X}\left(UV\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}+\frac{\partial}{\partial Y}\left(UV\frac{\partial\theta}{\partial X}\right)\tilde{\theta}$$
$$=UV\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial X}+UV\frac{\partial^2\theta}{\partial X\partial Y}\tilde{\theta}+U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}\tilde{\theta}+\frac{\partial U}{\partial X}V\frac{\partial\theta}{\partial Y}\tilde{\theta}$$
$$+UV\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial Y}+UV\frac{\partial^2\theta}{\partial Y\partial X}\tilde{\theta}+U\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial X}\tilde{\theta}+\frac{\partial U}{\partial Y}V\frac{\partial\theta}{\partial X}\tilde{\theta}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial \theta}{\partial Y}\right)\tilde{\theta} + \frac{\partial}{\partial Y}\left(UV\frac{\partial \theta}{\partial X}\right)\tilde{\theta}$$
$$= UV\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial X} + UV\frac{\partial^2 \theta}{\partial X \partial Y}\tilde{\theta} + U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y}\tilde{\theta} + \frac{\partial U}{\partial X}V\frac{\partial \theta}{\partial Y}\tilde{\theta}$$
$$+ UV\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial Y} + UV\frac{\partial^2 \theta}{\partial Y \partial X}\tilde{\theta} + U\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial X}\tilde{\theta} + \frac{\partial U}{\partial Y}V\frac{\partial \theta}{\partial X}\tilde{\theta}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial \theta}{\partial Y}\right)\tilde{\theta} + \frac{\partial}{\partial Y}\left(UV\frac{\partial \theta}{\partial X}\right)\tilde{\theta} - UV\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y}\tilde{\theta}$$
$$- \frac{\partial U}{\partial X}V\frac{\partial \theta}{\partial Y}\tilde{\theta} - UV\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial X}\tilde{\theta} - \frac{\partial U}{\partial Y}V\frac{\partial \theta}{\partial X}\tilde{\theta}$$
$$= UV\frac{\partial^2 \theta}{\partial X \partial Y}\tilde{\theta} + UV\frac{\partial^2 \theta}{\partial Y \partial X}\tilde{\theta}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial \theta}{\partial Y}\right)\tilde{\theta} + \frac{\partial}{\partial Y}\left(UV\frac{\partial \theta}{\partial X}\right)\tilde{\theta} - UV\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y}\tilde{\theta} -$$

$$\frac{\partial U}{\partial X}V\frac{\partial \theta}{\partial Y}\tilde{\theta} - UV\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial X}\tilde{\theta} - \frac{\partial U}{\partial Y}V\frac{\partial \theta}{\partial X}\tilde{\theta} = 2UV\frac{\partial^2 \theta}{\partial X \partial Y}\tilde{\theta}.$$

So equation (4.35) becomes:

$$\frac{\partial \theta}{\partial t}\tilde{\theta} + \left(U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y}\right)\tilde{\theta} = \frac{\partial^2 \theta}{\partial X^2}\tilde{\theta} + \frac{\partial^2 \theta}{\partial Y^2}\tilde{\theta}$$

$$+ A_5\left[U\frac{\partial U}{\partial X}\frac{\partial \theta}{\partial X} + V\frac{\partial U}{\partial Y}\frac{\partial \theta}{\partial X} + U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y} + V\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial Y}\right]\tilde{\theta}$$

$$+ A_5\left(\frac{\partial}{\partial X}\left(U^2\frac{\partial \theta}{\partial X}\tilde{\theta}\right) - U^2\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial \theta}{\partial X}\tilde{\theta}\right)$$

$$+ A_5\left(\frac{\partial}{\partial Y}\left(V^2\frac{\partial \theta}{\partial Y}\tilde{\theta}\right) - V^2\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial Y}\tilde{\theta}\right)$$

$$+ A_5\Big(\frac{\partial}{\partial X}\left(UV\frac{\partial \theta}{\partial Y}\right)\tilde{\theta} + \frac{\partial}{\partial Y}\left(UV\frac{\partial \theta}{\partial X}\right)\tilde{\theta} - UV\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y}\tilde{\theta}$$

$$- \frac{\partial U}{\partial X}V\frac{\partial \theta}{\partial Y}\tilde{\theta} - UV\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial X}\tilde{\theta} - \frac{\partial U}{\partial Y}V\frac{\partial \theta}{\partial X}\tilde{\theta}\Big) \tag{4.36}$$

$$\int\limits_{\Omega^n} \frac{\partial \theta}{\partial t}\tilde{\theta}\, d\Omega + \int\limits_{\Omega^n} \left(U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right) \tilde{\theta}\, d\Omega$$

$$= \int\limits_{\Omega^n} \left(\frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} \right) \tilde{\theta}\, d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(U\frac{\partial U}{\partial X}\frac{\partial \theta}{\partial X} + V\frac{\partial U}{\partial Y}\frac{\partial \theta}{\partial X} + U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y} + V\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial Y} \right) \tilde{\theta}\, d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(\frac{\partial}{\partial X}\left(U^2\frac{\partial \theta}{\partial X}\tilde{\theta} \right) - U^2\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial \theta}{\partial X}\tilde{\theta} \right) d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(\frac{\partial}{\partial Y}\left(V^2\frac{\partial \theta}{\partial Y}\tilde{\theta} \right) - V^2\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial Y}\tilde{\theta} \right) d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(\frac{\partial}{\partial X}\left(UV\frac{\partial \theta}{\partial Y} \right)\tilde{\theta} + \frac{\partial}{\partial Y}\left(UV\frac{\partial \theta}{\partial X} \right)\tilde{\theta} - UV\frac{\partial \theta}{\partial Y}\frac{\partial \tilde{\theta}}{\partial X} - UV\frac{\partial \theta}{\partial X}\frac{\partial \tilde{\theta}}{\partial Y} \right.$$

$$\left. - U\frac{\partial V}{\partial X}\frac{\partial \theta}{\partial Y}\tilde{\theta} - \frac{\partial U}{\partial X}V\frac{\partial \theta}{\partial Y}\tilde{\theta} - U\frac{\partial V}{\partial Y}\frac{\partial \theta}{\partial X}\tilde{\theta} - \frac{\partial U}{\partial Y}V\frac{\partial \theta}{\partial X}\tilde{\theta} \right) d\Omega$$

as $\int\limits_{\Omega^n} \frac{\partial \theta}{\partial t}\tilde{\theta} d\Omega + \int\limits_{\Omega^n} \left[U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right] \tilde{\theta} d\Omega$

$$= \int\limits_{\Omega^n} \frac{(\theta^{n+1} - \theta^n)}{\delta t}\tilde{\theta} d\Omega + \int\limits_{\Omega^n} \left[U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right] \tilde{\theta} d\Omega$$

$$= \frac{1}{\delta t}\int\limits_{\Omega^n} \theta^{n+1}\tilde{\theta} d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^n} \theta^{n}\tilde{\theta} d\Omega + \int\limits_{\Omega^n} \left[U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} \right] \tilde{\theta} d\Omega$$

$$= \frac{1}{\delta t}\int\limits_{\Omega^n} \theta^{n+1}\tilde{\theta} d\Omega - \frac{1}{\delta t}\Big(\theta^n \ o \ X^n\Big)\tilde{\theta} d\Omega$$

$$\frac{1}{\delta t}\int\limits_{\Omega^n} \theta^{n+1}\tilde{\theta}\, d\Omega - \frac{1}{\delta t}\left(\theta^n \circ X^n\right)\tilde{\theta}\, d\Omega$$

$$= \int\limits_{\Omega^n} \left(\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2}\right)\tilde{\theta}\, d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X} + V\frac{\partial U}{\partial Y}\frac{\partial\theta}{\partial X} + U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y} + V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}\, d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(\frac{\partial}{\partial X}\left(U^2\frac{\partial\theta}{\partial X}\tilde{\theta}\right) - U^2\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\tilde{\theta}\right) d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \left(\frac{\partial}{\partial Y}\left(V^2\frac{\partial\theta}{\partial Y}\tilde{\theta}\right) - V^2\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\tilde{\theta}\right) d\Omega$$

$$+ A_5 \int\limits_{\Omega^n} \Big(\frac{\partial}{\partial X}\left(UV\frac{\partial\theta}{\partial Y}\right)\tilde{\theta} + \frac{\partial}{\partial Y}\left(UV\frac{\partial\theta}{\partial X}\right)\tilde{\theta}$$
$$- UV\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}\tilde{\theta} - \frac{\partial U}{\partial X}V\frac{\partial\theta}{\partial Y}\tilde{\theta}$$
$$- UV\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial X}\tilde{\theta} - \frac{\partial U}{\partial Y}V\frac{\partial\theta}{\partial X}\tilde{\theta}\Big) d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n} \theta^{n+1}\tilde{\theta}\, d\Omega - \frac{1}{\delta t}\left(\theta^n \circ X^n\right)\tilde{\theta}\, d\Omega$$
$$= \int_{\Omega^n} \left(\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2}\right)\tilde{\theta}\, d\Omega$$
$$+ A_5 \int_{\Omega^n} \left(U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X} + V\frac{\partial U}{\partial Y}\frac{\partial\theta}{\partial X} + U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y} + V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}\, d\Omega$$
$$+ A_5 \int_{\Omega^n} \left(\cancelto{0}{\frac{\partial}{\partial X}\left(U^2\frac{\partial\theta}{\partial X}\tilde{\theta}\right)} - U^2\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\tilde{\theta}\right) d\Omega$$
$$+ A_5 \int_{\Omega^n} \left(\cancelto{0}{\frac{\partial}{\partial Y}\left(V^2\frac{\partial\theta}{\partial Y}\tilde{\theta}\right)} - V^2\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\tilde{\theta}\right) d\Omega$$
$$+ A_5 \int_{\Omega^n} \Big(\cancelto{0}{\frac{\partial}{\partial X}\left(UV\frac{\partial\theta}{\partial Y}\right)}\tilde{\theta} + \cancelto{0}{\frac{\partial}{\partial Y}\left(UV\frac{\partial\theta}{\partial X}\right)}\tilde{\theta}$$
$$- UV\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}\tilde{\theta} - \frac{\partial U}{\partial X}V\frac{\partial\theta}{\partial Y}\tilde{\theta}$$
$$- UV\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial X}\tilde{\theta} - \frac{\partial U}{\partial Y}V\frac{\partial\theta}{\partial X}\tilde{\theta}\Big) d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n}\theta^{n+1}\tilde{\theta}\,d\Omega-\frac{1}{\delta t}\left(\theta^n\circ X^n\right)\tilde{\theta}\,d\Omega=\int_{\Omega^n}\left(\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right)\tilde{\theta}\,d\Omega$$

$$+\frac{A_5}{3}\int_{\Omega^n}\left(3U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}+3V\frac{\partial U}{\partial Y}\frac{\partial\theta}{\partial X}+3U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}+3V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}\,d\Omega$$

$$+A_5\int_{\Omega^n}\left(-U^2\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X}-2U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\tilde{\theta}\right)d\Omega$$

$$+A_5\int_{\Omega^n}\left(-V^2\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y}-2V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\tilde{\theta}\right)d\Omega$$

$$+A_5\int_{\Omega^n}\Bigg(-UV\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial X}-U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}\tilde{\theta}-\frac{\partial U}{\partial X}V\frac{\partial\theta}{\partial Y}\tilde{\theta}$$

$$-UV\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial Y}-U\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial X}\tilde{\theta}-\frac{\partial U}{\partial Y}V\frac{\partial\theta}{\partial X}\tilde{\theta}\Bigg)d\Omega$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega}\psi\Delta\phi\;d\Omega=-\int\limits_{\Omega}\nabla\phi\nabla\psi\;d\Omega+\int\limits_{\Omega}\psi(\nabla\phi n)\;d\Gamma.$$

Here, $\psi=\tilde{\theta},\ \Delta\phi=\dfrac{\partial^2\theta}{\partial X^2},\ \Delta\phi=\dfrac{\partial^2\theta}{\partial Y^2},\ \nabla\phi=\dfrac{\partial\theta}{\partial X},\ \nabla\phi=\dfrac{\partial\theta}{\partial Y},$
$\nabla\psi=\dfrac{\partial\tilde{\theta}}{\partial X},\ \nabla\psi=\dfrac{\partial\tilde{\theta}}{\partial Y}.$

As $$\nabla\phi n=\frac{\partial\phi}{\partial n}=n_x\frac{\partial\phi}{\partial X}+n_y\frac{\partial\phi}{\partial Y}\qquad(\phi=\theta).$$

So,

$$\int\limits_{\Omega^n}\left[\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}\right]\tilde{\theta}d\Omega$$

$$=\left[-\int\limits_{\Omega^n}\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X}d\Omega+\oint\limits_{\Gamma}\tilde{\theta}\left(n_x\frac{\partial\theta}{\partial X}\right)d\Gamma\right]+\left[-\int\limits_{\Omega^n}\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y}d\Omega+\oint\limits_{\Gamma}\tilde{\theta}\left(n_y\frac{\partial\theta}{\partial Y}\right)d\Gamma\right].$$

Now (4.36) becomes as

$$\frac{1}{\delta t}\int_{\Omega^n}\theta^{n+1}\tilde{\theta}\,d\Omega-\frac{1}{\delta t}\left(\theta^n\circ X^n\right)\tilde{\theta}\,d\Omega=-\int_{\Omega^n}\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X}\,d\Omega+\oint_{\Gamma}\tilde{\theta}\left(n_x\frac{\partial\theta}{\partial X}\right)d\Gamma$$

$$-\int_{\Omega^n}\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y}\,d\Omega+\oint_{\Gamma}\tilde{\theta}\left(n_y\frac{\partial\theta}{\partial Y}\right)d\Gamma$$

$$+\frac{A_5}{3}\int_{\Omega^n}\left(3U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\right)\tilde{\theta}\,d\Omega+\frac{A_5}{3}\int_{\Omega^n}\left(3V\frac{\partial U}{\partial Y}\frac{\partial\theta}{\partial X}\right)\tilde{\theta}\,d\Omega$$

$$+\frac{A_5}{3}\int_{\Omega^n}\left(3U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}\,d\Omega+\frac{A_5}{3}\int_{\Omega^n}\left(3V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\right)\tilde{\theta}\,d\Omega$$

$$+A_5\int_{\Omega^n}\left(-U^2\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial X}-2U\frac{\partial U}{\partial X}\frac{\partial\theta}{\partial X}\tilde{\theta}\right)d\Omega$$

$$+A_5\int_{\Omega^n}\left(-V^2\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial Y}-2V\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial Y}\tilde{\theta}\right)d\Omega$$

$$+A_5\int_{\Omega^n}\Bigg(-UV\frac{\partial\theta}{\partial Y}\frac{\partial\tilde{\theta}}{\partial X}-U\frac{\partial V}{\partial X}\frac{\partial\theta}{\partial Y}\tilde{\theta}-\frac{\partial U}{\partial X}V\frac{\partial\theta}{\partial Y}\tilde{\theta}$$

$$-UV\frac{\partial\theta}{\partial X}\frac{\partial\tilde{\theta}}{\partial Y}-U\frac{\partial V}{\partial Y}\frac{\partial\theta}{\partial X}\tilde{\theta}-\frac{\partial U}{\partial Y}V\frac{\partial\theta}{\partial X}\tilde{\theta}\Bigg)d\Omega.$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} \theta^{n+1}\tilde{\theta}\, d\Omega - \frac{1}{\delta t}\left(\theta^{n}\circ X^{n}\right)\tilde{\theta}\, d\Omega = -\int_{\Omega^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{\theta}}{\partial X^{n+1}}\, d\Omega$$
$$+\oint_{\Gamma}\tilde{\theta}\left(n_x\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\right) d\Gamma - \int_{\Omega^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{\theta}}{\partial Y^{n+1}}\, d\Omega + \oint_{\Gamma}\tilde{\theta}\left(n_y\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\right) d\Gamma$$
$$+ A_5\int_{\Omega^{n+1}}\left(U^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\right)\cdot 3\tilde{\theta}\, d\Omega$$
$$+ A_5\int_{\Omega^{n+1}}\left(V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\right)\cdot 3\tilde{\theta}\, d\Omega$$
$$+ A_5\int_{\Omega^{n+1}}\left(U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\right)\cdot 3\tilde{\theta}\, d\Omega$$
$$+ A_5\int_{\Omega^{n+1}}\left(V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\right)\cdot 3\tilde{\theta}\, d\Omega$$
$$+ A_5\int_{\Omega^{n+1}}\left(-U^{n}U^{n+1}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{\theta}}{\partial X^{n+1}} - 2U^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\tilde{\theta}\right) d\Omega$$
$$+ A_5\int_{\Omega^{n+1}}\left(-V^{n}V^{n+1}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{\theta}}{\partial Y^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\tilde{\theta}\right) d\Omega$$
$$+ A_5\int_{\Omega^{n+1}}\Bigg(-U^{n+1}V^{n+1}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{\theta}}{\partial X^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\tilde{\theta}$$
$$-\frac{\partial U^{n+1}}{\partial X^{n+1}}V^{n+1}\frac{\partial \theta^{n+1}}{\partial Y^{n+1}}\tilde{\theta} - U^{n+1}V^{n+1}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{\theta}}{\partial Y^{n+1}}$$
$$-U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\tilde{\theta} - \frac{\partial U^{n+1}}{\partial Y^{n+1}}V^{n+1}\frac{\partial \theta^{n+1}}{\partial X^{n+1}}\tilde{\theta}\Bigg) d\Omega.$$

which is weak form of energy equation.

The oxygen conservation equation (4.29) in the same way becomes:

$$\frac{\partial \phi}{\partial t}+U\frac{\partial \phi}{\partial X}+V\frac{\partial \phi}{\partial Y} = A_6\left(\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\right)-A_7 N,$$

Multiplying by test function $\tilde{\phi}$ first and then integrate over computational domain,

$$\frac{\partial \phi}{\partial t}\tilde{\phi}+\left[\left(U\frac{\partial \phi}{\partial X}\right)+\left(V\frac{\partial \phi}{\partial Y}\right)\right]\tilde{\phi} = A_6\left(\frac{\partial^2 \phi}{\partial X^2}+\frac{\partial^2 \phi}{\partial Y^2}\right)\tilde{\phi}-A_7 N\tilde{\phi}$$

$$\int_{\Omega^n} \frac{\partial \phi}{\partial t}\tilde{\phi}\, d\Omega + \int_{\Omega^n} \left(U\frac{\partial \phi}{\partial X} + V\frac{\partial \phi}{\partial Y}\right)\tilde{\phi}\, d\Omega = A_6 \int_{\Omega^n} \left(\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2}\right)\tilde{\phi}\, d\Omega - A_7 \int_{\Omega^n} N\tilde{\phi}\, d\Omega \tag{4.37}$$

As $\int\limits_{\Omega^n} \frac{\partial \phi}{\partial t}\tilde{\phi} d\Omega + \int\limits_{\Omega^n} \left[U\frac{\partial \phi}{\partial X} + V\frac{\partial \phi}{\partial Y}\right]\tilde{\phi} d\Omega$

$$= \int\limits_{\Omega^n} \frac{(\phi^{n+1} - \phi^n)}{\delta t}\tilde{\phi} d\Omega + \int\limits_{\Omega^n} \left[U\frac{\partial \phi}{\partial X} + V\frac{\partial \phi}{\partial Y}\right]\tilde{\phi} d\Omega$$

$$= \frac{1}{\delta t}\int\limits_{\Omega^n} \phi^{n+1}\tilde{\phi} d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^n} \phi^{n}\tilde{\phi} d\Omega + \int\limits_{\Omega^n} \left[U\frac{\partial \phi}{\partial X} + V\frac{\partial \phi}{\partial Y}\right]\tilde{\phi} d\Omega$$

$$= \frac{1}{\delta t}\int\limits_{\Omega^n} \phi^{n+1}\tilde{\phi} d\Omega - \frac{1}{\delta t}\left(\phi^n \; o \; X^n\right)\tilde{\phi} d\Omega$$

So (4.37), becomes:

$$\frac{1}{\delta t}\int\limits_{\Omega^n} \phi^{n+1}\tilde{\phi} d\Omega - \frac{1}{\delta t}\left(\phi^n \; o \; X^n\right)\tilde{\phi} d\Omega = A_6 \int\limits_{\Omega^n} \left(\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial Y^2}\right)\tilde{\phi} d\Omega - A_7 \int\limits_{\Omega^n} N\tilde{\phi} d\Omega \tag{4.38}$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega} \psi \Delta \phi \; d\Omega = - \int\limits_{\Omega} \nabla \phi \nabla \psi \; d\Omega + \int\limits_{\Omega} \psi (\nabla \phi n) \; d\Gamma$$

Here, $\psi = \tilde{\phi}$, $\Delta \phi = \frac{\partial^2 \phi}{\partial X^2}$, $\Delta \phi = \frac{\partial^2 \phi}{\partial Y^2}$, $\nabla \phi = \frac{\partial \phi}{\partial X}$, $\nabla \phi = \frac{\partial \phi}{\partial Y}$, $\nabla \psi = \frac{\partial \tilde{\phi}}{\partial X}$, $\nabla \psi =$

$\frac{\partial \tilde{\phi}}{\partial Y}$

As $$\nabla \phi n = \frac{\partial \phi}{\partial n} = n_x \frac{\partial \phi}{\partial X} + n_y \frac{\partial \phi}{\partial Y} \qquad (\phi = \phi)$$

So,

$$\int\limits_{\Omega^n} \left[\frac{\partial^2\phi}{\partial X^2} + \frac{\partial^2\phi}{\partial Y^2}\right]\tilde{\phi}d\Omega$$
$$= \left[-\int\limits_{\Omega^n} \frac{\partial\phi}{\partial X}\frac{\partial\tilde{\phi}}{\partial X}d\Omega + \oint\limits_{\Gamma} \tilde{\phi}\left(n_x\frac{\partial\phi}{\partial X}\right)d\Gamma - \int\limits_{\Omega^n} \frac{\partial\phi}{\partial Y}\frac{\partial\tilde{\phi}}{\partial Y}d\Omega + \oint\limits_{\Gamma} \tilde{\phi}\left(n_y\frac{\partial\phi}{\partial Y}\right)d\Gamma\right],$$

Now (4.38), becomes:

$$\frac{1}{\delta t}\int\limits_{\Omega^n} \phi^{n+1}\tilde{\phi}d\Omega - \frac{1}{\delta t}\left(\phi^n \; o \; X^n\right)\tilde{\phi}d\Omega = A_6\left[-\int\limits_{\Omega^n} \frac{\partial\phi}{\partial X}\frac{\partial\tilde{\phi}}{\partial X}d\Omega + \oint\limits_{\Gamma} \tilde{\phi}\left(n_x\frac{\partial\phi}{\partial X}\right)d\Gamma\right.$$
$$\left. -\int\limits_{\Omega^n} \frac{\partial\phi}{\partial Y}\frac{\partial\tilde{\phi}}{\partial Y}d\Omega + \oint\limits_{\Gamma} \tilde{\phi}\left(n_y\frac{\partial\phi}{\partial Y}\right)d\Gamma\right] - A_7\int\limits_{\Omega^n} N\tilde{\phi}d\Omega.$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \phi^{n+1}\tilde{\phi}d\Omega - \frac{1}{\delta t}\left(\phi^n \; o \; X^n\right)\tilde{\phi}d\Omega = A_6\left[-\int\limits_{\Omega^{n+1}} \frac{\partial\phi^{n+1}}{\partial X^{n+1}}\frac{\partial\tilde{\phi}}{\partial X^{n+1}}d\Omega + \right.$$
$$\left.\oint\limits_{\Gamma} \tilde{\phi}\left(n_x\frac{\partial\phi^{n+1}}{\partial X^{n+1}}\right)d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial\phi^{n+1}}{\partial Y^{n+1}}\frac{\partial\tilde{\phi}}{\partial Y^{n+1}}d\Omega + \oint\limits_{\Gamma} \tilde{\phi}\left(n_y\frac{\partial\phi^{n+1}}{\partial Y^{n+1}}\right)d\Gamma\right]$$
$$- A_7\int\limits_{\Omega^{n+1}} N^{n+1}\tilde{\phi}d\Omega$$

which is weak form of oxygen conservation equation.

The cell conservation equation (4.30) in the same way becomes:

$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial X}(UN)+\frac{\partial}{\partial Y}(VN)+A_8\left(\frac{\partial}{\partial X}\left(N\frac{\partial\phi}{\partial X}\right)+\frac{\partial}{\partial Y}\left(N\frac{\partial\phi}{\partial Y}\right)\right) = A_9\left[\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right].$$

Multiplying by test function $\tilde{N}$ first and then integrate over computational domain,

$$\frac{\partial N}{\partial t}\tilde{N} + \left(U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N} + \left(V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N} + A_8\left(N\frac{\partial^2\phi}{\partial X^2} + N\frac{\partial^2\phi}{\partial Y^2}\right)\tilde{N}$$
$$= A_9\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}, \tag{4.39}$$

$$as \qquad \frac{\partial}{\partial X}\left(N\frac{\partial\phi}{\partial X}\tilde{N}\right) = N\frac{\partial\phi}{\partial X}\frac{\partial}{\partial X}(\tilde{N}) + N\frac{\partial}{\partial X}\left(\frac{\partial\phi}{\partial X}\right)\tilde{N} + \frac{\partial}{\partial X}(N)\frac{\partial\phi}{\partial X}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) = N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} + N\frac{\partial^2 \phi}{\partial X^2}\tilde{N} + \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} = N\frac{\partial^2 \phi}{\partial X^2}\tilde{N}$$

$$\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) = N\frac{\partial \phi}{\partial Y}\frac{\partial}{\partial Y}(\tilde{N}) + N\frac{\partial}{\partial Y}\left(\frac{\partial \phi}{\partial Y}\right)\tilde{N} + \frac{\partial}{\partial Y}(N)\frac{\partial \phi}{\partial Y}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) = N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} + N\frac{\partial^2 \phi}{\partial Y^2}\tilde{N} + \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N} = N\frac{\partial^2 \phi}{\partial Y^2}\tilde{N}$$

So (4.39), becomes:

$$\frac{\partial N}{\partial t}\tilde{N} + \frac{1}{2}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N} + \frac{1}{2}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y}\right.$$
$$\left. + N\frac{\partial V}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N} + A_8\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}\right.$$
$$\left. + \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right) = A_9\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}$$

$$\int_{\Omega^n}\frac{\partial N}{\partial t}\tilde{N}d\Omega + \frac{1}{2}\int_{\Omega^n}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega + \frac{1}{2}\int_{\Omega^n}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y},\right.$$
$$\left. + N\frac{\partial V}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega + A_8\int_{\Omega^n}\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right) - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}\right.$$
$$\left. + \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega = A_9\int_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega,$$

$$\text{as} \int_{\Omega^n}\frac{\partial N}{\partial t}\tilde{N}d\Omega = \int_{\Omega^n}\frac{(N^{n+1} - N^n)}{\delta t}\tilde{N}d\Omega = \frac{1}{\delta t}\int_{\Omega^n}N^{n+1}\tilde{N}d\Omega - \frac{1}{\delta t}\int_{\Omega^n}N^n\tilde{N}d\Omega,$$

$$\frac{1}{\delta t}\int_{\Omega^n}N^{n+1}\tilde{N}d\Omega - \frac{1}{\delta t}\int_{\Omega^n}N^n\tilde{N}d\Omega + \frac{1}{2}\int_{\Omega^n}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega$$
$$+ \frac{1}{2}\int_{\Omega^n}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega + A_8\int_{\Omega^n}\left(\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right)\right.$$
$$\left. - N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} + \frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right) - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega$$
$$= A_9\int_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega$$

$$\frac{1}{\delta t}\int\limits_{\Omega^n} N^{n+1}\tilde{N}d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^n} N^{n}\tilde{N}d\Omega + \frac{1}{2}\int\limits_{\Omega^n}\left(U\frac{\partial N}{\partial X} + U\frac{\partial N}{\partial X} + N\frac{\partial U}{\partial X} + N\frac{\partial U}{\partial X}\right)\tilde{N}d\Omega$$
$$+\frac{1}{2}\int\limits_{\Omega^n}\left(V\frac{\partial N}{\partial Y} + V\frac{\partial N}{\partial Y} + N\frac{\partial V}{\partial Y} + N\frac{\partial V}{\partial Y}\right)\tilde{N}d\Omega + A_8\int\limits_{\Omega^n}\Bigg(\cancelto{0}{\frac{\partial}{\partial X}\left(N\frac{\partial \phi}{\partial X}\tilde{N}\right)}$$

$$- N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} + \cancelto{0}{\frac{\partial}{\partial Y}\left(N\frac{\partial \phi}{\partial Y}\tilde{N}\right)} - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\Bigg)d\Omega$$
$$= A_9\int\limits_{\Omega^n}\left[\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right]\tilde{N}d\Omega,$$

$$\frac{1}{\delta t}\int_{\Omega^n} N^{n+1}\tilde{N}\,d\Omega - \frac{1}{\delta t}\int_{\Omega^n} N^{n}\tilde{N}\,d\Omega$$

$$+\frac{1}{2}\int_{\Omega^n}\left(2U\frac{\partial N}{\partial X} + 2N\frac{\partial U}{\partial X}\right)\tilde{N}\,d\Omega + \frac{1}{2}\int_{\Omega^n}\left(2V\frac{\partial N}{\partial Y} + 2N\frac{\partial V}{\partial Y}\right)\tilde{N}\,d\Omega$$

$$+A_8\int_{\Omega^n}\left(-N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X} - \frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N} - N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y} - \frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\right)d\Omega$$

$$= A_9\int_{\Omega^n}\left(\frac{\partial^2 N}{\partial X^2} + \frac{\partial^2 N}{\partial Y^2}\right)\tilde{N}\,d\Omega \tag{4.40}$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega}\psi\Delta\phi\ d\Omega = -\int\limits_{\Omega}\nabla\phi\nabla\psi\ d\Omega + \int\limits_{\Omega}\psi(\nabla\phi n)\ d\Gamma$$

Here,

$$\psi = \tilde{N},\ \Delta\phi = \frac{\partial^2 N}{\partial X^2},\ \Delta\phi = \frac{\partial^2 N}{\partial Y^2},\ \nabla\phi = \frac{\partial N}{\partial X},\ \nabla\phi = \frac{\partial N}{\partial Y},\ \nabla\psi = \frac{\partial \tilde{N}}{\partial X},\ \nabla\psi = \frac{\partial \tilde{N}}{\partial Y}$$

As $$\nabla\phi n = \frac{\partial \phi}{\partial n} = n_x\frac{\partial N}{\partial X} + n_y\frac{\partial N}{\partial Y} \qquad (\phi = N)$$

So,

$$\int_{\Omega^n}\left(\frac{\partial^2 N}{\partial X^2}+\frac{\partial^2 N}{\partial Y^2}\right)\tilde{N}\,d\Omega = -\int_{\Omega^n}\frac{\partial N}{\partial X}\frac{\partial \tilde{N}}{\partial X}\,d\Omega + \oint_{\Gamma}\tilde{N}\left(n_x\frac{\partial N}{\partial X}\right)d\Gamma$$

$$-\int_{\Omega^n}\frac{\partial N}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}\,d\Omega + \oint_{\Gamma}\tilde{N}\left(n_y\frac{\partial N}{\partial Y}\right)d\Gamma$$

Now (4.40), becomes:

$$\frac{1}{\delta t}\int_{\Omega^n} N^{n+1}\tilde{N}\,d\Omega - \frac{1}{\delta t}\int_{\Omega^n} N^{n}\tilde{N}\,d\Omega$$

$$+\frac{1}{2}\int_{\Omega^n}\left(U\frac{\partial N}{\partial X}+U\frac{\partial N}{\partial X}+N\frac{\partial U}{\partial X}+N\frac{\partial U}{\partial X}\right)\tilde{N}\,d\Omega$$

$$+\frac{1}{2}\int_{\Omega^n}\left(V\frac{\partial N}{\partial Y}+V\frac{\partial N}{\partial Y}+N\frac{\partial V}{\partial Y}+N\frac{\partial V}{\partial Y}\right)\tilde{N}\,d\Omega$$

$$+A_8\int_{\Omega^n}\Bigg(-N\frac{\partial \phi}{\partial X}\frac{\partial \tilde{N}}{\partial X}-\frac{\partial N}{\partial X}\frac{\partial \phi}{\partial X}\tilde{N}$$

$$-N\frac{\partial \phi}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}-\frac{\partial N}{\partial Y}\frac{\partial \phi}{\partial Y}\tilde{N}\Bigg)\,d\Omega$$

$$=A_9\Bigg(-\int_{\Omega^n}\frac{\partial N}{\partial X}\frac{\partial \tilde{N}}{\partial X}\,d\Omega+\oint_{\Gamma}\tilde{N}\left(n_x\frac{\partial N}{\partial X}\right)d\Gamma$$

$$-\int_{\Omega^n}\frac{\partial N}{\partial Y}\frac{\partial \tilde{N}}{\partial Y}\,d\Omega+\oint_{\Gamma}\tilde{N}\left(n_y\frac{\partial N}{\partial Y}\right)d\Gamma\Bigg)$$

Taking Domain as current time step value, we have

$$
\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N^{n+1}\tilde{N}\,d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N^{n}\tilde{N}\,d\Omega \\
&+ \frac{1}{2}\int\limits_{\Omega^{n+1}} \left(U^{n+1}\frac{\partial N^{n+1}}{\partial X^{n+1}} + U^{n+1}\frac{\partial N^{n+1}}{\partial X^{n+1}} + N^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}} + N^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\right)\tilde{N}\,d\Omega \\
&+ \frac{1}{2}\int\limits_{\Omega^{n+1}} \left(V^{n+1}\frac{\partial N^{n+1}}{\partial Y^{n+1}} + V^{n+1}\frac{\partial N^{n+1}}{\partial Y^{n+1}} + N^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}} + N^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\right)\tilde{N}\,d\Omega \\
&+ A_8\int\limits_{\Omega^{n+1}} \Bigg(-N^{n+1}\frac{\partial \phi^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{N}}{\partial X^{n+1}} - \frac{\partial N^{n+1}}{\partial X^{n+1}}\frac{\partial \phi^{n+1}}{\partial X^{n+1}}\tilde{N} \\
&\qquad - N^{n+1}\frac{\partial \phi^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{N}}{\partial Y^{n+1}} - \frac{\partial N^{n+1}}{\partial Y^{n+1}}\frac{\partial \phi^{n+1}}{\partial Y^{n+1}}\tilde{N}\Bigg)\,d\Omega \\
&= A_9\Bigg(-\int\limits_{\Omega^{n+1}} \frac{\partial N^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{N}}{\partial X^{n+1}}\,d\Omega + \oint\limits_{\Gamma}\tilde{N}\left(n_x\frac{\partial N^{n+1}}{\partial X^{n+1}}\right)\,d\Gamma \\
&\qquad - \int\limits_{\Omega^{n+1}} \frac{\partial N^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{N}}{\partial Y^{n+1}}\,d\Omega + \oint\limits_{\Gamma}\tilde{N}\left(n_y\frac{\partial N^{n+1}}{\partial Y^{n+1}}\right)\,d\Gamma\Bigg)
\end{aligned}
\tag{4.41}
$$

Find $(U,\ V,\ \theta,\ \phi,\ N)\ \varepsilon\ W$ and $P\ \varepsilon\ Q$ such that

$$
\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U^{n+1}\tilde{U}\,d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U^{n}\tilde{U}\,d\Omega \\
&+ A_1\int\limits_{\Omega^{n+1}} U^{n+1}\tilde{U}\,d\Omega + q^{n+1}\int\limits_{\Omega^{n+1}} U^{n+1}\frac{\partial \tilde{U}}{\partial X^{n+1}}\,d\Omega
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V}\,d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V^{n}\tilde{V}\,d\Omega + A_2\int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V}\,d\Omega \\
&+q^{n+1}\int\limits_{\Omega^{n+1}} V^{n+1}\frac{\partial \tilde{V}}{\partial X^{n+1}}\,d\Omega - P^{n+1}\int\limits_{\Omega^{n+1}} \frac{\partial \tilde{V}}{\partial Y^{n+1}}\,d\Omega \\
&+A_3A_4\int\limits_{\Omega^{n+1}} N^{n+1}\tilde{V}\,d\Omega - A_3\int\limits_{\Omega^{n+1}} \theta^{n+1}\tilde{V}\,d\Omega = 0
\end{aligned}
\tag{4.42}
$$

$$
\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \theta^{n+1}\tilde{\theta}\,d\Omega - \frac{1}{\delta t}\left(\theta^{n}\circ X^{n}\right)\tilde{\theta}\,d\Omega
$$

$$
\begin{aligned}
&+ \int\limits_{\Omega^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\theta}}{\partial X^{n+1}} \, d\Omega - \oint\limits_{\Gamma} \tilde{\theta} \left(n_x \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \right) \, d\Gamma \\
&+ \int\limits_{\Omega^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\theta}}{\partial Y^{n+1}} \, d\Omega - \oint\limits_{\Gamma} \tilde{\theta} \left(n_y \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \right) \, d\Gamma \\
&- \frac{A_5}{3} \int\limits_{\Omega^{n+1}} \Big(U^{n+1} \frac{\partial U^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} + U^{n+1} \frac{\partial U^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \\
&\qquad\qquad + U^{n+1} \frac{\partial U^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \Big) \tilde{\theta} \, d\Omega \\
&- \frac{A_5}{3} \int\limits_{\Omega^{n+1}} \Big(V^{n+1} \frac{\partial U^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} + V^{n+1} \frac{\partial U^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \\
&\qquad\qquad + V^{n+1} \frac{\partial U^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \Big) \tilde{\theta} \, d\Omega \\
&- \frac{A_5}{3} \int\limits_{\Omega^{n+1}} \Big(U^{n+1} \frac{\partial V^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} + U^{n+1} \frac{\partial V^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \\
&\qquad\qquad + U^{n+1} \frac{\partial V^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \Big) \tilde{\theta} \, d\Omega \\
&- \frac{A_5}{3} \int\limits_{\Omega^{n+1}} \Big(V^{n+1} \frac{\partial V^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} + V^{n+1} \frac{\partial V^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \\
&\qquad\qquad + V^{n+1} \frac{\partial V^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \Big) \tilde{\theta} \, d\Omega \\
&- A_5 \int\limits_{\Omega^{n}} \Big(- U^n U^{n+1} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\theta}}{\partial X^{n+1}} - 2U^{n+1} \frac{\partial U^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \tilde{\theta} \Big) \, d\Omega \\
&- A_5 \int\limits_{\Omega^{n+1}} \Big(- V^n V^{n+1} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\theta}}{\partial Y^{n+1}} - 2V^{n+1} \frac{\partial V^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \tilde{\theta} \Big) \, d\Omega \\
&- A_5 \int\limits_{\Omega^{n+1}} \Big(- U^{n+1} V^{n+1} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\theta}}{\partial X^{n+1}} - U^{n+1} \frac{\partial V^{n+1}}{\partial X^{n+1}} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \tilde{\theta} \\
&\qquad\qquad - \frac{\partial U^{n+1}}{\partial X^{n+1}} V^{n+1} \frac{\partial \theta^{n+1}}{\partial Y^{n+1}} \tilde{\theta} - U^{n+1} V^{n+1} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\theta}}{\partial Y^{n+1}} \\
&\qquad\qquad - U^{n+1} \frac{\partial V^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \tilde{\theta} - \frac{\partial U^{n+1}}{\partial Y^{n+1}} V^{n+1} \frac{\partial \theta^{n+1}}{\partial X^{n+1}} \tilde{\theta} \Big) \, d\Omega = 0 \qquad (4.43)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\delta t} \int\limits_{\Omega^{n+1}} \phi^{n+1} \tilde{\phi} d\Omega - \frac{1}{\delta t} \Big(\phi^n \ o \ X^n \Big) \tilde{\phi} d\Omega - A_6 \Big[- \int\limits_{\Omega^{n+1}} \frac{\partial \phi^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\phi}}{\partial X^{n+1}} d\Omega + \oint\limits_{\Gamma} \tilde{\phi} \Big(n_x \\
&\frac{\partial \phi^{n+1}}{\partial X^{n+1}} \Big) d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial \phi^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\phi}}{\partial Y^{n+1}} d\Omega + \oint\limits_{\Gamma} \tilde{\phi} \Big(n_y \frac{\partial \phi^{n+1}}{\partial Y^{n+1}} \Big) d\Gamma \Big] + A_7 \int\limits_{\Omega^{n+1}} N^{n+1} \tilde{\phi} d\Omega = 0
\end{aligned}
\qquad (4.44)
$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N^{n+1}\tilde{N}\,d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N^{n}\tilde{N}\,d\Omega \\
&+\frac{1}{2}\int\limits_{\Omega^{n+1}}\Big(U^{n+1}\frac{\partial N^{n+1}}{\partial X^{n+1}}+U^{n+1}\frac{\partial N^{n+1}}{\partial X^{n+1}}+N^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}+N^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)\tilde{N}\,d\Omega \\
&+\frac{1}{2}\int\limits_{\Omega^{n+1}}\Big(V^{n+1}\frac{\partial N^{n+1}}{\partial Y^{n+1}}+V^{n+1}\frac{\partial N^{n+1}}{\partial Y^{n+1}}+N^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}+N^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\Big)\tilde{N}\,d\Omega \\
&+A_8\int\limits_{\Omega^{n+1}}\Big(-N^{n+1}\frac{\partial \phi^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{N}}{\partial X^{n+1}}-\frac{\partial N^{n+1}}{\partial X^{n+1}}\frac{\partial \phi^{n+1}}{\partial X^{n+1}}\tilde{N} \\
&\qquad -N^{n+1}\frac{\partial \phi^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{N}}{\partial Y^{n+1}}-\frac{\partial N^{n+1}}{\partial Y^{n+1}}\frac{\partial \phi^{n+1}}{\partial Y^{n+1}}\tilde{N}\Big)\,d\Omega \\
&-A_9\Bigg(-\int\limits_{\Omega^{n+1}}\frac{\partial N^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{N}}{\partial X^{n+1}}\,d\Omega+\oint\limits_{\Gamma}\tilde{N}\left(n_x\frac{\partial N^{n+1}}{\partial X^{n+1}}\right)d\Gamma \\
&\qquad -\int\limits_{\Omega^{n+1}}\frac{\partial N^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{N}}{\partial Y^{n+1}}\,d\Omega+\oint\limits_{\Gamma}\tilde{N}\left(n_y\frac{\partial N^{n+1}}{\partial Y^{n+1}}\right)d\Gamma\Bigg)=0
\end{aligned}
\tag{4.45}
$$

for all $\tilde{U}$,$\tilde{V}$,$\tilde{\theta}$,$\tilde{\phi}$,$\tilde{N}$ ε W and P ε Q.

For the Galerkin discretization, the infinite dimensional test and trial spaces are approximated by finite dimensional spaces. In particular, following are the trial and test spaces

Trial Spaces:

$$U\approx U_h,\quad V\approx V_h,\quad \theta\approx\theta_h,\quad \phi\approx\phi_h,\quad N\approx N_h\quad\text{and}\quad P\approx P_h.$$

Test Spaces:

$$\mathrm{W}\approx \mathrm{W}_h\quad\text{and}\quad \mathrm{Q}\approx \mathrm{Q}_h.$$

Find $(U_h,\ V_h,\ \theta_h,\phi_h,N_h)\ \varepsilon\ W_h$ and $\mathrm{P}_h\ \varepsilon\ Q_h$ such that

$$
\begin{aligned}
&\frac{1}{\delta t}\int_{\Omega^{n+1}} U_h^{n+1}\tilde{U}_h\,d\Omega-\frac{1}{\delta t}\int_{\Omega^{n+1}} U_h^{n}\tilde{U}_h\,d\Omega+A_1\int_{\Omega^{n+1}} U_h^{n+1}\tilde{U}_h\,d\Omega \\
&+q_h^{n+1}\int_{\Omega^{n+1}} U_h^{n+1}\frac{\partial \tilde{U}_h}{\partial X^{n+1}}\,d\Omega-P_h^{n+1}\int_{\Omega^{n+1}}\frac{\partial \tilde{U}_h}{\partial X^{n+1}}\,d\Omega=0
\end{aligned}
\tag{4.46}
$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} V_h^{n+1}\tilde{V}_h\,d\Omega-\frac{1}{\delta t}\int_{\Omega^{n+1}} V_h^{n}\tilde{V}_h\,d\Omega+A_2\int_{\Omega^{n+1}} V_h^{n+1}\tilde{V}_h\,d\Omega$$

$$+ q_h^{n+1} \int_{\Omega^{n+1}} V_h^{n+1} \frac{\partial \tilde{V}_h}{\partial X^{n+1}}\, d\Omega - P_h^{n+1} \int_{\Omega^{n+1}} \frac{\partial \tilde{V}_h}{\partial Y^{n+1}}\, d\Omega$$
$$+ A_3 A_4 \int_{\Omega^{n+1}} N_h^{n+1} \tilde{V}_h\, d\Omega - A_3 \int_{\Omega^{n+1}} \theta_h^{n+1} \tilde{V}_h\, d\Omega = 0 \tag{4.47}$$

$$\begin{aligned}
&\frac{1}{\delta t} \int_{\Omega^{n+1}} \theta_h^{n+1} \tilde{\theta}_h\, d\Omega - \frac{1}{\delta t} \left(\theta_h^n \circ X^n\right) \tilde{\theta}_h\, d\Omega + \int_{\Omega^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\theta}_h}{\partial X^{n+1}}\, d\Omega \\
&- \oint_{\Gamma} \tilde{\theta}_h \left(n_x \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \right) d\Gamma + \int_{\Omega^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\theta}_h}{\partial Y^{n+1}}\, d\Omega - \oint_{\Gamma} \tilde{\theta}_h \left(n_y \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \right) d\Gamma \\
&- \frac{A_5}{3} \int_{\Omega^{n+1}} \left(3U_h^{n+1} \frac{\partial U_h^{n+1}}{\partial X^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \right) \tilde{\theta}_h\, d\Omega \\
&- \frac{A_5}{3} \int_{\Omega^{n+1}} \left(3V_h^{n+1} \frac{\partial U_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \right) \tilde{\theta}_h\, d\Omega \\
&- \frac{A_5}{3} \int_{\Omega^{n+1}} \left(3U_h^{n+1} \frac{\partial V_h^{n+1}}{\partial X^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \right) \tilde{\theta}_h\, d\Omega \\
&- \frac{A_5}{3} \int_{\Omega^{n+1}} \left(3V_h^{n+1} \frac{\partial V_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \right) \tilde{\theta}_h\, d\Omega \\
&- A_5 \int_{\Omega^{n}} \left(-U_h^n U_h^{n+1} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\theta}_h}{\partial X^{n+1}} - 2U_h^{n+1} \frac{\partial U_h^{n+1}}{\partial X^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \tilde{\theta}_h \right) d\Omega \\
&- A_5 \int_{\Omega^{n+1}} \left(-V_h^n V_h^{n+1} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\theta}_h}{\partial Y^{n+1}} - 2V_h^{n+1} \frac{\partial V_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \tilde{\theta}_h \right) d\Omega \\
&- A_5 \int_{\Omega^{n+1}} \Bigg(-U_h^{n+1} V_h^{n+1} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\theta}_h}{\partial X^{n+1}} - U_h^{n+1} \frac{\partial V_h^{n+1}}{\partial X^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \tilde{\theta}_h \\
&- \frac{\partial U_h^{n+1}}{\partial X^{n+1}} V_h^{n+1} \frac{\partial \theta_h^{n+1}}{\partial Y^{n+1}} \tilde{\theta}_h - U_h^{n+1} V_h^{n+1} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\theta}_h}{\partial Y^{n+1}} - U_h^{n+1} \frac{\partial V_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \tilde{\theta}_h \\
&- \frac{\partial U_h^{n+1}}{\partial Y^{n+1}} V_h^{n+1} \frac{\partial \theta_h^{n+1}}{\partial X^{n+1}} \tilde{\theta}_h \Bigg) d\Omega = 0
\end{aligned} \tag{4.48}$$

$$\frac{1}{\delta t} \int_{\Omega^{n+1}} \phi_h^{n+1} \tilde{\phi}_h\, d\Omega - \frac{1}{\delta t} \int_{\Omega^{n+1}} \left(\phi_h^n \circ X^n\right) \tilde{\phi}_h\, d\Omega$$

$$- A_6 \Bigg[\int_{\Omega^{n+1}} \frac{\partial \phi_h^{n+1}}{\partial X^{n+1}} \frac{\partial \tilde{\phi}_h}{\partial X^{n+1}}\, d\Omega - \oint_{\Gamma} \tilde{\phi}_h \left(n_x \frac{\partial \phi_h^{n+1}}{\partial X^{n+1}} \right) d\Gamma$$

$$+ \int_{\Omega^{n+1}} \frac{\partial \phi_h^{n+1}}{\partial Y^{n+1}} \frac{\partial \tilde{\phi}_h}{\partial Y^{n+1}}\, d\Omega - \oint_{\Gamma} \tilde{\phi}_h \left(n_y \frac{\partial \phi_h^{n+1}}{\partial Y^{n+1}} \right) d\Gamma \Bigg]$$

$$+ A_7 \int\limits_{\Omega^{n+1}} N_h^{n+1}\tilde{\phi}_h \, d\Omega = 0 \tag{4.49}$$

$$\begin{aligned}
&\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N_h^{n+1}\tilde{N}_h d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N_h^{n}\tilde{N}_h d\Omega + \frac{1}{2}\int\limits_{\Omega^{n+1}} \Bigg(U_h^{n+1}\frac{\partial N_h^{n+1}}{\partial X^{n+1}} + U_h^{n+1}\frac{\partial N_h^{n+1}}{\partial X^{n+1}} \\
&+ N_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}} + N_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\Bigg)\tilde{N}_h d\Omega + \frac{1}{2}\int\limits_{\Omega^{n+1}} \Bigg(V_h^{n+1}\frac{\partial N_h^{n+1}}{\partial Y^{n+1}} + V_h^{n+1}\frac{\partial N_h^{n+1}}{\partial Y^{n+1}} \\
&+ N_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} + N_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}\Bigg)\tilde{N}_h d\Omega + A_8\int\limits_{\Omega^{n+1}} \Bigg(- N_h^{n+1}\frac{\partial \phi_h^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{N}_h}{\partial X^{n+1}} \\
&- \frac{\partial N_h^{n+1}}{\partial X^{n+1}}\frac{\partial \phi_h^{n+1}}{\partial X^{n+1}}\tilde{N}_h - N_h^{n+1}\frac{\partial \phi_h^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{N}_h}{\partial Y^{n+1}} - \frac{\partial N_h^{n+1}}{\partial Y^{n+1}}\frac{\partial \phi_h^{n+1}}{\partial Y^{n+1}}\tilde{N}_h\Bigg)d\Omega \\
&- A_9\Bigg(\Bigg[-\int\limits_{\Omega^{n+1}} \frac{\partial N_h^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{N}_h}{\partial X^{n+1}}d\Omega \oint\limits_{\Gamma} \tilde{N}_h\Bigg(n_x\frac{\partial N_h^{n+1}}{\partial X^{n+1}}\Bigg)d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial N_h^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{N}_h}{\partial Y^{n+1}}d\Omega \\
&+ \oint\limits_{\Gamma} \tilde{N}_h\Bigg(n_y\frac{\partial N_h^{n+1}}{\partial Y^{n+1}}\Bigg)d\Gamma\Bigg]\Bigg) = 0,
\end{aligned} \tag{4.50}$$

for all $\tilde{U}_h$,$\tilde{V}_h$,$\tilde{\theta}_h$,$\tilde{\phi}_h$,$\tilde{N}_h$ ε W_h and P_h ε Q_h.

FEM approximation is achieved by using the approximate trial solution functions and trial test functions. These functions are the linear combination of nodal unknowns and shape functions which are linearly independent. Given below are the trial solution functions:

$$\begin{aligned}
&U_h = \sum_{j=1}^{m} U_j\xi_j,\ V_h = \sum_{j=1}^{m} V_j\xi_j,\ \theta_h = \sum_{j=1}^{m} \theta_j\xi_j,\ \phi_h = \sum_{j=1}^{m} \phi_j\xi_j, \\
&N_h = \sum_{j=1}^{m} N_j\xi_j,\ P_h = \sum_{j=1}^{l} P_j\eta_j.
\end{aligned}$$

Similarly following trial approximated functions are defined for test spaces:

$$\tilde{U}_h, \tilde{V}_h, \tilde{\theta}_h, \tilde{\phi}_h, \tilde{N}_h = \sum_{i=1}^{m}(\tilde{U}_i, \tilde{V}_i, \tilde{\theta}_i, \tilde{\phi}_i, \tilde{N}_i)\xi_i, \qquad q_h = \sum_{i=1}^{l} q_i\xi_i.$$

In all above relations ξ_j and η_j are the shape functions. By using these approximations in Eqs. (4.46) to (4.50), weak formulation can be expressed as

(4.46)$\Rightarrow$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\tilde{U}_i\xi_i d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)^{n}\sum_{i=1}^{m}\tilde{U}_i\xi_i d\Omega$$
$$+A_1\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}U_j\xi_j\sum_{i=1}^{m}\tilde{U}_i\xi_i d\Omega+\sum_{i=1}^{l}q_i\xi_i\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{U}_i\xi_i d\Omega$$
$$-\sum_{j=1}^{l}P_j\eta_j\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{U}_i\xi_i d\Omega=0.$$

By Galerkins,

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)^{n+1}\left(\sum_{i=1}^{m}\xi_i\right)d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)^{n}\left(\sum_{i=1}^{m}\xi_i\right)d\Omega$$
$$+A_1\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)\left(\sum_{i=1}^{m}\xi_i\right)d\Omega+\sum_{i=1}^{l}q_i\xi_i\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{i=1}^{m}\xi_i\right)d\Omega$$
$$-\sum_{j=1}^{l}P_j\eta_j\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X}\left(\sum_{i=1}^{m}\xi_i\right)d\Omega=0$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)^{n+1}\left(\sum_{i=1}^{m}\xi_i\right)d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)^{n}\left(\sum_{i=1}^{m}\xi_i\right)d\Omega$$
$$+A_1\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)\left(\sum_{i=1}^{m}\xi_i\right)d\Omega$$

$$+\sum_{i=1}^{l}q_i\xi_i\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\right)\left(\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial X}\right)d\Omega$$
$$-\sum_{j=1}^{l}P_j\eta_j\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial X}\right)d\Omega=0$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(U_j\xi_j\right)^{n+1}\xi_i\,d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(U_j\xi_j\right)^{n}\xi_i\,d\Omega$$
$$+A_1\int\limits_{\Omega^{n+1}}U_j\xi_j\xi_i\,d\Omega+\xi_i\int\limits_{\Omega^{n+1}}U_j\xi_j\frac{\partial\xi_i}{\partial X}\,d\Omega-\eta_j\int\limits_{\Omega^{n+1}}\frac{\partial\xi_i}{\partial X}\,d\Omega=0$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U_j^{n}\xi_j^{n}\xi_i^{n+1}d\Omega$$
$$+ A_1\int\limits_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega + \xi_i^{n+1}\int\limits_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}d\Omega$$
$$- \eta_j^{n+1}\int\limits_{\Omega^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}d\Omega = 0. \tag{4.51}$$

(4.47)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j\xi_j\right)^{n}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega$$
$$+ A_2\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j\xi_j\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega + \sum_{i=1}^{l} q_i\xi_i\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega$$
$$-\sum_{j=1}^{l} P_j\eta_j\int\limits_{\Omega^{n+1}} \frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega + A_3A_4\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} N_j\xi_j\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega$$
$$- A_3\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} \theta_j\xi_j\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega = 0.$$

By Galerkins,

$$\tilde{V}_h = \sum_{i=1}^{m}\xi_i,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\xi_i d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j\xi_j\right)^{n}\sum_{i=1}^{m}\xi_i d\Omega$$
$$+ A_2\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega + \sum_{i=1}^{l} q_i\xi_i\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\xi_i d\Omega - \sum_{j=1}^{l} P_j\eta_j$$
$$\int\limits_{\Omega^{n+1}} \frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i d\Omega + A_3A_4\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} N_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega A_3\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m}\theta_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega = 0$$
$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\xi_i d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j\xi_j\right)^{n}\sum_{i=1}^{m}\xi_i d\Omega + A_2\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j\xi_j$$
$$\sum_{i=1}^{m}\xi_i d\Omega + \sum_{i=1}^{l} q_i\xi_i\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j\xi_j\sum_{i=1}^{m}\frac{\partial \xi_i}{\partial X}d\Omega - \sum_{j=1}^{l} P_j\eta_j\int\limits_{\Omega^{n+1}} \sum_{i=1}^{m}\frac{\partial \xi_i}{\partial Y}d\Omega$$
$$+ A_3A_4\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} N_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega + A_3\int\limits_{\Omega^{n+1}} \sum_{j=1}^{m}\theta_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega = 0,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(V_j\xi_j\right)^{n+1}\xi_i d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(V_j\xi_j\right)^{n}\xi_i d\Omega+A_2\int\limits_{\Omega^{n+1}}V_j\xi_j\xi_i d\Omega$$
$$+\xi_i\int\limits_{\Omega^{n+1}}V_j\xi_j\frac{\partial\xi_i}{\partial X}d\Omega-\eta_j\int\limits_{\Omega^{n+1}}\frac{\partial\xi_i}{\partial Y}d\Omega+A_3A_4\int\limits_{\Omega^{n+1}}N_j\xi_j\xi_i d\Omega-A_3\int\limits_{\Omega^{n+1}}\theta_j\xi_j\xi_i d\Omega=0,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}V_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}V_j^{n}\xi_j^{n}\xi_i^{n+1}d\Omega+A_2\int\limits_{\Omega^{n+1}}V_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega$$
$$+\xi_i^{n+1}\int\limits_{\Omega^{n+1}}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega-\eta_j^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega+A_3A_4\int\limits_{\Omega^{n+1}}N_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega$$
$$-A_3\int\limits_{\Omega^{n+1}}\theta_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega=0. \tag{4.52}$$

(4.48)$\Rightarrow$

$$\int\limits_{\Omega^n}\Bigg(-\left(\sum_{j=1}^m U_j\xi_j\right)^2\frac{\partial}{\partial X}\left(\sum_{j=1}^m\theta_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-2\left(\sum_{j=1}^m U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^m U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^m\theta_j\xi_j\right)\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)\Bigg)d\Omega$$
$$-A_5\int\limits_{\Omega^{n+1}}\Bigg(-\left(\sum_{j=1}^m V_j\xi_j\right)^2\frac{\partial}{\partial Y}\left(\sum_{j=1}^m\theta_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-2\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^m\theta_j\xi_j\right)\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)\Bigg)d\Omega$$

$$-A_5\int\limits_{\Omega^{n+1}}\Bigg(-\left(\sum_{j=1}^m U_j\xi_j\right)\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^m\theta_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-\left(\sum_{j=1}^m U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^m\theta_j\xi_j\right)\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-\frac{\partial}{\partial X}\left(\sum_{j=1}^m U_j\xi_j\right)\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^m\theta_j\xi_j\right)\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-\left(\sum_{j=1}^m U_j\xi_j\right)\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^m\theta_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-\left(\sum_{j=1}^m U_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^m\theta_j\xi_j\right)\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)$$
$$-\frac{\partial}{\partial Y}\left(\sum_{j=1}^m U_j\xi_j\right)\left(\sum_{j=1}^m V_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^m\theta_j\xi_j\right)\left(\sum_{i=1}^m\tilde{\theta}_i\xi_i\right)\Bigg)d\Omega=0$$

By Galerkins,

$$\tilde{\theta}_h = \sum_{i=1}^{m} \xi_i$$

$$\tilde{\theta}_h = \sum_{i=1}^{m} \xi_i$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} \left(\sum_{j=1}^{m}\theta_j\xi_j\right)^{n+1}\left(\sum_{i=1}^{m}\xi_i\right) d\Omega - \frac{1}{\delta t}\left(\left(\sum_{j=1}^{m}\theta_j\xi_j\right)^{n}\circ X^n\right)\left(\sum_{i=1}^{m}\xi_i\right) d\Omega$$

$$+\int_{\Omega^{n+1}} \frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{i=1}^{m}\xi_i\right) d\Omega - \oint_{\Gamma}\left(\sum_{i=1}^{m}\xi_i\right)\left(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}\theta_j\xi_j\right) d\Gamma$$

$$+\int_{\Omega^{n+1}} \frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{i=1}^{m}\xi_i\right) d\Omega - \oint_{\Gamma}\left(\sum_{i=1}^{m}\xi_i\right)\left(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}\theta_j\xi_j\right) d\Gamma$$

$$-\frac{A_5}{3}\int_{\Omega^{n+1}}\left[\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\right.$$

$$+\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)$$

$$\left.+\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\right]\left(\sum_{i=1}^{m}\xi_i\right) d\Omega$$

$$-\frac{A_5}{3}\int_{\Omega^{n+1}}\left[\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\right.$$

$$+\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)$$

$$\left.+\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}U_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\right]\left(\sum_{i=1}^{m}\xi_i\right) d\Omega$$

$$-\frac{A_5}{3}\int_{\Omega^{n+1}}\left[\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\right.$$

$$+\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)$$

$$\left.+\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\left(\sum_{j=1}^{m}V_j\xi_j\right)\frac{\partial}{\partial Y}\left(\sum_{j=1}^{m}\theta_j\xi_j\right)\right]\left(\sum_{i=1}^{m}\xi_i\right) d\Omega$$

$$- \frac{A_5}{3} \int_{\Omega^{n+1}} \left[\sum_{j=1}^{m} V_j \xi_j \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \right.$$

$$+ \sum_{j=1}^{m} V_j \xi_j \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right)$$
$$\left. + \sum_{j=1}^{m} V_j \xi_j \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \right] \left(\sum_{i=1}^{m} \xi_i \right) \, d\Omega$$

$$- A_5 \int_{\Omega^{n}} \left[- \left(\sum_{j=1}^{m} U_j \xi_j \right)^2 \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \frac{\partial}{\partial X} \left(\sum_{i=1}^{m} \xi_i \right) \right.$$

$$\left. - 2 \sum_{j=1}^{m} U_j \xi_j \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} U_j \xi_j \right) \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \left(\sum_{i=1}^{m} \xi_i \right) \right] d\Omega$$

$$- A_5 \int_{\Omega^{n+1}} \left(- \left(\sum_{j=1}^{m} V_j \xi_j \right)^2 \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{i=1}^{m} \xi_i \right) \right.$$

$$\left. - 2 \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \left(\sum_{i=1}^{m} \xi_i \right) \right) d\Omega$$

$$- A_5 \int_{\Omega^{n+1}} \left(- \left(\sum_{j=1}^{m} U_j \xi_j \right) \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \frac{\partial}{\partial X} \left(\sum_{i=1}^{m} \xi_i \right) \right.$$

$$- \left(\sum_{j=1}^{m} U_j \xi_j \right) \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \left(\sum_{i=1}^{m} \xi_i \right)$$

$$- \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} U_j \xi_j \right) \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \left(\sum_{i=1}^{m} \xi_i \right)$$

$$- \left(\sum_{j=1}^{m} U_j \xi_j \right) \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{i=1}^{m} \xi_i \right)$$

$$- \left(\sum_{j=1}^{m} U_j \xi_j \right) \frac{\partial}{\partial Y} \left(\sum_{j=1}^{m} V_j \xi_j \right) \frac{\partial}{\partial X} \left(\sum_{j=1}^{m} \theta_j \xi_j \right) \left(\sum_{i=1}^{m} \xi_i \right)$$

$$- \frac{\partial}{\partial Y}\left(\sum_{j=1}^{m} U_j\xi_j\right)\left(\sum_{j=1}^{m} V_j\xi_j\right)\frac{\partial}{\partial X}\left(\sum_{j=1}^{m} \theta_j\xi_j\right)\left(\sum_{i=1}^{m} \xi_i\right)\Bigg) d\Omega = 0$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\theta_j\xi_j\right)^{n+1}\xi_i d\Omega - \frac{1}{\delta t}\left(\left(\theta_j\xi_j\right)^{n}\ o\ X^n\right)\xi_i d\Omega + \int\limits_{\Omega^{n+1}} \frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial X} d\Omega - \oint\limits_{\Gamma} \xi_i$$

$$\left(n_x\frac{\partial\xi_j}{\partial X}\right)d\Gamma + \int\limits_{\Omega^{n+1}} \frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial Y}d\Omega - \oint\limits_{\Gamma}\xi_i\left(n_y\frac{\partial\xi_j}{\partial Y}\right)d\Gamma - \frac{A_5}{3}\int\limits_{\Omega^{n+1}} \Bigg(U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial X}$$

$$+ U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial X} + U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial X}\Bigg)\xi_i d\Omega - \frac{A_5}{3}\int\limits_{\Omega^{n+1}} \Bigg(V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial X} + V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial X}$$

$$+ V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial X}\Bigg)\xi_i d\Omega - \frac{A_5}{3}\int\limits_{\Omega^{n+1}} \Bigg(U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial Y} + U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial Y} + U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial Y}\Bigg)\xi_i d\Omega$$

$$- \frac{A_5}{3}\int\limits_{\Omega^{n+1}} \Bigg(V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial Y} + V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial Y} + V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial Y}\Bigg)\xi_i d\Omega - A_5\int\limits_{\Omega^{n}} \Bigg(- U_j\xi_j U_j\xi_j$$

$$\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial X} - 2U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial X}\xi_i\Bigg)d\Omega - A_5\int\limits_{\Omega^{n+1}} \Bigg(- V_j\xi_j V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial Y} - 2V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial Y}$$

$$\xi_i\Bigg)d\Omega - A_5\int\limits_{\Omega^{n+1}} \Bigg(- U_j\xi_j V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial X} - U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial Y}\xi_i - \frac{\partial\xi_j}{\partial X}V_j\xi_j\frac{\partial\xi_j}{\partial Y}\xi_i - U_j\xi_j$$

$$V_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial Y} - U_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial X}\xi_i - \frac{\partial\xi_j}{\partial Y}V_j\xi_j\frac{\partial\xi_j}{\partial X}\xi_i\Bigg)d\Omega = 0,$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int_{\Omega^{n+1}} \theta_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}\,d\Omega - \frac{1}{\delta t}\left(\theta_j^n\xi_j^n\;o\,X^n\right)\xi_i^{n+1}\,d\Omega \\
&+\int_{\Omega^{n+1}}\left(\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}+\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}\right)d\Omega \\
&-A_5\int_{\Omega^{n+1}}\left(U_j^{n+1}\xi_j^{n+1}\left(\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\right)^2+V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\right. \\
&\qquad\left.+U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}+V_j^{n+1}\xi_j^{n+1}\left(\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\right)^2\right)\xi_i^{n+1}\,d\Omega \\
&-A_5\int_{\Omega^{n}}\left(-U_j^n\xi_j^nU_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}-2U_j^{n+1}\xi_j^{n+1}\left(\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\right)^2\xi_i^{n+1}\right)d\Omega \\
&-A_5\int_{\Omega^{n+1}}\left(-V_j^n\xi_j^nV_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}-2V_j^{n+1}\xi_j^{n+1}\left(\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\right)^2\xi_i^{n+1}\right)d\Omega \\
&-A_5\int_{\Omega^{n+1}}\left(-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}\right. \\
&\qquad-U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\xi_i^{n+1}-\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\xi_i^{n+1} \\
&\qquad-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}-U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\xi_i^{n+1} \\
&\qquad\left.-\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\xi_i^{n+1}\right)d\Omega \\
&\qquad\qquad=\oint_\Gamma \xi_i^{n+1}\left(n_x\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\right)d\Gamma+\oint_\Gamma \xi_i^{n+1}\left(n_y\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\right)d\Gamma.
\end{aligned}
\tag{4.53}
$$

(4.49)$\Rightarrow$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}\phi_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\tilde{\phi}_i\xi_i d\Omega-\frac{1}{\delta t}\left(\left(\sum_{j=1}^{m}\phi_j\xi_j\right)^{n} o\ X^n\right)\sum_{i=1}^{m}\tilde{\phi}_i\xi_i d\Omega$$
$$-A_6\Bigg[-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{\phi}_i\xi_i d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\tilde{\phi}_i\xi_i\left(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\right)d\Gamma$$
$$-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{\phi}_i\xi_i d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\tilde{\phi}_i\xi_i\left(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\right)d\Gamma\Bigg]$$
$$+A_7\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}N_j\xi_j\sum_{i=1}^{m}\tilde{\phi}_i\xi_i d\Omega=0.$$

By Galerkins,

$$\tilde{\phi}_h=\sum_{i=1}^{m}\xi_i,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}\phi_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\xi_i d\Omega-\frac{1}{\delta t}\left(\left(\sum_{j=1}^{m}\phi_j\xi_j\right)^{n} o\ X^n\right)\sum_{i=1}^{m}\xi_i d\Omega$$
$$-A_6\Bigg[-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\xi_i d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\xi_i\left(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\right)d\Gamma$$
$$-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\xi_i\left(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\right)d\Gamma\Bigg]$$
$$+A_7\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}N_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega=0,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}\phi_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\xi_i d\Omega-\frac{1}{\delta t}\left(\left(\sum_{j=1}^{m}\phi_j\xi_j\right)^{n} o\ X^n\right)\sum_{i=1}^{m}\xi_i d\Omega-A_6\Bigg[-\int\limits_{\Omega^{n+1}}$$
$$\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial X}\phi_j\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial X}d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\xi_i\left(n_x\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial X}\phi_j\right)d\Gamma-\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial Y}\phi_j\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial Y}d\Omega$$
$$+\oint\limits_{\Gamma}\sum_{i=1}^{m}\xi_i\left(n_y\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial Y}\phi_j\right)d\Gamma\Bigg]+A_7\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}N_j\xi_j\sum_{i=1}^{m}\xi_i d\Omega=0,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\phi_j\xi_j\right)^{n+1}\xi_i d\Omega-\frac{1}{\delta t}\left(\left(\phi_j\xi_j\right)^{n} o\ X^n\right)\xi_i d\Omega-A_6\Bigg[-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial X}d\Omega$$
$$+\oint\limits_{\Gamma}\xi_i\left(n_x\frac{\partial\xi_j}{\partial X}\right)d\Gamma-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial Y}d\Omega+\oint\limits_{\Gamma}\xi_i\left(n_y\frac{\partial\xi_j}{\partial Y}\right)d\Gamma\Bigg]+A_7\int\limits_{\Omega^{n+1}}N_j\xi_j\xi_i d\Omega=0,$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \phi_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\left(\phi_j^n\xi_j^n \ o\ X^n\right)\xi_i^{n+1}d\Omega - A_6\Bigg[-\int\limits_{\Omega^{n+1}} \frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega$$
$$-\int\limits_{\Omega^{n+1}} \frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega\Bigg] + A_7\int\limits_{\Omega^{n+1}} N_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega$$
$$= \oint\limits_{\Gamma}\xi_i^{n+1}\left(n_x\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\right)d\Gamma + \oint\limits_{\Gamma}\xi_i^{n+1}\left(n_y\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\right)d\Gamma. \tag{4.54}$$

(4.50)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}N_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\tilde{N}_i\xi_i d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}N_j\xi_j\right)^{n}\sum_{i=1}^{m}\tilde{N}_i\xi_i d\Omega$$
$$+\frac{1}{2}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j + \sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j + \sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\right.$$
$$\left.+\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\right)\sum_{i=1}^{m}\tilde{N}_i\xi_i d\Omega + \frac{1}{2}\int\limits_{\Omega^{n+1}}\left(\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j + \sum_{j=1}^{m}V_j\xi_j\right.$$
$$\left.\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j + \sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j + \sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\right)\sum_{i=1}^{m}\tilde{N}_i\xi_i d\Omega + A_8\int\limits_{\Omega^{n+1}}$$
$$\left(-\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{N}_i\xi_i - \frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\sum_{i=1}^{m}\tilde{N}_i\xi_i\right.$$
$$\left.-\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{N}_i\xi_i - \frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\sum_{i=1}^{m}\tilde{N}_i\xi_i\right)d\Omega$$
$$-A_9\Bigg[-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{N}_i\xi_i d\Omega + \oint\limits_{\Gamma}\sum_{i=1}^{m}\tilde{N}_i\xi_i\left(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j\right)d\Gamma$$
$$-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{N}_i\xi_i d\Omega + \oint\limits_{\Gamma}\sum_{i=1}^{m}\tilde{N}_i\xi_i\left(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j\right)d\Gamma\Bigg] = 0.$$

By Galerkins,

$$\tilde{N}_h = \sum_{i=1}^{m}\xi_i,$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\left(\sum_{j=1}^{m}N_j\xi_j\right)^{n+1}\sum_{i=1}^{m}\xi_i\,d\Omega-\frac{1}{\delta t}\int_{\Omega^{n+1}}\left(\sum_{j=1}^{m}N_j\xi_j\right)^{n}\sum_{i=1}^{m}\xi_i\,d\Omega$$
$$+\frac{1}{2}\int_{\Omega^{n+1}}\left(2\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j+2\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\right)\sum_{i=1}^{m}\xi_i\,d\Omega$$
$$+\frac{1}{2}\int_{\Omega^{n+1}}\left(2\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j+2\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\right)\sum_{i=1}^{m}\xi_i\,d\Omega$$
$$+A_8\int_{\Omega^{n+1}}\left(-\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\xi_i-\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}\phi_j\xi_j\sum_{i=1}^{m}\xi_i\right.$$
$$\left.-\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i-\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}\phi_j\xi_j\sum_{i=1}^{m}\xi_i\right)d\Omega$$
$$-A_9\left[-\int_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\xi_i\,d\Omega+\oint_{\Gamma}\sum_{i=1}^{m}\xi_i\left(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}N_j\xi_j\right)d\Gamma\right.$$
$$\left.-\int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i\,d\Omega+\oint_{\Gamma}\sum_{i=1}^{m}\xi_i\left(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}N_j\xi_j\right)d\Gamma\right]=0 \tag{4.55}$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(N_j\xi_j\right)^{n+1}\xi_i d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(N_j\xi_j\right)^{n}\xi_i d\Omega+\frac{1}{2}\int\limits_{\Omega^{n+1}}\left(U_j\xi_j\frac{\partial\xi_j}{\partial X}+U_j\xi_j\frac{\partial\xi_j}{\partial X}\right.$$
$$\left.+N_j\xi_j\frac{\partial\xi_j}{\partial X}+N_j\xi_j\frac{\partial\xi_j}{\partial X}\right)\xi_i d\Omega+\frac{1}{2}\int\limits_{\Omega^{n+1}}\left(V_j\xi_j\frac{\partial\xi_j}{\partial Y}+V_j\xi_j\frac{\partial\xi_j}{\partial Y}+N_j\xi_j\frac{\partial\xi_j}{\partial Y}+N_j\xi_j\frac{\partial\xi_j}{\partial Y}\right)$$
$$\xi_i d\Omega+A_8\int\limits_{\Omega^{n+1}}\left(-N_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial X}-\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial X}\xi_i-N_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial Y}-\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial Y}\xi_i\right)d\Omega$$
$$-A_9\left[-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial X}d\Omega+\oint\limits_{\Gamma}\xi_i\left(n_x\frac{\partial\xi_j}{\partial X}\right)d\Gamma-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial Y}d\Omega\right.$$
$$\left.+\oint\limits_{\Gamma}\xi_i\left(n_y\frac{\partial\xi_j}{\partial Y}\right)d\Gamma\right]=0,$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int_{\Omega^{n+1}} N_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}\,d\Omega - \frac{1}{\delta t}\int_{\Omega^{n+1}} N_j^{n}\xi_j^{n}\xi_i^{n+1}\,d\Omega \\
&+ \frac{1}{2}\int_{\Omega^{n+1}}\left(2U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} + 2N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right)\xi_i^{n+1}\,d\Omega \\
&+ \frac{1}{2}\int_{\Omega^{n+1}}\left(2V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} + 2N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right)\xi_i^{n+1}\,d\Omega \\
&+ A_8\int_{\Omega^{n+1}}\left(-N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} - \left(\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right)^2\xi_i^{n+1}\right. \\
&\qquad\qquad \left. -N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} - \left(\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right)^2\xi_i^{n+1}\right)d\Omega \\
&- A_9\left[\int_{\Omega^{n+1}}\left(\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} + \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}\right)d\Omega\right] \\
&= \oint_{\Gamma}\xi_i^{n+1}\left(n_x\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right)d\Gamma + \oint_{\Gamma}\xi_i^{n+1}\left(n_y\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right)d\Gamma
\end{aligned}
\tag{4.56}
$$

From Eqs. (4.51) to (4.56), we get the discretized system of nonlinear algebraic equations as

$$[A^*(U,V)]\{X^*\} = \{F^*\} + \{Q^*\}.$$

The matrix notation of $A^*(U,V)$, X^*, F^* and Q^* can be written as

$$
\underbrace{\begin{bmatrix}
K^{11} & K^{12} & B_1 & K^{14} & K^{15} & K^{16} \\
K^{21} & K^{22} & B_2 & K^{24} & K^{25} & K^{26} \\
B_1^T & B_2^T & K^{33} & K^{34} & K^{35} & K^{36} \\
K^{41} & K^{42} & K^{43} & K^{44} & K^{45} & K^{46} \\
K^{51} & K^{52} & K^{53} & K^{54} & K^{55} & K^{56} \\
K^{61} & K^{62} & K^{63} & K^{64} & K^{65} & K^{66}
\end{bmatrix}}_{A^*}
\underbrace{\begin{bmatrix} U \\ V \\ P \\ \theta \\ \phi \\ N \end{bmatrix}}_{X^*}
=
\underbrace{\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}}_{Q^*}.
\tag{4.57}
$$

Here A^*, X^* and Q^* are called block stiffness matrix, block solution vector and block boundary vector respectively. The local elemental entries of block stiffness matrix are given as

$$K^{11} = \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U_j^{n}\xi_j^{n}\xi_i^{n+1}d\Omega + A_1\int\limits_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega + \xi_i^{n+1}\int\limits_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}d\Omega,$$

$$K^{12} = 0,$$

$$K^{14} = 0,$$

$$K^{15} = 0,$$

$$K^{16} = 0,$$

$$K^{21} = 0,$$

$$K^{22} = \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V_j^{n}\xi_j^{n}\xi_i^{n+1}d\Omega + A_2\int\limits_{\Omega^{n+1}} V_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega + \xi_i^{n+1}\int\limits_{\Omega^{n+1}} V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}d\Omega,$$

$$K^{24} = -A_3\int\limits_{\Omega^{n+1}} \theta_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega,$$

$$K^{25} = 0,$$

$$\mathrm{K}^{26} = A_3A_4\int\limits_{\Omega^{n+1}} N_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega,$$

$$K^{33} = 0,$$

$$K^{34} = 0,$$

$$K^{35} = 0,$$

$$K^{36} = 0,$$

$$K^{41} = -\frac{A_5}{3}\int_{\Omega^{n+1}}\left(3U_j^{n+1}\xi_j^{n+1}\left(\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right)^2\right)\xi_i^{n+1}\,d\Omega$$

$$-\frac{A_5}{3}\int_{\Omega^{n+1}}\left(3U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right)\xi_i^{n+1}\,d\Omega$$

$$-A_5\int_{\Omega^{n}}\left(-U_j^{n}\xi_j^{n}U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} - 2U_j^{n+1}\xi_j^{n+1}\left(\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right)^2\xi_i^{n+1}\right)$$

$$d\Omega$$

$$+A_5\int_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}\,d\Omega$$

$$+A_5\int_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}\,d\Omega$$

$$+A_5\int_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\xi_i^{n+1}\,d\Omega$$

$$+A_5\int_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\xi_i^{n+1}d\Omega, \tag{4.58}$$

$$K^{42} = -\frac{A_5}{3}\int_{\Omega^{n+1}}\left(3V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right)\xi_i^{n+1}\,d\Omega$$

$$-\frac{A_5}{3}\int_{\Omega^{n+1}}\left(3V_j^{n+1}\xi_j^{n+1}\left(\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right)^2\right)\xi_i^{n+1}\,d\Omega$$

$$-A_5\int_{\Omega^{n+1}}\Bigg(-V_j^{n}\xi_j^{n}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}$$

$$-2V_j^{n+1}\xi_j^{n+1}\left(\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right)^2\xi_i^{n+1}\Bigg)d\Omega$$

$$+A_5\int_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}\,d\Omega$$

$$+A_5\int_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}\,d\Omega$$

$$+A_5\int_{\Omega^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\xi_i^{n+1}\,d\Omega$$

$$+A_5\int_{\Omega^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\xi_i^{n+1}\,d\Omega$$

$$K^{43} = 0,$$

$$K^{44} = \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \theta_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\Big(\theta_j^n\xi_j^n \ o\ X^n\Big)\xi_i^{n+1}d\Omega + \int\limits_{\Omega^{n+1}} \frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega$$
$$+ \int\limits_{\Omega^{n+1}} \frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega,$$

$$K^{45} = 0,$$

$$K^{46} = 0,$$

$$K^{51} = 0,$$

$$K^{52} = 0,$$

$$K^{53} = 0,$$

$$K^{54} = 0,$$

$$K^{55} = \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \phi_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\Big(\phi_j^n\xi_j^n \ o\ X^n\Big)\xi_i^{n+1}d\Omega$$
$$- A_6\Big[-\int\limits_{\Omega^{n+1}} \frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega - \int\limits_{\Omega^{n+1}} \frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega\Big],$$

$$K^{56} = A_7\int\limits_{\Omega^{n+1}} N_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega,$$

$$K^{61} = \frac{1}{2}\int\limits_{\Omega^{n+1}} \Big(U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}} + U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}} + N_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}$$
$$+ N_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Big)\xi_i^{n+1}d\Omega,$$

$$K^{62} = \frac{1}{2}\int\limits_{\Omega^{n+1}} \Big(V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} + V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} + N_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}$$
$$+ N_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Big)\xi_i^{n+1}d\Omega,$$

$$K^{63} = 0,$$

$$K^{64} = 0,$$

$$K^{65} = A_8 \int\limits_{\Omega^{n+1}} \Bigg(- N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} - \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\xi_i^{n+1}$$
$$- N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} - \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\xi_i^{n+1}\Bigg) d\Omega,$$

$$K^{66} = \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N_j^{n+1}\xi_j^{n+1}\xi_i^{n+1} d\Omega - \frac{1}{\delta t}\int\limits_{\Omega^{n+1}} N_j^{n}\xi_j^{n}\xi_i^{n+1} d\Omega + \frac{1}{2}\int\limits_{\Omega^{n+1}} \Bigg(U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}$$
$$+ U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} + N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} + N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\xi_i^{n+1} d\Omega$$
$$+ \frac{1}{2}\int\limits_{\Omega^{n+1}} \Bigg(V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} + V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} + N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}$$
$$+ N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\Bigg)\xi_i^{n+1} d\Omega + A_8\int\limits_{\Omega^{n+1}} \Bigg(- N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}$$
$$- \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\xi_i^{n+1} - N_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} - \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\xi_i^{n+1}\Bigg) d\Omega$$
$$- A_9\Bigg[- \int\limits_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} d\Omega - \int\limits_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} d\Omega\Bigg],$$

The entries $K^{13},\ K^{23}$ and $K^{31},\ K^{32}$ are the pressure matrices with their respective transposes can be written as

$$B_1^{ij} = -\eta_j^{n+1}\int\limits_{\Omega^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} d\Omega,$$

$$B_2^{ij} = -\eta_j^{n+1}\int\limits_{\Omega^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} d\Omega,$$

$$(B_1^{ij})^t = -\xi_i^{n+1}\int\limits_{\Omega^{n+1}} \frac{\partial \eta_j^{n+1}}{\partial X^{n+1}} d\Omega,$$

$$(B_2^{ij})^t = -\xi_i^{n+1}\int\limits_{\Omega^{n+1}} \frac{\partial \eta_j^{n+1}}{\partial Y^{n+1}} d\Omega,$$

$$Q_1 = 0,$$

$$Q_2 = 0,$$

$$Q_3 = 0,$$

$$Q_4 = \oint\limits_{\Gamma} \xi_i^{n+1}\Bigg(n_x\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\Bigg) d\Gamma + \oint\limits_{\Gamma} \xi_i^{n+1}\Bigg(n_y\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\Bigg) d\Gamma,$$

$$Q_5 = \oint_\Gamma \xi_i^{n+1}\left(n_x \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right) d\Gamma + \oint_\Gamma \xi_i^{n+1}\left(n_y \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right) d\Gamma,$$

$$Q_6 = \oint_\Gamma \xi_i^{n+1}\left(n_x \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right) d\Gamma + \oint_\Gamma \xi_i^{n+1}\left(n_y \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right) d\Gamma.$$

The discrete system of non-linear algebraic equations in matrix form can be written as:

$$\begin{bmatrix} K^{11} & 0 & B_1 & 0 & 0 & 0 \\ 0 & K^{22} & B_2 & K^{24} & 0 & K^{26} \\ B_1^T & B_2^T & 0 & 0 & 0 & 0 \\ K^{41} & K^{42} & 0 & K^{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & K^{55} & K^{56} \\ K^{61} & K^{62} & 0 & 0 & K^{65} & K^{66} \end{bmatrix} \begin{bmatrix} U \\ V \\ P \\ \theta \\ \phi \\ N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} \tag{4.59}$$

An examination of the weak form Eqs. (4.31) to (4.35) and the finite element matrices in (??) shows that ξ_i should be atleast linear functions of x and y. Discretization of the domain is performed using selected two-dimensional finite elements. One of the simplest two-dimensional elements is the three-noded triangular element. This is also known as linear triangular element. The element is shown in Figure 4.4. It has three nodes at the vertices of the triangle and the variable interpolation within the element is linear in x and y like

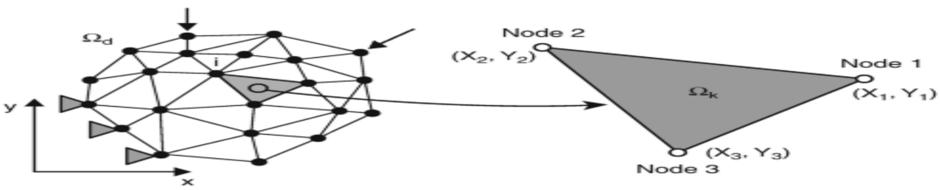


FIGURE 4.1: Systematic Computational Domain

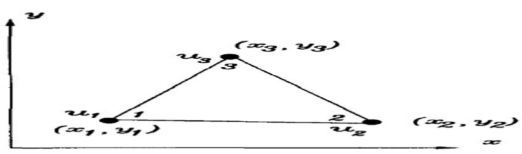


FIGURE 4.2: Linear Triangular Element

$$u = a_1 + a_2 x + a_3 y, \tag{4.60}$$

or

$$u = [1 \quad x \quad y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \tag{4.61}$$

where a_i is the constant to be determined. The interpolation function, Eq. (4.60) should represent the nodal variables at the three nodal points. Therefore, substituting the x and y values at each nodal point gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \tag{4.62}$$

here, x_i and y_i are the coordinate values at the i^{th} node and u_i is the nodal variable as seen in Figure 4.4. Inverting the matrix and rewriting Eq. (4.62) give

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix},$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{|A|} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}. \tag{4.63}$$

For the finite element computation, the element nodal sequence must be in the same direction for every element in the domain.

Substituting the Eq. (4.63) into Eq. (4.60), we obtain

$$u = [1 \quad x \quad y] \frac{1}{|A|} \begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix},$$

$$u = \frac{1}{|A|}\begin{bmatrix}\xi_1, & \xi_2, & \xi_3\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\\ u_3\end{Bmatrix} = \sum_{i=1}^{3}\xi_i u_i,$$

where,
$$\xi_1 = \frac{1}{|A|}[(x_2y_3 - x_3y_2) + x(y_2 - y_3) + y(x_3 - x_2)],$$
$$\xi_2 = \frac{1}{|A|}[(x_3y_1 - x_1y_3) + x(y_3 - y_1) + y(x_1 - x_3)],$$
$$\xi_3 = \frac{1}{|A|}[(x_1y_2 - x_2y_1) + x(y_1 - y_2) + y(x_2 - x_1)].$$

For a linear triangular element, Eq. (4.41) to (4.46), becomes

$$U = \sum_{i=1}^{3} U_i\xi_i, \quad V = \sum_{i=1}^{3} V_i\xi_i, \quad \theta = \sum_{i=1}^{3} \theta_i\xi_i, \quad \phi = \sum_{i=1}^{3} \phi_i\xi_i,$$
$$N = \sum_{i=1}^{3} N_i\xi_i, \quad P = \sum_{i=1}^{3} P_i\eta_i, \quad \tilde{U}, \tilde{V}, \tilde{\theta}, \tilde{\phi}, \tilde{N} = \sum_{i=1}^{3}(\tilde{U}_i, \tilde{V}_i, \tilde{\theta}_i, \tilde{\phi}_i, \tilde{N}_i)\xi_i,$$

(4.41)$\Rightarrow$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{U}_i\xi_i d\Omega - \frac{1}{\delta t}\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n}\sum_{i=1}^{3}\tilde{U}_i\xi_i d\Omega$$
$$+ A_1\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{U}_i\xi_i d\Omega + \left(\sum_{i=1}^{3} q_i\xi_i\right)^{n+1}\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}$$
$$\frac{\partial}{\partial X^{n+1}}\sum_{i=1}^{3}\tilde{U}_i\xi_i d\Omega - \left(\sum_{i=1}^{3} P_i\eta_i\right)^{n+1}\int_{\Omega^{n+1}}\frac{\partial}{\partial X^{n+1}}\sum_{i=1}^{3}\tilde{U}_i\xi_i d\Omega = 0. \tag{4.64}$$

(4.43)$\Rightarrow$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega - \frac{1}{\delta t}\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n}\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega$$
$$+ A_2\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega + \left(\sum_{i=1}^{3} q_i\xi_i\right)^{n+1}\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}$$
$$\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega - \left(\sum_{i=1}^{3} P_i\eta_i\right)^{n+1}\int_{\Omega^{n+1}}\frac{\partial}{\partial Y^{n+1}}\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega$$
$$+ A_3A_4\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} N_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega - A_3\int_{\Omega^{n+1}}\left(\sum_{i=1}^{3} \theta_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{V}_i\xi_i d\Omega = 0 \tag{4.65}$$

(4.44)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$-\frac{1}{\delta t}\left(\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n}\circ X^n\right)\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$+\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$-\oint\limits_{\Gamma}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)\left(n_x\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\right)d\Gamma$$

$$+\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$-\oint\limits_{\Gamma}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)\left(n_y\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\right)d\Gamma$$

$$-\frac{A_5}{3}\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$
$$+\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$
$$+\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$
$$-\frac{A_5}{3}\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$
$$+\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$
$$+\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$\int\limits_{\Omega^{n+1}} \left[\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\right]\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$-\frac{A_5}{3}\int\limits_{\Omega^{n+1}} \Bigg[\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$

$$+\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$

$$+\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\Bigg]\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$-\frac{A_5}{3}\int\limits_{\Omega^{n+1}} \Bigg[\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$

$$+\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}$$

$$+\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\Bigg]\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)d\Omega$$

$$-A_5\int\limits_{\Omega^{n}} \Bigg[-\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)$$

$$-2\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)\Bigg]d\Omega$$

$$-A_5\int\limits_{\Omega^{n+1}} \Bigg[-\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)$$

$$-2\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\theta_i\xi_i\right)^{n+1}\left(\sum_{i=1}^{3}\tilde{\theta}_i\xi_i\right)\Bigg]d\Omega$$

$$- A_5 \int\limits_{\Omega^{n+1}} \Bigg[$$

$$- \left(\sum_{i=1}^{3} U_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} V_i \xi_i\right)^{n+1} \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} \theta_i \xi_i\right)^{n+1} \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} \tilde{\theta}_i \xi_i\right)$$

$$- \left(\sum_{i=1}^{3} U_i \xi_i\right)^{n+1} \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} V_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} \tilde{\theta}_i \xi_i\right)$$

$$- \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} U_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} V_i \xi_i\right)^{n+1} \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} \theta_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} \tilde{\theta}_i \xi_i\right)$$

$$- \left(\sum_{i=1}^{3} U_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} V_i \xi_i\right)^{n+1} \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} \theta_i \xi_i\right)^{n+1} \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} \tilde{\theta}_i \xi_i\right)$$

$$- \left(\sum_{i=1}^{3} U_i \xi_i\right)^{n+1} \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} V_i \xi_i\right)^{n+1} \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} \theta_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} \tilde{\theta}_i \xi_i\right)$$

$$- \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} U_i \xi_i\right)^{n+1} \left(\sum_{i=1}^{3} V_i \xi_i\right)^{n+1} \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} \theta_i \xi_i\right)^{n+1} \Bigg] d\Omega = 0 \tag{4.66}$$

$$\frac{1}{\delta t} \int\limits_{\Omega^{n+1}} \left(\sum_{i=1}^{3} \phi_i \xi_i\right)^{n+1} \sum_{i=1}^{3} \tilde{\phi}_i \xi_i d\Omega - \frac{1}{\delta t} \left(\left(\sum_{i=1}^{3} \phi_i \xi_i\right)^{n} o \; X^n \right) \sum_{i=1}^{3} \tilde{\phi}_i \xi_i d\Omega$$

$$- A_6 \Bigg[- \int\limits_{\Omega^{n+1}} \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} \phi_i \xi_i\right)^{n+1} \frac{\partial}{\partial X^{n+1}} \sum_{i=1}^{3} \tilde{\phi}_i \xi_i d\Omega + \oint\limits_{\Gamma} \sum_{i=1}^{3} \tilde{\phi}_i \xi_i$$

$$\left(n_x \frac{\partial}{\partial X^{n+1}} \left(\sum_{i=1}^{3} \phi_i \xi_i\right)^{n+1} \right) d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} \phi_i \xi_i\right)^{n+1} \frac{\partial}{\partial Y^{n+1}} \sum_{i=1}^{3} \tilde{\phi}_i \xi_i d\Omega$$

$$+ \oint\limits_{\Gamma} \sum_{i=1}^{3} \tilde{\phi}_i \xi_i \left(n_y \frac{\partial}{\partial Y^{n+1}} \left(\sum_{i=1}^{3} \phi_i \xi_i\right)^{n+1} \right) d\Gamma \Bigg] + A_7 \int\limits_{\Omega^{n+1}}$$

$$\left(\sum_{i=1}^{3} N_i \xi_i\right)^{n+1} \sum_{i=1}^{3} \tilde{\phi}_i \xi_i d\Omega = 0 \tag{4.67}$$

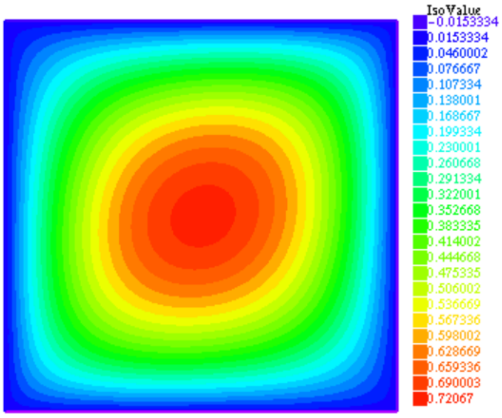
(4.46)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{N}_i\xi_i d\Omega-\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n}\sum_{i=1}^{3}\tilde{N}_i\xi_i d\Omega$$
$$+\frac{1}{2}\int\limits_{\Omega^{n+1}}\left(\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}+\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\right.$$
$$\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}+\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}+\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}$$
$$\left.\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}U_i\xi_i\right)^{n+1}\right)\sum_{i=1}^{3}\tilde{N}_i\xi_i d\Omega+\frac{1}{2}\int\limits_{\Omega^{n+1}}\left(\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\right.$$
$$+\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}+\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}$$
$$\left.+\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}V_i\xi_i\right)^{n+1}\right)\sum_{i=1}^{3}\tilde{N}_i\xi_i d\Omega+A_8\int\limits_{\Omega^{n+1}}\left(-\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\right.$$
$$\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\phi_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\sum_{i=1}^{3}\tilde{N}_i\xi_i-\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}\phi_i\xi_i\right)^{n+1}$$
$$\sum_{i=1}^{3}\tilde{N}_i\xi_i-\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\phi_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\sum_{i=1}^{3}\tilde{N}_i\xi_i-\frac{\partial}{\partial Y^{n+1}}$$
$$\left.\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}\phi_i\xi_i\right)^{n+1}\sum_{i=1}^{3}\tilde{N}_i\xi_i\right)d\Omega-$$
$$A_9\left(\left[-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\right.\right.$$
$$\frac{\partial}{\partial X^{n+1}}\sum_{i=1}^{3}\tilde{N}_i\xi_i d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{3}\tilde{N}_i\xi_i\left(n_x\frac{\partial}{\partial X^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\right)d\Gamma-\int\limits_{\Omega^{n+1}}\frac{\partial}{\partial Y^{n+1}}$$
$$\left.\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\frac{\partial}{\partial Y^{n+1}}\sum_{i=1}^{3}\tilde{N}_i\xi_i d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{3}\tilde{N}_i\xi_i\left(n_y\frac{\partial}{\partial Y^{n+1}}\left(\sum_{i=1}^{3}N_i\xi_i\right)^{n+1}\right)d\Gamma\right]$$
$$\left.\vphantom{\sum_{i=1}^{3}}\right)=0 \tag{4.68}$$

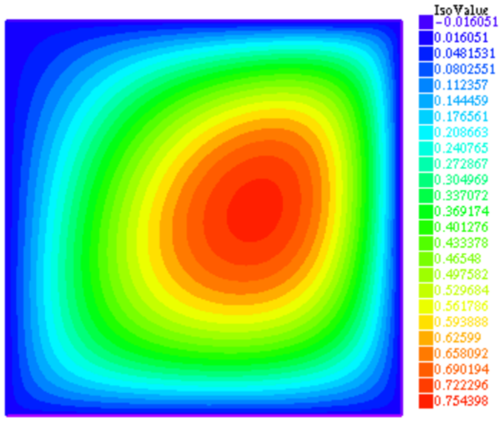
Eq. (4.64) to (4.68) are solved through FEM code Free Fem++. These equations are along with the given Boundary Conditions are implemented in Free Fem++, the code is used to compute the solution for variant parameter on the computational domain.

4.4 Results and Discussion

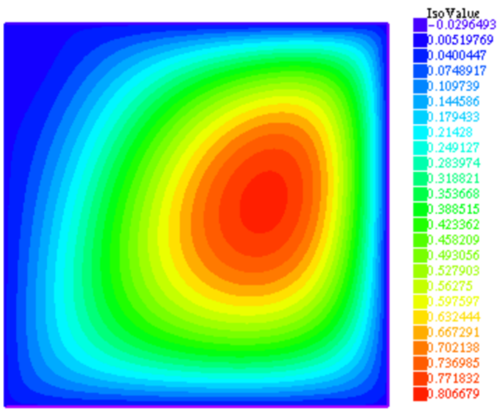
This section contains the results computed by the code implemented for the model problem presented.



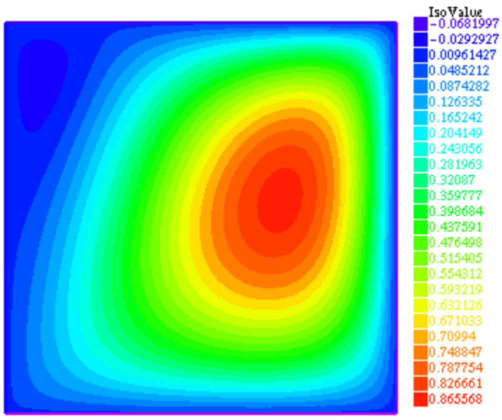
(a) $Pe = 0.1$



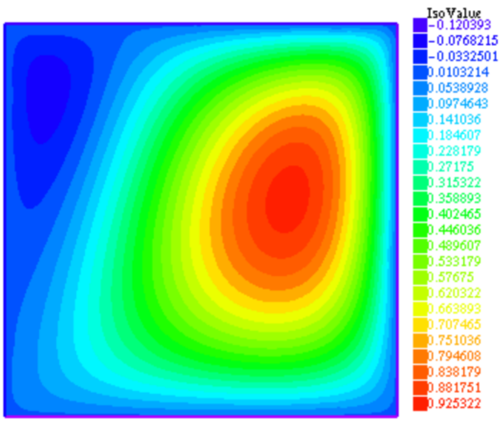
(b) $Pe = 3$



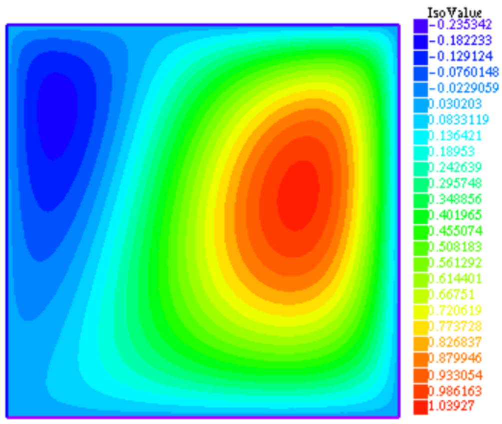
(c) $Pe = 5$



(d) $Pe = 7$



(e) $Pe = 9$



(f) $Pe = 13$

FIGURE 4.3: Streamline plots for varying values of Peclet number Pe.

In figure 4.3, the Peclet number increases, the dominance of advection over diffusion becomes more pronounced, leading to a stronger directional bias in the flow. This causes the streamlines to shift towards the right wall, with the velocity profiles becoming more skewed in that direction. In turn, the flow contours, which represent scalar fields like concentration or temperature, also migrate toward the right wall, further emphasizing the impact of advection. When the Peclet number is set to 13, as observed in the top left corner, the system exhibits a noticeable slowdown near the left sub-wall vortex. This results in the formation of a region with negative velocities, where the flow recirculates. The transition of the streamlines and contours with increasing Peclet number also highlights the reduced mixing effects due to the limited role of diffusion. In regions with high Peclet numbers, the flow becomes more stratified, with concentration or temperature gradients becoming sharper near the boundaries. This can lead to the formation of sharp interfaces or boundary layers, enhancing the formation of vortical structures.

In figure 4.4, the flow's shift towards the top edge of the domain results in asymmetric deformation, suggesting a non-uniform distribution of velocity and pressure across the flow field. Initially, the stream velocity reaches its maximum value of 1, but as the flow progresses, it begins to gradually decrease, reaching a minimum of 0.99. This velocity then continues to decline further to 0.87. At this point, the velocity begins to increase again, ultimately resulting in a reduction in the N values at the upper boundary, which drops to 0.77. This change in the velocity profile and N values can have significant implications in fluid dynamics, particularly when considering microorganisms in the flow. In the context of microorganisms, the decrease in velocity, particularly in the initial phase, can influence their swimming patterns and distribution. Additionally, changes in flow velocity, especially near the upper boundary, could impact the microorganisms' diffusion and distribution within the cavity. The variation in stream velocity may affect how microorganisms move through the fluid, influencing their ability to reach different parts of the cavity.

In figure 4.5, it is noted that, regardless of the Péclet number, oxygen consumption by microorganisms primarily occurs in the upper part of the cavity. This is because the concentration of dissolved oxygen is typically higher in the upper regions of the cavity, which facilitates microbial metabolic processes. As a result, microorganisms in this area consume oxygen at a higher rate compared to other regions of the cavity. This leads to a decrease in the concentration of motile bacteria in the upper part of the cavity. Furthermore, an increase in the average directional swimming velocity of microorganisms can lead to a reduction in the intensity of oxygen consumption in the upper part of the cavity. This phenomenon can be attributed to a more efficient use of the available oxygen. When microorganisms swim more efficiently, they are able to move through areas with lower oxygen concentrations. This behavior suggests that microorganisms, by adjusting their swimming velocity, can optimize their energy expenditure and oxygen consumption.

In figure 4.6, it is illustrated that the streamlines and isoconcentrations of motile microorganisms for various values of the Péclet number, providing insight into the effects of convective and diffusive transport on both the flow dynamics and microbial distribution. As the Péclet number increases, the core of the convective cell and the regions with low oxygen and motile microorganism concentration shift towards the right wall of the cavity. This shift reflects the growing dominance of convective effects over diffusive processes as the Péclet number increases, indicating that advection is driving the redistribution of both oxygen and microorganisms. As the Péclet number increases, the velocity profile changes, with the maximum velocity remaining at 1. However, the minimum velocity decreases significantly as the Péclet number increases. Initially, the minimum velocity decreases to 0.989, and at a Péclet number of 13, it further decreases to 0.915. When the Péclet number reaches 29, the minimum velocity continues to decrease, reaching 0.85. This progressive decrease in minimum velocity indicates a shift towards more uniform convective flow, which contributes to a symmetric redistribution of the velocity profile.

In Figure 4.7 (from a to c), as the Prandtl number increases, the overall thermal behavior of the fluid is altered significantly. With a higher Prandtl number, the relative importance of momentum diffusion compared to thermal diffusion increases. This results in a more stable and well-developed velocity profile, which confines most of the thermal variations to the boundary layer. Consequently, the thermal boundary layer becomes sharper and more pronounced, with temperature rising more steeply near the top boundary. Furthermore, with an increasing Prandtl number, the heat transfer near the surface becomes more efficient because of the thinner thermal boundary layer. As heat is concentrated in this narrower region, the system's overall thermal resistance is reduced, leading to a more localized and focused heat transfer near the boundary. From subfigure (d) to subfigure (f): when the Prandtl number is high, as in the case with a value of 5, the thermal diffusion becomes less efficient compared to momentum diffusion. This leads to a stronger resistance to temperature propagation in the fluid, particularly near the top boundary. The larger Prandtl number results in a thinner thermal boundary layer, which restricts the flow of heat into the bulk of the fluid. The temperature profile becomes more localized to the region near the boundary, further emphasizing the resistance to temperature movement across the cavity. The effect of the relaxation time in this context is significant. As the relaxation time increases, the time it takes for the fluid to reach thermal equilibrium is prolonged, further hindering the distribution of heat within the medium.

In Figure 4.8, as the Peclet number increases from 0.1 to 27, there is a notable change in the behavior of the isotherms, reflecting the changing dynamics of heat transfer in the fluid. For low Peclet numbers, heat transfer is dominated by thermal diffusion, and the isotherms are more spread out, indicating a less efficient transfer of heat throughout the medium. The temperature remains concentrated closer to the boundaries, with limited penetration into the interior of the fluid. At higher Peclet numbers, the fluid's advection effects help redistribute the heat across the system, promoting a more uniform temperature distribution throughout the fluid. As a result, the isotherms become more tightly packed near the top boundary, indicating a higher temperature gradient near the surface.

In Figure 4.9, as the Rayleigh number increases, the convective heat transfer becomes more dominant, leading to a more complex flow pattern within the fluid. Initially, at lower Rayleigh numbers, heat diffusion is the primary mode of temperature distribution, and the system behaves in a more symmetric manner, with temperature gradients distributed evenly across the medium. At higher Rayleigh numbers, the temperature distribution becomes increasingly asymmetric, with a pronounced difference in heat transfer efficiency between the top and bottom boundaries. The top boundary experiences a relatively easier and more efficient movement of heat, as the convective currents near the surface facilitate the transport of thermal energy. In contrast, at the bottom boundary, the resistance to heat movement increases, likely due to the development of a more stable thermal boundary layer or reduced convective activity near the colder regions of the medium.

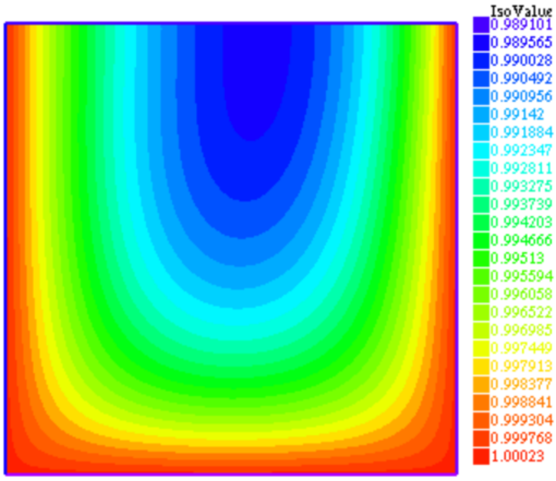
In Figure 4.10, as the Bio-convective Rayleigh number (Rb) increases, the microorganisms experience a stronger upward buoyant force due to the enhanced thermal gradients in the fluid. This causes the microorganisms to not only move towards the top boundary more quickly but also to align with the direction of the convective currents. At higher Rb values, the convection becomes more vigorous, resulting in a more dynamic and pronounced upward transport of microorganisms.

The increased buoyancy forces create a more turbulent environment within the fluid, facilitating the migration of microorganisms to regions where they are more likely to encounter favorable conditions, such as higher temperatures or nutrient concentrations near the top boundary.

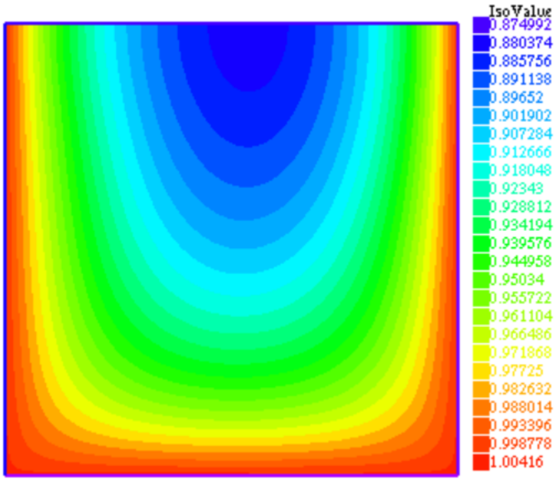
In Figure 4.11, as the relaxation parameter (Rt) increases further, the flow dynamics become more pronounced, with the streamlines exhibiting increasingly stretched and distorted shapes. Initially, the flow is more symmetric, with circular or near-circular streamlines near the boundaries. However, as the relaxation time lengthens, the effect of the relaxation parameter on the fluid's behavior becomes more evident, causing the streamlines to stretch towards both the left and right walls.

At higher values of Rt, the streamlines not only elongate but also show a more complex pattern, including diagonal stretching that forms critical shapes. This transition indicates a more significant influence of the relaxation effects on the thermal and momentum transport within the fluid.

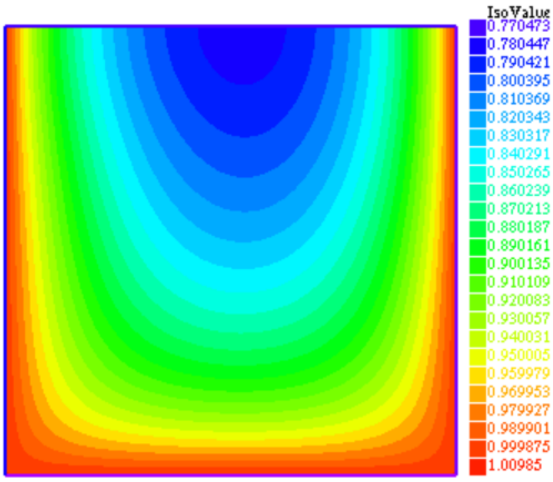
The flow is no longer symmetric, as it becomes increasingly skewed, with the streamlines aligning more with the boundaries, especially along the right wall, where the streamlines align parallel to the x-axis.



(a) N for $Pe = 1.0$

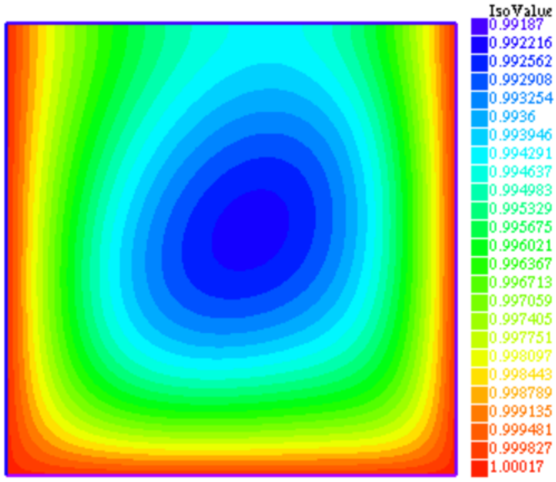


(b) N for $Pe = 13$

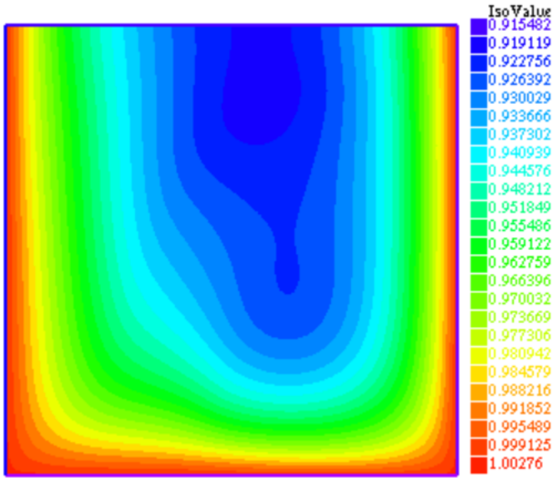


(c) N for $Pe = 29$

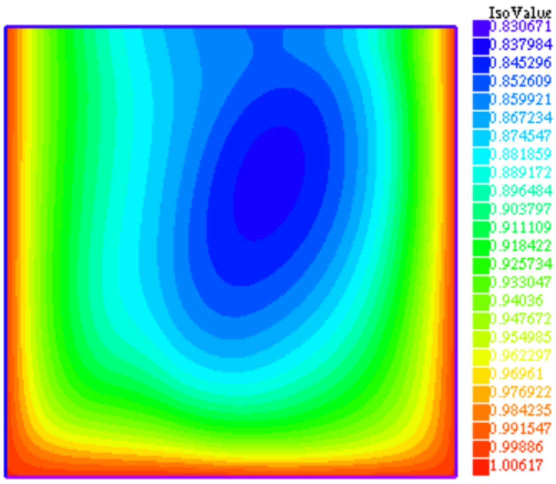
FIGURE 4.4: En plots for varying values of Peclet number Pe and fixed $Ra = 10$. Other parameters used are $Pr = 13$, $Da = 0.5$, $Rt = 0$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Ra = 10$, $Rb = 10$, $\lambda = 1$.



(a) N for $Pe = 1.0$

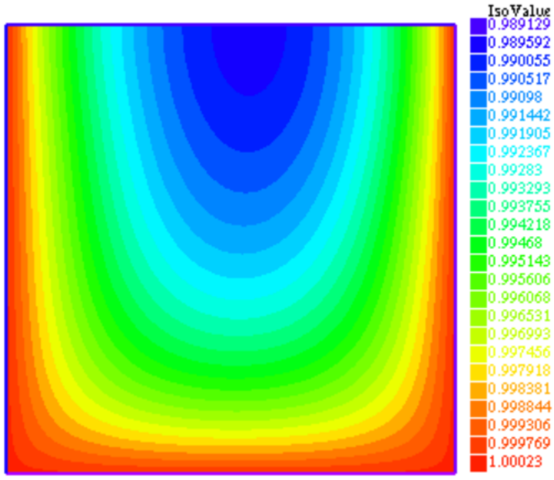


(b) N for $Pe = 13$

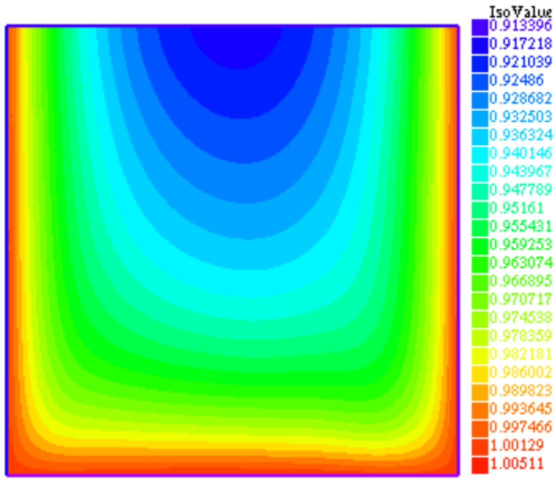


(c) N for $Pe = 29$

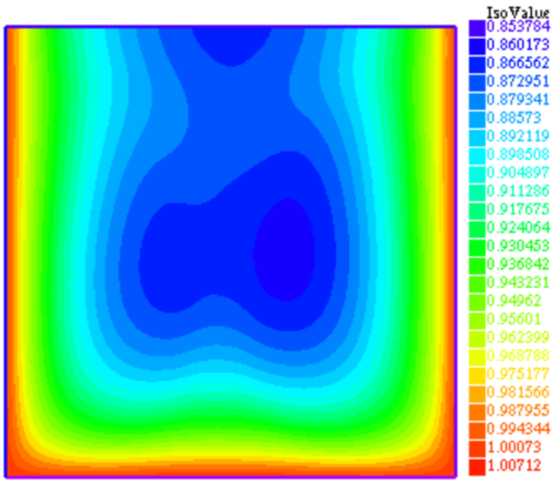
FIGURE 4.5: En plots for varying values of Peclet number Pe and fixed $Ra = 100$. Other parameters used are $Pr = 13$, $Da = 0.5$, $Rt = 0$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Rb = 10$, $\lambda = 1$.



(a) N for $Pe = 1.0$

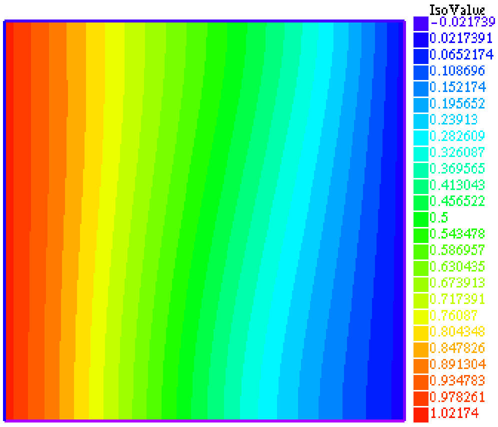


(b) N for $Pe = 13$

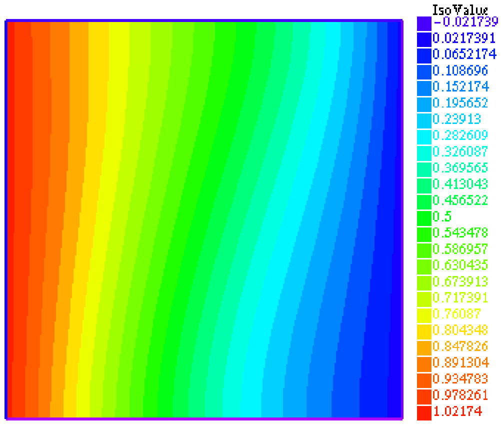


(c) N for $Pe = 29$

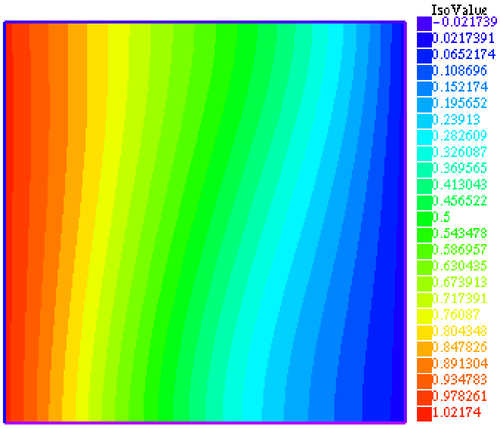
FIGURE 4.6: En plots for varying values of Peclet number Pe and fixed $Rb = 100$. Other parameters used are $Pr = 13$, $Da = 0.5$, $Rt = 0$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Ra = 10$, $Rb = 100$, $\lambda = 1$.



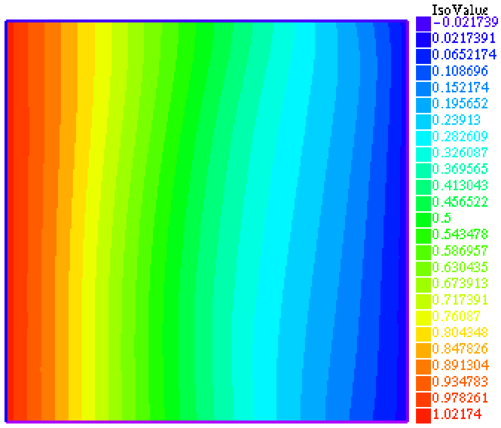
(a) Without relaxation for $Pr = 0.3$



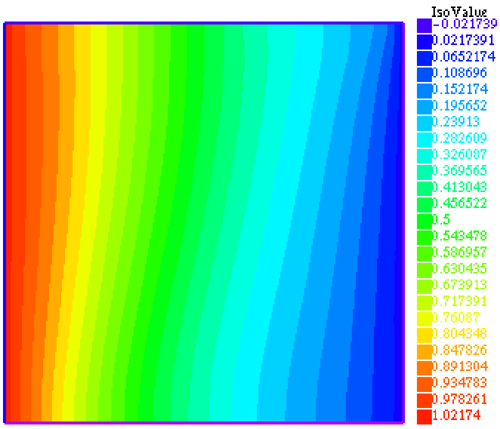
(b) Without relaxation for $Pr = 3$



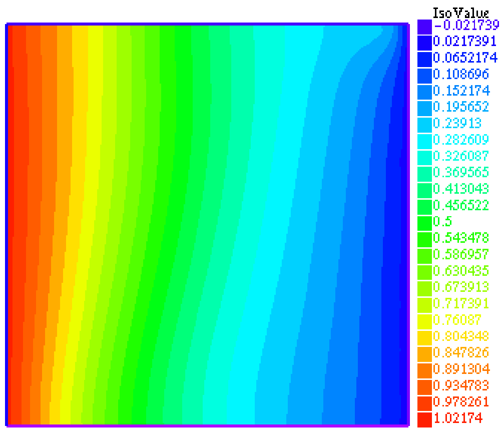
(c) Without relaxation for $Pr = 5$



(d) With relaxation for $Pr = 0.3$

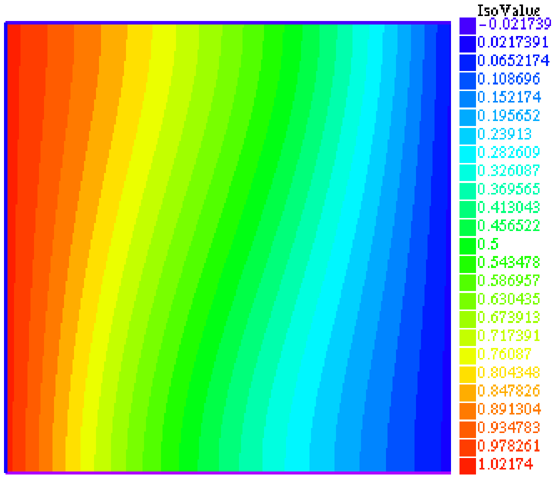


(e) With relaxation for $Pr = 3$

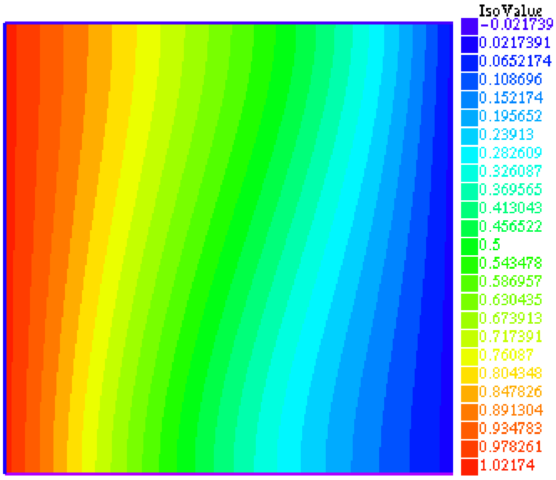


(f) With relaxation for $Pr = 5$

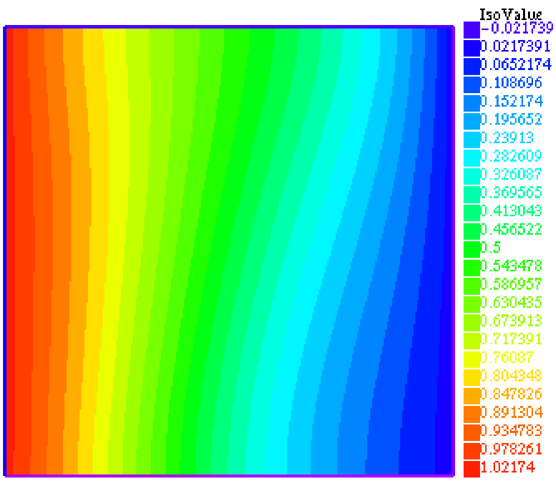
FIGURE 4.7: Isotherm plots for varying value of Prandtl number Pr. The relaxation time factor is chosen as $Rt = 0.085$. Other parameters are: $Pe = 1$, $Da = 0.5$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Ra = 10$, $Rb = 100$, $\lambda = 1$.



(a) $Pe = 0.1$

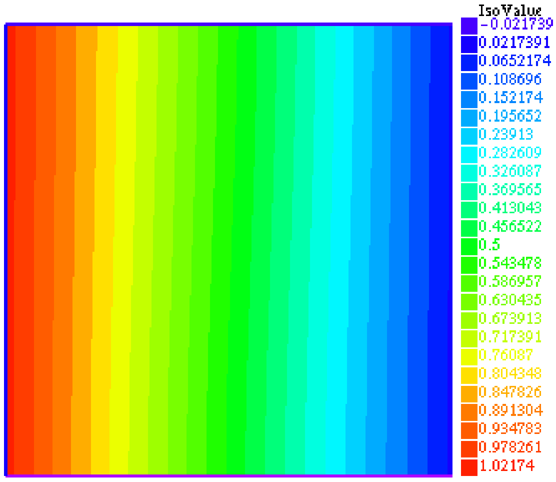


(b) $Pe = 5$

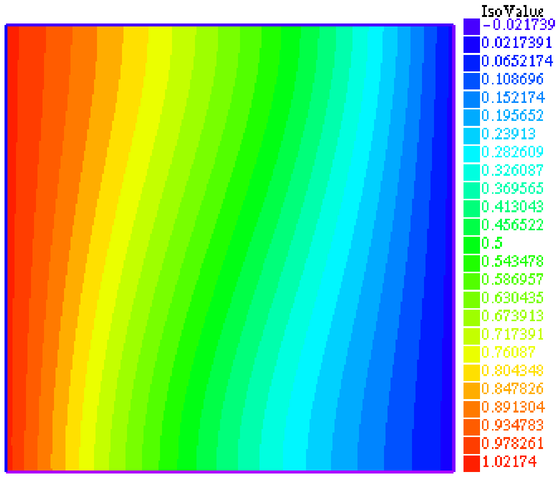


(c) $Pe = 7$

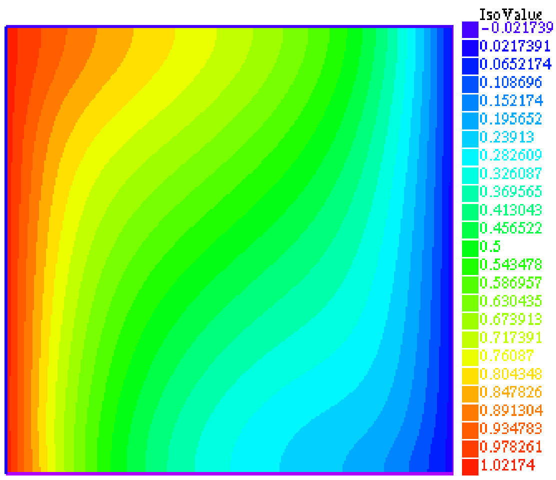
FIGURE 4.8: Isotherm plots for varying value of Peclet number Pe. Other parameters are: $Pr = 3$, $Da = 0.5$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Ra = 10$, $Rb = 10$, $\lambda = 1$.



(a) $Ra = 1$

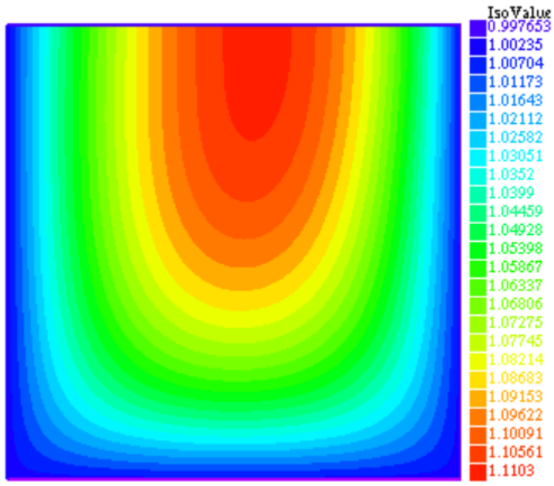


(b) $Ra = 10$

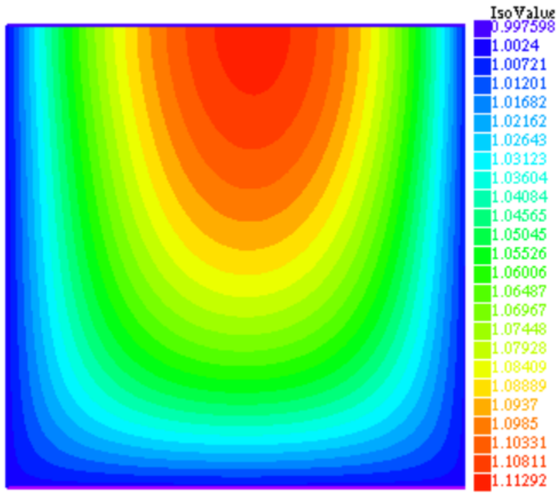


(c) $Ra = 100$

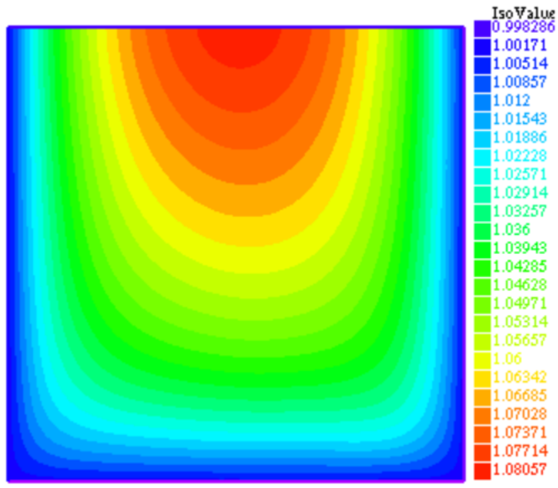
FIGURE 4.9: Isotherm plots for varying value of Rayleigh number Ra. Other parameters are: $Pe = 1$, $Pr = 3$, $Da = 0.5$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Rb = 10$, $\lambda = 1$.



(a) $Rb = 10$

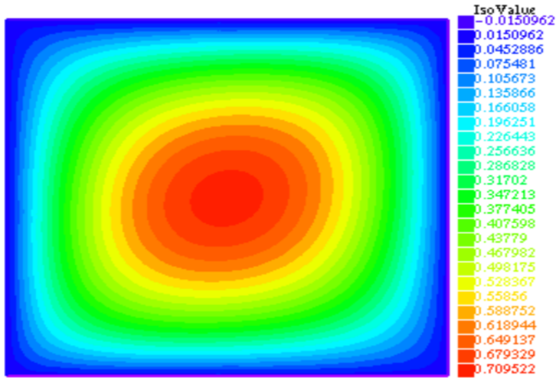


(b) $Rb = 100$

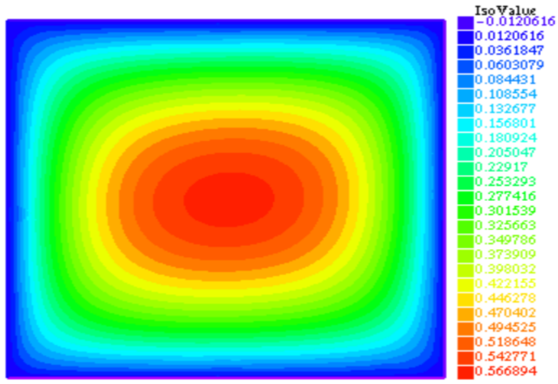


(c) $Rb = 1000$

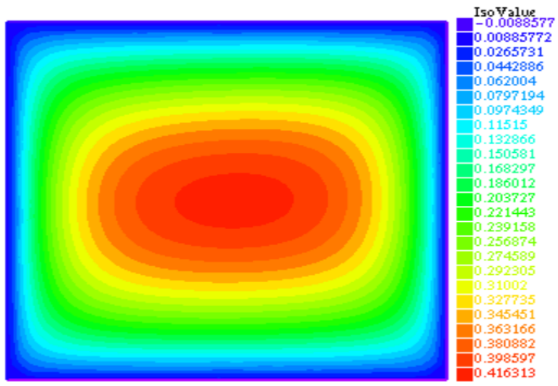
FIGURE 4.10: ϕ plots for varying value of Bio-convective Rayleigh number Rb. Other parameters are: $Pe = 1$, $Pr = 3$, $Da = 0.5$, $Le = 1$, $\kappa = 0.1$, $\sigma = 1$, $Ra = 10$, $\lambda = 1$.



(a) $Rt = 0$



(b) $Rt = 0.55$



(c) $Rt = 2.55$

FIGURE 4.11: Ψ plots for varying value of Relaxation parameter Rt. Other parameters are: $Pe = 1$, $Pr = 3$, $Da = 20.5$, $Le = 1$, $\kappa = 0.001$, $\sigma = 1$, $Ra = 10$, $Rb = 10$, $\lambda = 1$.

Chapter 5

Conclusion

This study numerically investigated steady thermo-bioconvection in a square porous cavity filled with oxytactic bacteria, employing the Cattaneo-Christov heat flux model. The finite element approach is used to numerically solve the governing equations, which include the conservation laws of mass, momentum, and energy. In order to illustrate the effects of different parameters, including the Rayleigh number Ra, bioconvection Rayleigh number Rb, Lewis number Le, and Peclet number Pe, on the fluid dynamics, heat transfer, and mass transfer behavior, the results are displayed using streamlines, isotherms and concentration profiles. The impact of these parameters on the Nusselt and Sherwood values at the vertical cavity walls is highlighted in the study. A Nusselt and Sherwood number table that contrasts the calculated solutions with the body of existing literature is provided in order to validate the calculated results. There is a high degree of consistency between the calculated numerical values and those found in the literature. This demonstrates how well the finite element code was implemented for the model problem that was supplied. The outcomes are calculated for different physical parameter values and thoroughly discussed. With an emphasis on the implications of important dimensionless quantities, the study offers insightful information about the intricate relationships between heat, fluid flow, and microbe concentration in a porous cavity.

These results are crucial for improving heat transmission and bioconvection processes in porous media, especially in biological or ecological systems where microorganisms are important to the dynamics.

Some of the future direction include:

1. Extend the current steady-state analysis to a transient study to investigate the dynamic evolution of thermo-bioconvection and the potential for oscillatory or chaotic behavior.

2. Explore the impact of different cavity shapes (e.g., rectangular, triangular) and porous structures on the heat and mass transfer characteristics in the presence of oxytactic bacteria.

3. Include the effects of thermal radiation, especially relevant in applications with significant temperature differences, to provide a more comprehensive understanding of the coupled heat transfer mechanisms.

Bibliography

[1] Irfan Anjum Badruddin, Abdullah Al-Rashed, NJ Salman Ahmed, Sarfaraz Kamangar, and K Jeevan. Natural convection in a square porous annulus. *International Journal of Heat and Mass Transfer*, 55(23-24):7175–7187, 2012.

[2] Irfan Anjum Badruddin, NJ Salman Ahmed, Ali E Anqi, and Sarfaraz Kamangar. Conjugate heat and mass transfer in a vertical porous cylinder. *Journal of Thermophysics and Heat Transfer*, 33(2):548–558, 2019.

[3] Payam Jalili, Ali Ahmadi Azar, Bahram Jalili, and Davood Domiri Ganji. Study of nonlinear radiative heat transfer with magnetic field for non-newtonian casson fluid flow in a porous medium. *Results in Physics*, 48: 106371, 2023.

[4] Lijun Zhang, Muhammad Mubashir Bhatti, Rahmat Ellahi, and Efstathios E Michaelides. Oxytactic microorganisms and thermo-bioconvection nanofluid flow over a porous riga plate with darcy–brinkman–forchheimer medium. *Journal of Non-Equilibrium Thermodynamics*, 45(3):257–268, 2020.

[5] AJ Hillesdon and TJ Pedley. Bioconvection in suspensions of oxytactic bacteria: linear theory. *Journal of Fluid Mechanics*, 324:223–259, 1996.

[6] S Mandal, GC Shit, S Shaw, and OD Makinde. Entropy analysis of thermo-solutal stratification of nanofluid flow containing gyrotactic microorganisms over an inclined radiative stretching cylinder. *Thermal Science and Engineering Progress*, 34:101379, 2022.

[7] Anas AM Arafa, Sameh A Hussein, and Sameh E Ahmed. Hydrothermal bio-convective bödewadt ternary composition nanofluids flow over a stretching

rotating disk through a heat generating porous medium. *Journal of Magnetism and Magnetic Materials*, 586:171174, 2023.

[8] AV Kuznetsov. The onset of thermo-bioconvection in a shallow fluid saturated porous layer heated from below in a suspension of oxytactic microorganisms. *European Journal of Mechanics-B/Fluids*, 25(2):223–233, 2006.

[9] AV Kuznetsov. The onset of nanofluid bioconvection in a suspension containing both nanoparticles and gyrotactic microorganisms. *International Communications in Heat and Mass Transfer*, 37(10):1421–1425, 2010.

[10] Farshad Moradi Kashkooli, M Soltani, and Kaamran Raahemifar. Fluid flow and heat transfer analysis of a nanofluid containing motile gyrotactic microorganisms passing a nonlinear stretching vertical sheet in the presence of a non-uniform magnetic field; numerical approach.

[11] Ponniah Meena Rajeswari and Poulomi De. Multi-stratified effects on stagnation point nanofluid flow with gyrotactic microorganisms over porous medium. *Journal of Porous Media*, 27(5), 2024.

[12] N Gomathi and Poulomi De. Impact of hall currents and ion slip on mixed convective casson williamson nanofluid flow with viscous dissipation through porous medium. *Nanoscience and Technology: An International Journal*, 15 (1), 2024.

[13] S Siraj Nisha and Poulomi De. Hall currents and ion slip effect on sisko nanofluid flow featuring chemical reaction over porous medium-a statistical approach. *Special Topics & Reviews in Porous Media: An International Journal*, 15(2), 2024.

[14] Munawar Abbas, Nargis Khan, MS Hashmi, Shahram Rezapour, Hameed Ullah, and Mustafa Inc. Impacts of stefan blowing on thermophoretic particle deposition in sisko nanofluid flow over a riga plate with soret and dufour effects. *Nano*, page 2450111, 2024.

[15] Jean Baptiste Joseph Fourier. *Théorie analytique de la chaleur*, volume 1. Gauthier-Villars, 1888.

[16] Carlo Cattaneo. On the conduction of heat. *Proceedings Sem. Matt. Phys. Univ. Modena*, 3:83–101, 1948.

[17] CI Christov. On frame indifferent formulation of the maxwell–cattaneo model of finite-speed heat conduction. *Mechanics research communications*, 36(4): 481–486, 2009.

[18] D Mohanty, G Mahanta, S Shaw, and P Sibanda. Thermal and irreversibility analysis on cattaneo–christov heat flux-based unsteady hybrid nanofluid flow over a spinning sphere with interfacial nanolayer mechanism. *Journal of Thermal Analysis and Calorimetry*, 148(21):12269–12284, 2023.

[19] D Mohanty, N Sethy, G Mahanta, and S Shaw. Impact of the interfacial nanolayer on marangoni convective darcy-forchheimer hybrid nanofluid flow over an infinite porous disk with cattaneo-christov heat flux. *Thermal Science and Engineering Progress*, 41:101854, 2023.

[20] Umar Nazir, Muhammad Sohail, Muhammad Bilal Hafeez, and Marek Krawczuk. Significant production of thermal energy in partially ionized hyperbolic tangent material based on ternary hybrid nanomaterials. *Energies*, 14(21):6911, 2021.

[21] Munawar Abbas, Nargis Khan, MS Hashmi, Hammad Alotaibi, Husna A Khan, Shahram Rezapour, and Mustafa Inc. Importance of thermophoretic particles deposition in ternary hybrid nanofluid with local thermal non-equilibrium conditions: Hamilton–crosser and yamada–ota models. *Case Studies in Thermal Engineering*, 56:104229, 2024.

[22] Sameh E Ahmed, Anas AM Arafa, Sameh A Hussein, and Zeinab Morsy. Time-dependent squeezing flow of variable properties ternary nanofluids between rotating parallel plates with variable magnetic and electric fields. *Numerical Heat Transfer, Part A: Applications*, pages 1–30, 2023.

[23] Munawar Abbas, Nargis Khan, MS Hashmi, Ferdous M Tawfiq, Shahram Rezapour, Muhammad Bilal, and Mustafa Inc. Combined effects of electrophoresis and thermophoresis in darcy forchheimer flow of trihybrid

nanofluid over a riga plate: Yamada-ota and xue models. *Case Studies in Thermal Engineering*, 59:104546, 2024.

[24] Sameh E Ahmed, Anas AM Arafa, and Sameh A Hussein. Cattaneo-christov double-diffusion and hydrothermal flow of sutterby tri, hybrid and mono-nanofluids with oxytactic microorganism. *International Journal of Ambient Energy*, 44(1):2200–2213, 2023.

[25] Ghulam Rasool, Anum Shafiq, Xinhua Wang, Ali J Chamkha, and Abderrahim Wakif. Numerical treatment of mhd al2o3–cu/engine oil-based nanofluid flow in a darcy–forchheimer medium: application of radiative heat and mass transfer laws. *International Journal of Modern Physics B*, 38(09): 2450129, 2024.

[26] Liaquat Ali Lund, Ubaidullah Yashkun, and Nehad Ali Shah. Multiple solutions of unsteady darcy–forchheimer porous medium flow of cu–al2o3/water based hybrid nanofluid with joule heating and viscous dissipation effect. *Journal of Thermal Analysis and Calorimetry*, 149(5):2303–2315, 2024.

[27] Anum Shafiq, Andaç Batur Çolak, and Tabassum Naz Sindhu. Comparative analysis to study the darcy–forchheimer tangent hyperbolic flow towards cylindrical surface using artificial neural network: an application to parabolic trough solar collector. *Mathematics and Computers in Simulation*, 216:213–230, 2024.

[28] Munawar Abbas, Nargis Khan, MS Hashmi, Ferdous M Tawfiq, Mustafa Inc, and KR Raghunatha. Numerical simulation of chemical reactive flow of boger fluid over a sheet with heat source and local thermal non-equilibrium conditions. *Case Studies in Thermal Engineering*, 59:104498, 2024.

[29] Debashis Mohanty, Ganeswar Mahanta, and Sachin Shaw. Analysis of irreversibility for 3-d mhd convective darcy–forchheimer casson hybrid nanofluid flow due to a rotating disk with cattaneo–christov heat flux, joule heating, and nonlinear thermal radiation. *Numerical Heat Transfer, Part B: Fundamentals*, 84(2):115–142, 2023.

[30] D Mohanty, G Mahanta, Ali J Chamkha, and S Shaw. Numerical analysis of interfacial nanolayer thickness on darcy-forchheimer casson hybrid nanofluid flow over a moving needle with cattaneo-christov dual flux. *Numerical Heat Transfer, Part A: Applications*, pages 1–25, 2023.

[31] R. Bansal. *A Textbook of Fluid Mechanics and Dydraulic Machines.* Laxmi publications, 2004.

[32] J. P. Hartnett W. M. Rohsenow and Y. I. Cho. *Handbook of heat transfer, vol. 3.* McGraw-Hill New York, 1998.

[33] J. N. Reddy and D. K. Gartling. *The Finite Element Method in Heat Transfer and Fluid Dynamics.* CRC press, 2010.

[34] J. Ahmed and M. S. Rahman. *Handbook of Food Process Design.* John Wiley Sons, 2012.

[35] P. A. Davidson and A. Thess. *Magnetohydrodynamics, vol. 418.* Springer Science Business Media, 2002.

[36] A. McDonald R. W. Fox and P. Pitchard. *Introduction to fluid mechanics, 2004.* 2006.

[37] P. Nithiarasu R. W. Lewis and K. N. Seetharamu. *Fundamentals of the finite element method for heat and fluid flow.* John Wiley Sons, 2004.

[38] DeWitt D.P. Bergman T.L. Lavine A.S. Incropera, F.P. *Fundamentals of Heat and Mass Transfer (6th ed.).* John Wiley Sons, 2007.

[39] J. Kunes. Dimensionless physical quantities in science and engineering. *Elsevier*, 2012.

[40] M. A. Sheremet and I. Pop. Thermo-bioconvection in a square porous cavity filled by oxytactic microorganisms. *Transp Porous Med*, 103:191–205, 2014.