CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Non-Linear Control Techniques for Stabilization of Underactuated Mechanical Systems

by

Nadir Mehmood

A thesis submitted in partial fulfillment for the degree of Master of Science

in the

Faculty of Engineering
Department of Electrical Engineering

2019

This thesis is dedicated to my Parents.

For their endless love, support and encouragement

CERTIFICATE OF APPROVAL

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Nadir Mehmood

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Acknowledgements

In the name of Allah SWT The Most Beneficent and The Most Merciful. All praises are to Allah SWT for all blesses so that the writer can accomplish this thesis. In addition, may peace and salutation be given to the prophet Muhammad (pbuh) who has taken all human being from the darkness to the lightness.

I would like to express my heartiest gratitude to my supervisor **Prof. Dr. Fazal-ur-Rehman**, Electrical Dept, Capital University of Science and Technology (CUST), for his efforts to establish research environments and activities at CUST. It was a wonderful experience and learning opportunity to work with him as a MS student. I am really indebted to his kind support and all out help to complete this research work successfully. His continuous encouragement, support and constructive criticism made me able to complete this task. In spite of his engagements, his dedication, as a supervisor, is highly appreciable.

I would also like to extend my gratitude to the **Prof. Dr. Aamer Iqbal Bhatti**, Dean Faculty of Engineering **Prof. Dr. Imtiaz Ahmad Taj** and Head of Department **Prof. Dr. Noor M. Khan** for their continuous encouragement and support during the entire span of my stay at CUST.

I feel honor to express my feeling of appreciation for the support I have received from the members of Research Group of control system and especially Yaseen, Fazal Noor and Muhsin Ullah who helped me in my research work and documentation of the thesis.

Financial support in the form of MS Scholarship by Capital University of Science and Technology is gratefully acknowledged.

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Abstract

Underactuated nonlinear systems are always equipped with less number of actuators than the degree of freedom. This feature offers certain benefits like reduction in weight and minimum energy usage. Majority of the robotic systems (including aerial, underwater and ground robotics) are found to be underactuated in nature. Therefore, research in such system is still quite demanding and challenging. It is also worthy to mention that the underactuation phenomenon, do not allow the direct design of control input as practiced in fully actuated systems. Majority of the techniques in literature lags the robust stabilization of underactuated mechanical systems.

In this work , the author suggests non-linear robust control techniques for underactuated mechanical systems which includes sliding mode control, adaptive backstepping, Input/Output feedback linearization. The proposed framework is applicable to n-DOF underactuated mechanical systems.

Firstly, input/output feedback linearization control technique is proposed for 2DOF systems regarding matched uncertainties. Secondly, adaptive backstepping algorithm is proposed for 2DOF systems. In the third scheme, the systems are transformed using transformation and a standard SMC design is proposed regarding 2DOF underactuated mechanical systems.

The first three strategies are applicable to 2DOF UMS systems. To overcome this hurdle, finally the adaptive sliding mode control strategy for n-DOF systems regarding the matched uncertainties is presented. The proposed control techniques are verified for following underactuated mechanical systems: cart-pole system, TORA (Translational Oscillator with Rotational Actuator), overhead crane system and double inverted Pendulum, to achieve the improved performance together with the added benefit of remarkable robustness to uncertainties.

Contents

List of Figures

Abbreviations

2DOF	Two Degree of Freedom
3DOF	Three Degree of Freedom
nDOF	n Degree of Freedom
IOFL	Input Output Feedback Linearization
AB	Adaptive Backstepping
FOSMC	First Order Sliding Mode Control
ASMC	Adaptive Sliding Mode Control
ISMC	Integral Sliding Mode Control
IDA-PBC	Interconnection and Damping Assignment-Passivity Based Control
SMC	Sliding Mode Control
TORA	Translational Oscillator with Rotational Actuator
UMS	Underactuated Mechanical System
UMSs	Underactuated Mechanical Systems

Symbols

m	Mass
g	Gravitational Acceleration
L	Length
F	Force
s	Sliding Surface
θ	Theta
ϕ	Phi
u	Control Input
L	Lagrangian

Chapter 1

Introduction

This chapter presents the research work carried out in this thesis. First, the author explains how inspiration for this work was developed and then clearly pinpoints the research problem and defines research objectives. The chapter also concludes having an overview regarding this thesis.

1.1 Background and Motivation

Mechanical Systems are among typically the oldest systems invented by humans to be employed as helping systems in their daily life. Nowadays, mechanical systems are applied in almost every part of our society. The variety of such systems is broad but their applications are different, such as; the basic types like a wheel, a pulley, and the on-off valve on the water line, simple devices like a sewing machine, bicycle, large systems such as the initial heavy steam engine, and the more advanced and sophisticated systems for example today's industrial systems, automobile, robots, aerospace systems and marine systems.

With the passage of time these systems became more common in daily life and their manual functioning became tiresome and less productive due to complex structure and functionality. Humans began thinking to somehow get a grip or automate these systems for better output each quantitatively and qualitatively.

This kind of need and realization involving regulation or automation brought to the development of Mechanical Control systems and hence the use of control Theory in the field of Mechanics. Historically water levels regulator for steam boiler by I. Pulzunov in 1765, and flyball governor for controlling the rate of steam engine, mechanical system by James Watt in 1769, could be cited as application of Automatic Feedback control in mechanical systems. State of the art research and developments inside control theory on single front and advances inside the technologies like electrical, digital and analog electronics, microprocessors, microcontrollers and computers on other front, caused it to design a more sophisticated mechanical systems. The results regarding these systems, achievements by human beings are numerous and their impacts are enormous reflected by the use of high performance in addition to quality systems in modern day practical life.
From control point of view, mechanical control systems can be classified into the following subclasses:

1. **Fully Actuated Mechanical:** Number of control inputs is equal to number of degrees of freedom to be controlled.

2. **Underactuated Mechanical Systems:** Number of control inputs is less as compared to the number of degree of freedom to be controlled.

3. **Nonholonomic Systems:** These Systems have non integrable first order constraints on their velocities.

Typically the control problem of fully actuated systems is not really a big issue because the matured nonlinear control techniques like feedback linearization are available [1]. Desire for the control of nonholonomic systems started in $70's$ and established be fully grown standalone discipline in the 90's [2]. Historically, nonholonomic systems are the most extensively studied systems evident from the vast quantity of literature and are still subject of active study in the control research groups. The most important purpose, which led to this kind of research along with other interesting control problems, is the fact that these systems fail to satisfy the necessary condition of Brockett [3] for existence of continuous time invariant state feedback control for stabilization. In fact, the beginning of research interest in underactuated mechanical systems could be traced returning to nonholonomic systems due to initial realization that nonholonomic systems obey first order constraints while underactuated mechanical systems obey second order constraints [4–6].

In the last ten years, research shifted from assumptive nature to practical when the usefulness of underactuated mechanical systems seemed to be realized in diverse applications of engineering and scientific importance. The wide range application areas of UMS systems include industry, robotics, aerospace systems and marine systems. Apart by practical applications, underactuated mechanical systems have great importance in education and research of control theory because prototype systems for higher order nonlinear systems. Both theoretical importance and practical usefulness have added to research activities concentrated on the control and analysis of UMS systems in the last twenty years [7, 8].

Underactuation arises due to fewer number of control actuators compared to the number of degrees in order to be controlled. Reasons associated with underactuation may be natural due to dynamics of system itself or even intentional/artificial for some beneficial practical purpose, for example:

- natural dynamics: helicopter, aircraft, underwater vehicle
- low cost, low weight
- low power consumption: important in applications like aerospace
- actuator failure
- low system level complexity

Some examples of practical UMSs include the following:

- **Robotics:** mobile robots and flexible-link joints.
- **Aerospace:** aircraft, helicopters, spacecrafts and satellites.
- **Marine:** underwater vehicles, ships and surface vessels
- **Education and Research:** The TORA System, The Acrobot, The double Inverted Pendulum, The Overhead Crane, The Beam-and-Ball System, The Cart-Pole System.

There are several excellent class based approaches for mechanical systems, for example, Controlled Lagrangian [9, 10], energy based [11], IDA-PBC [12], hybrid [13, 14], and input disturbance approach [15, 16], although these techniques lack of robustness. Because of absence of direct independent control actuators for a few number of the degree of freedom, UMS systems are more vulnerable to the disturbances. Standard SMC provides good measure the consequences of disturbance and to make system responce robust.
The aforementioned discussion and analysis makes clear that the benefits of UMS systems are numerous but their realization in practical applications is usually restricted due to difficult control design. Presently there are good research work in the literature but most are limited to the system by system approach or lack of robustness and hence there is a need to research a novel robust design approach applicable to UMS systems.

1.2 Modeling of Underactuated Mechanical Systems

Following a brief introduction now we are going to study about the dynamic modeling of UMS. For n degree mechanical systems the euler-lagrange representation is [17]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F(q)u \tag{1.1}$$

here;

$L(q,\dot{q})$ is Lagrangian

$F(q) \in R^{n\times m}$ Control input matrix

$u \in R^m$ Control input vector

$q \in R^n$ Configuration vector

In above, if *m=rank(F)=n* then such systems are called fully actuated systems, but If *m=rank(F)*<n then such systems are called underactuated systems [18]. The vector form of eq (1.1) can be written as;

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = F(q)u \tag{1.2}$$

here;

$M(q)$ is positive definite symmetric inertia matrix

$C(q,\dot{q})\dot{q}$ is centrifugal and coriolis term

$G(q)$ is gravitational term

The lagrangian of a system can be written as;

$$L(q,\dot{q}) = k(q,\ddot{q}) - v(q) = \frac{1}{2}{q_1}^T M(q) q_1 - v(q) \tag{1.3}$$

Lagrangian is the difference between kinetic energy and potential energy.

For general case $F(q) = [F_1(q), F_2(q)]^T$ and $q = [q_1, q_2]^T$ dynamics (1.2) can be written as;

$$\begin{aligned} m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 = F_1(q)u \\ m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 = F_2(q)u \end{aligned} \tag{1.4}$$

M(q)= $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ is positive definite symmetrical inertia matrix, $c_1(q,\dot{q}) \in R^{(n-m)}$ and $c_2(q,\dot{q}) \in R^m$ are the coriolis and centrifugal terms, $g_1(q) \in R^{n-m}$ and $g_2(q) \in R^m$ are the gravitational terms and $u \in R^m$ is the vector of control inputs produced by m actuators. For $F_1(q) = 0$ and $F_2(q) = 1$, one may have

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 = 0$$
$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 = u \tag{1.5}$$

Where as for $F_1(q) = 1$ and $F_2(q) = 0$, one may have

$$m_{11}\ddot{q}_1 + m_{12}\ddot{q}_2 + c_1 + g_1 = u$$
$$m_{21}\ddot{q}_1 + m_{22}\ddot{q}_2 + c_2 + g_2 = 0 \tag{1.6}$$

In this research work we are considering both the cases when $F_1(q) = 0$ and $F_2(q) = 1$ and $F_1(q) = 1$ and $F_2(q) = 0$. Solving the first system in (1.5) for $\ddot{q}_1$ and $\ddot{q}_2$, and then substituting the result in the second equation, (1.5) can be written as:

$$\bar{m_{11}}\ddot{q}_1 + \bar{c_1} + \bar{g_1} = u \tag{1.7a}$$

$$\bar{m_{22}}\ddot{q}_2 + \bar{c_2} + \bar{g_2} = u \tag{1.7b}$$

where

$$\begin{aligned}
\bar{m}_{11}(q) &= m_{21} - m_{22}m_{12}^{-1}m_{11} \\
\bar{c_1}(q,\dot{q}) &= c_2 - m_{22}m_{12}^{-1}c_1 \\
\bar{g}_1(q) &= g_2 - m_{22}m_{12}^{-1}g_1 \\
\bar{m}_{22}(q) &= m_{22} - m_{21}m_{11}^{-1}m_{12} \\
\bar{c_2}(q,\dot{q}) &= c_2 - m_{21}m_{11}^{-1}c_1 \\
\bar{g}_2(q) &= g_2 - m_{21}m_{11}^{-1}g_1
\end{aligned} \tag{1.8}$$

Since $q_1 \in R^{n-m}$ and $q_2 \in R^m$, dynamics (1.7) is a set of two second order systems in state variables. The state space representation of (1.7) can be written as [19].

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_1 + b_2(x)u \\
&\vdots \\
\dot{x}_{2n-1} &= x_2 \\
\dot{x}_{2n} &= f_n + b_n(x)u
\end{aligned} \tag{1.9}
$$

Here $x = [x_1, x_2, x_3 \ldots, x_{2n-1}, x_{2n}]^T$ is the state vector $f_i(x)$ and $b_i(x)$, $i = 1, 2, 3, 4..$
....n are the nonlinear functions of the states and u is the single control input. For $n = 2$, the equation (1.9) can be written as:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_1 + b_2(x)u
\end{aligned} \tag{1.10}
$$

state space models of the Pendubot and inverted pendulum systems are represented by (1.10). For $n = 3$ the equation (1.9) gives:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3 + b_3(x)u
\end{aligned} \tag{1.11}
$$

1.3 Problem Statement and Research Objectives

Underactuated mechanical systems have assumptive and practical importance along with the added benefits of underactuation. But the advantages associated with underactuation come at a higher cost of difficult control design due to complicated nonlinearity and control coupling. Lack of direct actuators for some of the degrees of freedom makes UMS systems more susceptible to model uncertainties and external disturbances. Greater possibility of mismatch between the real plant and their mathematical model on which control synthesis is based results in external disturbances and model uncertainties are common in practical applications. So designing robust nonlinear control techniques in order to effectively control the complex nonlinear behavior of UMS systems in presence of model uncertainties and external disturbances becomes an obviously significant control problem. Solving this challenging problem will certainly help in the realization of full advantages in addition to usefulness of underactuated mechanical systems in high performance control applications. Non-linear control techniques like backstepping and Sliding mode control [20, 21] can efficiently control higher order complex nonlinear dynamics and provides robustness to model uncertainties and external disturbances. There are other nonlinear control techniques developed for mechanical systems for example energy based control [11], Controlled Lagrangian method [9, 10], IDA-PBC [12], hybrid control [13, 14], and equivalent-input-disturbance control technique [15] although these techniques lack of robustness.
The research goal in this work is to investigate, using backstepping, feedback linearization and sliding mode control theory, a comprehensive and unified yet simple to apply high performance robust control design framework for underactuated mechanical systems. Finally, the framework will be numerically validated for the following benchmark UMS systems.

1. Double Inverted Pendulum

2. The TORA System

3. The Overhead Crane system

4. Inverted Pendulum

5. Cart-Pole System

1.4 Thesis Organization

This thesis consists of six chapters. The focus of this research work is to propose novel solution to the stabilization problem of underactuated systems. This chapter explain introduction, motivation for work, thesis objective and thesis organization.

Chapter 2-Literature Review:

This chapter takes into account the literature review regarding stabilization of underactuated mechanical systems.

Chapter 3-Stabilization of Underactuated Systems: Feedback Linearization Technique:

This chapter presents the proposed control techniques based on input/output feedback linearization base control. The algorithms is applied to underactuated cart-pole system, overhead crane system.

Chapter 4-Stabilization of Underactuated Systems: Adaptive Backstepping Technique:

This chapter presents adaptive backstepping based stabilization of the underactuated mechanical systems. The algorithm is applied to underactuated cart pole system and overhead crane system.

Chapter 5-Stabilization of Underactuated Systems: Sliding Mode Control:

This chapter presents the proposed control algorithm based sliding mode control and adaptive sliding mode control. The algorithms is applied to underactuated cart-pole system, overhead crane system, TORA, Double inverted pendulum.

Chapter 6-Conclusion and Future Work:

This chapter summarizes the overall thesis and draws conclusions. The significance of the proposed research is emphasized. Future directions have also been proposed for further research.

Chapter 2

Literature Review

This chapter presents a literature review of underactuated mechanical systems from control perspective. Theoretical challenges in control of UMS systems are mentioned. Different analytical tools and control techniques designed over the past years are reviewed.

2.1 Introduction

Control of underactuated mechanical systems remains active areas of research in the last decade. Research interest in this field started with the analysis and study of nonholonomic mechanical systems. Analysis and control of nonholonomic systems [2, 22] started in early 80's and became a full grown and established area of research in the middle of the 90's. Study of nonholonomic systems developed some interesting control problems. Being non-holonomic nature, it had been proved that these systems are not stabilizable by smooth continuous time invariant state feedback control techniques [3, 23].
As few of these nonholonomic mechanical systems were inherently underactuated, the interest of researchers shifted towards the analysis and control of UMS systems in the 90's together with the first applications mainly within robot manipulators [4, 24–32]. This kind of interest got momentum when

use of underactuated mechanical systems became increasingly popular in scientific and research applications such as robotics, marine systems and aerospace systems. These developments led to the establishment of research in UMS systems as one of the most effective field both from technical and theoretical point of view and research in the control of UMS systems started as the field [5, 6, 11, 17, 33].

Nonholonomic mechanical systems have 1st order (velocity) constraints. As parts of the dynamics of an UMS systems can be written as second order (acceleration) constraints contrast to non-holonomic systems, underactuated mechanical also called mechanical systems with 2nd order nonholonomic constraints[4, 6].

2.2 Theoretical Challenges in the Control of Underactuated Mechanical Systems

Underactuation, i.e. fewer quantity of independent control actuators compared to configuration variables being controlled, fundamentally makes control problem of underactuated mechanical systems an extremely challenging task. Moreover, higher non linear behavior, input coupling and nonholonomic constraints adds extra difficulties for this challenging control problem. Feedback linearization is an important design tool used an initial step in design of nonlinear control law for nonlinear systems [1]. However owing to underactuatoin. exact feedback linearization for UMS systems is not achievable. This could be seem from the dynamical equations of underactuated mechanical systems in which the control input matrix is non invertible, and hence, an explicit change of control is not possible that indicates the exact feedback linearization does not exist.

In [11, 25], it had been shown that for the certain class of underactuated mechanical systems, the dynamics could be partitioned into the unactuated subsystem and the actuated subsystem and a partial feedback linearization of the actuated subsystem is achievable. But still, the unactuated subsystem has nonlinear behaviour and coupled to the linearized actuated subsystem.

Afterwards, in [17], it had been shown that an explicit change is possible to decouple the two subsystems in a actuated one and an unactuated one. But still, decoupled systems are highly nonlinear and, as mentioned, this is for a specific class of UMS not all the systems.

2.3 Control Design Approaches for UMS

Different control design approaches, designed and used over the years for underactuated mechanical systems, are reviewed in this section.

- **Energy and Passivity Based Control:**
  In energy and Passivity Based control (PBC) methods the total energy is regulated towards the equivalent value of the desired equilibrium state as a result achieving regulation of the system states to desired values. This technique is primarily used for the set-point regulation of underactuated mechanical systems. Application of these methods could be found in the works [11, 14, 34–37].

- **Controlled Lagrangian:**
  In Controlled Lagrangian method, Lagrangian of UMS systems is regulated, simply by modifying the inertia matrix and potential energy matrix, towards the desired equilibrium state by using control input and then guarantee the passivity of the system, damping is injected into the system. Application of Controlled Lagrangian to underactuated mechanical systems could be found in the works [9, 10, 38].

- **IDA-PBC:**
  In IDA-PBC method the Hamiltonian of UMS systems is regulated, by changing the inertia matrix, the potential energy function and interconnection matrix, towards the desired equilibrium state by using control input and then guarantee the passivity of the system, damping is

injected into the system. Application of Controlled Lagrangian to underactuated mechanical systems could be found in the works [12, 39, 40].

- **Optimal Control:**
  In Optimal Control, the procedure is based on finding a control algorithm that minimizes or maximizes a cost function. Optimization of energy or time are two approaches in optimal Control. Application of Controlled Lagrangian to underactuated mechanical systems could be found in the works [41–44].

- **Sliding Mode Control:**
  Sliding Mode Control is most powerful robust technique against uncertainties, unmodeled plant dynamics and disturbances. In sliding mode control, 1st, a sliding manifold is designed with desired dynamics and then control law is chosen to force system towards the sliding surface. After reaching the surface, system states slide, alongside the surface, to desired values and remain there under the control action. Robustness to external disturbance is ensured through discontinuous term in control law or through estimation and after that cancellation by the control law. Aside from robustness, SMC can easily control higher order and complex nonlinear models. Because of these promising control features, SMC has been used by the researchers for the control of UMS systems. Application SMC to underactuated mechanical systems could be found in the works [45–51].

In conclusion, there are many outstanding research work on the subject. Most works address the control problem of the specic UMS systems. There is strong requirement for class based control design approaches that are robust.

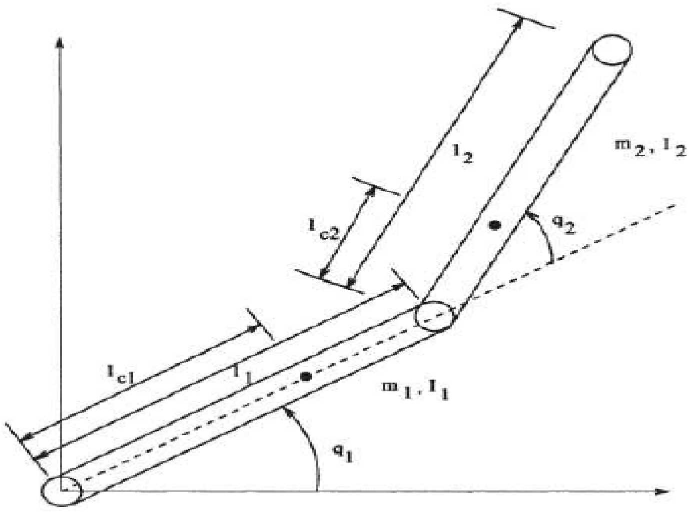
2.4 Examples Underactuated Mechanical Systems

The UMS include The Translational Oscillator with Rotational Actuator (TORA) system [52], the beam and ball system [53], the Acrobot [54], the Pendubot [36], the cart-pole system [55], the crane system [56], and the double inverted pendulum [57].

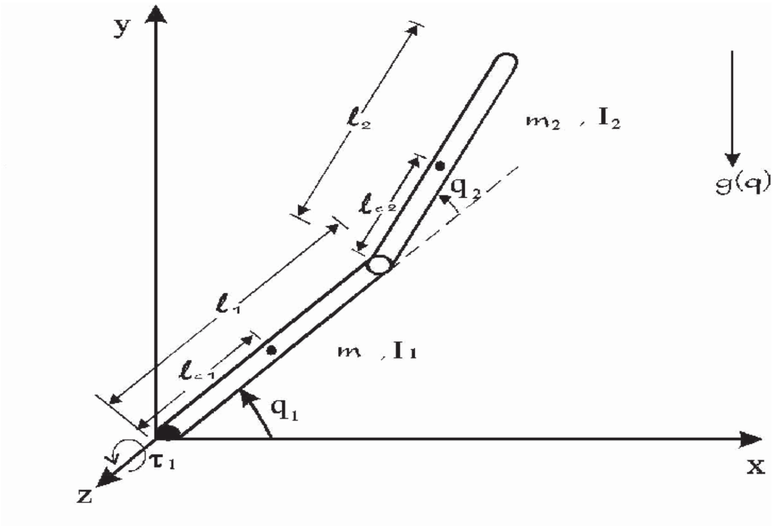
2.4.1 Acrobot and Pendubot

Acrobot [54] and Pendubot [36] are two-link manipulators with a single actuator at elbow and shoulder respectively as shown in Fig (2.1). Both manipulators have identical equations of motion and are graphically alike. The stabilization of the two-link manipulator to its upright equilibrium point ($q_1 = \frac{\pi}{2}$ *and* $q_2 = 0$) from any initial condition is their control task.

Energy-based control is one of the famous control approach used to swing up the system from its stable downward spot to precarious upright spot, and swap to a linear controller for stabilization [36]. Lai in [58] to give a complete unified control technique. An impulse momentum approach provided a new idea on swing up control by Albahkali in [59] and Jafari in [60].



(a) Acrobot



(b) Pendubot

FIGURE 2.1: Acrobot and Pendubot Systems

2.4.2 Cart-Pole System

The cart-pole system in Fig (2.2) is a benchmark underactuated system. It used as a testbed for nonlinear control study. The control job is to swingup the pendulum from its steady downward equilibrium state. ($q_1 = 0$ *and* $q_2 = \pi$) to vertical unbalanced equilibrium point ($q_2 = 0$), while retains the cart at its original point ($q_1 = 0$). Considerable work has been done in the past from fuzzy control (FC) and energy based prespective for the under consideration cart-pole system in [11].

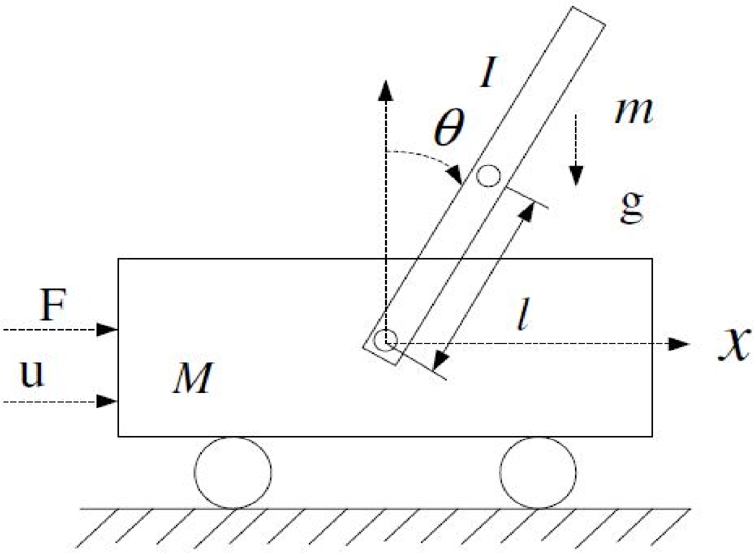


FIGURE 2.2: Cart-Pole System

2.4.3 Ball and Beam System

The ball and beam system [53] consists of a beam able to move up and downward via motor connected at one end (whereas the other end of the beam is fixed) as shown in Fig (2.3). As this beam is made of metal and iron ball is allowed to move freely on it, control task is to stabilize the ball on the desired position on the beam, starting from any initial condition on the beam. The Lyapunov-based method [61] control works on the ball and beam system.

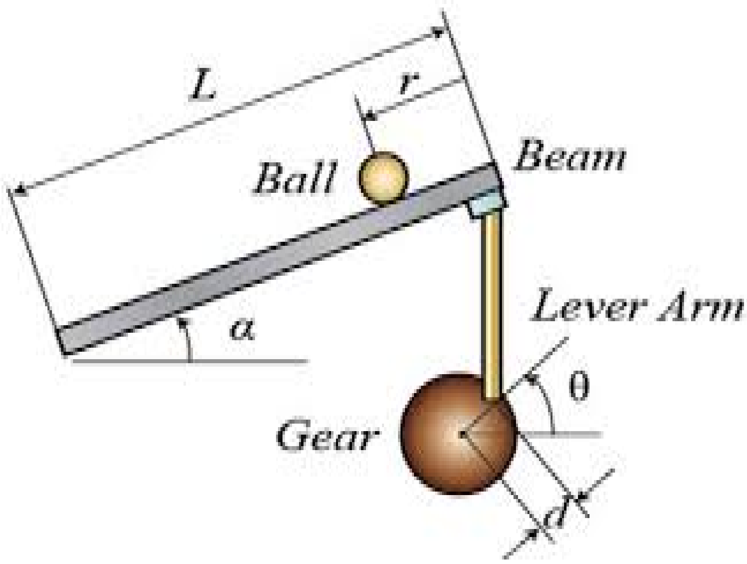


FIGURE 2.3: Ball and Beam System

2.4.4 Translational Oscillator with Rotational Actuator System (TORA)

The TORA system in Fig (2.4) is a non-linear benchmark example for different control techniques. The system contains an oscillating translational stage and an eccentric revolving pendulum. To make sure the horizontal displacement $q_1 \to 0$ in the occurrence of any exterior disturbance is the control task of TORA [62].

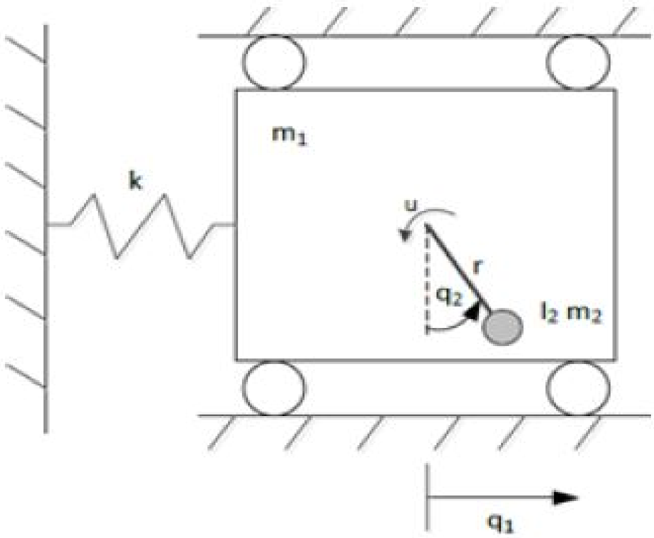


Figure 2.4: Translational Oscillator with a Rotational Actuator System

Chapter 3

Stabilization of Underactuated Systems: Feedback Linearization Technique

This chapter presents an input/output feedback linearization algorithm for the stabilization of underactuated mechanical systems. Feedback linearization is one of the common approach used in controlling nonlinear systems. The approach involves coming up with a transformation of a non linear system to equivalent linear system or at least closely to it. This problem deserves a lot of attention, its positive solution directly or indirectly extends the applicability of the linear methods to a more general nonlinear class of systems. First we define the suitable control u in term of u_1 and u_2 to transform the system into particular structure. In first method The u_1 is designed using input/output feedback linearization technique. On the basis of Lyapunov stability, the control u_2 is derived. In second method u_2 is derived using integral sliding mode control. The control algorithms is applied to two systems, namely; cart-pole system and overhead crane system. The effectiveness of proposed algorithm is verified through numerical simulations.

3.1 Problem Statement

For a given desired point $x_{des} \in R^n$, a control input u is constructed in such a way that x_{des} is an attractive point $t \to \infty$ leads to $x \to x_{des} = 0$.

3.2 The Proposed Control Algorithms

3.2.1 First Method

Step 1:

Write the system (1.10) as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1 + b_1(x)u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= f_2 + b_2(x)u \end{aligned} \tag{3.1}$$

where, f_i and b_i are nonlinear functions.

Step 2:

Choose the input u;

$$u = \frac{-f_2 + u_1}{b_2(x)} + u_2 \tag{3.2}$$

the system (3.1) can be written the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1 + b_1(x)(\frac{-f_2 + u_1}{b_2(x)} + u_2) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u_1 + b_2(x)u_2 \end{aligned} \tag{3.3}$$

where u_i are the new inputs.

Step 3:

Assume the system have two subparts, $x_i,\ i = 3, 4$ are stabilized via u_1 and

$x_i,\ i = 1, 2$ are stabilized using u_2, choose output of the system $y = x_3$. Design u_1 using the input/output feedback linearization technique, while u_2 design using lyapunov stability.

$$\begin{aligned} y &= x_3 \\ \dot{y} &= \dot{x_3} = x_4 \\ \ddot{y} &= \dot{x_4} = u_1 + b_2(x)u_2 \end{aligned} \tag{3.4}$$

Define the error:

$$\begin{aligned} e &= x_3 - x_{3d} \\ \dot{e} &= \dot{x_3} = x_4 \\ \ddot{e} &= \dot{x_4} = u_1 + b_2(x)u_2 \end{aligned} \tag{3.5}$$

where $x_{3d} = 0$, choose $u_1 = -k_1 e - k_2 \dot{e}$, after some manipulation the error dynamics can be written as:

$$\ddot{e} + k_1\dot{e} + k_2 e - b_2(x)u_2 = 0 \tag{3.6}$$

In eq (3.6), if $u_2 \to 0$ as $t \to \infty$, then error dynamics e, $\dot{e}$ and $\ddot{e}$ must $\to 0$ as $t \to \infty$, to stabilize the x_3 and x_4. For designing u_2, choose a lyapunov function as:

$$\begin{aligned} V &= \frac{1}{2}c_1 x_1^2 + \frac{1}{2}c_2 x_2^2 \\ \dot{V} &= c_1 x_1 \dot{x_1} + c_2 x_2 \dot{x_2} \\ \dot{V} &= c_1 x_1 x_2 + c_2 x_2 \dot{x_2} \\ \dot{V} &= x_2(c_1 x_1 + c_2 \dot{x_2}) \\ \dot{V} &= -c_3 x_2^2 \leqslant 0 \end{aligned} \tag{3.7}$$

where, $\dot{x_2} = (-c_1 x_1 - c_3 x_2)/c_2$

$$\begin{aligned} \frac{-c_1 x_1 - c_3 x_2}{c_2} &= f_1 + b_1(x)(\frac{-f_2 + u_1}{b_2(x)} + u_2) \\ u_2 b_1(x) &= \frac{-c_1 x_1 - c_3 x_2}{c_2} - f_1 - b_1(x)(\frac{-f_2 + u_1}{b_2(x)}) \\ u_2 &= \frac{(-c_1 x_1 - c_3 x_2 - c_2 f_1)b_2(x) - c_2 b_1(x)(u_1 - f_2)}{c_1 b_1 b_2} \\ u_2 &= \frac{(-c_1 x_1 - c_3 x_2 - c_2 f_1)b_2(x) - c_2 b_1(x)(-k_1 x_3 - k_2 x_4 - f_2)}{c_1 b_1 b_2} \end{aligned} \tag{3.8}$$

putt u_2 in dynamics (3.3) to get:

$$\dot{x}_2 = f_1 + b_1(x)(\frac{-f_2 + u_1}{b_2(x)}) - x_1 + \frac{(-c_3x_2 - c_2f_1)}{c_1} - \frac{c_2b_1(x)(-k_1x_3 - k_2x_4 - f_2)}{c_1b_2} \tag{3.9}$$

LaSalle's theorem:

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$ and and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

From eq (3.7):

$$\dot{V}(x) = 0 \;\; for \;\; x_2 = 0 \;\; irrespective \; of \; the \; value \; of \;\; x_1$$

$$x_2(t) \equiv 0 \;\; \Rightarrow \;\; \dot{x}_2(t) \equiv 0 \;\; \Rightarrow \;\; x_1(t) \equiv 0, \;\; using \; eq \; (3.9)$$

Thus, the origin is asymptotically stable.

Step 4:

The eq (3.2) as:

$$u = \frac{-f_2 + u_1}{b_2(x)} + u_2$$

$$u = \frac{-f_2 + -k_1e - k_2\dot{e}}{b_2(x)} + \frac{(-c_1x_1 - c_3x_2 - c_2f_1)b_2(x)}{c_1b_1b_2} - \frac{c_2b_1(x)(-k_1x_3 - k_2x_4 - f_2)}{c_1b_1b_2}$$

3.3 Application to Underactuated Mechanical Systems

3.3.1 Cart-Pole System

The proposed control scheme is now used to stabilize an cart-pole system as considered in [63]. It is an underactuated mechanical system with 2DOF. The dynamic of the system as given in [63] is:

$$
\begin{aligned}
&(M + m\sin^2\theta_1)\ddot{x}_1 - m\sin\theta_1(l\dot{\theta}_1^2 - g\cos\theta_1) = F + u \\
&(M + m\sin^2\theta_1)l\ddot{\theta}_1 + l\dot{\theta}_1^2\sin\theta_1\cos\theta_1 - (M+m)g\sin\theta_1 = -\cos\theta_1(F_1 + u)
\end{aligned}
\tag{3.10}
$$

define the state vector $x = [x_1 \;\; \dot{x}_1 \;\; \theta_1 \;\; \dot{\theta}_1]^T = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]$, the state space representation becomes;

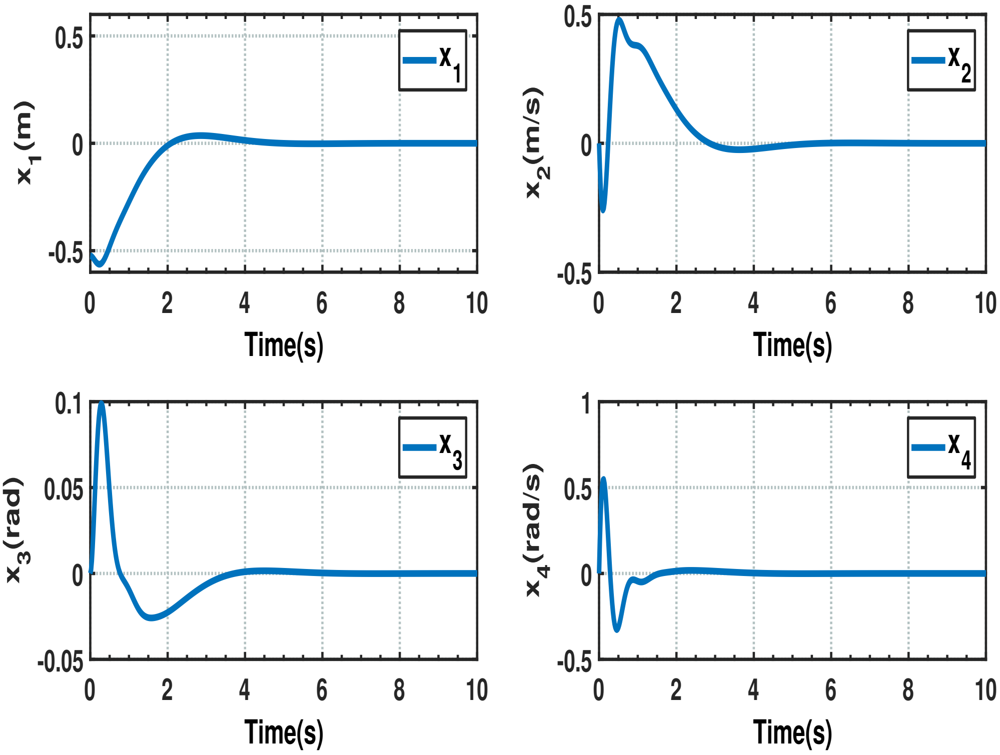
$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u
\end{aligned}
\tag{3.11}
$$

where

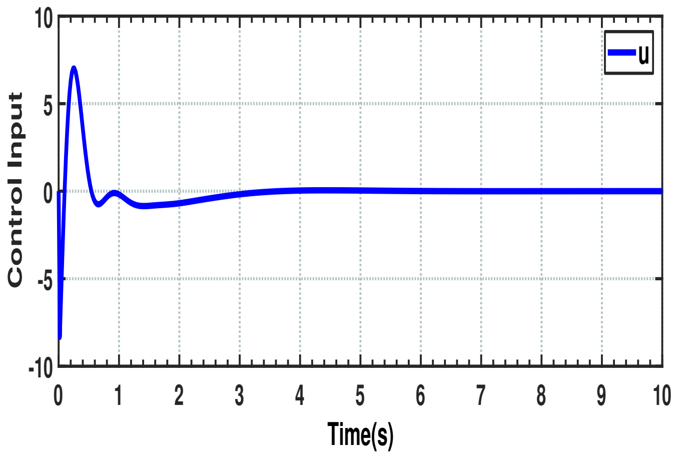
$$
\begin{aligned}
f_1 &= \frac{mlx_4^2\sin x_3 - mg\sin x_3\cos x_3 + F}{M + m\sin^2 x_3} \\
f_2 &= \frac{-mlx_4^2\sin x_3\cos x_3 + (M+m)g\sin x_3 - F\cos x_3}{Ml + ml\sin^2 x_3} \\
b_1(x) &= \frac{1}{M + m\sin^2 x_3} \\
b_2(x) &= \frac{-\cos x_3}{Ml + ml\sin^2 x_3}
\end{aligned}
\tag{3.12}
$$

The actual values of the system parameters in the simulation are:

$$
\begin{aligned}
F &= \sin x_3 \\
l &= 0.5(m) \\
m &= 1(kg) \\
M &= 2(kg)
\end{aligned}
$$

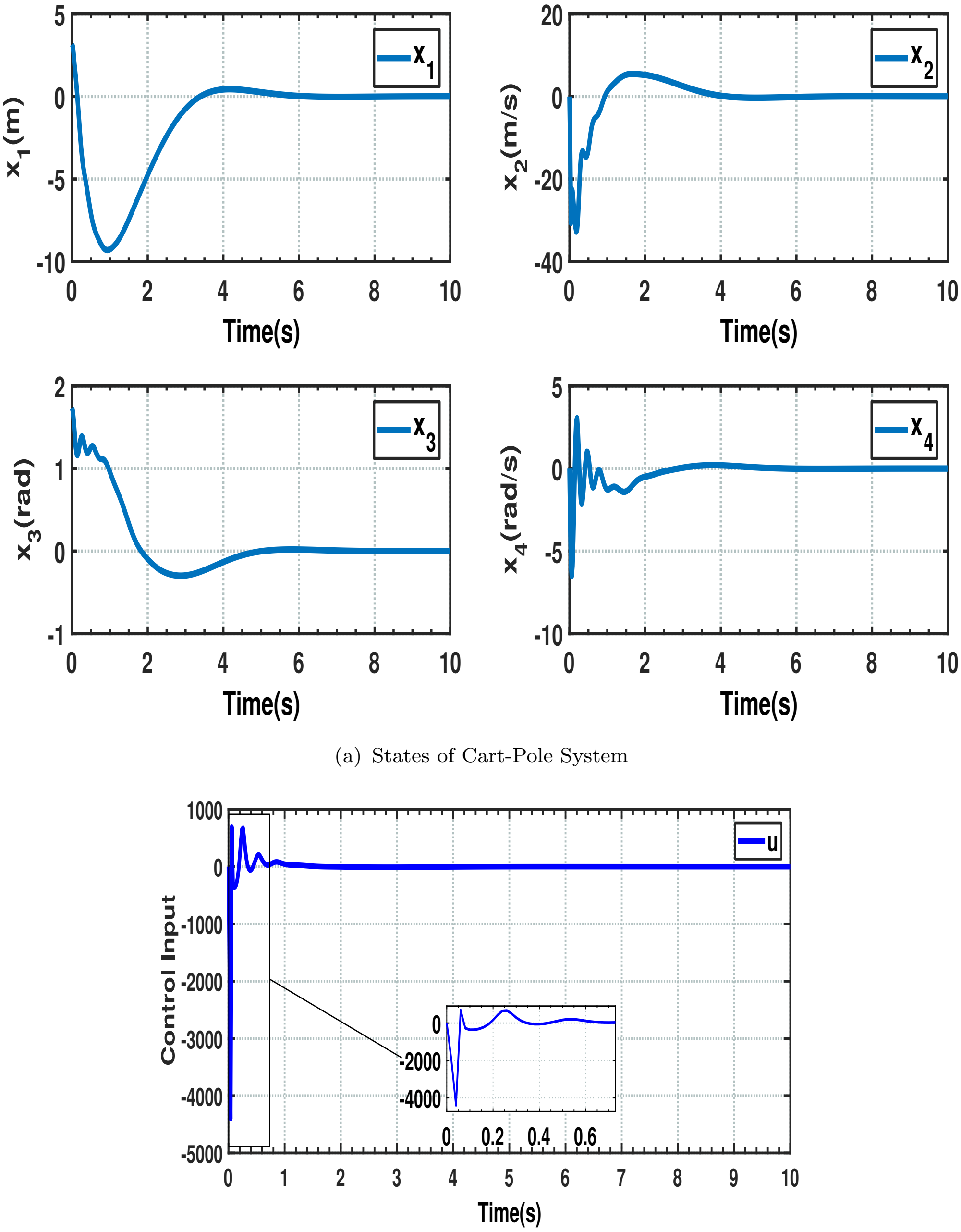


(a) States of Cart-Pole System



(b) Control Effort

Figure 3.1: Closed loop response of cart-pole system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (-\pi/6, 0, 0, 0)$, (b) Time history of control input u



(a) States of Cart-Pole System

(b) Control Effort

FIGURE 3.2: Closed loop response of cart-pole system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (2.5 + 0.5\ln(\frac{\sqrt{3}+1}{\sqrt{3}+1}), 0, \sqrt{3}, 0)$, (b) Time history of control input u

3.3.2 Second Method

Step 1:

Write the system (1.10) as:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u + d_1(x,t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u + d_2(x,t)
\end{aligned} \tag{3.13}
$$

where, f_i and b_i are nonlinear functions and suppose $d_1(x,t) = 0$, $d_2(x,t) = 0$.

Step 2:

Choose the input u;

$$
u = \frac{-f_2 + u_1}{b_2(x)} + u_2 \tag{3.14}
$$

the system (3.13) can be written the following form:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)(\frac{-f_2 + u_1}{b_2(x)} + u_2) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= u_1 + b_2(x)u_2
\end{aligned} \tag{3.15}
$$

where u_i are the new inputs.

Step 3:

Assume the system have two subparts, $x_i,\ i = 3,4$ are stabilized via u_1 and $x_i,\ i = 1,2$ are stabilized using u_2, choose output of the system $y = x_3$. Design u_1 using the input/output feedback linearization technique, while u_2 design using integral sliding mode control.

$$
\begin{aligned}
y &= x_3 \\
\dot{y} &= \dot{x_3} = x_4 \\
\ddot{y} &= \dot{x_4} = u_1 + b_2(x)u_2
\end{aligned} \tag{3.16}
$$

Define the error:

$$\begin{aligned} e &= x_3 - x_{3d} \\ \dot{e} &= \dot{x_3} = x_4 \\ \ddot{e} &= \dot{x_4} = u_1 + b_2(x)u_2 \end{aligned} \tag{3.17}$$

where $x_{3d} = 0$, choose $u_1 = -k_1 e - k_2 \dot{e}$, after some manipulation the error dynamics can be written as:

$$\ddot{e} + k_1\dot{e} + k_2 e - b_2(x)u_2 = 0 \tag{3.18}$$

In eq (3.18), if $u_2 \to 0$ as $t \to \infty$, then error dynamics e, $\dot{e}$ and $\ddot{e}$ must $\to 0$ as $t \to \infty$, to stabilize the x_3 and x_4. For designing u_2 using a integral sliding mode control. In this method the required control law is of the nature.

$$u_2 = u_{2o} + u_{21} \tag{3.19}$$

where u_{2o} being the ideal control and u_{21} is designed to reject perturbation term. Choose a integral sliding surface as [20]:

$$\sigma(x) = \sigma_o(x) + z \tag{3.20}$$

The first term in the right hand side of (3.20) indicates the contribution of conventional sliding surface which can be written as:

$$\begin{aligned} \sigma_o(x) &= c_1 x_1 + x_2 \\ \sigma(x_1, x_2) &= c_1 x_1 + x_2 + z \end{aligned} \tag{3.21}$$

Now, taking the derivative of (3.21)

$$\dot{\sigma} = c_1 x_2 + f_1 + b_1(x)\left(\frac{-f_2 + u_1}{b_2(x)} + u_{2o} + u_{21}\right) + \dot{z} \tag{3.22}$$

Choosing $\dot{z} = -(c_1 x_2 + f_1 + b_1(x)(\frac{-f_2+u_1}{b_2(x)} + u_{2o}))$ with $z(0) = \sigma(x(0))$, the above equation (3.22) becomes:

$$\dot{\sigma} = b_1(x)u_{21} \tag{3.23}$$

Comparing (3.23) with $\dot{\sigma} = -M_1 sign(\sigma)$, one has

$$u_{21} = -Msign(\sigma) \tag{3.24}$$

Where $M = \frac{M_1}{b_1(x)}$ is the gain of the discontinuous component and $u_{2o} = -k_3 x_1 - k_4 x_4$. The final control law can be obtained by substituting the designed expression of continuous and discontinuous components in (3.19). This control law eliminate the reaching phase and results in the robust regulation of the states to the origin.

Choose a Lyapunov function to ensure the stability:

$$\begin{aligned} V &= \frac{1}{2}\sigma^T\sigma \\ \dot{V} &= \sigma\dot{\sigma} \\ \dot{V} &= -M_1|\sigma| \leq 0 \end{aligned} \tag{3.25}$$

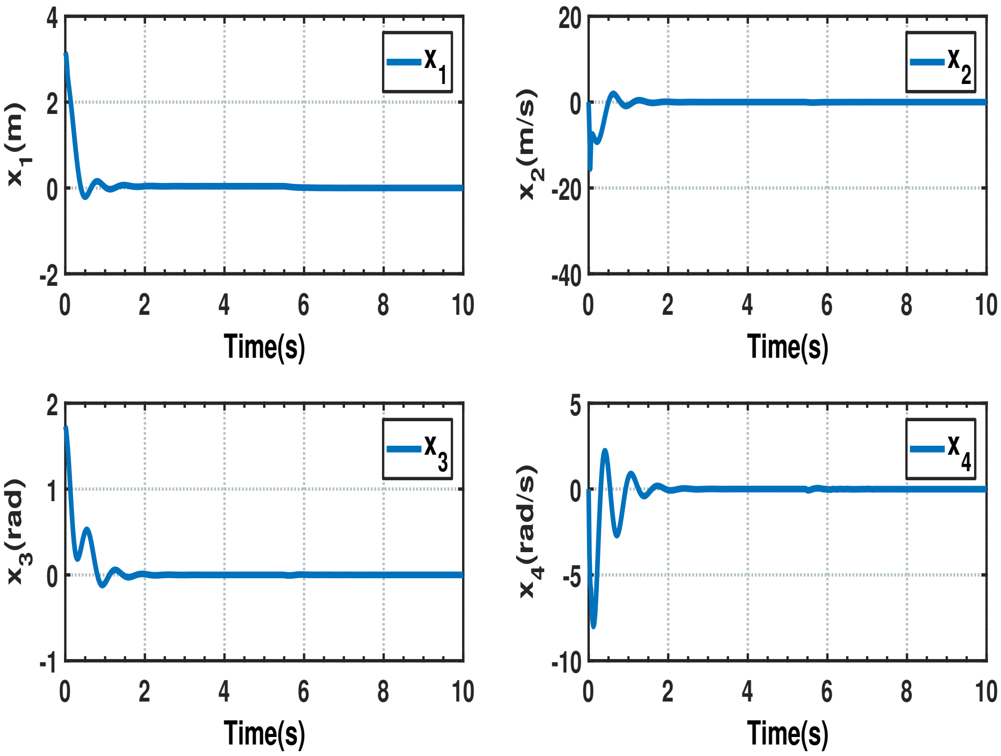
Eq (3.25) ensured that $\sigma \to 0$ in finite time.

3.4 Application to Underactuated Mechanical Systems

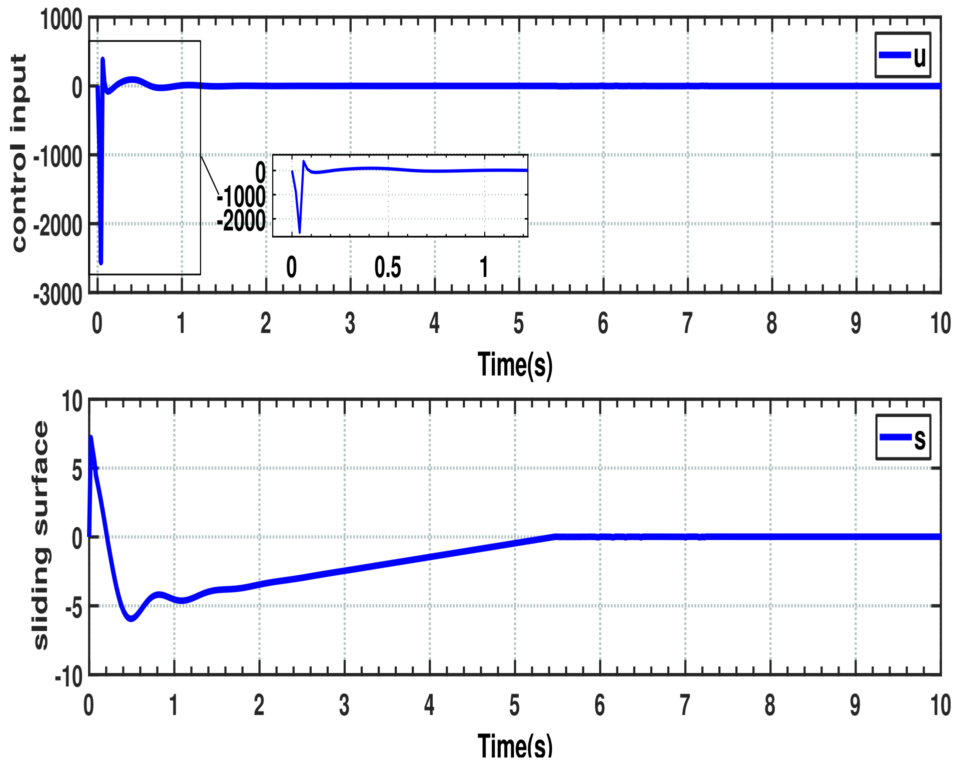
Case 1: Simulation results without external disturbances.

3.4.1 Cart-Pole System

The proposed control scheme is now used to stabilize an cart-pole system as considered in [63]. It is an underactuated mechanical system with 2DOF. The dynamic of the system discussed in (3.3.1).



(a) States of Cart-Pole System

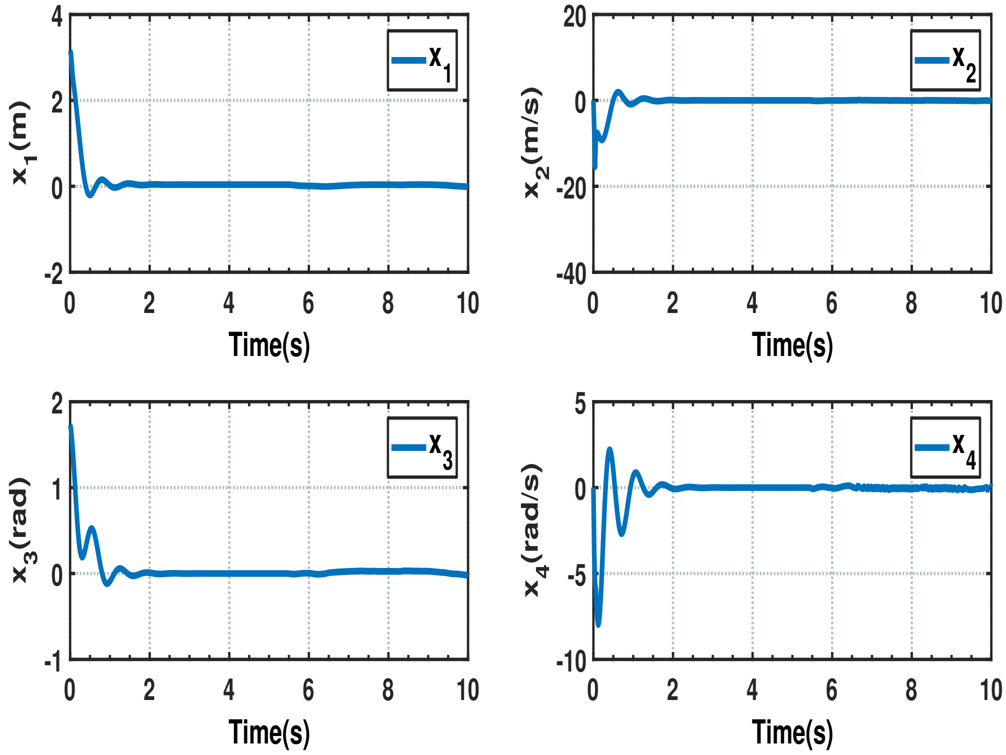


(b) Control Effort and Sliding Surface

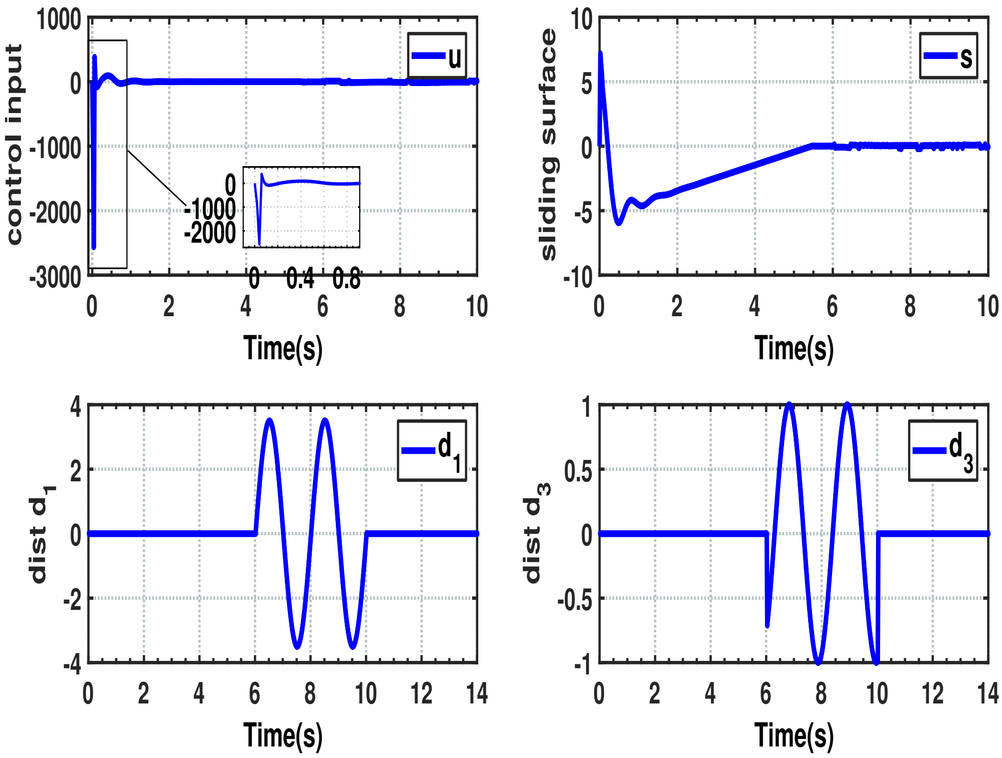
FIGURE 3.3: Closed loop response of cart-pole system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (2.5 + 0.5\ln(\frac{\sqrt{3}+1}{\sqrt{3}+1}), 0, \sqrt{3}, 0)$, (b) sliding surface s, Time history of control input u

Case 2: Simulation results with external disturbances.

3.4.2 Cart-Pole System



(a) States of the Cart-Pole system



(b) Control Input, Sliding Surface and Applied Disturbances

FIGURE 3.4: Closed loop response of cart-pole system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (2.5 + 0.5 \ln(\frac{\sqrt{3}+1}{\sqrt{3}+1}), 0, \sqrt{3}, 0)$, (b) sliding surface s, Time history of control input u and applied disturbances from t=6sec to t=10sec

Chapter 4

Stabilization of Underactuated Systems: Adaptive Backstepping Technique

In this chapter, we propose adaptive backstepping based design technique for underactuated systems. Our objective is to construct the stabilizing control algorithm for the class of 2DOF underactuated systems represented by (4.1). The model of 2DOF underactuated systems can be described as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1 + b_1(x)u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= f_2 + b_2(x)u \end{aligned} \tag{4.1}$$

In order to stabilize system (4.1), an adaptive backstepping based controller is proposed, which yields asymptotic stabilization of the closed-loop system. This is achieved by first transforming the original system into a new system that can be made asymptotically stable. After the stabilization of the transformed system, the stability of the original system can be easily established. Numerical results show the effectiveness of the proposed control algorithm when compared to existing methods.

4.1 Control Problem

In the presence of suitable feedback strategy, a control law is designed such that as $t \to \infty$, $x \to x_{des}$ from any initial condition x_o. It is further supposed that $x_{des} = 0$ can be achieved by suitable transformation of the system.

4.2 The Proposed Control Algorithm

Step 1:

Consider the system (4.1) as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u
\end{aligned} \tag{4.2}$$

By choosing $u = \frac{1}{b_1(x)}(-f_1 + \theta_1\phi + x_3)$, here θ_1 is unknown and we will find it adaptively, $\theta_1 = \widehat{\theta}_1 + \widetilde{\theta}_1$, system (4.2) is written as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + \widehat{\theta}_1\phi + \widetilde{\theta}_1\phi \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \beta + \alpha x_3 + \alpha\widehat{\theta}_1\phi + \alpha\widetilde{\theta}_1\phi
\end{aligned} \tag{4.3}$$

In equation (4.3) $\alpha = \frac{b_1}{b_2}$ and $\beta = f_2 - \frac{b_1}{b_2}f_1$. Now with the help of below-mentioned steps, the system (4.3) can be transformed from $x \to z$ domain and then back from $z \to x$ by using inverse transformation.

Step 2:

Consider Eq. (4.3): $\dot{x}_1 = x_2$, consider x_2 state as a virtual input and α_1 as a stabilizing function, then the error variable $z_1 = x_2 - \alpha_1$, Eq (43a) becomes:

$$\dot{x_1} = z_1 + \alpha_1 \tag{4.4}$$

Choose a layapunov function $V_1 = \frac{1}{2}x_1^2$. For computing the stabilizing function:

$$\dot{V_1} = x_1\dot{x_1} = x_1(z_1 + \alpha_1) \tag{4.5}$$

Choose $\alpha_1 = -x_1$, then $\dot{V}$ becomes:

$$\dot{V_1} = -x_1^2 + x_1 z_1 \tag{4.6}$$

The term $x_1 z_1$ will be canceled in the next step. Eq (4.4), becomes:

$$\dot{x_1} = z_1 - x_1 \tag{4.7}$$

Take a derivative of z_1:

$$\begin{aligned}
\dot{z_1} &= \dot{x_2} - \dot{\alpha_1} \\
\dot{z_1} &= x_3 + \widehat{\theta}_1\phi + \widetilde{\theta}_1\phi + \dot{x_1} \\
\dot{z_1} &= x_3 + \widehat{\theta}_1\phi + \widetilde{\theta}_1\phi + z_1 - x_1
\end{aligned} \tag{4.8}$$

Consider x_3 state as a virtual input and α_2 as a stabilizing function, then the error variable $z_2 = x_3 - \alpha_2$, Eq (4.8) becomes:

$$\dot{z_1} = z_2 + \alpha_2 + \widehat{\theta}_1\phi + \widetilde{\theta}_1\phi + z_1 - x_1 \tag{4.9}$$

Consider the Lyapunov function $V_2 = V_1 + \frac{1}{2}z_1^2$ for computing the stabilizing function α_2. Then:

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + z_1\dot{z}_1 \\
\dot{V}_2 &= -x_1^2 + x_1 z_1 + z_1(z_2 + \alpha_2 + \widehat{\theta}_1\phi + \widetilde{\theta}_1\phi + z_1 - x_1) \\
\dot{V}_2 &= -x_1^2 + z_1(z_2 + \alpha_2 + \widehat{\theta}_1\phi + z_1) + z_1\widetilde{\theta}_1\phi
\end{aligned} \tag{4.10}$$

Choose $\alpha_2 = -2z_1 - \widehat{\theta}_1\phi$, then $\dot{V}_2 = -x_1^2 - z_1^2 + z_1 z_2 + z_1\widetilde{\theta}_1\phi$, The term $z_1 z_2$ will be canceled in the next step. Eq (4.9) becomes:

$$\dot{z}_1 = -x_1 - z_1 + z_2 + \widetilde{\theta}_1\phi \tag{4.11}$$

Take a derivative of z_2:

$$\begin{aligned}
\dot{z}_2 &= \dot{x}_3 - \dot{\alpha}_2 \\
\dot{z}_2 &= x_4 - (-2\dot{z}_1 - \dot{\widehat{\theta}}_1\phi - \widehat{\theta}_1\dot{\phi}) \\
\dot{z}_2 &= x_4 - 2x_1 - 2z_1 + 2z_2 + 2\widetilde{\theta}_1\phi + \dot{\widehat{\theta}}_1\phi + \dot{\phi}\widehat{\theta}_1
\end{aligned} \tag{4.12}$$

Consider x_4 state as a virtual input and α_3 as a stabilizing function, then the error variable $z_3 = x_4 - \alpha_3$, Eq (4.12) becomes:

$$\dot{z}_2 = z_3 + \alpha_3 - 2x_1 - 2z_1 + 2z_2 + 2\widetilde{\theta}_1\phi + \dot{\widehat{\theta}}_1\phi + \dot{\phi}\widehat{\theta}_1 \tag{4.13}$$

Consider the Lyapunov function $V_3 = V_2 + \frac{1}{2}z_2^2$ for computing the stabilizing function α_3. Then:

$$\begin{aligned}
\dot{V}_3 &= \dot{V}_2 + z_2\dot{z}_2 \\
\dot{V}_3 &= -x_1^2 - z_1^2 + z_1 z_2 + z_1\widetilde{\theta}_1\phi + z_2(z_3 + \alpha_3 - 2x_1 - 2z_1 + 2z_2 + 2\widetilde{\theta}_1\phi + \dot{\widehat{\theta}}_1\phi + \dot{\phi}\widehat{\theta}_1) \\
\dot{V}_3 &= -x_1^2 - z_1^2 + z_1\widetilde{\theta}\phi + z_2(z_3 + \alpha_3 - 2x_1 - z_1 + 2z_2 + 2\widetilde{\theta}_1\phi + \dot{\widehat{\theta}}_1\phi + \dot{\phi}\widehat{\theta}_1)
\end{aligned} \tag{4.14}$$

Choose $\alpha_3 = 2x_1 + z_1 - 3z_2 - \dot{\widehat{\theta}}_1\phi - \dot{\phi}\widehat{\theta}_1$, then $\dot{V}_3 = -x_1^2 - z_1^2 - z_2^2 + z_2 z_3 + z_1\widetilde{\theta}_1\phi + 2z_2\widetilde{\theta}_1\phi$, Eq (4.13) becomes:

$$\dot{z}_2 = -z_1 - z_2 + z_3 + 2\widetilde{\theta}_1\phi \tag{4.15}$$

Add and subtract $\widehat{\theta}_2\phi$ in α_3;

$$\alpha_3 = 2x_1 + z_1 - 3z_2 - \dot{\widehat{\theta}}_1\phi - \dot{\phi}\widehat{\theta}_1 + \widehat{\theta}_2\phi - \widehat{\theta}_2\phi \tag{4.16}$$

Let, $w = \dot{\widehat{\theta}}_1\phi + \dot{\phi}\widehat{\theta}_1 + \widehat{\theta}_2\phi$, then Eq (4.16) becomes:

$$\alpha_3 = 2x_1 + z_1 - 3z_2 - w + \widehat{\theta}_2\phi \tag{4.17}$$

Take a derivative of z_3;

$$\begin{aligned}
\dot{z}_3 &= \dot{x}_4 - \dot{\alpha}_3 \\
\dot{z}_3 &= \beta + \alpha x_3 + \alpha\widehat{\theta}_1\phi + \alpha\widetilde{\theta}_1\phi - (2\dot{x}_1 + \dot{z}_1 - 3\dot{z}_2 - \dot{w} + \dot{\widehat{\theta}}_2\phi) + \widehat{\theta}_2\dot{\phi} \\
\dot{z}_3 &= \beta + \alpha x_3 + \alpha\widehat{\theta}_1\phi + \alpha\widetilde{\theta}_1\phi + 5\widetilde{\theta}_1\phi + 3x_1 - 4z_1 - 4z_2 + 3z_3 + \dot{w} - \dot{\widehat{\theta}}_2\phi - \widehat{\theta}_2\dot{\phi}
\end{aligned} \tag{4.18}$$

Consider the layapunov function;

$$V_4 = V_3 + \frac{1}{2}z_3^2 + \frac{1}{2}\widetilde{\theta}_1^{\,2} + \frac{1}{2}\widetilde{\theta}_2^{\,2} \tag{4.19}$$

$$\dot{V}_4 = \dot{V}_3 + z_3\dot{z}_3 + \widetilde{\theta}_1\dot{\widetilde{\theta}}_1 + \widetilde{\theta}_2\dot{\widetilde{\theta}}_2 \tag{4.20}$$

Choose $\dot{w} = u_1 = -\beta - \alpha x_3 - \alpha\widehat{\theta}_1\phi - 3x_1 + 4z_1 + 3z_2 - 4z_3 + \dot{\widehat{\theta}}_2\phi + \widehat{\theta}_2\dot{\phi} - \widetilde{\theta}_2\phi$, $\dot{\widetilde{\theta}}_1 = -z_1\phi - 2z_2\phi - 5z_3\phi - \alpha z_3\phi - k_1\widetilde{\theta}_1$ and $\dot{\widetilde{\theta}}_2 = -k_2\widetilde{\theta}_2$, Eq (4.20) becomes:

$$\dot{V}_4 = -x_1^2 - z_1^2 - z_2^2 - z_3^2 - k_1\widetilde{\theta}_1^{\,2} - k_2\widetilde{\theta}_1^{\,2} \leq 0 \tag{4.21}$$

and;

$$\begin{aligned}
\dot{\widehat{\theta}}_1 &= -\dot{\widetilde{\theta}}_1 \\
\dot{\widehat{\theta}}_2 &= -\dot{\widetilde{\theta}}_2
\end{aligned} \tag{4.22}$$

Eq (4.18) becomes:

$$\dot{z}_3 = -z_2 - z_3 + \widetilde{\theta}_1(\alpha + 5) - \widetilde{\theta}_2\phi \tag{4.23}$$

Step 3:

So the transformed system can be written as:

$$
\begin{aligned}
\dot{x}_1 &= z_1 - x_1 \\
\dot{z}_1 &= -x_1 - z_1 + z_2 + \widetilde{\theta}_1\phi \\
\dot{z}_2 &= -z_1 - z_2 + z_3 + 2\widetilde{\theta}_1\phi \\
\dot{z}_3 &= -z_2 - z_3 + \widetilde{\theta}_1(\alpha + 5) - \widetilde{\theta}_2\phi
\end{aligned}
\tag{4.24}
$$

Define $z = [x_1 \ \ z_1 \ \ z_2 \ \ z_3]^T$ and $\widetilde{\theta} = [\theta_1 \ \ \theta_2]^T$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \phi & 0 \\ 2\phi & 0 \\ \alpha + 5 & -\phi \end{bmatrix} \begin{bmatrix} \widetilde{\theta}_1 \\ \widetilde{\theta}_2 \end{bmatrix}
$$

$$
\dot{z} = Mz + N\widetilde{\theta} \tag{4.25}
$$

where M is negative definite. Since the derivative of Lyapunov function given by (4.21) is negative, therefore, we conclude that the transformed system (4.24) is asymptotically stable, therefore, it implies that $x_1, \ z_1, \ z_2, \ z_3 \ \ \to 0$ and $\widetilde{\theta}_1, \ \widetilde{\theta}_2 \ \to 0$.

The states x_i in term of z_i can be written as;

$$
\begin{aligned}
x_1 &= x_1 \\
x_2 &= -x_1 + z_1 \\
x_3 &= -2z_1 + z_2 + \widehat{\theta}_1\phi \\
x_4 &= z_3 + 2x_1 + z_1 - 3z_2 - f(\widehat{\theta}_1, \widehat{\theta}_2, \phi)
\end{aligned}
\tag{4.26}
$$

4.3 Application to Underactuated Mechanical Systems

4.3.1 Overhead Crane System

The proposed control scheme is now used to stabilize an overhead crane system as considered in [63]. It is an underactuated mechanical system with 2 DOF. The dynamic of the system as given in [63] is:

$$\begin{aligned}
&(M + m\sin^2\theta_1)\ddot{x}_1 - m\sin\theta_1(l\dot{\theta}_1^2 + g\cos\theta_1) = u \\
&(M + m\sin^2\theta_1)l\ddot{\theta}_1 + l\dot{\theta}_1^2\sin\theta_1\cos\theta_1 + (M+m)g\sin\theta_1 = -u\cos\theta_1
\end{aligned} \tag{4.27}$$

define the state vector $x = [x_1 \;\; \dot{x}_1 \;\; \theta_1 \;\; \dot{\theta}_1]^T = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]$ the state space representation becomes;

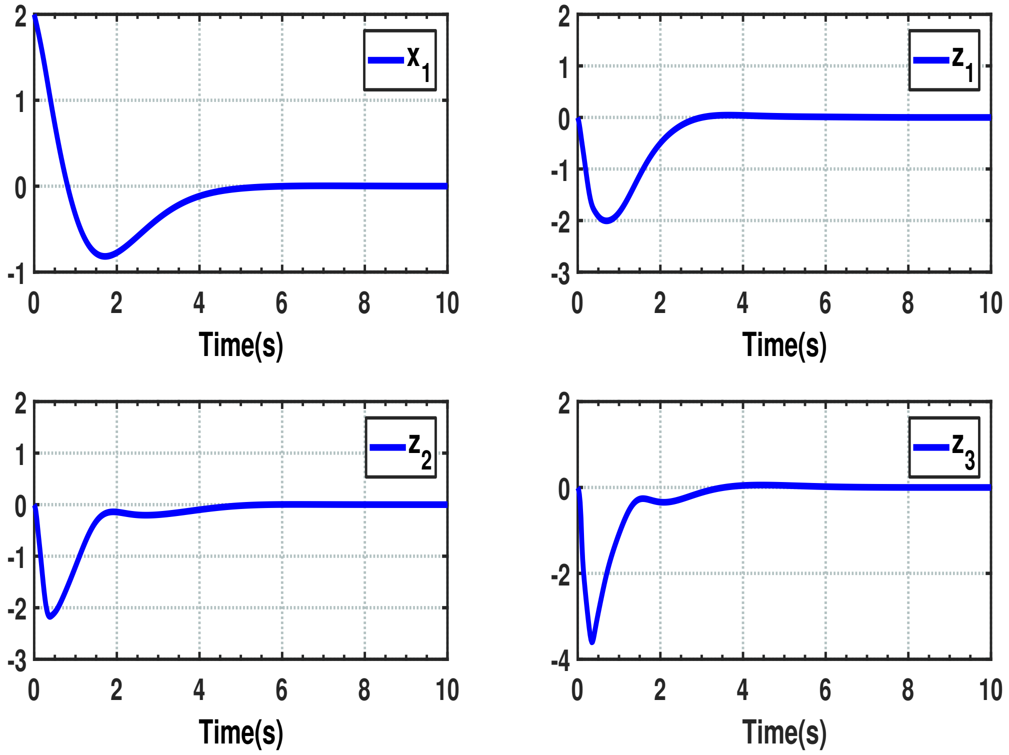
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u
\end{aligned} \tag{4.28}$$

where

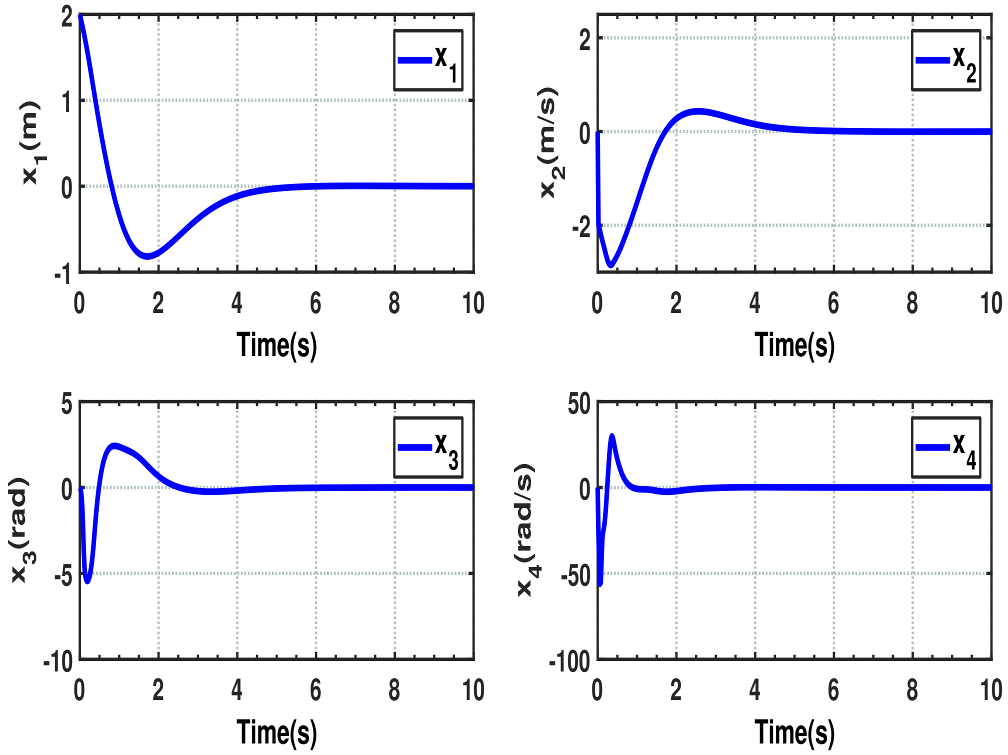
$$\begin{aligned}
f_1 &= \frac{mlx_4^2\sin x_3 - mg\sin x_3\cos x_3}{M + m\sin^2 x_3} \\
f_2 &= \frac{-mlx_4^2\sin x_3\cos x_3 + (M+m)g\sin x_3}{Ml + ml\sin^2 x_3} \\
b_1(x) &= \frac{1}{M + m\sin^2 x_3} \\
b_2(x) &= \frac{-\cos x_3}{Ml + ml\sin^2 x_3}
\end{aligned} \tag{4.29}$$

The actual values of the system parameters in the simulation are:

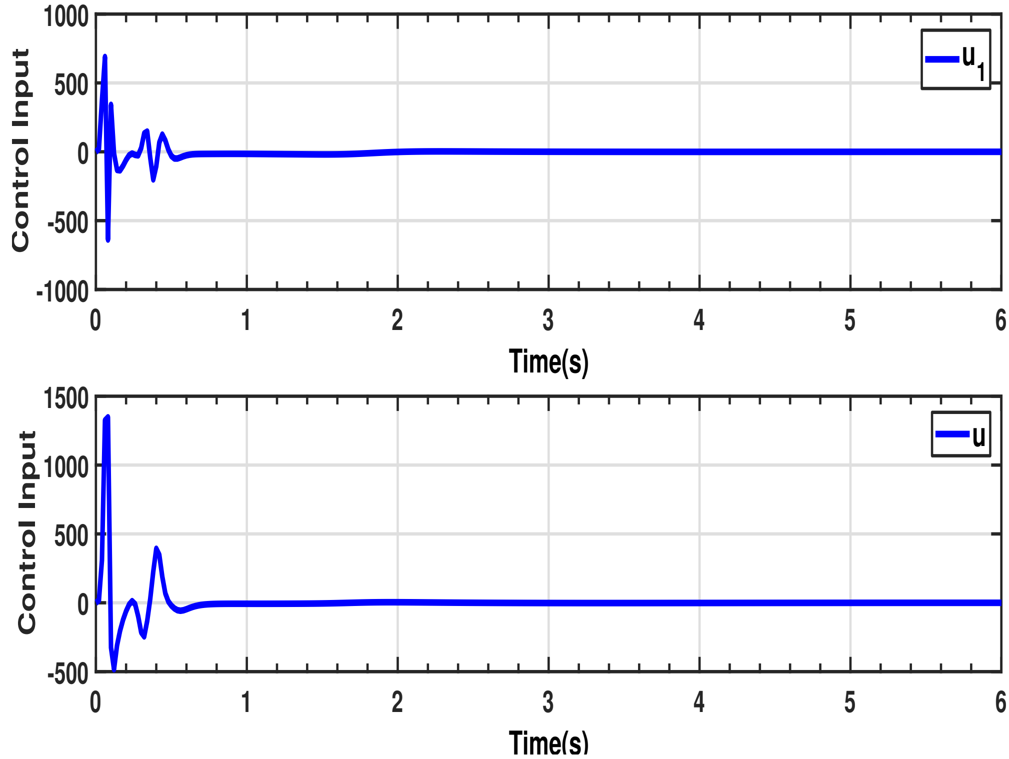
$$\begin{aligned}
m &= 1(kg) \\
l &= 3(m) \\
M &= 2(kg)
\end{aligned}$$


x1
Time(s)
z1
Time(s)
z2
Time(s)
z3
Time(s)

(a)


x1
x1(m)
Time(s)
x2
x2(m/s)
Time(s)
x3
x3(rad)
Time(s)
x4
x4(rad/s)
Time(s)

(b)



(c)

FIGURE 4.1: Closed loop response of overhead crane system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (2, 0, 0, 0)$, (c) Time history of control input u_1 and u

4.3.2 TORA System

The proposed control scheme is now used to stabilize an cart-pole system as considered in [63]. It is an underactuated mechanical system with 2DOF. The dynamic of the system as given in [63] is:

$$\begin{aligned} &(M+m)\ddot{x} + me\cos\theta\ddot{\theta} - me\dot{\theta}^2\sin\theta + kx = F \\ &(me^2 + I)\ddot{\theta} + me\cos\theta\ddot{\theta}\ddot{x} = N \end{aligned} \tag{4.30}$$

Define the normalized state $p = \sqrt{\frac{M+m}{I+me^2}}x$ normalized time, $\tau = \sqrt{\frac{k}{M+m}}t$, dimensionless control $u = \frac{M+m}{k(I+me^2)}$ and disturbance $w = \frac{1}{k}\sqrt{\frac{M+m}{I+me^2}}F$, then (4.30) becomes:

$$\begin{aligned} &\ddot{p} + p = \alpha(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) + w \\ &\ddot{\theta} = \alpha\dot{p}\cos\theta + u \end{aligned} \tag{4.31}$$

where the differentiations are with respect to the normalized time and α represents the coupling between the translational and rotational motions and can be defined as:

$$\alpha = \frac{me_1}{\sqrt{(I + me_1^2)(M + m)}} \tag{4.32}$$

define the state vector $x = [p \ \ \dot{p} \ \ \theta \ \ \dot{\theta}]^T = [x_1 \ \ x_2 \ \ x_3 \ \ x_4]$, the state space representation becomes;

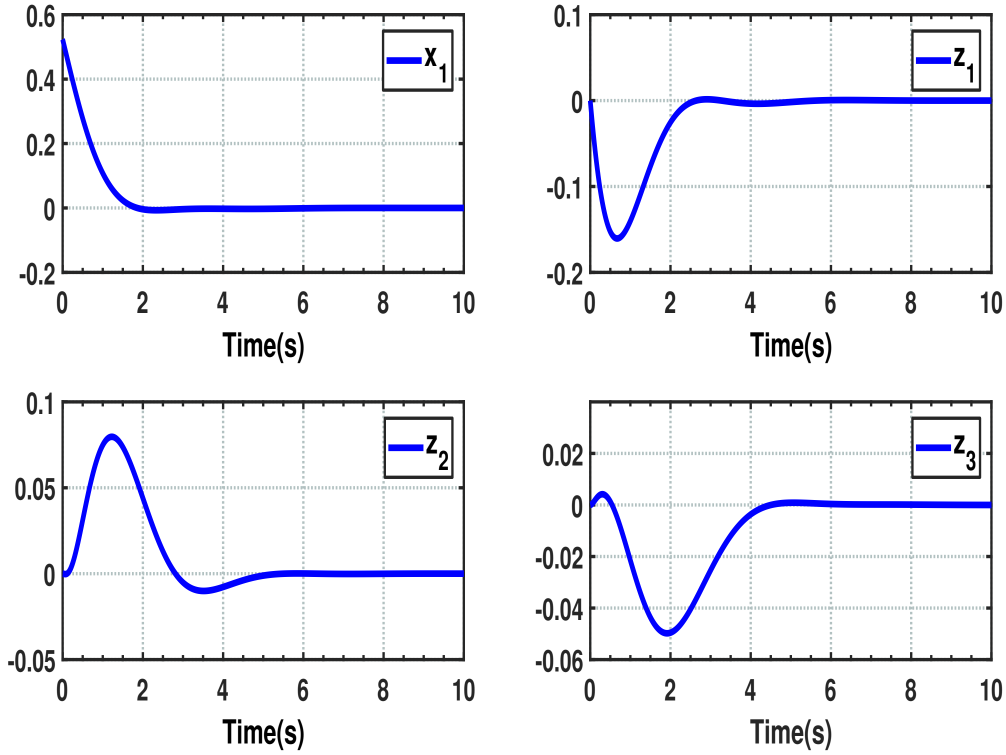
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u
\end{aligned} \tag{4.33}$$

where

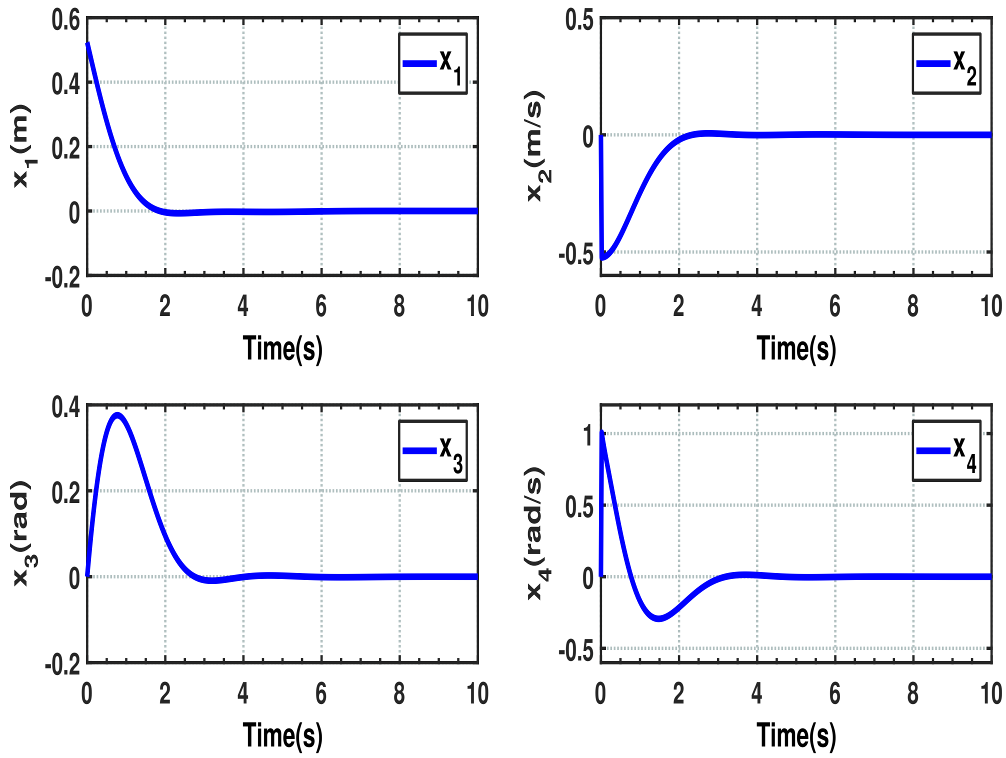
$$\begin{aligned}
f_1 &= \frac{-x_1 + \alpha x_4^2 \sin x_3 + w}{1 - \alpha^2 \cos^2 x_3} \\
f_2 &= \frac{\alpha x_1 \cos x_3 - \alpha^2 x_4^2 \sin x_3 - \alpha \cos x_3 w}{1 - \alpha^2 \cos^2 x_3} \\
b_1(x) &= \frac{-\alpha \cos x_3}{1 - \alpha^2 \cos^2 x_3} \\
b_2(x) &= \frac{1}{1 - \alpha^2 \cos^2 x_3}
\end{aligned} \tag{4.34}$$

The actual values of the system parameters in the simulation are:

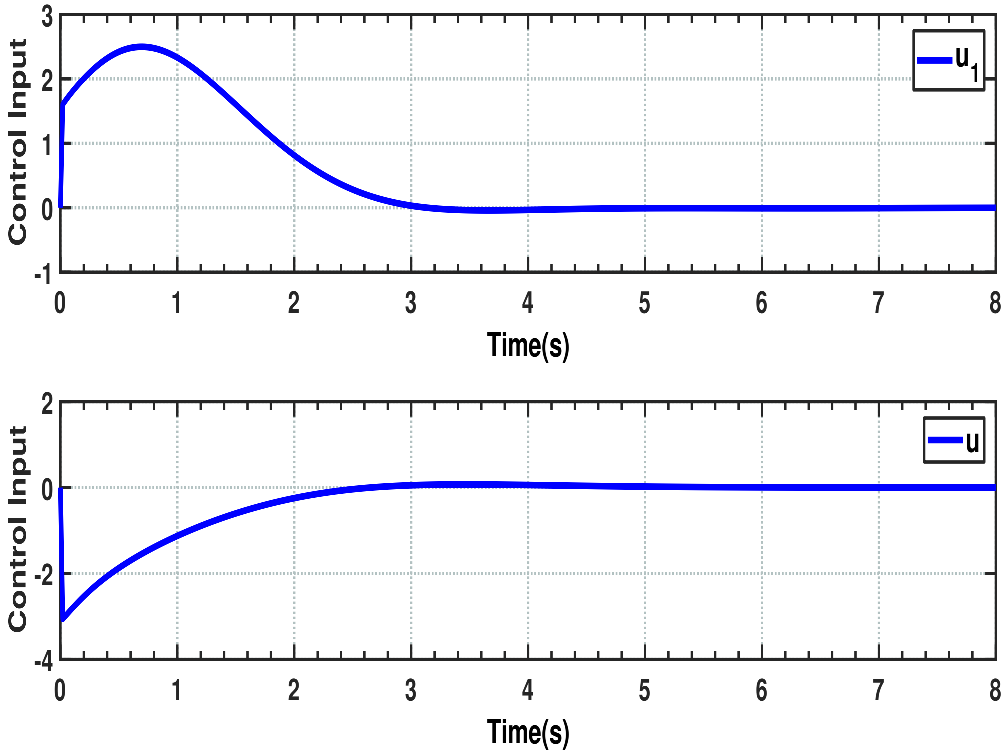
$$\begin{aligned}
m &= 0.5(kg) \\
l &= 0.3(m) \\
M &= 2(kg) \\
\alpha &= 2(m)
\end{aligned}$$


x1
z1
z2
z3
Time(s)
Time(s)
Time(s)
Time(s)

(a)


x1
x2
x3
x4
x1(m)
x2(m/s)
x3(rad)
x4(rad/s)
Time(s)
Time(s)
Time(s)
Time(s)

(b)



(c)

FIGURE 4.2: Closed loop response of TORA system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (\pi/6, 0, 0, 0)$, (c) Time history of control input u_1 and u

Chapter 5

Stabilization of Underactuated Systems: Sliding Mode Control

5.1 Introduction

This chapter presents a sliding mode control algorithm for the stabilization of underactuated mechanical systems. Two methods are proposed: first method, transforming the original system into a new system using suitable transformation. In this method the dimension of the system increased. The transformed system is then stabilized using first order sliding mode control.
Second method: The system is transformed into a particular structure containing a nominal part and some unknown terms, which are computed adaptively. The transformed system is then stabilized using adaptive integral sliding mode control.

5.2 Problem Statement

For a given desired point $x_{des} \in R^n$, a control input u is constructed in such a way that x_{des} is an attractive point $t \to \infty$ leads to $x \to x_{des} = 0$.

5.3 The Proposed Control Algorithms

5.3.1 First Order Sliding Mode Control

5.3.1.1 2DOF Systems

Step 1:

The model of 2DOF underactuated systems can be described as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u
\end{aligned} \tag{5.1}$$

Define the transformation:

$$\begin{aligned}
z_1 &= x_1 - \iint (f_1 + b_1 u - x_3)dt \\
z_2 &= x_2 - \int (f_1 + b_1 u - x_3)dt \\
z_3 &= x_3 \\
z_4 &= x_4
\end{aligned} \tag{5.2}$$

Using the transformation (5.2), the system (5.1) can be written in the following form:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= f_2 + b_2 u
\end{aligned} \tag{5.3}$$

Step 2:

Choose the sliding manifold $s_1 = z_1 + 3z_2 + 3z_3 + z_4$, the time derivative of sliding manifold becomes:

$$\begin{aligned}
\dot{s}_1 &= \dot{z}_1 + 3\dot{z}_2 + 3\dot{z}_3 + \dot{z}_4 \\
\dot{s}_1 &= z_2 + 3z_3 + 3z_4 + f_2 + b_2 u
\end{aligned} \tag{5.4}$$

Choose $u = \frac{1}{b_2}(-z_2 - 3z_3 - 3z_4 - f_2 - k_1 sign(s_1) - k_2 s_1)$, then $\dot{s}_1 = -k_1 sign(s_1) - k_2 s_1$, and lyapunov stability:

$$\begin{aligned} V &= \frac{1}{2}s_1^2 \\ \dot{V} &= -k_1|s_1| - k_2 s_1^2 \leq 0 \end{aligned} \tag{5.5}$$

Step 3:

Let $z_5 = \iint (f_1 + b_1 u - x_3)dt$ and $z_6 = \int (f_1 + b_1 u - x_3)dt$, then:

$$\begin{aligned} \dot{z}_5 &= z_6 \\ \dot{z}_6 &= f_1 + b_1 u - z_3 - v + v \\ \dot{z}_6 &= f_1 + b_1 u - z_3 + v - \widetilde{v} - \widehat{v} \end{aligned} \tag{5.6}$$

Choose the sliding manifold $s_2 = z_5 + z_6$, the time derivative of sliding manifold becomes:

$$\begin{aligned} \dot{s}_2 &= \dot{z}_5 + \dot{z}_6 \\ \dot{s}_2 &= z_6 + f_1 + b_1 u - z_3 + v - \widetilde{v} - \widehat{v} \end{aligned} \tag{5.7}$$

Choose $v = -z_6 - f_1 - b_1 u + z_3 - k_3 sign(s_2) + \widehat{v}$ then $\dot{s}_2 = -k_3 sign(s_2) - \widetilde{v}$, and lyapunov stability:

$$V_1 = \frac{1}{2}s_2^2 + \frac{1}{2}\widetilde{v}^2 \tag{5.8}$$

After some manipulation:

$$\begin{aligned} \dot{V} &= -k_3|s_2| - k_4\widetilde{v}^2 \\ \dot{\widetilde{v}} &= s_2 - k_4\widetilde{v} \\ \dot{\widehat{v}} &= -(s_2 - k_4\widetilde{v}) \end{aligned} \tag{5.9}$$

Step 4:

The overall system can be written as:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= f_2 + b_2 u \\
\dot{z}_5 &= z_6 \\
\dot{z}_6 &= f_1 + b_1 u - z_3 + v - \widetilde{v} - \widehat{v}
\end{aligned} \tag{5.10}$$

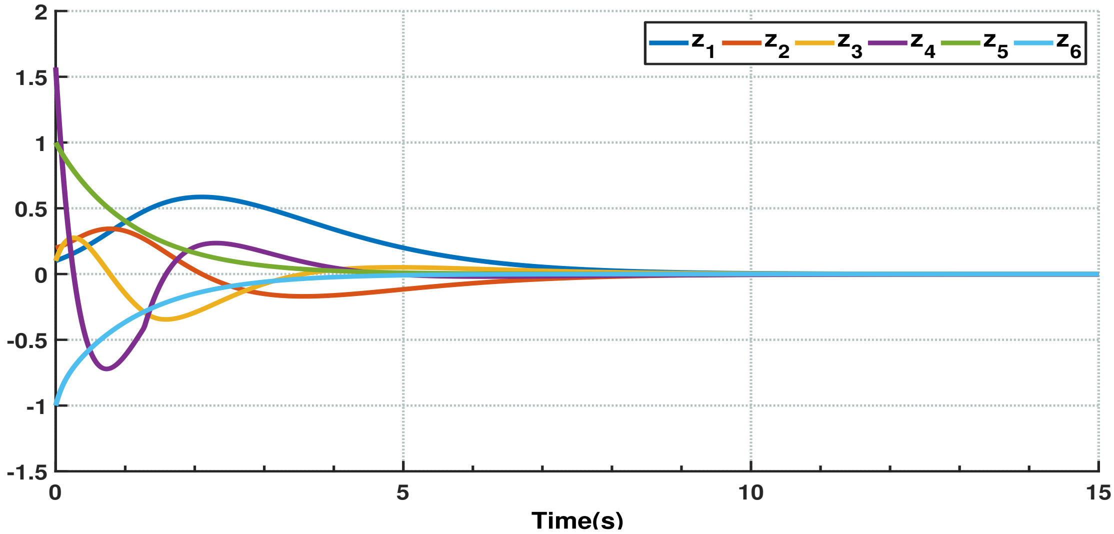
The states x_h $(h = 1, 2, 3, 4)$ in the term of z_i $(i = 1, 2...6)$ can be written as:

$$\begin{aligned}
x_1 &= z_1 + z_5 \\
x_2 &= z_2 + z_6 \\
x_3 &= z_3 \\
x_4 &= z_4
\end{aligned} \tag{5.11}$$

5.4 Application to Underactuated Mechanical Systems

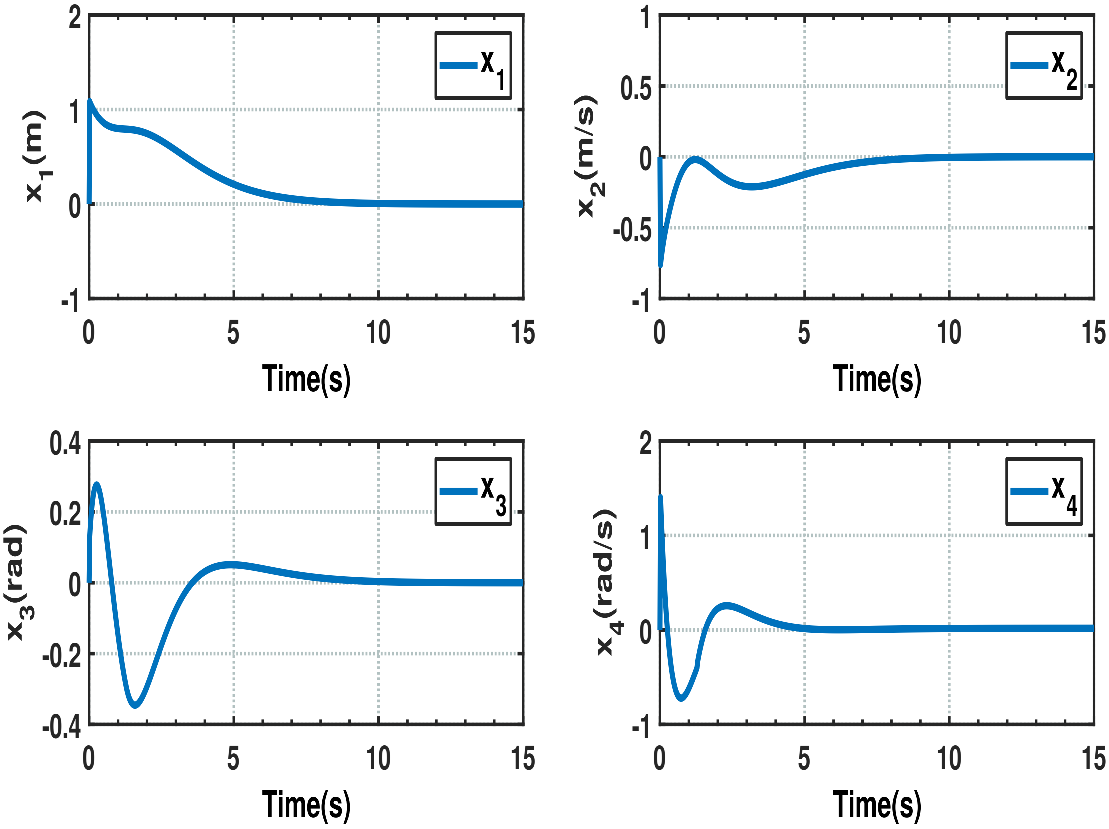
5.4.1 Cart-Pole System

The proposed control scheme is now used to stabilize an cart-pole system as considered in [63]. It is an underactuated mechanical system with 2DOF. The dynamic of the system discussed in (3.3.1).



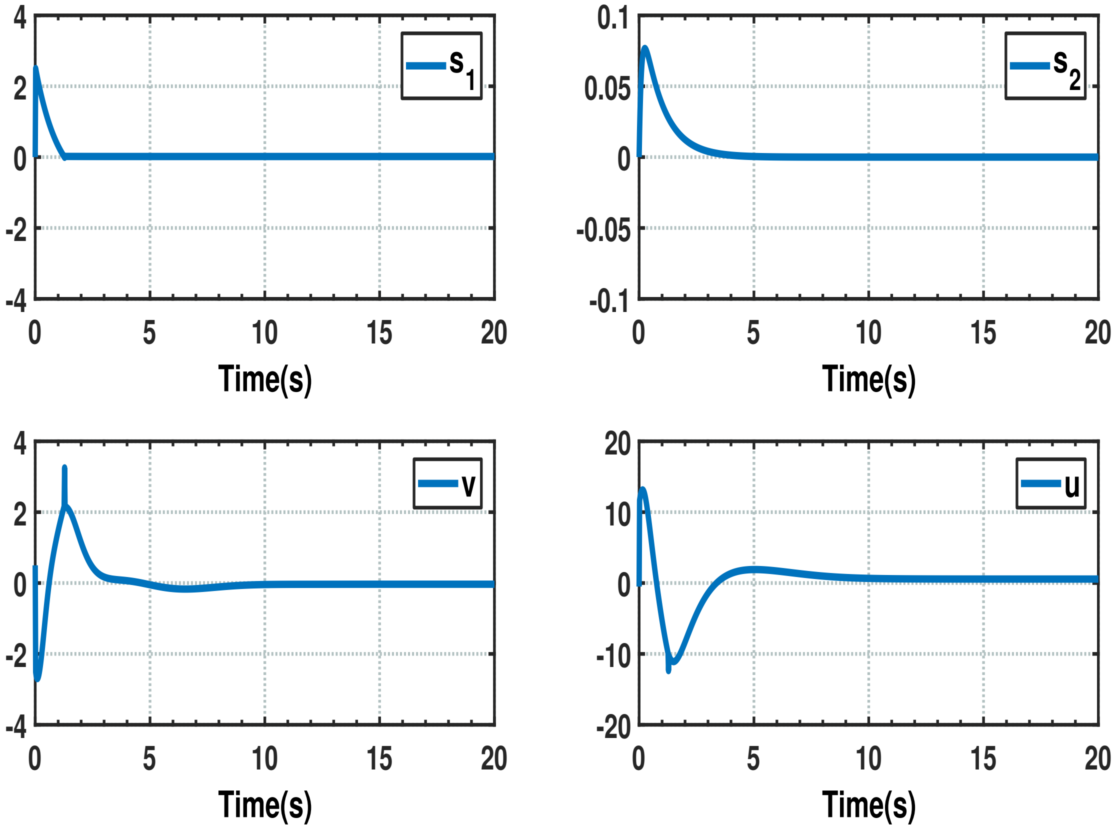
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
5
10
15
Time(s)
z_1
z_2
z_3
z_4
z_5
z_6

(a)



x_1(m)
x_1
Time(s)
x_2(m/s)
x_2
Time(s)
x_3(rad)
x_3
Time(s)
x_4(rad/s)
x_4
Time(s)

(b)



(c)

FIGURE 5.1: Closed loop response of cart-pole system corresponds to initial condition $(x_1(0), ..., x_4(0)) = (0.1, 0.2, 0.1, \pi/2)$, (c) Time history of control input u, v and sliding surfaces s_1, s_2

5.4.2 Adaptive Sliding Mode Control

5.4.2.1 2DOF Systems

Step 1:

The model of 2DOF underactuated systems can be described as:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u + d_1(x,t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u + d_3(x,t)
\end{aligned}
\tag{5.12}
$$

Eq (5.12) can be written as:

$$\begin{aligned}\ddot{x}_1 &= f_1 + b_1(x)u + d_1(x,t)\\ \ddot{x}_3 &= f_2 + b_2(x)u + d_3(x,t)\end{aligned} \tag{5.13}$$

we define $Q = [x_1\ x_3]^T$, $\dot{Q} = [\dot{x}_1\ \dot{x}_3]^T$, $\ddot{Q} = [\ddot{x}_1\ \ddot{x}_3]^T$
$F(x) = [f_1(x)\ f_2(x)]^T$, $B(x) = [b_1(x)\ b_2(x)]^T$, the system (5.13) can be written as:

$$\ddot{Q} = F(x) + B(x)u + d_i(x,t) \tag{5.14}$$

add and subtract $v = [0\ u_2]^T$ in (5.14) to get:

$$\ddot{Q} = F(x) + H(x)w - v + d_i(x,t) \tag{5.15}$$

where, H(x)= $\begin{bmatrix} b_1 & 0 \\ b_2 & 1 \end{bmatrix}$ such that $H^{-1}(x)$ exist and w = $\begin{bmatrix} u \\ u_2 \end{bmatrix}$.

$v = [0\ u_2]^T$ is unknown input vector and will computed it adaptively, $\hat{v}$ be the estimate of v and $\tilde{v}$ be the error of estimation, $\tilde{v} = v - \hat{v}$. Then system (5.15) can be written as:

$$\ddot{Q} = F(x) + H(x)w - \tilde{v} - \hat{v} + d_i(x,t) \tag{5.16}$$

Step 2:

Choose the sliding surface: S = $\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \end{bmatrix}$=$Q + \dot{Q}$

Then $\dot{S} = \dot{Q} + \ddot{Q} = \dot{Q} + F(x) + H(x)w - \tilde{v} - \hat{v} + d_i(x,t)$

Choose $w = -\{H^{-1}(x)\dot{Q} + F(x) - \hat{v} + KS + Ksign(S)\}$

so:

$w_1 = -\{H^{-1}(1,1)x_2 + H^{-1}(1,2)x_4 + f_1(x) - \hat{v_1} + K_1 s_1 + K_2 sign(s_1)\}$

$w_2 = -\{H^{-1}(2,1)x_2 + H^{-1}(2,2)x_4 + f_2(x) - \hat{v_2} + K_3 s_2 + K_4 sign(s_2)\}$

$K = diag\{k_1, k_2\}$

$$\dot{S} = -KS - Ksign(S) - \tilde{v} + d_i(x,t) \tag{5.17}$$

and:

$$\begin{aligned} \dot{s_1} &= -K_1 s_1 - K_2 sign(s_1) - \tilde{v_1} + d_1(x,t) \\ \dot{s_2} &= -K_3 s_2 - K_4 sign(s_2) - \tilde{v_2} + d_3(x,t) \end{aligned} \tag{5.18}$$

Step 3:

Choose a Lyapunov function $V = \frac{1}{2}S^T S + \frac{1}{2}\tilde{v}^T L^{-1}\tilde{v}$, where L is 2×2 positive definite matrix. Then

$\dot{V} = S^T\dot{S} + \tilde{v}^T L^{-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\}+\tilde{v}^T L^{-1}\dot{\tilde{v}}$

$\dot{V} = -KS^T S - K|S| + \tilde{v}^T\{L^{-1}\dot{\tilde{v}} - S\} + S^T d_i(x,t)$

By choosing

$\dot{\tilde{v}} = LS$

$\dot{\hat{v}} = -LS$

We get

$\dot{V} = S^T\dot{S} + \tilde{v}^T\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\}+\tilde{v}^{T-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^T\{^{-1}\dot{\tilde{v}} - S\}$

$$\dot{V} = -KS^T S - K|S| + S^T D_o \leq 0 \tag{5.19}$$

If $K > D_o$ then Eq.(5.19) confirms that $S \to 0$. Since S $= \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \end{bmatrix} \to 0$ and S is Hurwitz

5.4.2.2 3DOF Systems

Step 1:

The model of 3DOF underactuated systems can be described as:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u + d_1(x,t) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u + d_2(x,t) \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3 + b_3(x)u + d_3(x,t)
\end{aligned} \tag{5.20}$$

Eq (5.20) can be written as:

$$\begin{aligned}
\ddot{x}_1 &= f_1 + b_1(x)u + d_1(x,t) \\
\ddot{x}_3 &= f_2 + b_2(x)u + d_2(x,t) \\
\ddot{x}_5 &= f_3 + b_3(x)u + d_3(x,t)
\end{aligned} \tag{5.21}$$

we define $Q = [x_1\ x_3\ \ x_5]^T,\ \ \dot{Q} = [\dot{x}_1\ \dot{x}_3\ \ \dot{x}_5]^T,\ \ \ddot{Q} = [\ddot{x}_1\ \ddot{x}_3\ \ \ddot{x}_5]^T$ $F(x) = [f_1(x)\ f_2(x)\ \ f_3(x)]^T$, $B(x) = [b_1(x)\ b_2(x)\ \ b_3(x)]^T$, the system (5.21) can be written as:

$$\ddot{Q} = F(x) + B(x)u + d_i(x,t) \tag{5.22}$$

add and subtract $v = [0\ u_2\ \ u_3]^T$ in (5.22) to get:

$$\ddot{Q} = F(x) + H(x)w - v + d_i(x,t) \tag{5.23}$$

where, H(x)= $\begin{bmatrix} b_1 & 0 & 0 \\ b_2 & 1 & 0 \\ b_3 & 0 & 1 \end{bmatrix}$ such that $H^{-1}(x)$ exist and w = $\begin{bmatrix} u \\ u_2 \\ u_3 \end{bmatrix}$.

$v = [0\ u_2\ \ u_3]^T$ is unknown inputs vector and will computed it adaptively, $\hat{v}$ be the estimate of v and $\tilde{v}$ be the error of estimation, $\tilde{v} = v - \hat{v}$. Then system (5.23) can be written as:

$$\ddot{Q} = F(x) + H(x)w - \tilde{v} - \hat{v} + d_i(x,t) \tag{5.24}$$

Step 2:

Choose the sliding surface: S = $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_5 + x_6 \end{bmatrix} = Q + \dot{Q}$

Then $\dot{S} = \dot{Q} + \ddot{Q} = \dot{Q} + F(x) + H(x)w - \tilde{v} - \hat{v} + d_i(x,t)$

Choose $w = -\{H^{-1}(x)\dot{Q} + F(x) - \hat{v} + KS + Ksign(S)\}$

so:

$w_1 = -\{H^{-1}(1,1)x_2 + H^{-1}(1,2)x_4 + H^{-1}(1,3)x_6 + f_1(x) - \hat{v_1}\}$
$\quad -K_1 s_1 - K_2 sign(s_1)$

$w_2 = -\{H^{-1}(2,1)x_2 + H^{-1}(2,2)x_4 + H^{-1}(2,3)x_6 + f_2(x) - \hat{v_2}\}$
$\quad -K_3 s_2 - K_4 sign(s_2)$

$w_3 = -\{H^{-1}(3,1)x_2 + H^{-1}(3,2)x_4 + H^{-1}(3,3)x_6 + f_3(x) - \hat{v_3}\}$
$\quad -K_5 s_3 - K_6 sign(s_3)$

$K = diag\{k_1, k_2, k_3\}$

$$\dot{S} = -KS - Ksign(S) - \tilde{v} + d_i(x,t) \tag{5.25}$$

and:

$$\begin{aligned} \dot{s_1} &= -K_1 s_1 - K_2 sign(s_1) - \tilde{v_1} + d_1(x,t) \\ \dot{s_2} &= -K_3 s_2 - K_4 sign(s_2) - \tilde{v_2} + d_2(x,t) \\ \dot{s_3} &= -K_5 s_3 - K_6 sign(s_3) - \tilde{v_3} + d_3(x,t) \end{aligned} \tag{5.26}$$

Step 3:

Choose a Lyapunov function $V = \frac{1}{2}S^T S + \frac{1}{2}\tilde{v}^T L^{-1}\tilde{v}$, where L is 3×3 positive definite matrix. Then

$\dot{V} = S^T \dot{S} + \tilde{v}^T L^{-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^T L^{-1}\dot{\tilde{v}}$

$\dot{V} = -KS^T S - K|S| + \tilde{v}^T\{L^{-1}\dot{\tilde{v}} - S\} + S^T d_i(x,t)$

By choosing

$\dot{\tilde{v}} = LS$

$\dot{\hat{v}} = -LS$

We get

$\dot{V} = S^T\dot{S} + \tilde{v}^T\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^{T-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^T\{^{-1}\dot{\tilde{v}} - S\}$

$$\dot{V} = -KS^TS - K|S| + S^TD_o \leq 0 \tag{5.27}$$

If $K > D_o$ then Eq.(5.27) confirms that $S \to 0$. Since S = $\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ x_5 + x_6 \end{bmatrix}$ $\to 0$ and S is Hurwitz.

Similarly for nDOF:

$$\ddot{Q} = F(x) + H(x)w - \tilde{v} - \hat{v} + d_i(x,t) \tag{5.28}$$

where, H(x)= $\begin{bmatrix} h_1 \\ h_3 \\ h_5 \\ \vdots \\ h_{2n-1} \end{bmatrix} = \begin{bmatrix} b_1(x) & 0 & 0 & \dots & 0 \\ b_3(x) & 1 & 0 & \dots & 0 \\ b_5(x) & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \\ b_{2n-1}(x) & 0 & 0 & 0 & 1 \end{bmatrix}$ such that $H^{-1}(z)$ exist.

Choose the sliding surface: S = $\begin{bmatrix} s_1 \\ s_3 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ \vdots \\ x_{2n-1} + x_{2n} \end{bmatrix}$ $= Y + \dot{Y}$

Then $\dot{S} = \dot{Q} + \ddot{Q} = \dot{Q} + F(x) + H(x)w - \tilde{v} - \hat{v} + d_i(x,t)$

Choose $w = -\{H^{-1}(x)\dot{Q} + F(x) - \hat{v} + KS + Ksign(S)\}$

so:

$w_1 = -\{H^{-1}(1,1)x_2 + H^{-1}(1,2)x_4 + \ldots + H^{-1}(1,n)x_{2n} + f_1(x)$

$-\hat{v_1} + K_1 s_1 + K_2 sign(s_1)\}$

$w_2 = -\{H^{-1}(2,1)x_2 + H^{-1}(2,2)x_4 + \ldots + H^{-1}(2,n)x_2 n + f_2(x)$

$-\hat{v_2} + K_3 s_2 + K_4 sign(s_2)\}$

$\vdots$

$w_n = -\{H^{-1}(n,1)x_2 + H^{-1}(n,2)x_4 + \ldots + H^{-1}(n,n)x_2 n + f_n(x)$

$-\hat{v_n} + K_{2n-1} s_n + K_{2n} sign(s_n)\}$

$K = diag\{k_1, k_2, \ldots, k_n\}$

$$\dot{S} = -KS - Ksign(S) - \tilde{v} + d_i(x,t) \tag{5.29}$$

and:

$$\begin{aligned}
\dot{s_1} &= -K_1 s_1 - K_2 sign(s_1) - \tilde{v_1} + d_1(x,t) \\
\dot{s_2} &= -K_3 s_2 - K_4 sign(s_2) - \tilde{v_2} + d_2(x,t) \\
&\vdots \\
\dot{s_n} &= -K_{2n-1} s_n - K_{2n} sign(s_n) - \tilde{v_n} + d_n(x,t)
\end{aligned} \tag{5.30}$$

Choose a Lyapunov function $V = \frac{1}{2}S^T S + \frac{1}{2}\tilde{v}^T L^{-1}\tilde{v}$, where L is $n \times n$ positive definite matrix. Then

$\dot{V} = S^T \dot{S} + \tilde{v}^T L^{-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^T L^{-1}\dot{\tilde{v}}$

$\dot{V} = -KS^T S - K|S| + \tilde{v}^T\{L^{-1}\dot{\tilde{v}} - S\} + S^T d_i(x,t)$

By choosing

$\dot{\tilde{v}} = LS$

$\dot{\hat{v}} = -LS$

We get

$\dot{V} = S^T \dot{S} + \tilde{v}^T \dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^{T-1}\dot{\tilde{v}}$

$\dot{V} = S^T\{-KS - Ksign(S) - \tilde{v} + d_i(x,t)\} + \tilde{v}^T\{^{-1}\dot{\tilde{v}} - S\}$

$$\dot{V} = -KS^T S - K|S| + S^T D_o \leq 0 \tag{5.31}$$

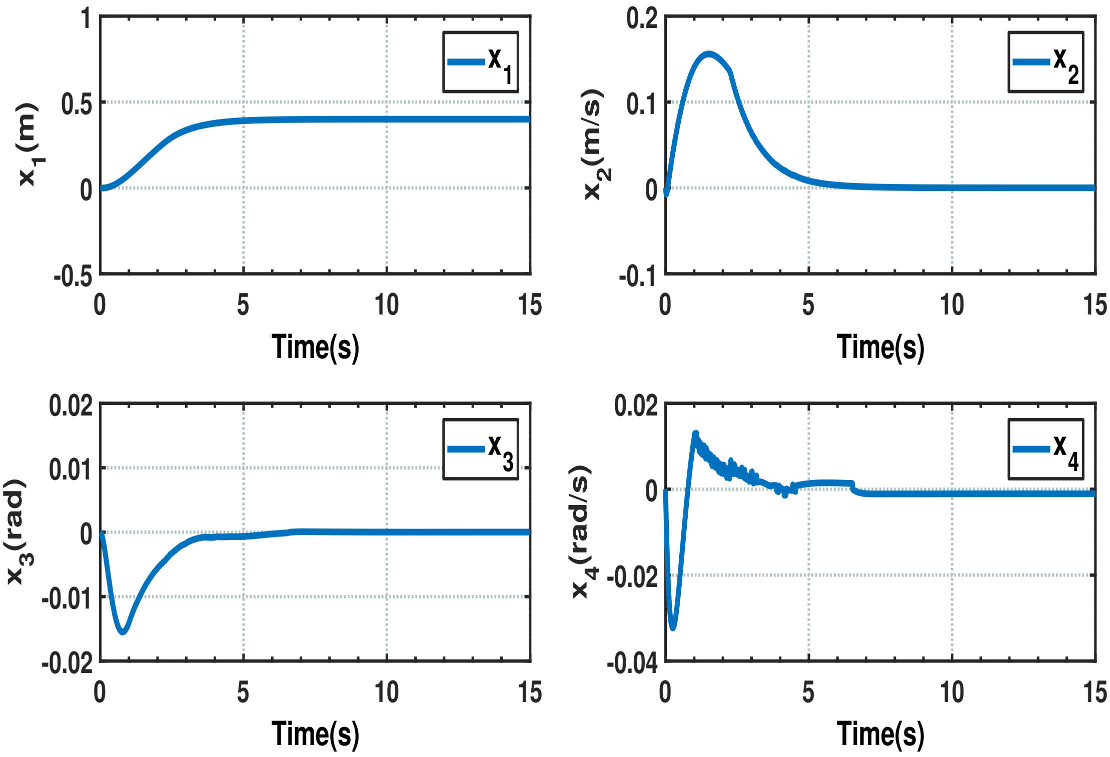
If $K > D_o$ then Eq.(5.31) confirms that $S \rightarrow 0$. Since S $= \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_4 \\ \vdots \\ x_{2n-1} + x_{2n} \end{bmatrix} \rightarrow 0$ and S is Hurwitz.

5.5 Application to Underactuated Mechanical Systems

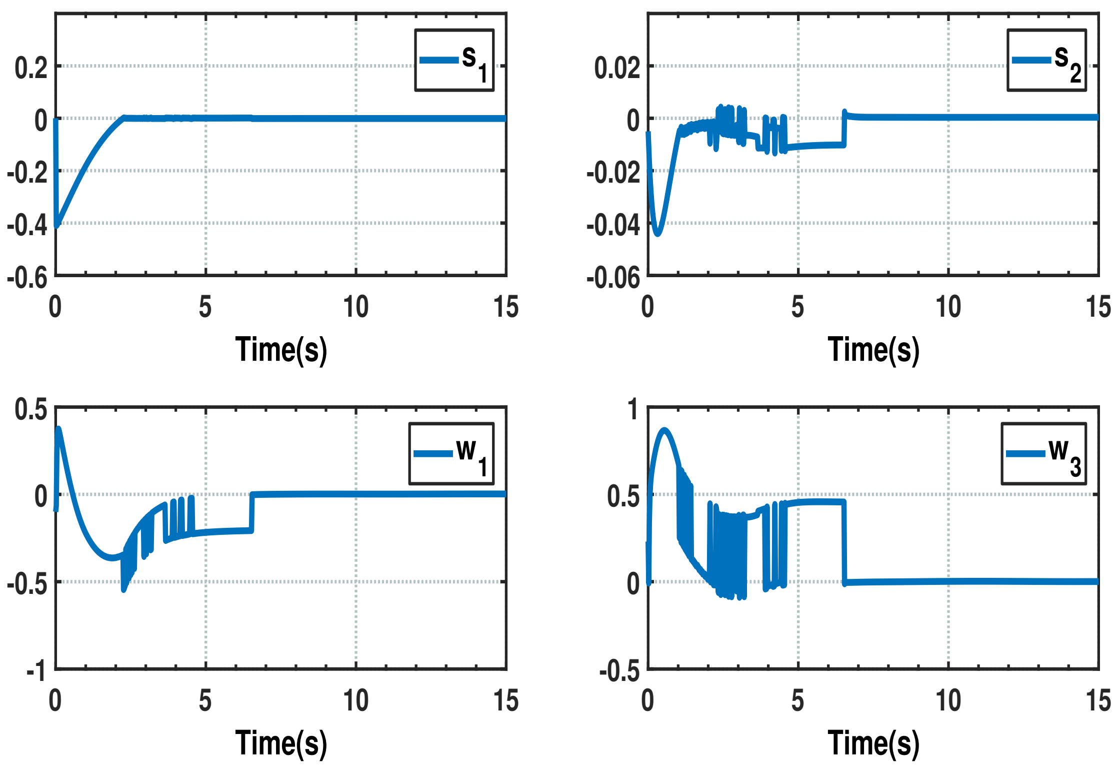
Case 1: Simulation results without external disturbances.

5.5.1 Overhead Crane System

The proposed control scheme is now used to stabilize an overhead crane system as considered in [63]. It is an underactuated mechanical system with 2DOF.



(a) System states



(b) Sliding surfaces and Control effort

FIGURE 5.2: Closed loop response of overhead crane system ,(a) Time response of system states corresponds to initial condition $(x_1(0), ..., x_4(0)) = (0, 0, 0, 0)$ (b) Time response of sliding surfaces s_1, s_2 and Time history of control inputs $w_1 = u$ and w_3

5.5.2 Double Inverted Pendulum System

The proposed control scheme is now used to stabilize an double inverted pendulum system as considered in [64]. It is an underactuated mechanical system with 3DOF. This system consist of two pendulums link together on a moving cart as is shown in Fig (5.3). The whole system consist of three subsystems, which has pendulum 1, pendulum 2 and cart. The dynamic of the system as given in [64].

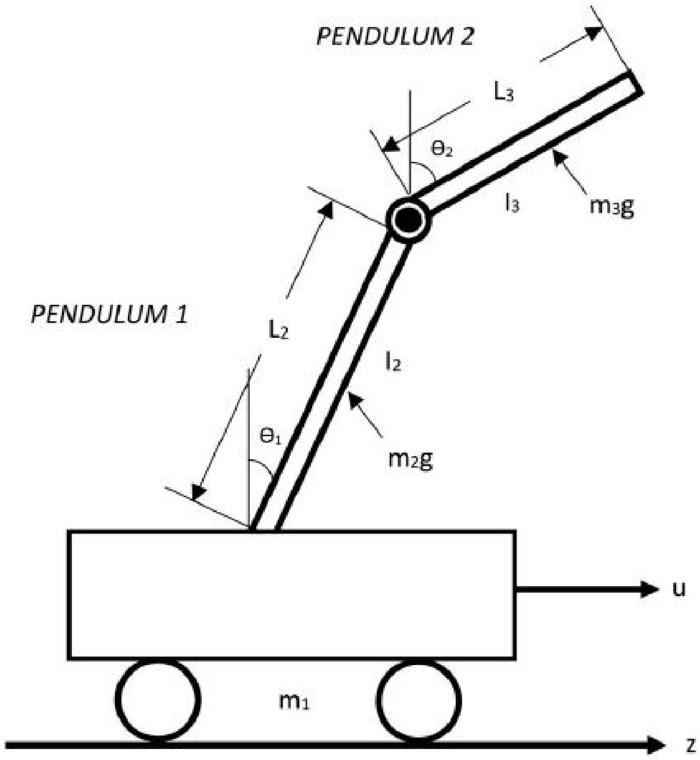


FIGURE 5.3: Double Inverted Pendulum System

From Fig (5.3), θ_1 is the angle of pendulum 1 and θ_2 is the angle of pendulum 2, $y = z$ is the cart position, u is the control force, here m_1, m_2, m_3 is the cart, the pendulum 1 and the pendulum 2 masses respectively and L_2, L_3 is the length of the lower and upper pendulums respectively and l_1, l_2 is the respective lengths from their center of masses . Let I_2, I_3 is the respective inertia of pendulum 1 and pendulum 2 respectively.

define the coordinates of center of masses: $y_1 = \begin{bmatrix} y \\ 0 \end{bmatrix}$, $y_2 = \begin{bmatrix} y + l_2 \sin\theta_1 \\ l_2 \cos\theta_1 \end{bmatrix}$, $y_3 =$

$\begin{bmatrix} y + L_2 \sin\theta_1 + l_3 \sin(\theta_1 + \theta_2) \\ L_2 \cos\theta_1 l_3 \cos(\theta_1 + \theta_2) \end{bmatrix}$ and $\theta = [y \ \ \theta_1 \ \ \theta_2]^T$, the total kinetic energy can be written as: $K = \frac{1}{2}\dot{\theta}^T\dot{\theta}$, where M is the 3×3 symmetric matrix.

$M_{11} = m_1 + m_2 + m_3,$

$M_{22}(\theta_2) = I_2 + I_3 + m_2 l_2^2 + m_3 L_2^2 + m_3 l_3^2 + 2m_3 L_2 l_3 \cos\theta_2$

$M_{33} = I_3 + m_3 l_3^2$

$M_{12}(\theta_1, \theta_2) = M_{21} = (m_2 l_2 + m_3 L_2)\cos\theta_1 + m_3 l_3 \cos(\theta_1 + \theta_2)$

$M_{13}(\theta_1, \theta_2) = M_{31} = m_3 l_3 \cos(\theta_1 + \theta_2)$

$M_{23}(\theta_2) = m_{32} = (M_3 l_3^2 + m_3 l_3 L_2)\cos\theta_2 + I_3$

Therefore:

$K = \frac{1}{2}M_{11}\dot{z}^2 + \frac{1}{2}M_{22}(\theta_2)\dot{\theta_1}^2 + \frac{1}{2}M_{33}\dot{\theta_2}^2 + M_{12}(\theta_1, \theta_2)\dot{x}\dot{\theta}_1 + M_{13}(\theta_1, \theta_2)\dot{x}\dot{\theta}_2$
$+ M_{23}(\theta_2)\dot{\theta}_1\dot{\theta}_2$

The total potential energy is:

$V = (m_2 l_2 + m_3 L_2) g \cos\theta_1 + m_3 l_3 g \cos(\theta_1 + \theta_2)$

Then Lagrangian is:

$L = K - V = K = \frac{1}{2}M_{11}\dot{z}^2 + \frac{1}{2}M_{22}(\theta_2)\dot{\theta_1}^2 + \frac{1}{2}M_{33}\dot{\theta_2}^2 + M_{12}(\theta_1, \theta_2)\dot{x}\dot{\theta}_1$
$+ M_{13}(\theta_1, \theta_2)\dot{x}\dot{\theta}_2 + M_{23}(\theta_2)\dot{\theta}_1\dot{\theta}_2 - (m_2 l_2 + m_3 L_2) g \cos\theta_1 + m_3 l_3 g \cos(\theta_1 + \theta_2)$

Then Euler-Lagrange equation becomes: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = F$ gives that:

$$M_{11}\ddot{z} + M_{12}\ddot{\theta}_1 + M_{13}\ddot{\theta}_2 = u \tag{5.32}$$

$$\begin{aligned} &M_{12}\ddot{z} + M_{22}\ddot{\theta}_1 + M_{23}\ddot{\theta}_2 + \{k\sin\theta_1 + m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{z}\dot{\theta}_1 \\ &+ \{m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{z}\dot{\theta}_2 + kg\sin\theta_1 + m_3 l_3 g \sin(\theta_1 + \theta_2) = 0 \\ &where \quad k = (m_2 l_2 + m_3 L_2) \end{aligned} \tag{5.33}$$

$$\begin{aligned} &M_{13}\ddot{z} + M_{23}\ddot{\theta}_1 + M_{33}\ddot{\theta}_2 + \{2m_3 l_3 L_2 \sin\theta_2\}\dot{\theta}_1^2 + \{m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{z}\dot{\theta}_1 \\ &+ \{m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{z}\dot{\theta}_2 + \{m_3 l_3 \sin(\theta_1 + \theta_2)\}\dot{\theta}_1\dot{\theta}_2 + m_3 l_3 g \sin(\theta_1 + \theta_2) = 0 \end{aligned} \tag{5.34}$$

Equation (5.32)-(5.34) can be expressed as:

$$\begin{aligned}
&M_{11}\ddot{z} + M_{12}\ddot{\theta}_1 + M_{13}\ddot{\theta}_2 = u \\
&M_{12}\ddot{z} + M_{22}\ddot{\theta}_1 + M_{23}\ddot{\theta}_2 + d_2 + h2 = 0 \\
&m_{13}\ddot{z} + M_{23}\ddot{\theta}_1 + M_{33}\ddot{\theta}_2 + d_3 + h3 = 0
\end{aligned} \tag{5.35}$$

Where

$d_2 = \{k\sin\theta_1 + m_3 l_3 \sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}1 + \{m_3 l_3 \sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_2$

$h_2 = kg\sin\theta_1 + m_3 l_3 g\sin(\theta_1+\theta_2$, where, $k = (m_2 l_2 + m_3 L_2)$

$d_3 = \{2m_3 l_3 L_2 \sin\theta_2\}\dot{\theta}_1^2 + \{m_3 l_3 \sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_1 + \{m_3 l_3 \sin(\theta_1+\theta_2)\}\dot{z}\dot{\theta}_2$
$+ \{m_3 l_3 \sin(\theta_1+\theta_2)\}\dot{\theta}_1\dot{\theta}_2$

$h_3 = m_3 l_3 g \sin(\theta_1+\theta_2)$

Solving the equation (5.35) we have

$$\begin{aligned}
\ddot{z} &= -\frac{m_{12}}{m_{11}}\ddot{\theta}_1 - \frac{M_{13}}{M_{11}}\ddot{\theta}_2 + \frac{1}{M_{11}}u \\
\bar{M}_{11}\ddot{\theta}_1 + \bar{M}_{12}\ddot{\theta}_2 &= -\frac{M_{12}}{M_{11}}u - d_2 - h_2 \\
\bar{M}_{12}\ddot{\theta}_1 + \bar{M}_{22}\ddot{\theta}_2 &= -\frac{M_{13}}{M_{11}}u - d_3 - h_3
\end{aligned} \tag{5.36}$$

where $\bar{M}_{11} = (M_{22} - \frac{{M_{12}}^2}{M_{11}})$, $\bar{M}_{12} = (M_{23} - \frac{M_{12}M_{13}}{M_{11}})$ and $\bar{M}_{22} = (M_{33} - \frac{{M_{13}}^2}{M_{11}})$

Equation (5.36) can be written further as:

$$\begin{aligned}
\ddot{z} &= -\frac{M_{12}}{M_{11}}\ddot{\theta}_1 - \frac{M_{13}}{M_{11}}\ddot{\theta}_2 + \frac{1}{M_{11}}u \\
\bar{\bar{M}}_{11}\ddot{\theta}_1 &= u(\frac{M_{12}\bar{M}_{22} - \bar{M}_{12}M_{13}}{M_{11}}) + \bar{d}_2 + \bar{h}_2 - \bar{d}_3 - \bar{h}_3 \\
\bar{\bar{M}}_{11}\ddot{\theta}_2 &= u(\frac{\bar{M}_{11}M_{13} - \bar{M}_{12}M_{12}}{M_{11}}) - \bar{\bar{d}}_2 - \bar{\bar{h}}_2 + \bar{\bar{d}}_3 + \bar{\bar{h}}_3
\end{aligned} \tag{5.37}$$

where

$\bar{d}_2 = \bar{M}_{22}d_2$, $\bar{h}_2 = \bar{M}_{22}h_2$, $\bar{d}_3 = \bar{M}_{12}d_3$ and $\bar{h}_3 = \bar{M}_{12}h_3$

$\bar{\bar{d}}_2 = \bar{M}_{11}d_2$, $\bar{\bar{h}}_2 = \bar{M}_{12}h_2$, $\bar{\bar{d}}_3 = \bar{M}_{11}d_3$, $\bar{\bar{d}}_3 = \bar{M}_{11}d_3$, $\bar{\bar{h}}_3 = \bar{M}_{11}h_3$ and $\bar{\bar{M}}_{11} = \bar{M}_{12}^2 - \bar{M}_{22}\bar{M}_{11}$

From (5.37) we have the following expressions:

$$
\begin{aligned}
\ddot{y} &= \frac{M_{12}}{M_{11}\bar{\bar{M}}_{11}}\{-\bar{d}_2 - \bar{h}_2 + \bar{d}_3 + \bar{h}_3\} + \frac{M_{13}}{M_{11}\bar{\bar{M}}_{11}}\{\bar{\bar{d}}_2 + \bar{\bar{h}}_2 - \bar{\bar{d}}_3 - \bar{\bar{h}}_3\} \\
&+ \frac{1}{M_{11}}\{1 + M_{12}\bar{M}_{22} - \bar{M}_{12}M_{13} + \bar{M}_{11}M_{13} - \bar{M}_{12}M_{12}\}u \\
\ddot{\theta}_1 &= \frac{1}{\bar{\bar{M}}_{11}}\{\bar{d}_2 + \bar{h}_2 - \bar{d}_3 - \bar{h}_3\} + u(\frac{M_{12}\bar{M}_{22} - \bar{M}_{12}M_{13}}{M_{11}\bar{\bar{M}}_{11}}) \\
\ddot{\theta}_2 &= \frac{1}{\bar{\bar{M}}_{11}}\{-\bar{\bar{d}}_2 - \bar{\bar{h}}_2 + \bar{\bar{d}}_3 + \bar{\bar{h}}_3\} + u(\frac{\bar{M}_{11}M_{13} - \bar{M}_{12}M_{12}}{M_{11}\bar{\bar{M}}_{11}})
\end{aligned}
\tag{5.38}
$$

we define the state matrix: $x_1 = y,\ x_2 = \dot{y},\ x_3 = \theta,\ x_4 = \dot{\theta}_1,\ x_5 = \theta_2\ and\ x_6 = \dot{\theta}_2$ $x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6]^T$. so;

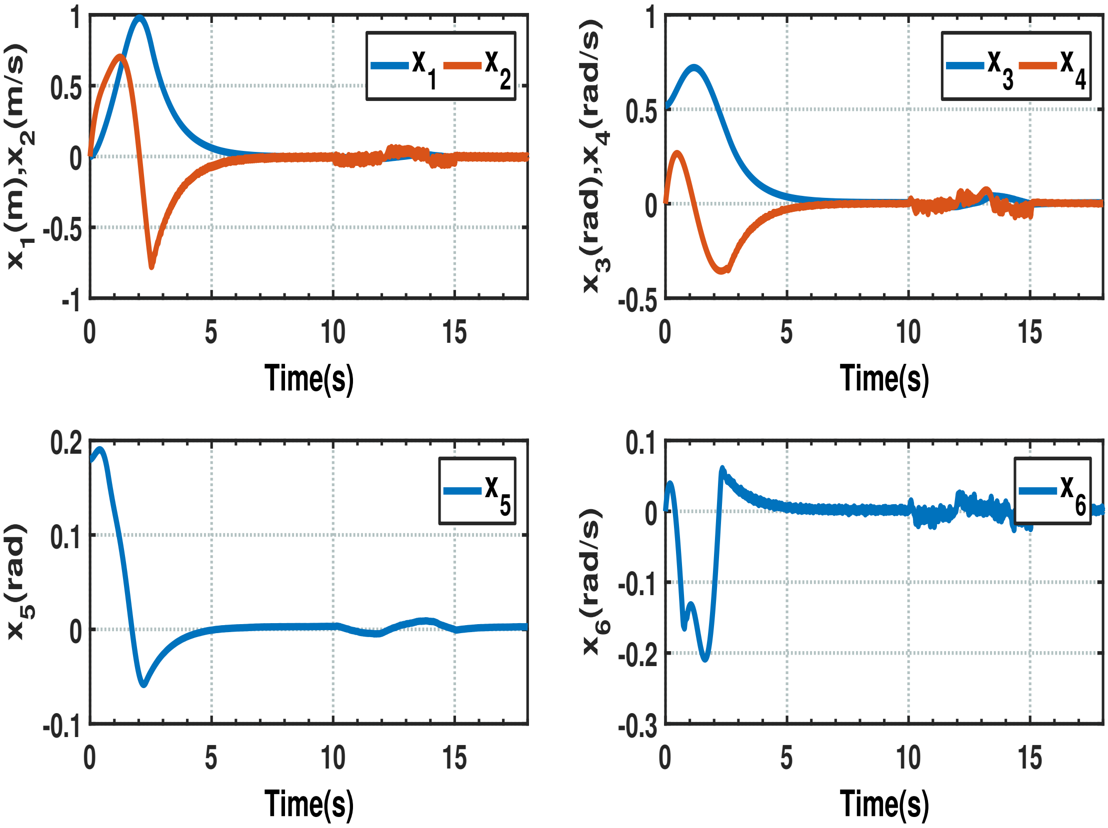
$$
\begin{aligned}
f_1 &= \frac{M_{12}}{M_{11}\bar{\bar{M}}_{11}}\{-\bar{d}_2 - \bar{h}_2 + \bar{d}_3 + \bar{h}_3\} + \frac{M_{13}}{M_{11}\bar{\bar{M}}_{11}}\{\bar{\bar{d}}_2 + \bar{\bar{h}}_2 - \bar{\bar{d}}_3 - \bar{\bar{h}}_3\} \\
b_1(x) &= \frac{1}{M_{11}}\{1 + M_{12}\bar{M}_{22} - \bar{M}_{12}M_{13} + \bar{M}_{11}M_{13} - \bar{M}_{12}M_{12}\} \\
f_2 &= \frac{1}{\bar{\bar{M}}_{11}}\{\bar{d}_2 + \bar{h}_2 - \bar{d}_3 - \bar{h}_3\} \\
b_2(x) &= (\frac{M_{12}\bar{M}_{22} - \bar{M}_{12}M_{13}}{M_{11}\bar{\bar{M}}_{11}}) \\
f_3 &= \frac{1}{\bar{\bar{M}}_{11}}\{-\bar{\bar{d}}_2 - \bar{\bar{h}}_2 + \bar{\bar{d}}_3 + \bar{\bar{h}}_3\} \\
b_3(x) &= (\frac{\bar{M}_{11}M_{13} - \bar{M}_{12}M_{12}}{M_{11}\bar{\bar{M}}_{11}})
\end{aligned}
\tag{5.39}
$$

The state space representation can be written as:

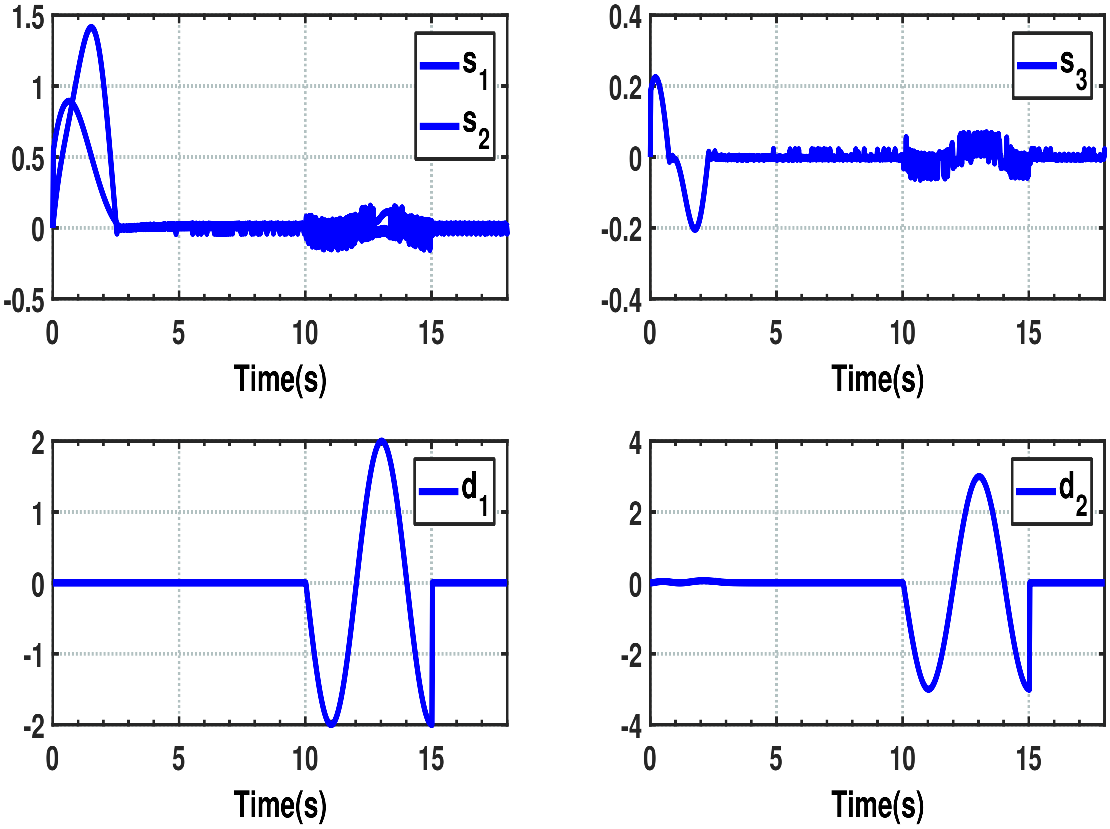
$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_1 + b_1(x)u \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= f_2 + b_2(x)u \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= f_3 + b_3(x)u
\end{aligned}
\tag{5.40}
$$

The physical parameters of the Double Inverted Pendulum as:

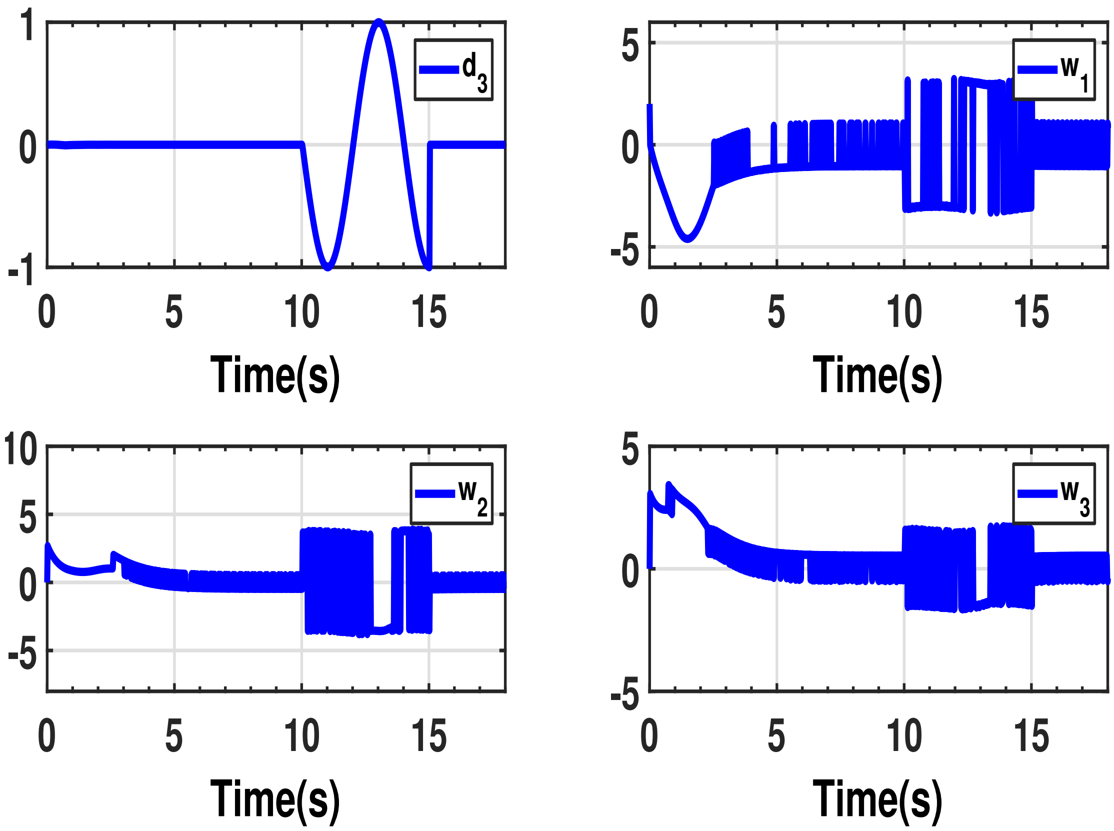
$m_1 = m_2 = m_3 = 1,\ l_2 = l_3 = 0.75,\ I_2 = \frac{4}{3}m_2l_2^2, I_3 = \frac{4}{3}m_3l_3^2, g = 9.8$



(a) System states



(b) Sliding surfaces and disturbances



(c) Control effort and disturbances

FIGURE 5.4: Closed loop response of Double Inverted Pendulum system ,(a) Time response of system states corresponds to initial condition $(x_1(0), ..., x_6(0)) = (-0.5, 1, 0.2, 0.4, -0.8, 0.2)$ (b) Applied disturbances $d_1 = 2sin(0.5\pi t) + 0.1x_2$, $d_2 = 3sin(0.5\pi t) + sin(x_3)x_4^2$ and $d_3 = 3sin(0.4\pi t) + sin(x_5)x_5x_6$ from $t = 10(s)$ to $t = 15(s)$ (c) Time response of sliding surfaces s_1, s_2 , s_3 and Time history of control inputs $w_1 = u$, w_2 and w_3

Chapter 6

Conclusion and Future Work

In this research work comprehensive robust control techniques proposed for UMS on the basis of backstepping, feedback linearization, sliding mode control. the proposed techniques provides a control problem for UMS.

1. Standard First Order SMC for UMSs (FOSMC).
2. Adaptive SMC for UMSs (ASMC).
3. Adaptive Backstepping for UMSs (AB).
4. Input/Output Feedback linearization Control based on lyapunov theory (IOFL-LT) and based on integral sliding mode control (IOFL-ISMC) for UMSs.

The above control techniques applied to the following underactuated mechanical systems;

1. Cart Pole system.
2. Overhead Crane system.

3. TORA system.

4. Double Inverted Pendulum system.

6.1 Performance Analysis

The overall performance analysis is summarized in the form of Table 6.1, based on different features in the simulation results. Having analyzed, it was decided that ASMC carries substantial marks in case of robustness and applicable to nDOF underactuated mechanical systems. However, in case of fast convergence, the IOFL-LT and IOFL-ISMC can be preferred whereas IOFL-LT suffers from robustness issues. The named AB exhibits high control effort. In case of FOSMC dur to transformation the dimension of system increases. According to the attributes presented in Table 6.1, it can be claimed that ASMC proves itself to be an appealing control protocol for the class of underactuated systems.

Table 6.1: Comparative analysis IOFL-LT, IOFL-ISMC, AB, FOSMC and ISMC.

Attributes	**IOFL-LT**	**IOFL-ISMC**	**AB**	**FOSMC**	**ISMC**
Settling Time	2.5-4.5 sec	1.8-1.9 sec	3-5 sec	5-10 sec	4-6 sec Increase when DOF increase
Overshoot	Medium	Low	High	Medium	Lowest
Control Effort	Low	Medium	High	Medium	Lowest
Sliding surface convergence	No sliding surface	sliding surface converge to zero	No sliding surface	sliding surface converge to zero	sliding surface converge to zero
Robustness	No	Yes	Yes	Yes	Yes
Application	2DOF	2DOF	2DOF	2DOF	nDOF
Computational Complexity	Low	Medium	High	Highest	High

6.2 Conclusion

In the past two decades the interest increasing in area of underactuated mechanical systems. These systems have many diverse applications in the field of aerospace, mechatronics, robotics, industry etc. This thesis presents a stabilization of UMSs.The proposed methodologies is based on first order SMC, adaptive SMC, adaptive backstepping, feedback linearization. In first order SMC system is transformed, the dimensions of the system increases. In adaptive SMC system is transformed through input transformation, contains a nominal part and unknown term, unknown term is adaptively computed. In adaptive backstepping we use the backstepping approach plus some unknown terms adaptively computed. In feedback linearization control problem is computed through feedback linearization plus layapunov function. The proposed techniques is applied to UMS systems with 2 DOF. The adaptive SMC technique is applicable to nDOF systems.

6.3 Future Research Directions

Based on such work, certain directions are suggested for future research.

1. Extension of the proposed techniques to the higher order UMSs.
2. Application of proposed algorithms to other UMSs.
3. Apply sliding mode observation techniques to UMSs.
4. Practical implementation of the proposed algorithms.

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