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Some Applications of Fixed Point Results for $(\alpha - \theta)$ Meir-Keeler Contractive Mappings in Bipolar b-Metric Spaces

by

Tehreem Rafique

A thesis submitted in partial fulfillment for the
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Dedicated to my ***Family*** *and* ***Teachers***

CERTIFICATE OF APPROVAL

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Abstract

In this thesis some results are proved by using contravariant, generalized and covariant $(\alpha - \theta)$ Meir-Keeler contracion, within the framework of bipolar b-metric space. Our findings extend and generalize the results with several known results emerging as special cases. To illustrate the proposed approach, various examples are provided. Additionally, as a practical application we establish the existence and uniqueness for fredholm integral equation.

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Abbreviations

BCP	Banach Contraction Principle
BbMS	bipolar b-metric space
BMS	bipolar metric space
bMS	b-metric space
CF	Continuous functions
Cbs	Cauchy bisequence
cont	contravariant
cov	covariant
FP	Fixed point
M-K	Meir-Keeler
MS	Metric space

Symbols

$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
∞	Infinity
$\forall$	for all
$\Rightarrow$	Implies
$\rightrightarrows$	Covariant mapping
$\rightleftarrows$	Contravariant mapping

Chapter 1

Introduction

1.1 Background

Mathematics is very important for scientific knowledge, which is why it is often called the ”mother of all sciences.” It is applicable in almost every part of life. Mathematics has many branches, each with its own importance, and one of these is functional analysis. It provides tools for understanding functions and spaces that can have infinitely many dimensions, connecting theoretical mathematics with practical applications. It also draws from areas like approximation theory, calculus of variations, and the study of differential and integral equations. Consequently, it plays a key role in advancing knowledge and technology. Initially, functional analysis was used to solve differential equations, but over time it has been applied to many other problems, including nonlinear ones. In nonlinear analysis, metric fixed point (FP) theory serves as one of the key tools. Metric space (MS) is important in mathematics because they help us study distances, spatial relationships, and limits. They extend the idea of Euclidean space, enabling the study of more general and abstract spaces. The concept of a MS was introduced by Maurice Frechet [1].

This powerful framework is essential for proving the existence and uniqueness of solutions to differential and integral equations, supporting various areas of mathematics and their applications in many fields.

In the 1880s, Poincaré [2] used the FP method to study differential equations in celestial mechanics. A FP theorem states that, under certain conditions, a function $\varphi : \mathfrak{M} \to \mathfrak{M}$ has at least one point r where $\varphi(r) = r$. This point r, which stays unchanged by the function, is called a FP. In 1912, Brouwer [3] proved a famous result in algebraic topology called Brouwer's FP theorem. This theorem says that any continuous function that maps a compact convex set to itself in Euclidean space will always have at least one FP. This is a very useful result in areas like game theory, where it helps find equilibrium points, and in economics, where it is used to show that solutions to some optimization problems exist. Other notable FP theorems includes Schauder FP theorem [4] and Kirk's FP theorem, which broaden the concepts of Brouwer to apply in more general contexts. The Schauder FP Theorem is a generalization of the Brouwer Fixed-Point Theorem. It applies to continuous functions that map a convex set in a Banach space to itself and ensures that a FP exists under these condition. The collection of FP theorems keeps expanding, with new discoveries and uses appearing regularly.

In 1922, Stefan Banach [5] presented a crucial result known as the Banach Contraction Principle (BCP). It states that a function $\xi : \mathfrak{M} \to \mathfrak{M}$ on a complete MS $(\mathfrak{M}, \rho)$ gives a unique FP, meaning $\xi(\mu) = \mu$, if it satisfies the contraction condition

$$\rho(\xi(\mu_1), \xi(\mu_2)) \leq k.d(\mu_1, \mu_2) \quad \forall \quad \mu_1, \mu_2 \in \mathfrak{M},$$

where $k \in (0, 1)$. This principle is viewed as one of the foundational results in FP theory. It assures the existence and uniqueness of FP. It also offers a way to find FP of mapping.

The BCP can be used to prove the Picard-Lindelöf theorem [6], which helps show that solutions to certain differential equations exist and are unique. In FP theory, researchers have explored two main ways to extend the BCP and find more uses for it. One way is by changing the space, and the other is by adjusting or relaxing the contraction condition. A key contribution in this area came from Berinde and Pacurar [7] in 2008, who introduced a new group of generalized contractions, including types like usual contractions, Kannan mappings[8], and Chatterjea mapping[9]. Kannan mapping state that a fuction $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$ on a MS $(\mathfrak{M}, \rho)$ is called Kannan if there exist a constant $\alpha \in [0, \frac{1}{2})$ such that

$$\rho(\mathsf{S}\mu, \mathsf{S}\nu) \leq \alpha[\rho(\mu, \mathsf{S}\nu) + \rho(\nu, \mathsf{S}\nu)]$$

for all $\mu, \nu \in \mathfrak{M}$. Later on, he proved that if $(\mathfrak{M}, \rho)$ is complete MS then S has a unique FP in $\mathfrak{M}$. Chatterjea mapping states that a function $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$, where $\mathfrak{M}$ is a MS with a distance function ρ, is refferd to as a Chatterjea-type contraction if there exist a constant α in the interval $[0, 1)$ such that for all $\mu, \nu \in \mathfrak{M}$, the following inequality hold:

$$\rho(\mathsf{S}\mu, \mathsf{S}\nu) \leq \alpha[\rho(\mu, \mathsf{S}\nu) + \rho(\nu, \mathsf{S}\nu)]$$

Bakhtin [10] introduced the concept of bMS, which Czerwik [11] later formally defined to extend the BCP. It is an important extension of classical MS theory, offered a wider framework for solving problems in areas like mathematical analysis, geometry, and real-world applications. Recently, E.Ameer [12] introduce a new type of contraction, called the generalized $(\alpha - \theta)$ Geraghty contraction, for multivalued mappings. Then prove the FP theorems for these mappings in an α-complete bMS. In 2012, Samet et al. [13] introduced α-admissible mappings and used to prove FP theorems. In 2014, A.Mukheimer [14] establish the idea of $(\alpha - \theta - \phi)$ contractive self mapping in ordered partial bMS.

T.Abdeljawad [15] proposed generalized Meir-Keeler (M-K) α-contractive functions and pairs, and derive FP and common FP theorems for them. Additionally, he presented generalized M-K $(\alpha - f)$ contractive maps that commute with f and study their coincidence and common FP theorems. Then, J.H.Asl [16] presented the notion of $(\alpha_* - \theta)$ contractive multifunctions which states that the function $\mathsf{S} : \mathfrak{M} \to 2^{\mathfrak{M}}, \theta \in \Psi$ and $\alpha : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ on a MS $(\mathfrak{M}, \rho)$ then S is said to be a $(\alpha_* - \theta)$ contractive multifunctions when it satifies

$$\alpha_*(\mathsf{S}\mu, \mathsf{S}\nu)H(\mathsf{S}\mu, \mathsf{S}\nu) \leq \theta(\rho(\mu, \nu))$$

for all $\mu, \nu \in \mathfrak{M}$, where H is the Hausdorff generalized metric and the set $2^{\mathfrak{M}}$ contains all non-empty subsets of $\mathfrak{M}$. In 2014, Popescu [17] proposed α-orbital admissible mappings to obtain FP theorems.

The idea of bipolar metric space (BMS) was inroduced by A.Mutlu and U.Gurda [18] in 2016. A BMS is an extended version of a MS. While a metric space measures distances between points within the same set, a BMS measured of distances between points from different sets. This made it useful for more complex situations where multiple sets are involved. Since then researchers have explored various contractive conditions to establish FP theorems in BMSs. In 2020, A Mutlu [19] introduced the idea of $(\alpha - \theta)$ contractive cov and cont mapping in BMS, that offered a framework to study distances between different objects. He then prove the existance and uniqueness of FP for these mappings in complete BMS. PP.Murthy [20] in 2022 establish a new FP theorem in BMS by using the Boyd-Wong-type contraction. Later on, in 2023 M Kumar [21] introduce $(\alpha - \theta)$ M-K contractive mappings by defining Ψ function that is monotonically decreasing and assumed to be continuous. He also used the α-admissible and α-orbital admissible mapping to establish FP theorems in the environment of BMS.The M-K contraction is an extension of the BCP with more flexible conditions. Recently, researchers have combined $(\alpha - \theta)$ conditions with M-K contractions to further generalize the concept. These advancements have created exciting new possibilities for research in this area. It helps to solve problems in broader spaces where traditional FP methods cannot be applied.

In our research study, we propose a new concept of cont $(\alpha - \theta)$ M-K contractive mappings by introducing α-orbital admissible mappings and cov M-K contractions in the environment of bipolar b-metric space (BbMS). Several FP theorems for these mappings are established and derive related corollaries. To support our findings, an illustrative example is provided, and we conclude by solving an integral equation using our main results.

The thesis is further divided into four chapters, which are organized in the following manner:

Chapter 2 provides the fundamental definitions and concepts, including MS, bMS, BCP, BMS, BbMS. Some examples related to these concepts are also given.

Chapter 3 the notion of α-admissible and α-orbital admissible mapping in the framework of BMS has been introduced. Several FP results using cont$(\alpha - \theta)$ M-K and cov($\alpha - \theta$) M-k contractive mappings has been established. An example is provided to support our findings.

Chapter 4 The main idea of this chapter is to introduced ($\alpha - \theta$) M-K contraction mapping in BbMS. Some results are proved using these contractions. The chapter includes the corollaries and an illustrative example that demonstrates the efficency and validity of the obtained results. For applicability purpose, the solution of fredholm integral equation is presented by using this idea.

Chapter 5 provides the conclusion of the thesis.

Chapter 2

Basic Definitions

This chapter is about some basic definitions and results that will be used throughout the thesis. The first section covers some basics of MS and examples. The second section covers the basic concept of bMS with some fundamental results and examples. The third section deals with the FP and related concept. In fourth section BMS is introduced and some examples are given to elaborate the idea. In fifth section the idea of BbMS is introduced with an example.

2.1 Metric Space

In this section, the definition of MS with suitable examples is presented and also the idea of convergence, Cauchyness, and completeness has been discussed in the frame work of MS.

Definition 2.1.1. (**Metric Space**)
"A MS is a pair $(\mathfrak{M}, \rho)$, where $\mathfrak{M}$ is a set and ρ is a metric on $\mathfrak{M}$ (or distance function on $\mathfrak{M}$), that is, a function defined on $\mathfrak{M} \times \mathfrak{M}$ such that for all $\mu, \nu, r \in \mathfrak{M}$ we have:

(**M1**) ρ is real-valued, finite and non-negative.

(**M2**) $\rho(\mu, \nu) = 0$ if and only if $\mu = \nu$.

(M3) $\rho(\mu,\nu) = \rho(\nu,\mu)$ (Symmetry).

(M4) : $\rho(\mu,\nu) \leq \rho(\mu,r) + \rho(r,\nu)$ (Triangular inequality)."[6]

Example 2.1.1. Let $\mathfrak{M} = \mathbb{R}$, define $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\rho(\mu,\nu) = |\mu - \nu|$$

then, the function ρ satisfies the properties of a metric on $\mathbb{R}$ and ρ is called the usual metric.

Example 2.1.2. Let $\mathfrak{M} = \mathbb{R}^2$; define $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\rho(\mu,\nu) = \sqrt{(\mu_1 - \nu_1)^2 + (\mu_2 - \nu_2)^2}$$

then, ρ is a metric on $\mathbb{R}^2$ and $(\mathbb{R}^2, \rho)$ is called Eucledian plane.

Example 2.1.3. Let $\mathfrak{M} = \mathsf{C}[\mathsf{a},\mathsf{b}]$ be the set of all real-valued CF on a closed interval [a,b]. Choosing the metric defined by

$$\rho(\mathsf{f},\mathsf{g}) = \max_{s\in[a,b]} |\mathsf{f}(\mathsf{s}) - \mathsf{g}(\mathsf{s})|, \qquad \forall\, \mathsf{f},\mathsf{g} \in \mathrm{C[a,b]}$$

then, ρ is a metric on $\mathsf{C}[\mathsf{a},\mathsf{b}]$.

Definition 2.1.2. (Convergent sequence)

"A sequence (μ_{n}) in a MS $\mathfrak{M} = (\mathfrak{M}, \rho)$ is said to converge or to be convergent if there is a $\mu \in \mathfrak{M}$ such that

$$\lim_{\mathsf{n}\to\infty} \rho(\mu_{\mathsf{n}}, \mu) = 0,$$

μ is called the limit of (μ_{n}) and we write

$$\lim_{\mathsf{n}\to\infty} \mu_{\mathsf{n}} = \mu$$

or, simply,

$$\mu_{\mathsf{n}} \to \mu.$$

We say that (μ_{n}) converges to μ has the limit μ. If (μ_{n}) is not convergent, it is said to be divergent."[6]

Example 2.1.4. (i) Let $\mathfrak{M} = \mathbb{R}$ and ρ be a usual metric, on $\mathbb{R}$. Then the sequence $(\frac{1}{\mathsf{n}})$ converges to 0 in $\mathfrak{M}$ as $\mathsf{n} \to \infty$.

(ii) Let $\mathfrak{M} = (0, 1)$ be an open interval on $\mathbb{R}$ with the usual metric $\rho(\mu, \nu) = |\mu - \nu|$, then the sequence $(\frac{1}{\mathsf{n}})$ is not convergent since $0 \notin \mathfrak{M}$.

Definition 2.1.3. **(Cauchy sequence)**
"A sequence (μ_{n}) in a MS $\mathfrak{M} = (\mathfrak{M}, \rho)$ is said to be a Cauchy (or fundamental) if for every $\epsilon > 0$ there is an $\mathsf{N} = \mathsf{N}(\epsilon)$ such that

$$\rho(\mu_{\mathsf{m}}, \mu_{\mathsf{n}}) < \epsilon$$

for all $\mathsf{m}, \mathsf{n} > \mathsf{N}$.
The space $\mathfrak{M}$ is said to be complete if every Cauchy sequence in $\mathfrak{M}$ converges (that is, has a limit which is an element of $\mathfrak{M}$)."[6]

Example 2.1.5. Consider the sequence (μ_{n}) in usual MS $(\mathbb{R}, \rho)$ and $(\mu_{\mathsf{n}}) = (\frac{1}{\mathsf{n}})$. This sequence is Cauchy, because for any $\epsilon > 0$, we can choose $\mathsf{N} = \frac{1}{\epsilon}$, and then for all $\mathsf{n}, \mathsf{m} \geq \mathsf{N}$ we have $|\mu_{\mathsf{n}} - \mu_{\mathsf{m}}| = |\frac{1}{\mathsf{n}} - \frac{1}{\mathsf{m}}| < \frac{1}{\epsilon}$.

2.2 b-Metric Space

The idea of bMS was initiated from the works of Boubaki [22] and Bakhtin [10]. The concept of bMS was first introduced by Czewik [11] as a generalization of MS.

Definition 2.2.1. "Let $\mathfrak{M}$ be a non-empty set and let $\mathsf{b} \geq 1$ be a given real number. A function $\rho : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ is called a b-metric. If it satisfies the following properties for each $\mu_1, \mu_2, \mu_3 \in \mathfrak{M}$.

(B1) $\rho(\mu_1, \mu_2) \geq 0$ and $\rho(\mu_1, \mu_2) = 0$ if and only if $\mu_1 = \mu_2$;

(B2) $\rho(\mu_1, \mu_2) = \rho(\mu_2, \mu_1)$;

$(\mathsf{B3}):\ \rho(\mu_1,\mu_3)\leq \mathsf{b}[\rho(\mu_1,\mu_2)+\rho(\mu_2,\mu_3)].$

The pair $(\mathfrak{M},\rho)$ is called bMS." [23]

Example 2.2.1. The set of real numbers $\mathbb{R}$ is a bMS with metric defined as:

$$\rho_{\mathsf{b}}(\mu,\nu)=(\mu-\nu)^2 \quad \forall\ \mu,\nu\in\mathbb{R} \text{ with } \mathsf{b}=2.$$

Example 2.2.2. Let $\mathfrak{M}=\ell_p(\mathbb{R})$ with $0<p<1$, where $\ell_p(\mathbb{R})=\{(\mu_{\mathsf{n}})\subseteq\mathbb{R}:\sum_{n=1}^{\infty}|\mu_{\mathsf{n}}|^p<\infty\}$. Define $\rho:\mathfrak{M}\times\mathfrak{M}\to\mathbb{R}^+$ as:

$$\rho(\mu,\nu)=(\sum_{n=1}^{\infty}|\mu_{\mathsf{n}}-\nu_{\mathsf{n}}|^p)^{\frac{1}{p}}$$

where $\mu=(\mu_{\mathsf{n}})$, $\nu=(\nu_{\mathsf{n}})$. Then ρ is a bMS with coefficient $\mathsf{b}=2^{\frac{1}{p}}$. [23]

Example 2.2.3. Let $\mathfrak{M}=\{0,1,2\}$ and let $\rho_{\mathsf{b}}:\mathfrak{M}\times\mathfrak{M}\to\mathbb{R}$, defined as

$$\begin{aligned}\rho_{\mathsf{b}}(0,0)=\rho_{\mathsf{b}}(1,1)=\rho_{\mathsf{b}}(2,2)&=0\\ \rho_{\mathsf{b}}(0,1)=\rho_{\mathsf{b}}(1,0)=\rho_{\mathsf{b}}(1,2)=\rho_{\mathsf{b}}(2,1)&=1\\ \rho_{\mathsf{b}}(0,2)=\rho_{\mathsf{b}}(2,0)&=s\end{aligned}$$

where $s\in\mathbb{R}$ and $s\geq 2$,

$(\mathsf{B1})$ and $(\mathsf{B2})$ are obvious. For $(\mathsf{B3})$ we have,

$$\rho_{\mathsf{b}}(\mu_1,\mu_2)\leq\tfrac{s}{2}[\rho_{\mathsf{b}}(\mu_1,\mu_3)+\rho_{\mathsf{b}}(\mu_3,\mu_2)] \qquad \forall\ \mu_1,\mu_2,\mu_3\in\mathfrak{M}.$$

Hence ρ_{b} is a b-metric on $\mathfrak{M}$ with $\mathsf{b}=\frac{s}{2}$

Definition 2.2.2. "Let $(\mathfrak{M},\rho)$ be a bMS. A sequence (μ_{n}) is said to be:

(i) Cauchy if and only if $\rho(\mu_{\mathsf{n}},\mu_{\mathsf{m}})\to 0$ as $\mathsf{n},\mathsf{m}\to\infty$.

(ii) Convergent if and only if there exist $\mu\in\mathfrak{M}$ such that $\rho(\mu_{\mathsf{n}},\mu)\to 0$ as $\mathsf{n}\to\infty$ and we write $\lim_{\mathsf{n}\to\infty}\mu_{\mathsf{n}}=\mu$.

(iii) The bMS $(\mathfrak{M},\rho)$ is complete if every Cauchy sequence is convergent." [23]

2.3 Banach Contraction Principle

In mathematics, the Banach FP theorem (also known as contracting mapping theorem) is an important tool in the theory of MS. It guarantees the existence and uniqueness of FP of certain self-maps of MS and provides a constructive method to find those FP.

The theorem is named after Stefan Banach (1892-1945) who first stated it in 1922. Many extensions and generalization on Banach contraction principle are made by a lot of author essentially, [24–26].

Definition 2.3.1. **(Fixed Point)**
"A FP of a mapping $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$ of a set $\mathfrak{M}$ into itself is a $\mu \in \mathfrak{M}$ which is mapped onto itself (is "kept fixed" by S), that is,

$$\mathsf{S}(\mu) = \mu$$

the image $\mathsf{S}\mu$ concides with μ."[6]

Example 2.3.1. Let $\mathfrak{M} = \mathbb{R}$ and $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$ be a mapping such that,

$$\mathsf{S}(\mu) = \mu + 1,$$

then S has no FP.

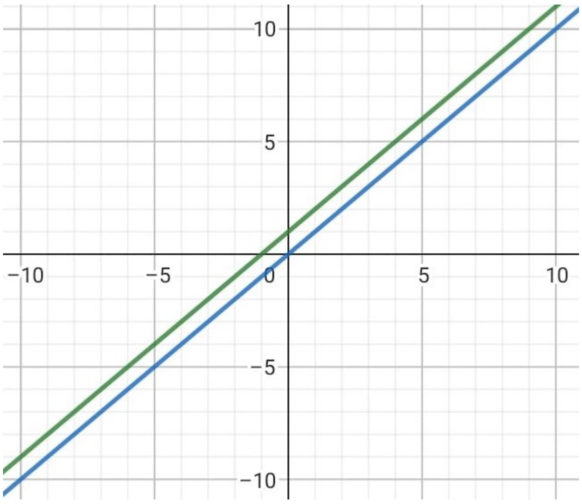

Figure 2.1: No FP

Example 2.3.2. Let $\mathfrak{M} = \mathbb{R}$ and $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$ be a mapping such that,

$$\mathsf{S}(\mu) = 2\mu + 1$$

then S has a unique FP.

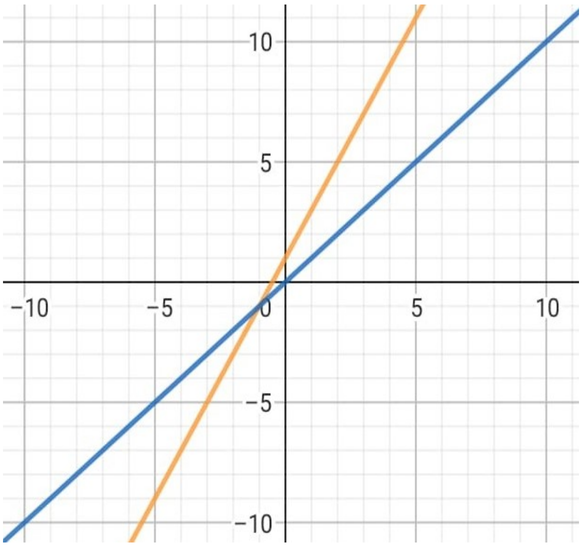

FIGURE 2.2: Unique FP

Example 2.3.3. Let $\mathfrak{M} = \mathbb{R}$ and $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$ be a mapping such that,

$$\mathsf{S}(\mu) = \mu^3 + 4\mu^2 - 3\mu - 16$$

then S has three FP.

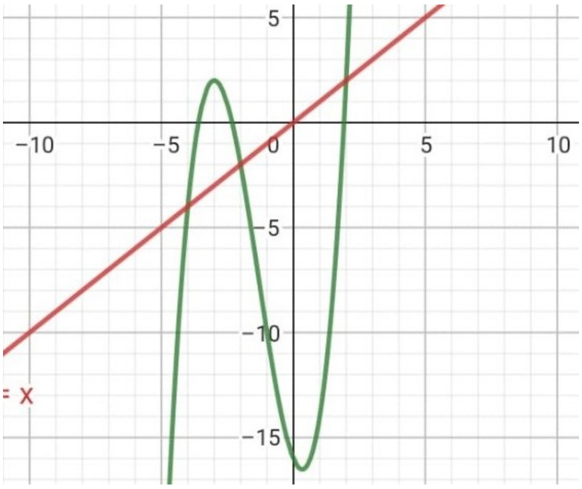

FIGURE 2.3: Three FP

Definition 2.3.2. (**Contraction Mapping**)

“Let $\mathfrak{M} = (\mathfrak{M}, \rho)$ be a MS. A mapping $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$, is called a contraction on $\mathfrak{M}$ if there is a positive real number $\alpha < 1$ such that for all $\mu_1, \mu_2 \in \mathfrak{M}$

$$\rho(\mathsf{S}(\mu_1), \mathsf{S}(\mu_2)) \leq \alpha\rho(\mu_1, \mu_2).\text{''}[6]$$

Example 2.3.4. Consider usual MS $(\mathbb{R}, \rho)$. Then the function $\mathsf{S} : \mathbb{R} \to \mathbb{R}$ defined as

$$\mathsf{S}(\mu) = \tfrac{\mu}{e} + b.$$

Consider,

$$\begin{aligned} |\mathsf{S}\mu - \mathsf{S}\nu| &= |\frac{\mu}{e} + b - (\frac{\nu}{e} + b)| \\ &= |\frac{\mu}{e} - \frac{\nu}{e}| \\ &= \frac{1}{e}|\mu - \nu|. \end{aligned}$$

Since $e > 1$ then $\frac{1}{e} < 1$. Therefore S is a contraction, and its FP is $\mu = \frac{eb}{e-1}$.

Theorem 2.3.1. (**Banach Contraction Principle**)

Consider a MS $\mathfrak{M} = (\mathfrak{M}, \rho)$, where $\mathfrak{M} \neq \varnothing$. Suppose that $\mathfrak{M}$ is complete and let $\mathsf{S} : \mathfrak{M} \to \mathfrak{M}$ be a contraction on $\mathfrak{M}$. Then S has unique FP.[6]

2.4 Bipolar Metric Space

In this section, we introduce some basic concept of BMS with suitable examples to understand the structure of BMS.

Definition 2.4.1. (**Bipolar Metric Space**)

“Let $\mathfrak{M}$ and $\mathfrak{N}$ be two nonempty sets and $\rho : \mathfrak{M} \times \mathfrak{N} \to [0, \infty)$ be a map satisfying the following conditions:

(i) $\rho(\mu, \nu) = 0$ if and only if $\mu = \nu$ for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$;

(ii) $\rho(\mu, \nu) = \rho(\nu, \mu)$ for all $\mu, \nu \in \mathfrak{M} \cap \mathfrak{N}$;

(iii) $\rho(\mu_1, \nu_2) \leq \rho(\mu_1, \nu_1) + \rho(\mu_2, \nu_1) + \rho(\mu_2, \nu_2)$; for all $\mu_1, \mu_2 \in \mathfrak{M}$ and $\nu_1, \nu_2 \in \mathfrak{N}$.

Then,ρ is called a bipolar metric and $(\mathfrak{M}, \mathfrak{N}, \rho)$ is called BMS. If $\mathfrak{M} \cap \mathfrak{N} = \emptyset$, then the space is called disjoint; otherwise, it is called joint. The set $\mathfrak{M}$ is called left pole, and the set $\mathfrak{N}$ is called the right pole of $(\mathfrak{M}, \mathfrak{N}, \rho)$. The elements of $\mathfrak{M}, \mathfrak{N}$ and $\mathfrak{M} \cap \mathfrak{N}$ are called left, right and central elements, respectively." [21]

Example 2.4.1. Let $\mathfrak{M} = [0, 1]$ and $\mathfrak{N} = [2, 3]$. Define $\rho : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}^+$ by $\rho(\mu, \nu) = 3$ for any (μ, ν) in $\mathfrak{M} \times \mathfrak{N}$.

Then ρ satisfies the conditions of BMS on $(\mathfrak{M}, \mathfrak{N})$. Hence $(\mathfrak{M}, \mathfrak{N}, \rho)$ is a BMS.

Example 2.4.2. If $\mathfrak{M}, \mathfrak{N}$ are two non-empty subsets of a MS $(\mathbb{R}, \rho)$ and suppose that $\gamma : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}$ be one to one function.

Define

$\Delta : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}^+$ by $\Delta(\mu, \nu) = \rho(\gamma(\mu), \gamma(\nu))$ for any $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$.

Then Δ is a BMS.

Proof. (i) If

$$\begin{aligned} \Delta(\mu, \nu) &= 0 \\ \Leftrightarrow \rho(\gamma(\mathsf{u}), \gamma(\mathsf{v})) &= 0 \\ \Leftrightarrow \gamma(\mathsf{u}) &= \gamma(\mathsf{v}) \\ \Leftrightarrow \mu &= \nu. \quad (\gamma \text{ is one - one}) \end{aligned}$$

(ii) If μ, ν are two arbitrary points in $\mathfrak{M} \times \mathfrak{N}$ then,

$$\begin{aligned} \Delta(\mu, \nu) &= \rho(\gamma(\mu), \gamma(\nu)) \\ &= \rho(\gamma(\nu), \gamma(\mu)), \\ &= \Delta(\nu, \mu), \quad \text{for any } \mu, \nu \text{ in } \mathfrak{M} \times \mathfrak{N}. \end{aligned}$$

(iii) : Consider,

$$\begin{aligned}\Delta(\mu_1,\nu_2) &= \rho(\gamma(\mu_1),\gamma(\nu_2))\\ &\leq \rho(\gamma(\mu_1),\gamma(\nu_1)) + \rho(\gamma(\nu_1),\gamma(\mu_2)) + \rho(\gamma(\mu_2),\gamma(\nu_2))\\ &= \Delta(\mu_1,\nu_1) + \Delta(\nu_1,\mu_2) + \Delta(\mu_2,\nu_2)\end{aligned}$$

for all μ_1, μ_2 in $\mathfrak{M}$ and ν_1, ν_2 in $\mathfrak{N}$.

Hence $(\mathfrak{M}, \mathfrak{N}, \Delta)$ is a BMS. □

Definition 2.4.2. "Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BMS. Then, any sequence $(\mu_n) \subseteq \mathfrak{M}$ is called left sequence and is said to be convergent to the right element say ν, if $\rho(\mu_n, \nu) \to 0$ as $\mathsf{n} \to \infty$. Similarly, a right sequence $(\nu_n) \subseteq \mathfrak{N}$ is said to be convergent to a left element say μ, if $\rho(\mu, \nu_n) \to 0$ as $\mathsf{n} \to \infty$."[21]

Example 2.4.3. Let $\mathfrak{M} = (1, \infty)$ and $\mathfrak{N} = [-1, 1]$ define $\rho : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}^+$ as

$$\rho(\mu, \nu) = |\mu^2 - \nu^2|$$

then, $(\mathfrak{M}, \mathfrak{N}, \rho)$ is BMS. Note that the left sequence $(1 + \frac{1}{\mathsf{n}})$ converges to right point 1 and -1.

Definition 2.4.3. "Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BMS.

(i) A sequence $\{\mu_n, \nu_n\} \subseteq \mathfrak{M} \times \mathfrak{N}$ is called a bisequence on $(\mathfrak{M}, \mathfrak{N}, \rho)$.

(ii) If both the sequences $\{\mu_n\}$ and $\{\nu_n\}$ converge, then the bisequence $\{\mu_n, \nu_n\}$ is said to be convergent. If both sequences $\{\mu_n\}$ and $\{\nu_n\}$ converge to the same point $\mu \in \mathfrak{M} \cap \mathfrak{N}$, then the bisequence $\{\mu_n, \nu_n\}$ is called biconvergent.

(iii) A bisequence $\{\mu_n, \nu_n\}$ on $(\mathfrak{M}, \mathfrak{N}, \rho)$ is said to be a Cauchy bisequence if for each $\epsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $\rho(\mu_n, \nu_m) < \epsilon$, for all $\mathsf{n}, \mathsf{m} \geq N$.

(iv) A BMS is said to be complete if every Cauchy bisequence is convergent in this space."[21]

Definition 2.4.4. "Let ($\mathfrak{M}_1$,$\mathfrak{N}_1$,ρ_1) and ($\mathfrak{M}_2$,$\mathfrak{N}_2$,ρ_2) be two BMS and $\mathsf{S} : \mathfrak{M}_1 \cup \mathfrak{N}_1 \to \mathfrak{M}_2 \cup \mathfrak{N}_2$ be a function:

(i) If $\mathsf{S}(\mathfrak{M}_1) \subseteq \mathfrak{M}_2$ and $\mathsf{S}(\mathfrak{N}_1) \subseteq \mathfrak{N}_2$, then S is called covariant mapping and is denoted by $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$.

(ii) If $\mathsf{S}(\mathfrak{M}_1) \subseteq \mathfrak{N}_2$ and $\mathsf{S}(\mathfrak{N}_1) \subseteq \mathfrak{M}_2$, then S is called contravariant mapping and is denoted by $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \leftrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$." [21]

Definition 2.4.5. "Let ($\mathfrak{M}_1$,$\mathfrak{N}_1$,ρ_1) and ($\mathfrak{M}_2$,$\mathfrak{N}_2$,ρ_2) be two BMS.

(i) A map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is called left continuous at a point $\mu_0 \in \mathfrak{M}$ if for every $\epsilon > 0$ there exist a $\delta > 0$ such that $\rho_2(\mathsf{S}\mu_0, \mathsf{S}\nu) < \epsilon$ whenever $\rho_1(\mu_0, \nu) < \delta$.

(ii) A map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is called right continuous at a point $\nu_0 \in \mathfrak{N}$ if for every $\epsilon > 0$ there exist $\delta > 0$ such that $\rho_2(\mathsf{S}\mu, \mathsf{S}\nu_0) < \epsilon$ whenever $\rho_1(\mu, \nu_0) < \delta$.

(iii) A map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is called continuous if and only if it is left continuous at each $\mu_0 \in \mathfrak{M}$ and right continuous at each $\nu_0 \in \mathfrak{N}$.

(iv) A map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightleftarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is called continuous if and only if it is continuous as covariant map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$." [21]

Definition 2.4.6. "Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ and $\alpha : \mathfrak{M} \times \mathfrak{N} \to [0, \infty)$. Then, S is called α- admissible if

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(\mathsf{S}\mu, \mathsf{S}\nu) \geq 1, \tag{2.1}$$

for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$." [21]

Definition 2.4.7. "Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ and $\alpha : \mathfrak{M} \times \mathfrak{N} \to [0, \infty)$. Then, S is called α- admissible if

$$\alpha(\mu, \nu) \geq 1 \Rightarrow \alpha(\mathsf{S}\nu, \mathsf{S}\mu) \geq 1, \tag{2.2}$$

for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$." [21]

Example 2.4.4. Let $\mathfrak{M} = [0, +\infty)$ and $\mathfrak{N} = (-\infty, 0]$. We define the covariant mapping $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ by $\mathsf{S}\mu = \mu$ and $\alpha : \mathfrak{M} \times \mathfrak{N} \to [0, +\infty)$ by

$$\alpha(\mu, \nu) = \begin{cases} 0, & \mu = \nu \\ 2, & \text{otherwise} \end{cases}$$

for all $\mu \in \mathfrak{M}$ and $\nu \in \mathfrak{N}$. Then S is α-admissible. Similarly, if we take contravariant mapping $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ which is defined by $\mathsf{S}\mu = -\mu$. Then S is also α-admissible.[27]

Definition 2.4.8. "Let Θ be the family of functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) θ is non- decreasing.

(ii) $\sum_{n=1}^{+\infty} \theta^n < \infty$ for all $s > 0$, where θ^n is n^{th} iterate of θ. These functions are known as (c)-comparison functions. It can be easily verified that $\theta(s) < s$ for any $s > 0$."[21]

Definition 2.4.9. "Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ and $\alpha : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}$. Then ,$\mathsf{S}$ is called an α-orbital admissible mapping if

$$\alpha(\mu, \mathsf{S}\mu) \geq 1 \Rightarrow \alpha(\mathsf{S}^2\mu, \mathsf{S}\mu) \geq 1, \tag{2.3}$$

and

$$\alpha(\mathsf{S}\nu, \nu) \geq 1 \Rightarrow \alpha(\mathsf{S}\nu, \mathsf{S}^2\nu) \geq 1, \tag{2.4}$$

for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$."[21]

Example 2.4.5. Let $\mathfrak{M} = \{0, 1, 2\}$ and $\mathfrak{N} = \{2, 3, 4\}$. We define the contravariant mapping $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ by $\mathsf{S}(0) = 0, \mathsf{S}(1) = 2, \mathsf{S}(2) = 2, \mathsf{S}(3) = 4$, $\mathsf{S}(4) = 4$, and $\alpha : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}$,

$$\alpha(\mu, \nu) = \begin{cases} 0, & \mu = \nu \\ 1, & \mu \neq \nu \end{cases}$$

for all $\mu \in \mathfrak{M}$ and $\nu \in \mathfrak{N}$. Then S is α-orbital admissible.

2.5 Bipolar b-Metric Space

In this section we introduce the definition of BbMS with an example and its related results.

Definition 2.5.1. "A BbMS is a triple $(\mathfrak{M}, \mathfrak{N}, \rho)$ such that $\mathfrak{M}, \mathfrak{N} \neq \phi$ and $\rho : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}^+$ is a function satisfying the following conditions:

(i) $\rho(\mu, \nu) = 0$ if and only if $\mu = \nu$;

(ii) if $\mu, \nu \in \mathfrak{M} \cap \mathfrak{N}$, then $\rho(\mu, \nu) = \rho(\nu, \mu)$;

(iii) $\rho(\mu_1, \nu_2) \leq \mathsf{b}[\rho(\mu_1, \nu_1) + \rho(\mu_2, \nu_1) + \rho(\mu_2, \nu_2)]$,
for all $(\mu, \nu), \mu_1, \nu_1, \mu_2, \nu_2 \in \mathfrak{M} \times \mathfrak{N}$ and $\mathsf{b} \geq 1$. We say ρ is a bipolar b-metric on the pair $(\mathfrak{M}, \mathfrak{N})$." [28]

Example 2.5.1. Let $\mathfrak{M} = \{(\mathsf{p}, 0) | \mathsf{p} \in \mathbb{R}\}$, $\mathfrak{N} = \{(\mathsf{q}, \mathsf{r}) | \mathsf{q}, \mathsf{r} \in \mathbb{R}\} = \mathbb{R}^2$ and

$$\rho(\mu, \nu) = (\mathsf{p} - \mathsf{q})^2 + |\mathsf{r}|,$$

for every $\mu = (\mathsf{p}, 0) \in \mathfrak{M}$ and $\nu = (\mathsf{q}, \mathsf{r}) \in \mathfrak{N}$. Obviously, $\mathfrak{M} \cap \mathfrak{N} = \mathfrak{M}$ and condition (i) and (ii) are satisfied.

For each $\mu = (\mathsf{p}, 0)$, $\mu_1 = (\mathsf{p}_1, 0) \in \mathfrak{M}$; $\nu = (\mathsf{q}, \mathsf{r}), \nu_1 = (\mathsf{q}_1, \mathsf{r}_1) \in \mathfrak{N}$ and $\mathsf{b} = 3$, we have

$$\begin{aligned}
\rho(\mu, \nu) &= (\mathsf{p} - \mathsf{q})^2 + |\mathsf{r}| \\
&= [(\mathsf{p} - \mathsf{q}_1) + (\mathsf{q}_1 - \mathsf{p}_1) + (\mathsf{p}_1 - \mathsf{q})]^2 + |\mathsf{r}| \\
&\leq 3(\mathsf{p} - \mathsf{q}_1)^2 + |\mathsf{r}_1| + 3(\mathsf{p}_1 - \mathsf{q}_1)^2 + |\mathsf{r}_1| + 3(\mathsf{p}_1 - \mathsf{q})^2 + |\mathsf{r}| \\
&= 3[\rho(\mu, \nu_1) + \rho(\mu_1, \nu_1) + \rho(\mu_1, \nu)].
\end{aligned}$$

So, condition (iii) is also satisfied and ρ is a bipolar b-metric with $\mathsf{b} = 3$. [28]

Definition 2.5.2. "Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BbMS

(i) A sequence $\{\mu_n\} \subseteq \mathfrak{M}$ is called left sequence, and a sequence $\{\nu_n\} \subseteq \mathfrak{N}$ is called a right sequence. In BbMS, a left or right sequence is simply called a sequence.

(ii) A sequence $\{\mu_n\}$ is said to be convergent to a point μ, if and only if $\{\mu_n\}$ is a left sequence, μ is a right point and $\lim_{\mathsf{n}\to\infty} \rho(\mu_{\mathsf{n}}, \mu) = 0$, or $\{\mu_n\}$ is a right sequence, μ is a left point and $\lim_{\mathsf{n}\to\infty} \rho(\mu, \mu_{\mathsf{n}}) = 0$.

(iii) A bi-sequence $\{\mu_n, \nu_n\}$ on $(\mathfrak{M}, \mathfrak{N}, \rho)$ is a sequence on the set $\mathfrak{M} \times \mathfrak{N}$. If the sequences $\{\mu_n\}$ and $\{\nu_n\}$ are convergent, then the bi-sequence $\{\mu_n, \nu_n\}$ is said to be convergent, and if $\{\mu_n\}$ and $\{\nu_n\}$ converge to a common fixed point then $\{\mu_n, \nu_n\}$ is said to be bi-convergent.

(iv) $\{\mu_n, \nu_n\}$ is called Cauchy bi-sequence if $\lim_{\mathsf{n},\mathsf{m}\to\infty} \rho(\mu_{\mathsf{n}}, \nu_{\mathsf{m}}) = 0$.

(v) A BbMS is called complete, if every Cauchy bi-sequence is convergent, hence bi-convergent." [28]

Definition 2.5.3. "Let $(\mathfrak{M}_1,\mathfrak{N}_1,\rho_1)$ and $(\mathfrak{M}_2,\mathfrak{N}_2,\rho_2)$ be BbMS.

(i) A map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is called left continuous at a point $\mu_0 \in \mathfrak{M}$ if for every $\epsilon > 0$, there exist a $\delta > 0$ such that $\rho_1(\mu_0, \nu) < \delta$ implies $\rho_2(\mathsf{S}\mu_0, \mathsf{S}\nu) < \epsilon$ for all $\nu \in \mathfrak{N}$.

(ii) A map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is called right continuous at a point $\nu_0 \in \mathsf{V}_1$, if for every $\epsilon > 0$, there exist $\delta > 0$ such that $\rho_1(\mu, \nu_0) < \delta$ implies $\rho_2(\mathsf{S}\mu, \mathsf{S}\nu_0) < \epsilon$ for all $\mu \in \mathfrak{M}$.

(iii) A map S is called continuous if it is left continuous at each point $\mu \in \mathfrak{M}_1$ and right continuous at each point $\nu \in \mathfrak{N}_1$.

(iv) A contravariant map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightleftarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$ is continuous if and only if it is continuous as covariant map $\mathsf{S} : (\mathfrak{M}_1, \mathfrak{N}_1, \rho_1) \rightrightarrows (\mathfrak{M}_2, \mathfrak{N}_2, \rho_2)$." [28]

Chapter 3

Fixed Point Theorems for $(\alpha - \theta)$ Meir-Keeler Contractions in Bipolar Metric Spaces

In this chapter we introduce the notion of M-K type contractions and proved certain FP results endowed with these contractions. In order to elaborate our results, an example is also provided.

3.1 Contravariant M-K Contraction

In this section we introduce $(\alpha - \theta)$ M-K contractions and α-orbital admissible mappings and prove FP theorems for these contractions in BMSs.

Definition 3.1.1. Let Θ be the family of functions $\theta : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) θ is non-decreasing.

(ii θ is subadditive.

(iii) $\theta(0) = 0$ at $s = 0$.

(iv) $\sum_{n=1}^{+\infty}\theta^n < \infty$ for all $\mathsf{s} > 0$, where θ^n is n^{th} iterate of θ.
These functions are known as (c)-comparison functions. It can be easily verified that $\theta(s) < s$ for any $s > 0$."

Definition 3.1.2. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a BMS and $\theta\in\Theta$. Suppose that $\mathsf{S}:(\mathfrak{M},\mathfrak{N})\rightleftarrows(\mathfrak{M},\mathfrak{N})$ is a cont mapping. Then, for each $\epsilon>0$ there exist $\delta>0$ such that

$$\epsilon\le\theta(\rho(\mu,\nu))<\epsilon+\delta\Rightarrow\alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu))<\epsilon \tag{3.1}$$

for all $(\mu,\nu)\in\mathfrak{M}\times\mathfrak{N}$ and $\alpha:\mathfrak{M}\times\mathfrak{N}\to\mathbb{R}$. Then, S be a cont$(\alpha-\theta)$ M-K contractive mapping.

Remark 1. By (3.1),
$\alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu))<\theta(\rho(\mu,\nu))$ when $\mu\neq\nu$.
and
$\alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu))\le\theta(\rho(\mu,\nu))$ when $\mu=\nu$

Theorem 3.1.1. Let $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BMS. Suppose that $\mathsf{S}:(\mathfrak{M},\mathfrak{N})\rightleftarrows(\mathfrak{M},\mathfrak{N})$ is a cont $(\alpha-\theta)$ M-K contractive mapping which satisfies the following conditions:

(i) S is α-orbital admissible;

(ii) there exist $\mu_0\in\mathfrak{M}$ such that $\alpha(\mu_0,\mathsf{S}\mu_0)\ge 1$;

(iii) S is continuous;

then S has a FP. Moreover, if $\mathsf{S}\mu=\mu$ implies $\alpha(\mu,\mathsf{S}\mu)\ge1$, then S has a unique FP.

Proof. Let $\mu_0\in\mathfrak{M}$ be such that $\alpha(\mu_0,\mathsf{S}\mu_0)\ge1$. Construct the sequences $\{\mu_n\}$ and $\{\nu_n\}$ taking $\nu_n=\mathsf{S}\mu_n$ and $\mu_{n+1}=\mathsf{S}\nu_n$ for all $\mathsf{n}\in\mathbb{N}$.. It follows that $\{\mu_n,\nu_n\}$ is a bisequence.

From condition (ii) $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$,

therefore,

$$\begin{aligned}
\alpha(\mu_0,\nu_0) = \alpha(\mu_0,\mathsf{S}\mu_0) \geq 1 &\Rightarrow \alpha(\mathsf{S}^2\mathsf{u}_0,\mathsf{S}\mu_0) = \alpha(\mu_1,\nu_0) \geq 1,\\
\alpha(\mu_1,\nu_0) = \alpha(\mathsf{S}\nu_0,\nu_0) \geq 1 &\Rightarrow \alpha(\mathsf{S}\nu_0,\mathsf{S}^2\mathsf{v}_0) = \alpha(\mu_1,\nu_1) \geq 1,\\
\alpha(\mu_1,\nu_1) = \alpha(\mu_1,\mathsf{S}\nu_1) \geq 1 &\Rightarrow \alpha(\mathsf{S}^2\mathsf{u}_1,\mathsf{S}\mu_1) = \alpha(\mu_2,\nu_1) \geq 1,\\
\alpha(\mu_2,\nu_1) = \alpha(\mathsf{S}\nu_1,\nu_1) \geq 1 &\Rightarrow \alpha(\mathsf{S}\nu_1,\mathsf{S}^2\mathsf{v}_1) = \alpha(\mu_2,\nu_2) \geq 1.
\end{aligned}$$

By continuing this process

$$\alpha(\mu_n,\nu_n) \geq 1 \text{ and } \alpha(\mu_{n+1},\nu_n) \geq 1 \quad \forall\, n \in \mathbb{N}. \tag{3.2}$$

By using Remark 1 and (3.2),

$$\begin{aligned}
\theta(\rho(\mu_n,\nu_n)) = \theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)) &\leq \alpha(\mu_n,\nu_n)\alpha(\mu_n,\nu_{n-1})\theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)),\\
&= \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_{n-1},\nu_{n-1})\theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)),\\
&< \theta(\rho(\mu_n,\nu_{n-1})).
\end{aligned}$$

Again using Remark 1 together with (3.2), we have

$$\begin{aligned}
\theta(\rho(\mu_{n+1},\nu_n)) = \theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)) &\leq \alpha(\mu_n,\nu_n)\alpha(\mu_{n+1},\nu_n)\theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)),\\
&= \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_n,\nu_n)\theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)),\\
&< \theta(\rho(\mu_n,\nu_n)).
\end{aligned}$$

Mathematical induction together with (3.3) and (3.3) gives

$$\theta(\rho(\mu_n,\nu_n)) < \theta(\rho(\mu_{n-1},\nu_{n-1})) \quad \forall\, n \in \mathbb{N} \tag{3.3}$$

and

$$\theta(\rho(\mu_{n+1},\nu_n)) < \theta(\rho(\mu_n,\nu_{n-1})) \quad \forall\, n \in \mathbb{N}. \tag{3.4}$$

From (3.3) and (3.4), it is clear that the sequences $\{\theta(\rho(\mu_n,\nu_n))\}$ and $\{\theta(\rho(\mu_{n+1},\nu_n))\}$ are monotonically decreasing and it consist positive real numbers. Therefore, the sequences must be convergent. Let $\{\theta(\rho(\mu_n,\nu_n))\} \to \mathsf{c}_1$ and $\{\theta(\rho(\mu_{n+1},\nu_n))\} \to \mathsf{c}_2$ as $\mathsf{n} \to \infty$, where $\mathsf{c}_1, \mathsf{c}_2 \geq 0$.

Now, we prove that $\mathsf{c}_1 = 0$ and $\mathsf{c}_2 = 0$.

Suppose to the contrary that $\mathsf{c}_1 > 0$. Clearly, $\theta(\rho(\mu_n, \nu_n)) \geq \mathsf{c}_1 > 0 \;\; \forall\, n \in \mathbb{N}$.

Let $\epsilon = \mathsf{c}_1$. Then, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\epsilon \leq \theta(\rho(\mu_{n_0}, \nu_{n_0})) < \epsilon + \delta.$$

From (3.1), we have

$$\begin{aligned}\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})) &\leq \alpha(\mu_{n_0+1}, \nu_{n_0+1})\alpha(\mu_{n_0+1}, \nu_{n_0})\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})),\\ &= \alpha(\mu_{n_0+1}, \mathsf{S}\mu_{n_0+1})\alpha(\mathsf{S}\nu_{n_0}, \nu_{n_0})\theta(\rho(\mathsf{S}\nu_{n_0}, \mathsf{S}\mu_{n_0+1}))\\ &< \epsilon = \mathsf{c}_1,\end{aligned}$$

which contradicts $\mathsf{c}_1 > 0$. So, $\mathsf{c}_1 = 0$.

Similarly, we can prove that $\mathsf{c}_2 = 0$.

Hence, $\theta(\rho(\mu_n, \nu_n)) \to 0$ and $\theta(\rho(\mu_{n+1}, \nu_n)) \to 0$ as $\mathsf{n} \to \infty$. As θ is continuous at $\mathsf{s} = 0$, we have

$$\rho(\mu_n, \nu_n) \to 0 \text{ and } \rho(\mu_{n+1}, \nu_n) \to 0 \text{ as } n \to \infty. \tag{3.5}$$

For any $\epsilon > 0$, there exist $\delta > 0$ that satisfies (3.1). For simplicity, we can take $\delta < \epsilon$. As $\theta(\rho(\mu_n, \nu_n)) \to 0$ and $\theta(\rho(\mu_{n+1}, \nu_n)) \to 0$, there exist $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{N}$ such that

$$\theta(\rho(\mu_{n-1}, \nu_{n-1})) < \frac{\delta}{3} \;\; \forall\, n \geq \mathcal{M}_1, \tag{3.6}$$

$$\theta(\rho(\mu_n, \nu_{n-1})) < \frac{\delta}{3} \;\; \forall\, n \geq \mathcal{M}_2. \tag{3.7}$$

Now, we will prove these inequalities

$$\theta(\rho(\mu_{n+l}, \nu_n)) < \epsilon \tag{3.8}$$

and

$$\theta(\rho(\mu_n, \nu_{n+l})) < \epsilon, \;\; \forall\, n \geq \mathcal{M}, \tag{3.9}$$

where $\mathcal{M} = max\{\mathcal{M}_1, \mathcal{M}_2\}$.

Now we can prove the inequality $\theta(\rho(\mu_{n+l}, \nu_n)) < \epsilon$ by mathematical induction. From (3.7), the result is true for $\ell = 1$.

Let the inequality hold for

$$\ell = \mathsf{q},$$

that is,

$$\theta(\rho(\mu_{n+q}, \nu_n)) < \epsilon, \ \forall\ n \geq \mathcal{M}. \tag{3.10}$$

Now, using the definition of BMS together with (3.6),(3.7) and (3.10),

$$\begin{aligned}
\theta(\rho(\mu_{n+q}, \nu_{n-1})) &\leq \theta(\rho(\mu_{n+q}, \nu_n) + \rho(\mu_n, \nu_n) + \rho(\mu_n, \nu_{n-1})),\\
&\leq \theta(\rho(\mu_{n+q}, \nu_n)) + \theta(\rho(\mu_n, \nu_n)) + \theta(\rho(\mu_n, \nu_{n-1}),\\
&< \frac{\delta}{3} + \frac{\delta}{3} + \epsilon\\
&= \frac{2\delta}{3} + \epsilon,\\
&< \epsilon + \delta.
\end{aligned}$$

If $\theta(\rho(\mu_{n+q}, \nu_{n-1})) \geq \epsilon$, then (3.1) follows,

$$\begin{aligned}
\theta(\rho(\mu_{n+q+1}, \nu_n)) &\leq \alpha(\mu_n, \nu_n)\alpha(\mu_{n+q+1}, \nu_{n+q})\theta(\rho(\mu_{n+q+1}, \nu_n)),\\
&= \alpha(\mu_n, \mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_{n+q}, \nu_{n+q})\theta(\rho(\mathsf{S}\nu_{n+q}, \mathsf{S}\mu_n)),\\
&< \epsilon.
\end{aligned}$$

So, (3.8) holds for $\ell = \mathsf{q} + 1$.

Hence,

$$\rho(\mu_n, \nu_m) < \epsilon \ \ \forall\ n > m \geq \mathcal{M}. \tag{3.11}$$

Similarly, we can prove (3.9).

Using the definition of BMS together with (3.6) and (3.7), to obtain

$$\begin{aligned}
\theta(\rho(\mu_n, \nu_{n+1})) &\leq \theta(\rho(\mu_n, \nu_n) + \rho(\mu_{n+1}, \nu_n) + \rho(\mu_{n+1}, \nu_{n+1}))\\
&\leq \theta(\rho(\mu_{n+1}, \nu_{n+1})) + \theta\rho(\mu_{n+1}, \nu_n)) + \theta\rho(\mu_n, \nu_n))\\
&\leq \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta < \epsilon.
\end{aligned}$$

So, (3.9) is true for $\ell = 1$.

Now, assume that this one is true for some $\ell = \mathsf{q}$, that is,

$$\theta(\rho(\mu_n, \nu_{n+q})) < \epsilon \quad \forall\ n \geq \mathcal{M}. \tag{3.12}$$

Using the definition of BMS, (3.6), (3.7), and (3.12), the following is obtained:

$$\begin{aligned}\theta(\rho(\mu_{n-1}, \nu_{n+q})) &\leq \theta(\rho(\mu_{n-1}, \nu_{n-1}) + \rho(\mu_n, \nu_{n-1}) + \rho(\mu_n, \nu_{n+q}))\\ &\leq \theta(\rho(\mu_{n-1}, \nu_{n-1})) + \theta(\rho(\mu_n, \nu_{n-1})) + \theta(\rho(\mu_n, \nu_{n+q}))\\ &< \frac{\delta}{3} + \frac{\delta}{3} + \epsilon\\ &= \frac{2\delta}{3} + \epsilon\\ &< \epsilon + \delta.\end{aligned} \tag{3.13}$$

When $\theta(\rho(\mu_{n-1}, \nu_{n+q})) \geq \epsilon$, then by (3.1),

$$\begin{aligned}\theta(\rho(\mu_n, \nu_{n+q+1})) &\leq \alpha(\mu_{n+q}, \nu_{n+q})\alpha(\mu_{n+1}, \nu_n)\theta(\rho(\mu_n, \nu_{n+q+1}))\\ &= \alpha(\mu_{n+q}, \mathsf{S}\mu_{n+q})\alpha(\mathsf{S}\nu_n, \nu_n)\theta(\rho(\mathsf{S}\mu_{n+q}, \mathsf{S}\nu_n))\\ &< \epsilon.\end{aligned}$$

Hence, (3.9) holds for $\ell = \mathsf{q} + 1$.

By using property of θ we say that

$$\rho(\mu_n, \nu_m) < \epsilon \ \ \forall\ m > n \geq \mathcal{M}. \tag{3.14}$$

From(3.11) and (3.14), we say that $\{\mu_n, \nu_n\}$ is a Cbs. As $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BMS, implies $\{\mu_n, \nu_n\}$ is biconvergent. Then, there exist $\mathfrak{r} \in \mathfrak{M} \cap \mathfrak{N}$ such that $\mu_n \to \mathfrak{r}$ and $\nu_n \to \mathfrak{r}$ as $n \to \infty$. As S is a continuous map.

$$\mu_n \to \mathfrak{r} \text{ implies that } \nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}.$$

To combine $\nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}$ with $\nu_n \to \mathfrak{r}$, we get $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.

Now we will prove the uniqueness of FP. Suppose that S has two distinct FPs $\mathfrak{r}_1$ and

$\mathfrak{r}_2$ such that ,

$\alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1), \alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2) \geq 1$.

Now, by Remark 1,

$$\begin{aligned}\rho(\mathfrak{r}_1, \mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2) &\leq \alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1)\alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2)\rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2)\\ &< \rho(\mathfrak{r}_1, \mathfrak{r}_2)\end{aligned}$$

which is a contradiction, so $\mathfrak{r}_1 = \mathfrak{r}_2$. ☐

Now, we exclude the continuity condition and introduced the new criteria to find FP in the following theorem.

Theorem 3.1.2. Suppose that $(\mathfrak{M},\mathfrak{N},\rho)$ be a complete BMS. Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ is a cont $(\alpha-\theta)$ M-K contractive mapping which satisfies the following conditions:

(i) S is α-orbital admissible;

(ii) there exist $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$;

(iii) if $\{\mu_n, \nu_n\}$ is a bisequence such that $\alpha(\mu_n, \nu_n) \geq 1$ for all n and $\nu_n \to \mathfrak{r} \in \mathfrak{M}\cap\mathfrak{N}$ as $n \to \infty$, then $\alpha(\mathsf{S}\mathfrak{r}, \mathfrak{r}) \geq 1$;

then S has a FP. Moreover, if $\mathsf{S}\mu = \mu$ implies $\alpha(\mu, \mathsf{S}\mu) \geq 1$ then S has a unique FP.

Proof. From previous Theorem 3.1.1 , $\{\mu_n, \nu_n\}$ is a Cbs. As $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BMS, then it follows that $\{\mu_n, \nu_n\}$ is biconvergent. So, there exist $\mathfrak{r} \in \mathfrak{M}\cap\mathfrak{N}$ such that $\mu_n \to \mathfrak{r}$, $\nu_n \to \mathfrak{r}$. From condition (iii), we have $\alpha(\mathsf{S}\mathfrak{r}, \mathfrak{r}) \geq 1$. Using the definition of BMS

$$\theta(\rho(\mathsf{S}\mathfrak{r}, \mathfrak{r})) \leq \theta(\rho(\mathsf{S}\mathfrak{r}, \mathsf{S}\mu_n) + \rho(\mathsf{S}\nu_n, \mathsf{S}\mu_n) + \rho(\mathsf{S}\nu_n, \mathfrak{r})).$$

From the definition of θ,

$$\theta(\rho(\mathsf{S}\mathfrak{r}, \mathfrak{r})) \leq \theta(\rho(\mathsf{S}\mathfrak{r}, \mathsf{S}\mu_n)) + \theta(\rho(\mathsf{S}\nu_n, \mathsf{S}\mu_n)) + \theta(\rho(\mathsf{S}\nu_n, \mathfrak{r})).$$

From (3.1), to obtain

$$\begin{aligned}\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) \leq\ & \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\mathfrak{r},\mathfrak{r})\theta(\rho(\mathsf{S}\mathfrak{r},\mathsf{S}\mu_n)),\\ &+\alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_n,\nu_n)\theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)),\\ &+\theta(\rho(\mathsf{S}\nu_n,\mathfrak{r})).\end{aligned}$$

Then by Remark 1

$$\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) \leq \theta(\rho(\mu_n,\mathfrak{r})) + \theta(\rho(\mu_n,\nu_n)) + \theta(\rho(\mu_{n+1},\mathfrak{r})).$$

Letting $\mathsf{n}\to\infty$ in above inequality and using (3.14),

$$\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) \leq 0.$$

By using the property of θ, we say that $\rho(\mathsf{S}\mathfrak{r},\mathfrak{r}) = 0$. Hence, $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.

Now we will prove its uniqueness. For this let us suppose that S has two distinct FPs $\mathfrak{r}_1$ and $\mathfrak{r}_2$ such that

$\alpha(\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_1), \alpha(\mathfrak{r}_2,\mathsf{S}\mathfrak{r}_2) \geq 1$.

Now, by Remark 1,

$$\begin{aligned}\rho(\mathfrak{r}_1,\mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_2) &\leq \alpha(\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_1)\alpha(\mathfrak{r}_2,\mathsf{S}\mathfrak{r}_2)\rho(\mathsf{S}\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_2)\\ &< \rho(\mathfrak{r}_1,\mathfrak{r}_2)\end{aligned}$$

which is a contradiction and so $\mathfrak{r}_1 = \mathfrak{r}_2$. □

3.2 Generalized Contravariant M-K Contraction

Definition 3.2.1. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a BMS and $\theta\in\Theta$, suppose that $\mathsf{S}:(\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ is a cont mapping. Then, for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \theta(\mathcal{R}(\mu,\nu)) < \epsilon+\delta \Rightarrow \alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu)) < \epsilon. \tag{3.15}$$

Here $\mathcal{R}(\mu, \nu) = \max\{\rho(\mu, \nu), \rho(\mu, \mathsf{S}\mu), \rho(\mathsf{S}\nu, \nu), \frac{\rho(\mu,\mathsf{S}\mu)+\rho(\mathsf{S}\nu,\nu)}{2}\}$, for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$. Then, S is a generalized form of cont $(\alpha - \theta)$ M-K contractive mapping.

Remark 2. Using (3.15), we conclude that
$\alpha(\mu, \mathsf{S}\mu)\alpha(\mathsf{S}\nu, \nu)\theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \theta(\mathcal{R}(\mu, \nu))$, when $\mu \neq \nu$.
If $\mu = \nu$ then $\alpha(\mu, \mathsf{S}\mu)\alpha(\mathsf{S}\nu, \nu)\theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) \leq \theta(\mathcal{R}(\mu, \nu))$.

Theorem 3.2.1. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a complete BMS. Suppose that $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ is a generalized cont $(\alpha - \theta)$ M-K contractive mapping. If the following conditions hold:

(i) S is α-orbital admissible;

(ii) there exist $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$;

(iii) S is orbital continuous;

then S has a FP.
Moreover,
if,

$$\mathsf{S}\mu = \mu \quad \text{implies} \quad \alpha(\mu, \mathsf{S}\mu) \geq 1$$

then S has a unique FP.

Proof. Let $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$. Construct sequences $\{\mu_n\}$ and $\{\nu_n\}$ by taking $\nu_n = \mathsf{S}\mu_n$ and $\mu_{n+1} = \mathsf{S}\nu_n$ for all $\mathsf{n} \in \mathbb{N}$. It follows that $\{\mu_n, \nu_n\}$ is a bisequence. Theorem 3.1.1, and condition (iii) follows the result

$$\alpha(\mu_n, \nu_n) \geq 1 \text{ and } \alpha(\mu_{n+1}, \nu_n) \geq 1; \quad \forall\ \mathsf{n} \in \mathbb{N}. \tag{3.16}$$

From Remark 2 and (3.16),

we construct the following equation,

$$\begin{aligned}\theta(\rho(\mu_n,\nu_n)) &= \theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n))\\ &\leq \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_{n-1},\nu_{n-1})\theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)),\\ &< \theta(\mathcal{R}(\mu_n,\nu_{n-1})),\\ &= \theta\Bigg(\max\{\rho(\mu_n,\nu_{n-1}),\rho(\mu_n,\mathsf{S}\mu_n),\rho(\mathsf{S}\nu_{n-1},\nu_{n-1}),\\ &\qquad \frac{\rho(\mu_n,\mathsf{S}\mu_n)+\rho(\mathsf{S}\nu_{n-1},\nu_{n-1})}{2}\}\Bigg),\end{aligned}$$

$$\begin{aligned}&= \theta(\max\{\rho(\mu_n,\nu_{n-1}),\rho(\mu_n,\nu_n),\rho(\mu_n,\nu_{n-1}),\frac{\rho(\mu_n,\nu_n)+\rho(\mu_n,\nu_{n-1})}{2}\}),\\ &\leq \theta(\max\{\rho(\mu_n,\nu_n),\rho(\mu_n,\nu_{n-1})\}).\end{aligned}$$

Since θ is a non-decreasing function, therefore

$$\rho(\mu_n,\nu_n) \leq \max\{\rho(\mu_n,\nu_n),\rho(\mu_n,\nu_{n-1})\}.$$

If

$$\rho(\mu_n,\nu_n) > \rho(\mu_n,\nu_{n-1}),$$

then

$\rho(\mu_n,\nu_n) < \rho(\mu_n,\nu_n)$, which is not possible. Hence,

$$\rho(\mu_n,\nu_n) \leq \rho(\mu_n,\nu_{n-1}),\ \forall\ n\in\mathbb{N}. \tag{3.17}$$

Similarly, we can say that

$$\rho(\mu_{n+1},\nu_n) \leq \rho(\mu_n,\nu_n),\ \forall\ n\in\mathbb{N}. \tag{3.18}$$

From (3.17) and (3.18), it is clear that $\{\rho(\mu_n,\nu_n)\}$ and $\{\rho(\mu_{n+1},\nu_n)\}$ are monotonically decreasing sequences of positive real numbers and hence convergent. Suppose that $\{\rho(\mu_n,\nu_n)\}\to \mathsf{c}_1$ and $\{\rho(\mu_{n+1},\nu_n)\}\to \mathsf{c}_2$ as $\mathsf{n}\to\infty$, where $\mathsf{c}_1,\mathsf{c}_2\geq 0$. It follows that

$$\begin{aligned}\lim_{n\to\infty}\{\theta(\rho(\mathsf{u_n},\mathsf{v_n}))\} &= \lim_{n\to\infty}\{\theta(\mathcal{R}(\mathsf{u_n},\mathsf{v_n}))\}\\ &= \theta(\mathsf{s}_1), \end{aligned}\tag{3.19}$$

and

$$\begin{aligned}\lim_{n\to\infty}\{\theta(\rho(\mathsf{u}_{\mathsf{n}+1},\mathsf{v}_{\mathsf{n}}))\} &= \lim_{n\to\infty}\{\theta(\mathcal{R}(\mathsf{u}_{\mathsf{n}+1},\mathsf{v}_{\mathsf{n}}))\}\\ &= \theta(\mathsf{s}_2).\end{aligned} \tag{3.20}$$

Now, we will prove that $\mathsf{c}_1 = 0$ and $\mathsf{c}_2 = 0$.

Suppose on contrary $\mathsf{c}_1 > 0$. Clearly, $\rho(\mu_n, \nu_n) \geq \mathsf{c}_1 > 0$ for all $n \in \mathbb{N}$.

Consider $\epsilon = \mathsf{c}_1$. Then, there exist $\delta > 0$ and $\mu_0 \in \mathbb{N}$ such that

$$\theta(\epsilon) \leq \theta(\mathcal{R}(\mu_{n_0}, \nu_{n_0})) < \theta(\epsilon) + \delta. \tag{3.21}$$

From (3.15),

$$\begin{aligned}\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})) &\leq \alpha(\mu_{n_0+1}, \nu_{n_0+1})\alpha(\mu_{n_0+1}, \nu_n)\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1}))\\ &= \alpha(\mathsf{S}\nu_{n_0}, \nu_{n_0})\alpha(\mu_{n_0+1}, \mathsf{S}\mu_{n_0+1})\theta(\rho(\mathsf{S}\nu_{n_0}, \mathsf{S}\mu_{n_0+1}))\\ &< \theta(\epsilon).\end{aligned}$$

From the definition of θ,

$$\rho(\mu_{n_0+1}, \nu_{n_0+1}) < \epsilon = \mathsf{s}_1, \tag{3.22}$$

a contradiction, so $\mathsf{c}_1 = 0$.

Similarly, we can prove that $\mathsf{c}_2 = 0$.

Now, we want to prove that $\{\mu_n, \nu_n\}$ is a Cbs; that is, $\lim_{n,m\to\infty} \rho(\mu_n, \nu_m) = 0$. For this we suppose on contrary that $\{\mu_n, \nu_n\}$ is not a Cbs, then there exist $\epsilon > 0$ and subsequences $\{n_{(r)}\}$, $\{n_{(r+1)}\} \in \mathbb{N}$ such that

$$\rho(\mu_{n_{(r)}}, \nu_{n_{(r+1)}}) > \epsilon \tag{3.23}$$

for every $\mathsf{r} \in \mathbb{N}$. It follows that, any given $\epsilon > 0$ there exist $\delta > 0$ such that (3.15) holds

Set $\mathsf{h} = \min\{\epsilon, \delta\}$.

Since $\rho(\mu_n, \nu_n)$ and $\rho(\mu_{n+1}, \nu_n) \to 0$ as $n \to \infty$, there exist $fn_1, n_2 \in \mathbb{N}$ such that

$$\rho(\mu_n, \nu_n) < \frac{h}{8} \quad \forall\ n \geq n_1, \tag{3.24}$$

and

$$\rho(\mu_{n+1}, \nu_n) < \frac{h}{8} \quad \forall\ n \geq n_2. \tag{3.25}$$

Choosing $\mathcal{R} = max\{n_1, n_2\}$. Then, 3.24 and 3.25 are satisfied for all $n \geq \mathcal{R}$. Let $\mathsf{n}_{(\mathsf{r})} > \mathcal{R}$. to obtain $\mathsf{n}_{(\mathsf{r})} \leq \mathsf{n}_{(\mathsf{r}+1)-1}$. If $\rho(\mu_{n_{(r)}}, \nu_{n_{(r+1)-1}}) \leq \epsilon + \frac{h}{2}$; then, using the definition of BMS, (3.24) and (3.25),

$$\begin{aligned}
\rho(\mu_{n_{(r)}}, \nu_{n_{(r+1)}}) &\leq \rho(\mu_{n_{(r)}}, \nu_{n_{(r+1)-1}}) + \rho(\mu_{n_{(r+1)}}, \nu_{n_{(r+1)-1}}) + \rho(\mu_{n_{(r+1)}}, \nu_{n_{(r+1)-1}}), \\
&< \epsilon + \frac{h}{2} + \frac{h}{8} + \frac{h}{8}, \\
&= \epsilon + \frac{3}{4}h, \\
&< \epsilon,
\end{aligned}$$

this contradict the result. So, there exist p such that $\mathsf{n}_{(\mathsf{r})} \leq p < \mathsf{n}_{(\mathsf{r}+1)}$ and $\rho(\mu_{n_{(r)}}, \nu_k) > \epsilon + \frac{h}{2}$.

Now if $\rho(\mu_{n_{(r+1)}}, \nu_{n_{(r)}}) \geq \epsilon + \frac{h}{2}$, then $\rho(\mu_{n_{(r+1)}}, \nu_{n_{(r)}}) \geq \epsilon + \frac{h}{2} > h + \frac{h}{2} > \frac{h}{8}$.
that contradicts 3.23 .

So, there exist an integer k such that $\mathsf{n}_{(\mathsf{r})} \leq \mathsf{k} \leq \mathsf{n}_{(\mathsf{r}+1)}$ implies $\rho(\mu_{n_{(r)}}, \nu_k) < \epsilon + \frac{h}{2}$.
Choose $\mathsf{k} \geq \mathsf{n}_{(\mathsf{r})}$ such that $\rho(\mathsf{u}_{\mathsf{n}_{(\mathsf{r})}}, \nu_k) \geq \epsilon + \frac{r}{2}$.
Thus, $\rho(\mu_{n_{(r)}}, \nu_{k-1}) < \epsilon + \frac{h}{2}$.
From the definition of BMS and (3.25),

$$\begin{aligned}
\rho(\mu_{n_{(r)}}, \nu_k) &\leq \rho(\mu_{n_{(r)}}, \nu_{k-1}) + \rho(\mu_k, \nu_{k-1}) + \rho(\mu_k, \nu_k), \\
&\leq \epsilon + \frac{h}{2} + \frac{h}{8} + \frac{h}{8}, \\
&= \epsilon + \frac{3}{4}h.
\end{aligned}$$

Now, we choose a number p that satisfied $\mathsf{n}_{(\mathsf{r})} \leq \mathsf{k} \leq \mathsf{n}_{(\mathsf{r}+1)}$ such that

$$\epsilon + \frac{h}{2} \leq \rho(\mu_{n_{(r)}}, \nu_k) < \epsilon + \frac{3}{4}h. \tag{3.26}$$

Therefore,

$$\rho(\mu_{n_{(r)}},\nu_k)\leq\epsilon+\frac{3}{4}h<\epsilon+\mathsf{h} \tag{3.27}$$

$$\rho(\mu_{n_{(r)}},\nu_{n_{r)}})\leq\frac{h}{8}<\epsilon+\mathsf{h} \tag{3.28}$$

$$\rho(\mu_{k+1},\nu_k)\leq\frac{h}{8}<\epsilon+\mathsf{h}. \tag{3.29}$$

Now, from (3.27)-(3.29) we have $\epsilon\leq\mathcal{R}(\mu_{n(r)},\nu_k)<\epsilon+\mathsf{r}\leq\epsilon+\delta$ and so $\theta(\epsilon)\leq\theta(\mathcal{R}(\mu_{n(r)},\nu_k))<\theta(\epsilon+\mathsf{r})\leq\theta(\epsilon+\delta)\leq\theta(\epsilon)+\theta(\delta)$.

As S be a generalized $(\alpha-\theta)$ M-K contractive mapping,

$$\begin{aligned}\theta(\rho(\mu_{k+1},\nu_{n_{(r)}}))&\leq\alpha(\mu_{n_{(r)}},\mathsf{S}\mu_{n_{(r)}})\alpha(\mathsf{S}\mu_k,\nu_k)\theta(\rho(\mathsf{S}\nu_k,\mathsf{S}\mu_{n_{(r)}}))\\&<\theta(\epsilon),\end{aligned}$$

which follows that

$$\rho(\mu_{k+1},\nu_{n(r)})<\epsilon. \tag{3.30}$$

Since ρ is a BMS,

$$\rho(\mu_{n_{(r)}},\nu_k)\leq\rho(\mu_{n_{(r)}},\nu_{n_{(r)}})+\rho(\mu_{k+1},\nu_{n_{(r)}})+\rho(\mu_{k+1},\nu_k),$$

which implies that

$$\rho(\mu_{n_{(r)}},\nu_k)-\rho(\mu_{n_{(r)}},\nu_{n_{(r)}})-\rho(\mu_{k+1},\nu_k)\leq\rho(\mu_{k+1},\nu_{n_{(r)}}),$$

$$\epsilon+\frac{h}{2}-\frac{h}{8}-\frac{h}{8}<\rho(\mu_{k+1},\nu_{n_{(r)}}).$$

This shows that

$$\epsilon<\rho(\mu_{k+1},\nu_{n_{(r)}}). \tag{3.31}$$

This contradicts (3.30).

So, $\{\mu_n,\nu_n\}$ is a Cbs. As we know that $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BMS, then we say $\{\mu_n,\nu_n\}$ is biconvergent. Hence, there exist $\mathfrak{r}\in\mathfrak{M}\cap\mathfrak{N}$ such that $\mu_n\to\mathfrak{r}$ and $\nu_n\to\mathfrak{r}$

as $\mathsf{n} \to \infty$. As S is an orbital continuous map,

$$\mu_n \to \mathfrak{r} \text{ implies that } \nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}.$$

Combining $\nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}$ with $\nu_n \to \mathfrak{r}$, we have $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.

Uniqueness: Suppose that S has two distinct FPs $\mathfrak{r}_1$ and $\mathfrak{r}_2$. Therefore $\alpha(\mathfrak{r}_1, \mathsf{Sw}_1), \alpha(\mathfrak{r}_2, \mathsf{Sw}_2) \geq 1$.

Now, by Remark 1,

$$\begin{aligned} \rho(\mathfrak{r}_1, \mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2) &\leq \alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1)\alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2)\rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2) \\ &< \rho(\mathfrak{r}_1, \mathfrak{r}_2) \end{aligned}$$

which is a contradiction and so $\mathfrak{r}_1 = \mathfrak{r}_2$. □

3.3 Covariant M-K Contraction

Definition 3.3.1. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BMS. Suppose $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ is a cov mapping. Then, for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \rho(\mu, \nu) < \epsilon + \delta \Rightarrow \rho(\mathsf{S}\mu, \mathsf{S}\nu) < \epsilon, \tag{3.32}$$

for all $\mu, \nu \in \mathfrak{M} \times \mathfrak{N}$.

Then, S be a cov M-K contractive mapping.

Remark 3. From(3.32), we say that $\rho(\mathsf{S}\mu, \mathsf{S}\nu) < \rho(\mu, \nu)$, whenever $\mu \neq \nu$. If $\mu = \nu$ then $\rho(\mathsf{S}\mu, \mathsf{S}\nu) \leq \rho(\mu, \nu)$.

Theorem 3.3.1. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a complete BMS. Suppose that $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ be a cov M-K contractive mapping. Then S has a unique FP.

Proof. Construct the sequences $\{\mu_n\}$ and $\{\nu_n\}$ by taking $\mu_n = \mathsf{S}\mu_{n-1}$ and $\nu_n = \mathsf{S}\nu_{n-1}$. Using Remark 3 and contractive condition, to obtain

$$\rho(\mu_n, \nu_n) = \rho(\mathsf{S}\mu_{n-1}, \mathsf{S}\nu_{n-1}) \leq \rho(\mu_{n-1}, \nu_{n-1}). \tag{3.33}$$

Now,

using Remark 3 together with contractive condition

$$\rho(\mu_n, \nu_{n+1}) = \rho(\mathsf{S}\mu_{n-1}, \mathsf{S}\nu_n) \leq \rho(\mu_{n-1}, \nu_n). \tag{3.34}$$

From (3.33) and (3.34), it is clear that

$$\{\rho(\mu_n, \nu_n)\} and \{\rho(\mu_n, \nu_{n+1})\}$$

are monotonically decreasing sequences of positive real numbers and hence convergent. Suppose that $\{(\rho(\mu_n, \nu_n))\} \to \mathsf{c}_1$ and $\{(\rho(\mu_{n+1}, \nu_n))\} \to \mathsf{c}_2$ as $\mathsf{n} \to \infty$, where $\mathsf{c}_1, \mathsf{c}_2 \geq 0$.
Now, we prove that $\mathsf{c}_1 = 0$ and $\mathsf{c}_2 = 0$.
Suppose on contrary that $\mathsf{c}_1 > 0$. Clearly, $\rho(\mu_n, \nu_n) \geq \mathsf{c}_1 > 0$ for all $n \in \mathbb{N}$.
Let $\epsilon = \mathsf{c}_1$. Then, there exist $\delta > 0$ and $\mathsf{n}_0 \in \mathbb{N}$ such that

$$\epsilon \leq \rho(\mu_0, \nu_0) < \epsilon + \delta. \tag{3.35}$$

From (3.32),

$$\begin{aligned} \rho(\mu_{n_0+1}, \nu_{n_0+1}) &\leq \rho(\mu_{n_0+1}, \nu_{n_0+1}), \\ &= \rho(\mathsf{S}\mu_{n_0}, \mathsf{S}\nu_{n_0}) < \epsilon = \mathsf{s}_1, \end{aligned}$$

a contradiction, So $\mathsf{c}_1 = 0$.
Similarly, prove that $\mathsf{c}_2 = 0$.
Hence,

$$\rho(\mu_n, \nu_n) \to 0 \text{ and } \rho(\mu_n, \nu_{n+1}) \to 0 \text{ as } n \to \infty. \tag{3.36}$$

For any given $\epsilon > 0$, there exist $\delta > 0$ such that (3.32) holds.
For simplicity, we assume that $\delta < \epsilon$. As

$$\rho(\mu_n, \nu_n) \to 0$$

and

$$\rho(\mu_n, \nu_{n+1}) \to 0,$$

there exist, $\mathcal{R}_1, \mathcal{R}_2 \in \mathbb{N}$ such that

$$\rho(\mu_{n-1}, \nu_{n-1}) < \frac{\delta}{3} \text{ for all } n \geq \mathcal{R}_1, \tag{3.37}$$

and

$$\rho(\mu_{n-1}, \nu_n) < \frac{\delta}{3} \text{ for all } n \geq \mathcal{R}_2. \tag{3.38}$$

Now, we will prove these inequalities

$$\rho(\mu_n, \nu_{n+l}) < \epsilon \tag{3.39}$$

and

$$\rho(\mu_{n+l}, \nu_n) < \epsilon, \text{ for all } n \geq \mathcal{R}. \tag{3.40}$$

where $\mathcal{R} \ = max\{\mathcal{R}_1, \mathcal{R}_2\}$. From mathematical induction, we prove (3.39), that is $\theta(\rho(\mu_{n+l}, \nu_n)) < \epsilon$. By (3.36), it holds for $\ell = 1$.

Let the inquality holds for some $\ell = \mathsf{q}$, that is

$$\rho(\mu_n, \nu_{n+k})) < \epsilon, \text{ for all } \mathsf{n} \geq \mathcal{R}. \tag{3.41}$$

From the definition of BMS

$$\rho(\mu_{n-1}, \nu_{n+q})) \leq \rho(\mu_{n-1}, \nu_n) + \rho(\mu_n, \nu_n) + \rho(\mu_n, \nu_{n+q}).$$

From (3.37)-(3.41),

$$\begin{aligned} \rho(\mu_{n-1}, \nu_{n+q})) &< \frac{\delta}{3} + \frac{\delta}{3} + \epsilon = \frac{2\delta}{3} + \epsilon \\ &< \epsilon + \delta. \end{aligned} \tag{3.42}$$

If $\rho(\mu_{n-1}, \nu_{n+q})) \geq \epsilon$, then by (3.32),

$$\rho(\mu_n, \nu_{n+q+1})) < \epsilon.$$

Hence, (3.39) holds.

If $\rho(\mu_{n+k},\nu_{n-1}))\leq\epsilon$, then

$$\rho(\mu_{n+k+1},\nu_n)<\epsilon.$$

So, (3.39) holds for $\ell=\mathsf{q}+1$.

Hence,

$$\rho(\mu_n,\nu_m)<\epsilon;\quad\forall\ \mathsf{n}>\mathsf{m}\geq\mathsf{N}. \tag{3.43}$$

Similarly, one can prove (3.40), from which we conclude that

$$\rho(\mu_n,\nu_m)<\epsilon;\quad\forall\ \mathsf{n}>\mathsf{m}\geq\mathsf{N}. \tag{3.44}$$

Therefore, $\{\mu_n,\nu_n\}$ is a Cbs. As $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BMS, implies $\{\mu_n,\nu_n\}$is biconvergent. That is, there exist $\mathfrak{r}\in\mathfrak{M}\cap\mathfrak{N}$ such that $\mu_n\to\mathfrak{r}$ and $\nu_n\to\mathfrak{r}$ as $\mathsf{n}\to\infty$. Since S is a continuous map

$$\{\mu_n\}\to\mathfrak{r}\text{ implies that }\mu_{n+1}=\mathsf{S}\mu_n\to\mathsf{S}\mathfrak{r}.$$

which gives $\mathsf{S}\mathfrak{r}=\mathfrak{r}$.

Now we will prove the uniqueness of FP. For this we suppose on contrary that $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are two different FPs of S. Then, by Remark 3,

$$\rho(\mathfrak{r}_1,\mathfrak{r}_2)=\rho(\mathsf{S}\mathsf{w}_1,\mathsf{T}\mathsf{w}_2)<\rho(\mathfrak{r}_1,\mathfrak{r}_2),$$

which holds only when $\mathfrak{r}_1=\mathfrak{r}_2$. □

Example 3.3.1. Let $\mathfrak{M}=(-\infty,0],\mathfrak{N}=[0,\infty)$

Define $\rho:(-\infty,0]\times[0,\infty)\to[0,\infty)$ as $\rho(\mathsf{u},\mathsf{v})=|\mu-\nu|$. Then, $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BMS.

Define $\mathsf{S}:(-\infty,0]\cup[0,\infty)\rightleftarrows(-\infty,0]\cup[0,\infty)$ by $\mathsf{S}\mu=\frac{-u}{3}$, for all $\mu\in(-\infty,0]\cup[0,\infty)$, and $\theta(\mathsf{t})=\frac{t}{2}$, $\alpha(\mu,\nu)=1$ for all $(\mu,\nu)\in\mathfrak{M}\times\mathfrak{N}$. $\mathsf{S}((-\infty,0])\subset[0,\infty)$ and $\mathsf{S}([0,\infty))\subset(-\infty,0]$.

Let $\mu \in (-\infty, 0]$, there exist $\mathsf{m} \in [0,\infty)$, such that $\mu = -\mathsf{m}$. Now,

$$\theta(\rho(\mu,\nu)) = \theta(|\mu-\nu|) = \theta(|-\mathsf{m}-\nu|) = \theta(\mathsf{m}+\nu) = \frac{\mathsf{m}+\nu}{2},$$
$$\theta(\rho(\mathsf{S}\mu,\mathsf{S}\nu)) = \theta(\rho(\frac{-v}{3},\frac{-u}{3})) = \theta(|\frac{-v}{3}-(\frac{-u}{3})|) = \frac{\mathsf{m}+\nu}{2}.$$

By choosing carefully $0<\epsilon<1$ and $\delta = \; 2\epsilon$, that follows (3.1). So, all the conditions of Theorem 3.1.1 must hold. From above it is clear that S is continuous. By choosing $\mu=-3$ we have

$$\begin{aligned}\alpha(-3,\mathsf{S}(-3)) &= \alpha(-3,1)\\ &= 1.\end{aligned}$$

implies

$$\begin{aligned}\alpha(\mathsf{S}(\mathsf{S}(-3),\mathsf{S}(-3)) &= \alpha(\mathsf{S}(1),\mathsf{S}(-3))\\ &= \alpha(\frac{-1}{3},1)\\ &= 1.\end{aligned}$$

It shows that α is orbital admissible. For condition (iii) there exist $\mu_0 = -1 \in \mathfrak{M}$ such that

$$\begin{aligned}\alpha(\mu_0,\mathsf{S}(\mu_0)) &= \alpha(-1,\mathsf{S}(-1))\\ &= \alpha(-1,\frac{1}{3})\\ &= 1\end{aligned}$$

Hence, all the condition are satisfied and S has a FP.

Clearly 0 is the FP of S.

Corollary 3.3.1. Suppose that $(\mathfrak{M},\mathfrak{N},\rho)$ be a BMS and $\theta\in\Theta$. Let $\mathsf{S}:(\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ be a cont mapping. Then, for each $\epsilon>0$ there exist $\delta>0$ such that

$$\epsilon \le \theta(\rho(\mu,\nu)) < \epsilon+\delta \Rightarrow \theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu)) < \frac{\epsilon}{A} \tag{3.45}$$

here $\theta\in\Theta$ and $A\ge 1$. Then, S has a FP.

Proof follows by letting $\alpha(\mu, \nu) = \sqrt{A}$ in Theorem 3.1.1.

Corollary 3.3.2. Suppose that $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BMS and $\theta \in \Theta$. Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ be a cont mapping. Then, for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \mathcal{R}(\rho(\mu, \nu)) < \epsilon + \delta \Rightarrow \theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \frac{\epsilon}{A} \tag{3.46}$$

here $\theta \in \Theta$ and $A \geq 1$. Then, S has a FP.

Proof follows by letting $\alpha(\mu, \nu) = \sqrt{A}$ in Theorem 3.1.1.

3.4 Application

Theorem 3.4.1. Let us consider the following integral equation

$$\mathfrak{p}(\varphi) = p(\varphi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \pi, \mathfrak{p}(\pi)) d\pi + \lambda_2 \int \mathfrak{B}_2(\varphi, \eta, \mathfrak{p}(\pi)) d\pi \tag{3.47}$$

$\varphi \in \mathsf{F}_1 \cup \mathsf{F}_2, \mathsf{F}_1 \cup \mathsf{F}_2$ is a Lebesgue measurable set with finite measures and λ_1, λ_2 are constants.

(1) Suppose that $\mathfrak{B} : \mathsf{F}_1^2 \cup \mathsf{F}_2^2 \times [0, \infty) \to [0, \infty)$ and $\mathfrak{B}_2 : \mathsf{F}_1^2 \cup \mathsf{F}_2^2 \times [0, \infty) \to [0, \infty)$.

(2) There is a CF $\sigma : \mathsf{F}_1^2 \cup \mathsf{F}_2^2 \times [0, \infty)$ and $s \in (0, 1)$ such that for all $(\varphi, \pi) \in \mathsf{F}_1^2 \cup \mathsf{F}_2^2$ and $p(\varphi) \in \mathsf{L}^\infty(F_1) \cup \mathsf{L}^\infty(F_2)$.

$$|\lambda_i(\mathfrak{B}_i(\varphi, \pi, \mathfrak{p}(\pi))) - \lambda_i(\mathfrak{B}_i(\varphi, \pi, \xi(\pi)))| \leq \frac{s}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|$$

for all $\mathsf{i} = 1, 2$ and $\|\sigma(\varphi, \pi) d\pi\| \leq 1$ that is $\sup_{\varphi \in \mathsf{F}_1 \cup \mathsf{F}_2} \int |\sigma(\varphi, \pi) d\pi| \leq 1$.

Then, (4.45) gives unique solution in $\mathsf{L}^\infty(\mathsf{F}_1) \cup \mathsf{L}^\infty(\mathsf{F}_2)$.

Proof. Let $\mathfrak{M} = \mathsf{L}^\infty(\mathsf{F}_1)$ and $\mathfrak{N} = \mathsf{L}^\infty(\mathsf{F}_2)$. be two normed linear spaces, where F_1 and F_2 be two Lebesgue measurable sets with $p(\mathsf{F}_1 \cup \mathsf{F}_2) < \infty$.
Consider $\rho : \mathfrak{M} \times \mathfrak{N} \to [0, \infty)$ as

$$\rho(\mu, \nu) = \|\mu - \nu\|_\infty.$$

Since $(\mathfrak{M}, \mathfrak{N}, \rho)$ is a complete BMS. Define a cov mapping as

$$\mathsf{S}(\mathfrak{p}(\varphi)) = p(\varphi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \pi, \mathfrak{p}(\pi))d\pi + \lambda_2 \int \mathfrak{B}_2(\varphi, \pi, \mathfrak{p}(\pi))d\pi.$$

Now, for each $\epsilon > 0$, there exist $\delta > 0$ such that $\epsilon \leq \rho(\mathfrak{p}(\pi), \xi(\pi)) < \epsilon + \delta$.

$$\begin{aligned}
\rho(\mathsf{S}\mathfrak{p}(\pi), \mathsf{S}\xi(\pi)) &= \|\mathsf{S}\mathfrak{p}(\pi), \mathsf{S}\xi(\pi)\| \\
&= \|p(\pi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \pi, \mathfrak{p}(\pi))d\pi + \lambda_2 \int \mathfrak{B}_2(\varphi, \pi, \mathfrak{p}(\pi))d\pi \\
&\quad - p(\pi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \eta, \mathfrak{p}(\pi))d\pi - \lambda_2 \int \mathfrak{B}_2(\varphi, \pi, \mathfrak{p}(\pi))d\pi\| \\
&\leq \lambda_1(\frac{s}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|) + \lambda_2(\frac{k}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|) \\
&\leq \frac{s}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)| + \frac{s}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)| \\
&= \frac{s}{2}\rho(\mathfrak{p}(\pi), \xi(\pi)) \\
&< \frac{1}{2}(\epsilon + \delta) \\
&< \epsilon.
\end{aligned}$$

Hence, all conditions of Theorems 3.3.1 holds. This imlpies,S has a FP, and (3.47) has a unique solution. □

Example 3.4.1. Consider

$$\mathfrak{p}(\varphi) = 0.001\varphi + 0.2 \int_0^{\varphi} (\frac{\pi}{4} - 0.2\varphi)\mathfrak{p}(\varphi)\mathrm{d}\pi + sin(0.1) \int_0^{\varphi} (-\varphi + \frac{\pi}{3} + 1)\mathfrak{p}(\varphi)\mathrm{d}\pi. \quad (3.48)$$

Here, φ is independent on π. The first integral gives the solution

$$\int_0^{\varphi} (\frac{\pi}{4} - 0.2\varphi)\mathfrak{p}(\gamma)\mathrm{d}\pi = \mathfrak{p}(\varphi)[\frac{\varphi^2 - 1.6\varphi}{8}] \quad (3.49)$$

The second integral gives

$$\int_0^{\varphi} (-\varphi + \frac{\pi}{3} + 1)\mathfrak{p}(\varphi)\mathrm{d}\pi = \mathfrak{p}(\varphi)[\frac{-6\varphi^2 + \varphi^2 + 6\varphi}{6}] \quad (3.50)$$

By putting 3.49 and 3.50 in 3.48, the solution is obtained

$$\mathfrak{p}(\varphi) = \frac{0.01\varphi}{0.0582\varphi^2 - 0.0598\varphi + 1}.$$

Figure illustrate the solution of integral equation with φ values on x-axis and numerical values of $\mathfrak{p}(\varphi)$ on y-axis.

$\mathfrak{p}(0) = 0$	$\mathfrak{p}(1) = 0.010016$
$\mathfrak{p}(3) = 0.022314$	$\mathfrak{p}(4) = 0.023641$
$\mathfrak{p}(5) = 0.023191$	$\mathfrak{p}(6) = 0.021926$
$\mathfrak{p}(7) = 0.020389$	$\mathfrak{p}(8) = 0.018839$
$\mathfrak{p}(9) = 0.017387$	$\mathfrak{p}(10) = 0.016072$
$\mathfrak{p}(11) = 0.01489$	$\mathfrak{p}(12) = 0.01385$
$\mathfrak{p}(13) = 0.01292$	$\mathfrak{p}(14) = 0.0121$
$\mathfrak{p}(15) = 0.01136$	$\mathfrak{p}(16) = 0.0107$
$\mathfrak{p}(17) = 0.01011$	$\mathfrak{p}(18) = 0.00958$
$\mathfrak{p}(19) = 0.0091$	$\mathfrak{p}(20) = 0.00866$
$\mathfrak{p}(21) = 0.00826$	$\mathfrak{p}(22) = 0.00789$
$\mathfrak{p}(23) = 0.00756$	$\mathfrak{p}(24) = 0.00725$
$\mathfrak{p}(25) = 0.00696$	$\mathfrak{p}(26) = 0.0067$
$\mathfrak{p}(27) = 0.00645$	$\mathfrak{p}(28) = 0.00622$
$\mathfrak{p}(29) = 0.00601$	$\mathfrak{p}(30) = 0.00581$
$\mathfrak{p}(31) = 0.00562$	$\mathfrak{p}(32) = 0.00545$
$\mathfrak{p}(33) = 0.00528$	$\mathfrak{p}(34) = 0.00513$
$\mathfrak{p}(35) = 0.00498$	$\mathfrak{p}(36) = 0.00484$
$\mathfrak{p}(37) = 0.00471$	$\mathfrak{p}(38) = 0.00459$
$\mathfrak{p}(39) = 0.00447$	$\mathfrak{p}(40) = 0.00436$

TABLE 3.1: Solution of Integral Equation

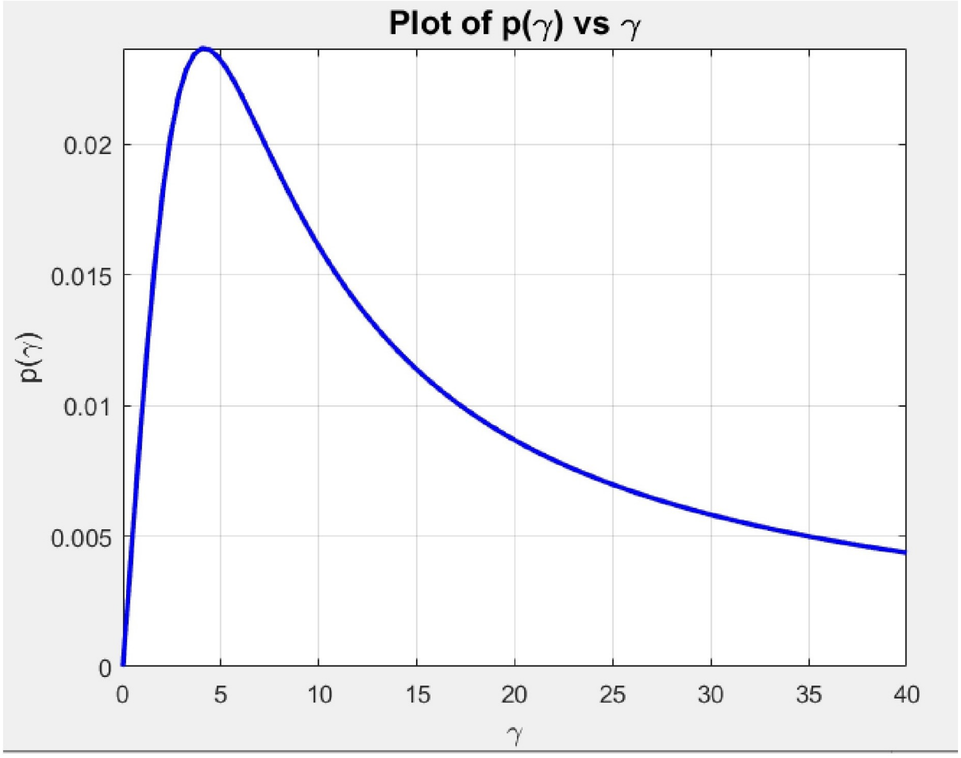

FIGURE 3.1: Convergence behaviour

Chapter 4

Fixed Point Theorems for $(\alpha - \theta)$ Meir-Keeler Contraction in Bipolar b-Metric Space

In this chapter we prove the FP results for M-K contractive mapping, specially in the environment of complete BbMS, using the (c)-comparison function. An example is presented to validate the result.

4.1 Contravariant M-K Contraction

In this section we prove the FPs theorems for M-K contraction in BbMS.

Definition 4.1.1. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BbMS with $b \geq 1$ and $\theta \in \Theta$. Consider $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ is a cont mapping. Then for each $\epsilon > 0$, there exist $\delta > 0$ such that

$$\epsilon \leq \psi(\rho(\mu, \nu)) < \epsilon + \delta \Rightarrow \alpha(\mu, \mathsf{S}\mu)\alpha(\mathsf{S}\nu, \nu)\kappa(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \frac{\epsilon}{\mathsf{b}}, \tag{4.1}$$

for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$ and $\alpha : \mathfrak{M} \times \mathfrak{N} \to \mathbb{R}$. Then, S cont $(\alpha - \theta)$ M-K contractive mapping.

Remark 4. By (4.1), we conclude that

$$\alpha(\mu, \mathsf{S}\mu)\alpha(\mathsf{S}\nu, \nu)\theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \frac{\theta(\rho(\mu, \nu))}{\mathsf{b}}$$

,

if $\mu \neq \nu$.

When $\mu = \nu$ we get

$\alpha(\mu, \mathsf{S}\mu)\alpha(\mathsf{S}\nu, \nu)\theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) \leq \frac{\theta(\rho(\mu,\nu))}{\mathsf{b}}$.

Note: From the Definition 3.1.1 we add the hypothesis that the function θ is linear throughout the chapter to prove the FP results.

Theorem 4.1.1. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a complete BbMS. Suppose that $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ is a cont $(\alpha - \theta)$ M-K contractive mapping which satisfies the following conditions:

(i) S is α-orbital admissible;

(ii) there exist $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$;

(iii) S is continuous;

(iv) if $\mathsf{S}\mu = \mu$ then $\alpha(\mu, \mathsf{S}\mu) \geq 1$;

then S has a unique FP.

Proof. Let $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$. Construct the sequences $\{\mu_n\}$ and $\{\nu_n\}$ by taking $\nu_n = \mathsf{S}\mu_n$ and $\mu_{n+1} = \mathsf{S}\nu_n$ for all $\mathsf{n} \in \mathbb{N}$. It follows, $\{\mu_n, \nu_n\}$ is a bisequence. By (ii) $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$, therefore

$$\begin{aligned}
\alpha(\mu_0, \nu_0) = \alpha(\mu_0, \mathsf{S}\mu_0) \geq 1 &\Rightarrow \alpha(\mathsf{S}^2\mathsf{u}_0, \mathsf{S}\mu_0) = \alpha(\mu_1, \nu_0) \geq 1, \\
\alpha(\mu_1, \nu_0) = \alpha(\mathsf{S}\nu_0, \nu_0) \geq 1 &\Rightarrow \alpha(\mathsf{S}\nu_0, \mathsf{S}^2\mathsf{v}_0) = \alpha(\mu_1, \nu_1) \geq 1, \\
\alpha(\mu_1, \nu_1) = \alpha(\mu_1, \mathsf{S}\nu_1) \geq 1 &\Rightarrow \alpha(\mathsf{S}^2\mathsf{u}_1, \mathsf{S}\mu_1) = \alpha(\mu_2, \nu_1) \geq 1, \\
\alpha(\mu_2, \nu_1) = \alpha(\mathsf{S}\nu_1, \nu_1) \geq 1 &\Rightarrow \alpha(\mathsf{S}\nu_1, \mathsf{S}^2\mathsf{v}_1) = \alpha(\mu_2, \nu_2) \geq 1.
\end{aligned}$$

By continuing this process

$$\alpha(\mu_n,\nu_n)\geq 1 \text{ and } \alpha(\mu_{n+1},\nu_n)\geq 1 \ \ \forall\ n\in\mathbb{N}. \tag{4.2}$$

By using Remark 4 and (4.2),

$$\begin{aligned}\theta(\rho(\mu_n,\nu_n)) = \theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)) &\leq \alpha(\mu_n,\nu_n)\alpha(\mu_n,\nu_{n-1})\theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)),\\ &= \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_{n-1},\nu_{n-1})\theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)),\\ &< \frac{\theta(\rho(\mu_n,\nu_{n-1}))}{\mathsf{b}}.\end{aligned}$$

Again using Remark 4 together with (4.2),

we have

$$\begin{aligned}\theta(\rho(\mu_{n+1},\nu_n)) = \theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)) &\leq \alpha(\mu_n,\nu_n)\alpha(\mu_{n+1},\nu_n)\theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)),\\ &= \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_n,\nu_n)\theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)),\\ &< \frac{\theta(\rho(\mu_n,\nu_n))}{\mathsf{b}}.\end{aligned}$$

Mathematical induction together with (4.3) and (4.3) gives

$$\theta(\rho(\mu_n,\nu_n)) < \frac{\theta(\rho(\mu_{n-1},\nu_{n-1}))}{\mathsf{b}} \quad \forall\ n\in\mathbb{N} \tag{4.3}$$

and

$$\theta(\rho(\mu_{n+1},\nu_n)) < \frac{\theta(\rho(\mu_n,\nu_{n-1}))}{\mathsf{b}} \quad \forall\ n\in\mathbb{N}. \tag{4.4}$$

From (4.3) and (4.4), it is clear that the sequences $\{\theta(\rho(\mu_n,\nu_n))\}$ and $\{\theta(\rho(\mu_{n+1},\nu_n))\}$ are monotonically decreasing and it consist positive real numbers. Therefore, the sequences must be convergent.

Let $\{\theta(\rho(\mu_n,\nu_n))\}\to\mathsf{c}_1$ and $\{\theta(\rho(\mu_{n+1},\nu_n))\}\to\mathsf{c}_2$

as $\mathsf{n}\to\infty$, where $\mathsf{c}_1,\mathsf{c}_2\geq 0$.

Now, we prove that $\mathsf{c}_1=0$ and $\mathsf{c}_2=0$.

Suppose on contrary that $\mathsf{c}_1>0$.

Clearly, $\theta(\rho(\mu_n, \nu_n)) \geq \mathsf{c}_1 > 0 \ \ \forall\, n \in \mathbb{N}$.

Let $\epsilon = \mathsf{c}_1$. Then, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\epsilon \leq \theta(\rho(\mu_{n_0}, \nu_{n_0})) < \epsilon + \delta. \tag{4.5}$$

From (4.1), we have

$$\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})) \leq \alpha(\mu_{n_0+1}, \nu_{n_0+1})\alpha(\mu_{n_0+1}, \nu_{n_0})\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})),$$

$$= \alpha(\mu_{n_0+1}, \mathsf{S}\mu_{n_0+1})\alpha(\mathsf{S}\nu_{n_0}, \nu_{n_0})\theta(\rho(\mathsf{S}\nu_{n_0}, \mathsf{S}\mu_{n_0+1}))$$
$$< \epsilon = \mathsf{c}_1,$$

which contradicts $\mathsf{c}_1 > 0$. So, $\mathsf{c}_1 = 0$.

Similarly, we can prove that $\mathsf{c}_2 = 0$.

Hence, $\theta(\rho(\mu_n, \nu_n)) \to 0$ and $\theta(\rho(\mu_{n+1}, \nu_n)) \to 0$ as $\mathsf{n} \to \infty$. As θ is continuous at $s = 0$, therefore

$$\rho(\mu_n, \nu_n) \to 0 \text{ and } \rho(\mu_{n+1}, \nu_n) \to 0 \text{ as } n \to \infty. \tag{4.6}$$

For any $\epsilon > 0$, there exist $\delta > 0$ that satisfies (4.1). For simplicity, we can take $\delta < \epsilon$. As $\theta(\rho(\mu_n, \nu_n)) \to 0$ and $\theta(\rho(\mu_{n+1}, \nu_n)) \to 0$, there exist $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{N}$ such that

$$\theta(\rho(\mu_{n-1}, \nu_{n-1})) < \frac{\delta}{3b} \ \ \forall\, n \geq \mathcal{M}_1, \tag{4.7}$$

$$\theta(\rho(\mu_n, \nu_{n-1})) < \frac{\delta}{3b} \ \ \forall\, n \geq \mathcal{M}_2. \tag{4.8}$$

Now, we will prove these inequalities

$$\theta(\rho(\mu_{n+l}, \nu_n)) < \frac{\epsilon}{b} \tag{4.9}$$

and

$$\theta(\rho(\mu_n, \nu_{n+l})) < \frac{\epsilon}{b}, \ \forall\, n \geq \mathcal{M}, \tag{4.10}$$

where

$$\mathcal{M} = \max\{\mathcal{M}_1, \mathcal{M}_2\}$$

.

By mathematical induction, we can prove that

$$\theta(\rho(\mu_{n+l},\nu_n)) < \frac{\epsilon}{b}.$$

From (4.8), the result is true for $\ell = 1$.

Let the inequality holds for

$$\ell = \mathsf{q},$$

such that

$$\theta(\rho(\mu_{n+q},\nu_n)) < \frac{\epsilon}{b},\ \forall\ n \geq \mathcal{M}. \tag{4.11}$$

Now, using the definition of bipolar metric space together with (4.7),(4.8) and (4.11),

$$\theta(\rho(\mu_{n+q},\nu_{n-1})) \leq \theta[b(\rho(\mu_{n+q},\nu_n)+\rho(\mu_n,\nu_n)+\rho(\mu_n,\nu_{n-1}))],$$

$$\begin{aligned}
&\leq\ b[\theta(\rho(\mu_{n+q},\nu_n))+\theta(\rho(\mu_n,\nu_n))+\theta(\rho(\mu_n,\nu_{n-1}))],\\
&< b(\frac{\delta}{3b}+\frac{\delta}{3b}+\frac{\epsilon}{b}) = b(\frac{2\delta}{3b}+\frac{\epsilon}{b})\\
&= \frac{2\delta}{3}+\epsilon\\
&< \epsilon+\delta.
\end{aligned}$$

If

$$\theta(\rho(\mu_{n+q},\nu_{n-1})) \geq \epsilon,$$

then by (4.1),

$$\begin{aligned}
\theta(\rho(\mu_{n+q+1},\nu_n)) &\leq \alpha(\mu_n,\nu_n)\alpha(\mu_{n+q+1},\nu_{n+q})\theta(\rho(\mu_{n+q+1},\nu_n)),\\
&= \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_{n+q},\nu_{n+q})\theta(\rho(\mathsf{S}\nu_{n+q},\mathsf{S}\mu_n)),\\
&< \frac{\epsilon}{b}.
\end{aligned}$$

So, (4.9) holds for $\ell = \mathsf{q}+1$.

Hence,

$$\rho(\mu_n,\nu_m) < \frac{\epsilon}{b} = \epsilon_1\ \ \forall\ n > m \geq \mathcal{M}. \tag{4.12}$$

Similarly, we can prove (4.10).

Using the definition of BMS together with (4.7) and (4.8), to obtain

$$\begin{aligned}\theta(\rho(\mu_n,\nu_{n+1})) &\leq \theta[b(\rho(\mu_n,\nu_n)+\rho(\mu_{n+1},\nu_n)+\rho(\mu_{n+1},\nu_{n+1}))]\\ &\leq\ b[\theta(\rho(\mu_{n+1},\nu_{n+1}))+\theta\rho(\mu_{n+1},\nu_n))+\theta\rho(\mu_n,\nu_n))]\\ &\leq\ b[\frac{\delta}{3b}+\frac{\delta}{3b}+\frac{\delta}{3b}]\\ &=\delta<\frac{\epsilon}{b}.\end{aligned}$$

That is (4.10) holds for $\ell=1$.

Now, suppose that the result is true for some $\ell=\mathsf{q}$, that is,

$$\theta(\rho(\mu_n,\nu_{n+k}))<\frac{\epsilon}{b}\ \ \forall\ n\geq\mathsf{N}. \tag{4.13}$$

By using the definition of BMS,(4.7), (4.8), and (4.13), the following is obtained

$$\begin{aligned}\theta(\rho(\mu_{n-1},\nu_{n+q})) &\leq \theta[b(\rho(\mu_{n-1},\nu_{n-1})+\rho(\mu_n,\nu_{n-1})+\rho(\mu_n,\nu_{n+q}))],\\ &\leq\ b[\theta(\rho(\mu_{n-1},\nu_{n-1}))+\theta\rho(\mu_n,\nu_{n-1}))+\theta\rho(\mu_n,\nu_{n+q}))],\\ &<b(\frac{\delta}{3b}+\frac{\delta}{3b}+\frac{\epsilon}{b})=\frac{2\delta}{3}+\epsilon,\\ &<\epsilon+\delta.\end{aligned}$$

If $\theta(\rho(\mu_{n-1},\nu_{n+q}))\geq\epsilon$, then by (4.1),

$$\begin{aligned}\theta(\rho(\mu_n,\nu_{n+q+1})) &\leq \alpha(\mu_{n+q},\nu_{n+q})\alpha(\mu_{n+1},\nu_n)\theta(\rho(\mu_n,\nu_{n+q+1})),\\ &=\alpha(\mu_{n+q},\mathsf{S}\mu_{n+q})\alpha(\mathsf{S}\nu_n,\nu_n)\theta(\rho(\mathsf{S}\mu_{n+q},\mathsf{S}\nu_n)),\\ &<\frac{\epsilon}{b}.\end{aligned}$$

Hence, (4.10) holds for $\ell=\mathsf{q}+1$.

By using property of θ we say that

$$\rho(\mu_n,\nu_m)<\frac{\epsilon}{b}\ \ \forall\ m>n\geq\mathsf{N}. \tag{4.14}$$

From(4.12) and (4.14), we say that $\{\mu_n,\nu_n\}$ is a Cbs. As $(\mathfrak{M}$,$\mathfrak{N}$,$\rho)$ is a complete BMS, implies $\{\mu_n,\nu_n\}$ is biconvergent. Then, there exist $\mathfrak{r}\in\mathfrak{M}\cap\mathfrak{N}$ such that $\mu_n\to\mathfrak{r}$ and

$\nu_n \to \mathfrak{r}$ as $n \to \infty$. As S is a continuous map.

$$\mu_n \to \mathfrak{r} \text{ implies that } \nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}.$$

To combine $\nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}$ with $\nu_n \to \mathfrak{r}$, we get $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.
Now we will prove the uniqueness of FP. Suppose that S has two distinct FPs $\mathfrak{r}_1$ and $\mathfrak{r}_2$. Then, from the condition (iv),
$\alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1), \alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2) \geq 1$.
Now, by Remark 4,

$$\begin{aligned}\rho(\mathfrak{r}_1, \mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2) &\leq \alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1)\alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2)\rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2)\\ &< \frac{\rho(\mathfrak{r}_1, \mathfrak{r}_2)}{b}\end{aligned}$$

which is a contradiction, so $\mathfrak{r}_1 = \mathfrak{r}_2$. □

Theorem 4.1.2. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a complete BbMS. Suppose that $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ is a cont $(\alpha - \theta)$ M-K contractive mapping which satisfies the following conditions:

- (i) S is α-orbital admissible;
- (ii) there exist $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0, \mathsf{S}\mu_0) \geq 1$;
- (iii) if $\{\mu_n, \nu_n\}$ is a bisequence such that $\alpha(\mu_n, \nu_n) \geq 1$ for all n and $\nu_n \to \mathfrak{r} \in \mathfrak{M} \cap \mathfrak{N}$ as $\mathsf{n} \to \infty$, then $\alpha(\mathsf{S}\mathfrak{r}, \mathfrak{r}) \geq 1$;
- (iv) if $\mathsf{S}\mu = \mu$ then $\alpha(\mu, \mathsf{S}\mu) \geq 1$;

then S has a unique FP.

Proof. From previous Theorem 4.1.1 , we say that $\{\mu_n, \nu_n\}$ is a Cbs. As $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BbMS, then it follows $\{\mu_n, \nu_n\}$ is biconvergent. So, there exist $\mathfrak{r} \in \mathfrak{M} \cap \mathfrak{N}$ such that $\mu_n \to \mathfrak{r}$, $\nu_n \to \mathfrak{r}$. From condition (iii), we have $\alpha(\mathsf{S}\mathfrak{r}, \mathfrak{r}) \geq 1$. By applying the definition of BbMS,

$$\theta(\rho(\mathsf{S}\mathfrak{r}, \mathfrak{r})) \leq \theta[\mathsf{b}((\rho(\mathsf{S}\mathfrak{r}, \mathsf{S}\mu_n) + \rho(\mathsf{S}\nu_n, \mathsf{S}\mu_n) + \rho(\mathsf{S}\nu_n, \mathfrak{r}))].$$

From the definition of θ

$$\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) \leq \mathsf{b}[\theta(\rho(\mathsf{S}\mathfrak{r},\mathsf{S}\mu_n)) + \theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n)) + \theta(\rho(\mathsf{S}\nu_n,\mathfrak{r}))].$$

Using (4.1), to obtain

$$\begin{aligned}\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) \leq{}& \mathsf{b}[\alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\mathfrak{r},\mathfrak{r})\theta(\rho(\mathsf{S}\mathfrak{r},\mathsf{S}\mu_n))],\\ &+\mathsf{b}[\alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_n,\nu_n)\theta(\rho(\mathsf{S}\nu_n,\mathsf{S}\mu_n))],\\ &+\mathsf{b}[\theta(\rho(\mathsf{S}\nu_n,\mathfrak{r}))].\end{aligned}$$

Then by Remark 4

$$\begin{aligned}\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) &\leq \mathsf{b}(\theta(\rho(\mu_n,\mathfrak{r}))) + \mathsf{b}(\theta(\rho(\mu_n,\nu_n))) + \mathsf{b}(\theta(\rho(\mu_{n+1},\mathfrak{r}))),\\ &\leq \mathsf{b}[\theta(\rho(\mu_n,\mathfrak{r})) + (\theta(\rho(\mu_n,\nu_n)) + (\theta(\rho(\mu_{n+1},\mathfrak{r}))].\end{aligned}$$

Letting $\mathsf{n}\to\infty$ in above inequality and using (4.14),

$$\theta(\rho(\mathsf{S}\mathfrak{r},\mathfrak{r})) \leq 0.$$

By using the property of θ, we say that $\rho(\mathsf{S}\mathfrak{r},\mathfrak{r}) = 0.$, hence, $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.

Now we will prove its uniqueness. For this let us suppose that S has two distinct FPs $\mathfrak{r}_1$ and $\mathfrak{r}_2$. Then, from the condition (iv),

$\alpha(\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_1), \alpha(\mathfrak{r}_2,\mathsf{S}\mathfrak{r}_2) \geq 1.$

Now, by Remark 4,

$$\begin{aligned}\rho(\mathfrak{r}_1,\mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_2) &\leq \alpha(\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_1)\alpha(\mathfrak{r}_2,\mathsf{S}\mathfrak{r}_2)\rho(\mathsf{S}\mathfrak{r}_1,\mathsf{S}\mathfrak{r}_2)\\ &< \frac{\rho(\mathfrak{r}_1,\mathfrak{r}_2)}{\mathsf{b}},\end{aligned}$$

which is a contradiction and so $\mathfrak{r}_1 = \mathfrak{r}_2$. □

4.2 Generalized Contravariant M-K Contraction

Definition 4.2.1. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a BbMS and $\theta \in \Theta$, Suppose $\mathsf{S} : (\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ be a cont mapping and that for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \theta(\mathcal{R}(\mu,\nu)) < \epsilon + \delta \Rightarrow \alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu)) < \frac{\epsilon}{\mathsf{b}}. \tag{4.15}$$

where $\mathcal{R}(\mu,\nu) = \max\{\rho(\mu,\nu), \rho(\mu,\mathsf{S}\mu), \rho(\mathsf{S}\nu,\nu), \frac{\rho(\mu,\mathsf{S}\mu)+\rho(\mathsf{S}\nu,\nu)}{2}\}$, for all $(\mu,\nu) \in \mathfrak{M}\times\mathfrak{N}$. Then, S is generalized cont $(\alpha-\theta)$ M-K contractive mapping.

Remark 5. Using (4.15), we conclude that
$\alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu)) < \frac{\theta(\mathsf{M}(\mu,\nu))}{\mathsf{b}}$, when $\mu \neq \nu$.
If $\mu = \nu$ we get $\alpha(\mu,\mathsf{S}\mu)\alpha(\mathsf{S}\nu,\nu)\theta(\rho(\mathsf{S}\nu,\mathsf{S}\mu)) \leq \frac{\theta(\mathsf{M}(\mu,\nu))}{\mathsf{b}}$.

Theorem 4.2.1. Let $(\mathfrak{M},\mathfrak{N},\rho)$ be a complete BbMS. Suppose that $\mathsf{S} : (\mathfrak{M},\mathfrak{N}) \rightleftarrows (\mathfrak{M},\mathfrak{N})$ is a generalized cont $(\alpha-\theta)$ M-K contractive mapping. If the following conditions hold,

(i) : S is α-orbital admissible;

(ii) : there exist $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0,\mathsf{S}\mu_0) \geq 1$;

(iii) : S is orbital continuous;

(iv) : if $\mathsf{S}\mu = \mu$ then $\alpha(\mu,\mathsf{S}\mu) \geq 1$;

then S has a unique FP.

Proof. Let $\mu_0 \in \mathfrak{M}$ such that $\alpha(\mu_0,\mathsf{S}\mu_0) \geq 1$. Construct sequences $\{\mu_n\}$ and $\{\nu_n\}$ by taking $\nu_n = \mathsf{S}\mu_n$ and $\mu_{n+1} = \mathsf{S}\nu_n$ for all $\mathsf{n} \in \mathbb{N}$. It follows $\{\mu_n,\nu_n\}$ is a bisequence. Theorem 4.1.1 and condition (iii) follows the result

$$\alpha(\mu_n,\nu_n) \geq 1 \text{ and } \alpha(\mu_{n+1},\nu_n) \geq 1 \ \forall \ \mathsf{n} \in \mathbb{N}. \tag{4.16}$$

From Remark 5 and (4.16)

$$\begin{aligned}
\theta(\rho(\mu_n,\nu_n)) &= \theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n))\\
&\leq \alpha(\mu_n,\mathsf{S}\mu_n)\alpha(\mathsf{S}\nu_{n-1},\nu_{n-1})\theta(\rho(\mathsf{S}\nu_{n-1},\mathsf{S}\mu_n)),\\
&< \frac{\theta(\mathcal{R}(\mu_n,\nu_{n-1}))}{\mathsf{b}},\\
&= \frac{1}{\mathsf{b}}\theta\bigg(\max\{\rho(\mu_n,\nu_{n-1}),\rho(\mu_n,\mathsf{S}\mu_n),\rho(\mathsf{S}\nu_{n-1},\nu_{n-1}),\\
&\qquad\qquad \frac{\rho(\mu_n,\mathsf{S}\mu_n)+\rho(\mathsf{S}\nu_{n-1},\nu_{n-1})}{2}\}\bigg),\\
&= \frac{1}{\mathsf{b}}\theta\bigg(\max\{\rho(\mu_n,\nu_{n-1}),\rho(\mu_n,\nu_n),\rho(\mu_n,\nu_{n-1}),\\
&\qquad\qquad \frac{\rho(\mu_n,\nu_n)+\rho(\mu_n,\nu_{n-1})}{2}\}\bigg),\\
&\leq \frac{\theta}{\mathsf{b}}(\max\{\rho(\mu_n,\nu_n),\rho(\mu_n,\nu_{n-1})\}).
\end{aligned}$$

Since θ is a non-decreasing function, so $\rho(\mu_n,\nu_n) \leq \frac{\max\{\rho(\mu_n,\nu_n),\rho(\mu_n,\nu_{n-1})\}}{\mathsf{b}}$.
If possible, suppose that $\rho(\mu_n,\nu_n) > \rho(\mu_n,\nu_{n-1})$, then $\rho(\mu_n,\nu_n) < \frac{\rho(\mu_n,\nu_n)}{\mathsf{b}}$, which is not possible. Hence,

$$\rho(\mu_n,\nu_n) \leq \frac{\rho(\mu_n,\nu_{n-1})}{\mathsf{b}},\ \forall\ \mathsf{n}\in\mathbb{N}. \tag{4.17}$$

Similarly, we can say that

$$\rho(\mu_{n+1},\nu_n) \leq \frac{\rho(\mu_n,\nu_n)}{\mathsf{b}},\ \forall\ \mathsf{n}\in\mathbb{N}. \tag{4.18}$$

From (4.17) and (4.18), it is clear that $\{\rho(\mu_n,\nu_n)\}$ and $\{\rho(\mu_{n+1},\nu_n)\}$ are monotonically decreasing sequences of positive real numbers and hence convergent.
Suppose that $\{\rho(\mu_n,\nu_n)\}\to \mathsf{c}_1$ and $\{\rho(\mu_{n+1},\nu_n)\}\to \mathsf{c}_2$ as $\mathsf{n}\to\infty$, where $\mathsf{c}_1,\mathsf{c}_2\geq 0$. It follows that

$$\begin{aligned}
\lim_{n\to\infty}\{\theta(\rho(\mu_{\mathsf{n}},\nu_{\mathsf{n}}))\} &= \lim_{n\to\infty}\{\theta(\mathcal{R}(\mu_{\mathsf{n}},\nu_{\mathsf{n}}))\}\\
&= \theta(\mathsf{c}_1),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}\{\theta(\rho(\mu_{\mathsf{n+1}},\nu_{\mathsf{n}}))\} &= \lim_{n\to\infty}\{\theta(\mathcal{R}(\mu_{\mathsf{n+1}},\nu_{\mathsf{n}}))\}\\
&= \theta(\mathsf{c}_2).
\end{aligned}$$

Now, we prove that $\mathsf{c}_1 = 0$ and $\mathsf{c}_2 = 0$.

Firstly, suppose that $\mathsf{c}_1 > 0$.

Clearly, $\rho(\mu_n, \nu_n) \geq \mathsf{c}_1 > 0$ for all $\mathsf{n} \in \mathbb{N}$.

Consider $\epsilon = \mathsf{c}_1$. then there exist $\delta > 0$ and $\mu_0 \in \mathbb{N}$ such that

$$\theta(\epsilon) \leq \theta(\mathcal{R}(\mu_{n_0}, \nu_{n_0})) < \theta(\epsilon) + \delta. \tag{4.19}$$

From (4.15),

$$\begin{aligned}\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})) &\leq \alpha(\mu_{n_0+1}, \nu_{n_0+1})\alpha(\mu_{n_0+1}, \nu_n)\theta(\rho(\mu_{n_0+1}, \nu_{n_0+1})) \\ &= \alpha(\mathsf{S}\nu_{n_0}, \nu_{n_0})\alpha(\mu_{n_0+1}, \mathsf{S}\mu_{n_0+1})\theta(\rho(\mathsf{S}\nu_{n_0}, \mathsf{S}\mu_{n_0+1})) \\ &< \theta(\frac{\epsilon}{\mathsf{b}}).\end{aligned}$$

From the definition of θ,

$$\rho(\mu_{n_0+1}, \nu_{n_0+1}) < \frac{\epsilon}{\mathsf{b}} = \frac{\mathsf{c}_1}{\mathsf{b}}, \tag{4.20}$$

a contradiction, therefore $\mathsf{c}_1 = 0$.

Similarly, we can prove that $\mathsf{c}_2 = 0$.

Now, we want to show that $\{\mu_n, \nu_n\}$ is a Cbs; that is

$\lim_{n,m\to\infty} \rho(\mu_n, \nu_m) = 0$. For this we suppose on contrary that $\{\mu_n, \nu_n\}$ is not a Cbs, then there exist $\epsilon > 0$ and subsequences $\{n_{(r)}\}$ and $\{n_{(r+1)}\}$ of $\mathbb{N}$ such that

$$\rho(\mu_{n_{(r)}}, \nu_{n_{(r+1)}}) > \epsilon, \tag{4.21}$$

for every $\mathsf{r} \in \mathbb{N}$. It follows, for any given $\epsilon > 0$ there exist $\delta > 0$ such that 4.15 holds. Set $\mathsf{h} = \min\{\epsilon, \delta\}$. Since $\rho(\mu_n, \nu_n)$ and $\rho(\mu_{n+1}, \nu_n) \to 0$ as $\mathsf{n} \to \infty$, there exist $\mathsf{n}_1, \mathsf{n}_2 \in \mathbb{N}$ such that

$$\rho(\mu_n, \nu_n) < \frac{h}{8b} \ \forall\ \mathsf{n} \geq \mathsf{n}_1, \tag{4.22}$$

and

$$\rho(\mu_{n+1}, \nu_n) < \frac{h}{8b} \ \forall\ \mathsf{n} \geq \mathsf{n}_2. \tag{4.23}$$

Choose $\mathcal{M} = max\{\mathsf{n}_1, \mathsf{n}_2\}$, then 4.22 and 4.23 holds for $\mathsf{n} \geq \mathcal{M}$. Let $\mathsf{n}_{(\mathsf{r})} > \mathcal{M}$. We get $\mathsf{n}_{(\mathsf{r})} \leq \mathsf{n}_{(\mathsf{r+1})} - 1$. If $\rho(\mu_{n_{(r)}}, \nu_{n_{(r+1)}-1}) \leq \frac{1}{\mathsf{b}}(\epsilon + \frac{h}{2})$; then, using the definition of BbMS, (4.22) and (4.23),

$$\begin{aligned}\rho(\mu_{n_{(r)}},\nu_{n_{(r+1)}}) &\leq \mathsf{b}[\rho(\mu_{n_{(r)}},\nu_{n_{(r+1)}-1})+\rho(\mu_{n_{(r+1)}},\nu_{n_{(r+1)}-1})+\rho(\mu_{n_{(r+1)}},\nu_{n_{(r+1)}-1})],\\ &< \mathsf{b}[\frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2})+\frac{h}{8b}+\frac{h}{8b}],\\ &= \mathsf{b}[\frac{\epsilon}{\mathsf{b}}+\frac{3h}{4b}],\\ &= \epsilon+\frac{3h}{4},\\ &< \epsilon,\end{aligned}$$

this contradict the result. So, there exist p such that $\mathsf{n}_{(\mathsf{r})} \leq p < \mathsf{n}_{(\mathsf{r+1})}$ and $\rho(\mu_{n_{(r)}},\nu_k) > \epsilon+\frac{h}{2}$.

Now if $\rho(\mu_{n_{(r+1)}},\nu_{n_{(r)}}) \geq \frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2})$, then $\rho(\mu_{n_{(r+1)}},\nu_{n_{(r)}}) \geq \frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2}) > \frac{1}{\mathsf{b}}(h+\frac{h}{2}) > \frac{h}{8b^2}$,

it contradicts the result.

So, there exist values of p such that $\mathsf{n}_(\mathsf{r}) \leq \mathsf{k} \leq \mathsf{n}_{(\mathsf{r+1})}$ implies $\rho(\mu_{n_{(r)}},\nu_k) < \frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2})$.

Select the integer p with $p \geq \mathsf{n}_{(\mathsf{r})}$ such that $\rho(\mathsf{u}_{\mathsf{n_r}},\nu_k) \geq \frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2})$. Thus, $\rho(\mu_{n_{(r)}},\nu_{k-1}) < \frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2})$.

Using the definition of BbMS and (4.23),

$$\begin{aligned}\rho(\mu_{n_{(r)}},\nu_k) &\leq \mathsf{b}[\rho(\mu_{n_{(r)}},\nu_{k-1})+\rho(\mu_k,\nu_{k-1})+\rho(\mu_k,\nu_k)],\\ &\leq \mathsf{b}[\frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2})+\frac{h}{8b}+\frac{h}{8b}],\\ &= \epsilon+\frac{3}{4}h.\end{aligned}$$

Now, we can select k satisfying $\mathsf{n}_{(\mathsf{r})} \leq \mathsf{k} \leq \mathsf{n}_{(\mathsf{r+1})}$ such that

$$\frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2}) \leq \rho(\mu_{n_{(r)}},\nu_k) < \epsilon+\frac{3}{4}h. \tag{4.24}$$

Therefore,

$$\rho(\mu_{n_{(r)}},\nu_k) \leq \epsilon+\frac{3}{4}h < \epsilon+\mathsf{h} \tag{4.25}$$

$$\rho(\mu_{n_{(r)}},\nu_{n_{(r)}}) \leq \frac{h}{8b} < \epsilon+\mathsf{h} \tag{4.26}$$

$$\rho(\mu_{k+1},\nu_k) \leq \frac{h}{8b} < \epsilon + \mathsf{h}. \tag{4.27}$$

Now, from (4.25)-(4.27) we have

$\epsilon \leq \mathcal{R}(\mu_{n(r)},\nu_k) < \epsilon + \mathsf{h} \leq \epsilon + \delta$ and so

$\theta(\epsilon) \leq \theta(\mathcal{R}(\mu_{n(r)},\nu_k)) < \theta(\epsilon+\mathsf{h}) \leq \theta(\epsilon+\delta) \leq \theta(\epsilon)+\theta(\delta)$.

As S be a generalized $(\alpha-\theta)$ M-K contractive mapping,

$$\begin{aligned}\theta(\rho(\mu_{k+1},\nu_{n_{(r)}})) &\leq \alpha(\mu_{n_{(r)}},\mathsf{S}\mu_{n_{(r)}})\alpha(\mathsf{S}\mu_k,\nu_k)\theta(\rho(\mathsf{S}\nu_k,\mathsf{S}\mu_{n_{(r)}}))\\ &< \theta(\frac{\epsilon}{\mathsf{b}})\\ &< \theta(\epsilon),\end{aligned}$$

which follows that

$$\rho(\mu_{k+1},\nu_{n(r)}) < \epsilon. \tag{4.28}$$

Since ρ is a BbMS,

$$\rho(\mu_{n_{(r)}},\nu_k) \leq \mathsf{b}[\rho(\mu_{n_{(r)}},\nu_{n_{(r)}}) + \rho(\mu_{k+1},\nu_{n_{(r)}}) + \rho(\mu_{k+1},\nu_k)],$$

which implies that

$$\begin{aligned}\rho(\mu_{n_{(r)}},\nu_k) - \mathsf{b}(\rho(\mu_{n_{(r)}},\nu_{n_{(r)}})) - \mathsf{b}(\rho(\mu_{k+1},\nu_k)) &\leq \mathsf{b}(\rho(\mu_{k+1},\nu_{n_{(r)}})),\\ \frac{1}{\mathsf{b}}(\epsilon+\frac{h}{2}) - \frac{h}{8b} - \frac{h}{8b} &< \rho(\mu_{k+1},\nu_{n_{(r)}}).\end{aligned}$$

This shows that

$$\epsilon < \rho(\mu_{k+1},\nu_{n_{(r)}}). \tag{4.29}$$

This contradicts (4.28).

So, $\{\mu_n,\nu_n\}$ is a Cbs.

As $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BbMS, then we say that $\{\mu_n,\nu_n\}$ biconverges. There exist $\mathfrak{r} \in \mathfrak{M}\cap\mathfrak{N}$ such that $\mu_n \to \mathfrak{r}$ and $\nu_n \to \mathfrak{r}$ as $\mathsf{n} \to \infty$.

As S is an orbital continuous map,

$$\mu_n \to \mathfrak{r} \text{ implies that } \nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}.$$

Combining $\nu_n = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}$ with $\nu_n \to \mathfrak{r}$, we have $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.

Uniqueness: Suppose that S has two distinct FPs $\mathfrak{r}_1$ and $\mathfrak{r}_2$. Then, from condition (iv),
$\alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1), \alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2) \geq 1$.
Now, by Remark 5,

$$\begin{aligned}\rho(\mathfrak{r}_1, \mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2) &\leq \alpha(\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_1)\alpha(\mathfrak{r}_2, \mathsf{S}\mathfrak{r}_2)\rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2)\\ &< \frac{\rho(\mathfrak{r}_1, \mathfrak{r}_2)}{\mathsf{b}},\end{aligned}$$

which is a contradiction and so $\mathfrak{r}_1 = \mathfrak{r}_2$. □

4.3 Covariant M-K Contraction

Definition 4.3.1. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BbMS. Suppose that $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ be a cov mapping then for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \rho(\mu, \nu) < \epsilon + \delta \Rightarrow \rho(\mathsf{S}\mu, \mathsf{S}\nu) < \frac{\epsilon}{\mathsf{b}}, \tag{4.30}$$

for all $\mu, \nu \in \mathfrak{M} \times \mathfrak{N}$.
Then, S is a cov M-K contractive mapping.

Remark 6. By(4.30), we say that $\rho(\mathsf{S}\mu, \mathsf{S}\nu) < \frac{\rho(\mu,\nu)}{\mathsf{b}}$, whenever $\mu \neq \nu$. If $\mu = \nu$ then $\rho(\mathsf{S}\mu, \mathsf{S}\nu) \leq \frac{\rho(\mu,\nu)}{\mathsf{b}}$.

Theorem 4.3.1. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a complete BbMS. Suppose that $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ be a cov M-K contractive mapping. Then S has a unique FP.

Proof. Construct the sequences $\{\mu_n\}$ and $\{\nu_n\}$ by taking $\mu_n = \mathsf{S}\mu_{n-1}$ and $\nu_n = \mathsf{S}\nu_{n-1}$. Using Remark 6 and contractive condition, to obtain

$$\rho(\mu_n, \nu_n) = \rho(\mathsf{S}\mu_{n-1}, \mathsf{S}\nu_{n-1}) \leq \frac{\rho(\mu_{n-1}, \nu_{n-1})}{\mathsf{b}}. \tag{4.31}$$

Again, using Remark 6 together with contractive condition

$$\rho(\mu_n, \nu_{n+1}) = \rho(\mathsf{S}\mu_{n-1}, \mathsf{S}\nu_n) \leq \frac{\rho(\mu_{n-1}, \nu_n)}{\mathsf{b}}. \tag{4.32}$$

From (4.31) and (4.32), it is clear that $\{\rho(\mu_n, \nu_n)\}$ and $\{\rho(\mu_n, \nu_{n+1})\}$ are monotonically decreasing sequences of positive reals and hence convergent. Suppose that $\{(\rho(\mu_n, \nu_n))\} \to \mathsf{c}_1$ and $\{(\rho(\mu_{n+1}, \nu_n))\} \to \mathsf{c}_2$ as $\mathsf{n} \to \infty$, where $\mathsf{c}_1, \mathsf{c}_2 \geq 0$.

Now, we claim that $\mathsf{c}_1 = 0$ and $\mathsf{c}_2 = 0$.

Suppose by contradiction that $\mathsf{c}_1 > 0$.

Clearly, $\rho(\mu_n, \nu_n) \geq \mathsf{c}_1 > 0$ for all $\mathsf{n} \in \mathbb{N}$.

Assume $\epsilon = \mathsf{c}_1$. Then, there exist $\delta > 0$ and $\mathsf{n}_0 \in \mathbb{N}$ such that

$$\epsilon \leq \rho(\mu_{n_0}, \nu_{n_0}) < \epsilon + \delta. \tag{4.33}$$

From (4.30),

$$\begin{aligned} \rho(\mu_{n_0+1}, \nu_{n_0+1}) &\leq \rho(\mu_{n_0+1}, \nu_{n_0+1}), \\ &= \rho(\mathsf{S}\mu_{n_0}, \mathsf{S}\nu_{n_0}) < \frac{\epsilon}{\mathsf{b}} = \frac{\mathsf{c}_1}{\mathsf{b}}, \end{aligned}$$

a contradiction, So $\mathsf{c}_1 = 0$.

Similarly, we can prove that $\mathsf{c}_2 = 0$.

Hence,

$$\rho(\mu_n, \nu_n) \to 0 \text{ and } \rho(\mu_n, \nu_{n+1}) \to 0 \ as \ \mathsf{n} \to \infty. \tag{4.34}$$

For any given $\epsilon > 0$, there exist $\delta > 0$ such that (4.30) holds. For simplicity, we assume that $\delta < \epsilon$. As $\rho(\mu_n, \nu_n) \to 0$ and $\rho(\mu_n, \nu_{n+1}) \to 0$, there exist $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{N}$ such that

$$\rho(\mu_{n-1}, \nu_{n-1}) < \frac{\delta}{3b} \text{ for all } \mathsf{n} \geq \mathcal{M}_1, \tag{4.35}$$

and

$$\rho(\mu_{n-1}, \nu_n) < \frac{\delta}{3b} \text{ for all } \mathsf{n} \geq \mathcal{M}_2. \tag{4.36}$$

Now, we want to prove that

$$\rho(\mu_n, \nu_{n+l}) < \frac{\epsilon}{\mathsf{b}} \tag{4.37}$$

and

$$\rho(\mu_{n+l}, \nu_n) < \frac{\epsilon}{\mathsf{b}}, \text{ for all } \mathsf{n} \geq \mathcal{M}. \tag{4.38}$$

where $\mathcal{M} = \max\{\mathcal{M}_1, \mathcal{M}_2\}$. Now we prove (4.37)Firstly by using mathematical induction, that is $\theta(\rho(\mu_{n+l}, \nu_n)) < \frac{\epsilon}{\mathsf{b}}$. From (4.34), the result is true for $\ell = 1$. Let this inequality holds $\ell = \mathsf{q}$, that is

$$\rho(\mu_n, \nu_{n+q})) < \frac{\epsilon}{\mathsf{b}}, \text{ forall } \mathsf{n} \geq \mathcal{M}. \tag{4.39}$$

Now, by using the definition of BbMS

$$\rho(\mu_{n-1}, \nu_{n+q})) \leq \mathsf{b}[\rho(\mu_{n-1}, \nu_n) + \rho(\mu_n, \nu_n) + \rho(\mu_n, \nu_{n+q})].$$

From (4.35)-(4.39),

$$\begin{aligned}\rho(\mu_{n-1}, \nu_{n+q})) &< \mathsf{b}(\frac{\delta}{3b} + \frac{\delta}{3b} + \frac{\epsilon}{\mathsf{b}}) \\ &= \frac{2\delta}{3} + \epsilon \\ &< \epsilon + \delta.\end{aligned}$$

If $\rho(\mu_{n-1}, \nu_{n+q})) \geq \epsilon$, then by (4.30),

$$\rho(\mu_n, \nu_{n+q+1})) < \frac{\epsilon}{\mathsf{b}}.$$

Hence, (4.37) holds.

If $\rho(\mu_{n+q}, \nu_{n-1})) \leq \epsilon$, then

$$\rho(\mu_{n+q+1}, \nu_n) < \frac{\epsilon}{\mathsf{b}}.$$

So, (4.37) holds for $\ell = \mathsf{q} + 1$.

Hence,

$$\rho(\mu_n, \nu_m) < \frac{\epsilon}{\mathsf{b}}; \quad \forall\ \mathsf{m} > \mathsf{n} \geq \mathcal{M}. \tag{4.40}$$

Similarly, one can prove (4.38), from which we conclude that

$$\rho(\mu_n, \nu_m) < \frac{\epsilon}{\mathsf{b}}; \quad \forall\ \mathsf{n} > \mathsf{m} \geq \mathcal{M}. \tag{4.41}$$

Therefore, $\{\mu_n, \nu_n\}$ is a Cbs. Since $(\mathfrak{M}, \mathfrak{N}, \rho)$ is a complete BbMS, hence $\{\mu_n, \nu_n\}$ biconverges. That is, there exist $\mathfrak{r} \in \mathfrak{M} \cap \mathfrak{N}$ such that $\mu_n \to \mathfrak{r}$ and $\nu_n \to \mathfrak{r}$ as $\mathsf{n} \to \infty$.

Since S is a continuous map

$$\mu_n \to \mathfrak{r} \text{ implies that } \mu_{n+1} = \mathsf{S}\mu_n \to \mathsf{S}\mathfrak{r}.$$

which gives $\mathsf{S}\mathfrak{r} = \mathfrak{r}$.

Now we will prove the uniqueness of FP. For this, we suppose on contrary that $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are two different FPs of S. Then, by Remark 6,

$$\rho(\mathfrak{r}_1, \mathfrak{r}_2) = \rho(\mathsf{S}\mathfrak{r}_1, \mathsf{S}\mathfrak{r}_2) < \frac{\rho(\mathfrak{r}_1, \mathfrak{r}_2)}{\mathsf{b}},$$

which holds only when $\mathfrak{r}_1 = \mathfrak{r}_2$. □

Example 4.3.1. Suppose that $\mathfrak{M} = (-\infty, 0], \mathfrak{N} = [0, \infty)$

Define $\rho : (-\infty, 0] \times [0, \infty) \to [0, \infty)$ as $\rho(\mathsf{u}, \mathsf{v}) = |\mu - \nu|^2$. Then, $(\mathfrak{M}, \mathfrak{N}, \rho)$ is a complete BbMS.

Define $\mathsf{S} : (-\infty, 0] \cup [0, \infty) \rightleftarrows (-\infty, 0] \cup [0, \infty)$ by $\mathsf{S}\mu = \frac{-\mu}{3}\ \forall\ \mu \in (-\infty, 0] \cup [0, \infty)$, and $\theta(\mathsf{t}) = \frac{t}{2}$, $\alpha(\mu, \nu) = 1$ for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$. $\mathsf{S}((-\infty, 0]) \subset [0, \infty)$ and $\mathsf{S}([0, \infty)) \subset (-\infty, 0]$.

Let $\mu \in (-\infty, 0]$, there exist $\mathsf{m} \in [0, \infty)$, such that $\mu = -\mathsf{m}$. Now,

$$\theta(\rho(\mu, \nu)) = \theta(|\mu - \nu|^2) = \theta(|-\mathsf{m} - \nu|^2) = \theta((\mathsf{m} + \nu)^2) = \frac{(\mathsf{m} + \nu)^2}{2},$$
$$\theta(\rho(\mathsf{S}\mu, \mathsf{S}\nu)) = \theta(\rho(\frac{-\nu}{3}, \frac{-\mu}{3})) = \theta(|\frac{-\nu}{3} - (\frac{-\mu}{3})|^2) = \frac{(\mathsf{m} + \nu)^2}{18}.$$

By taking $\delta = 2\epsilon$ and $0 < \epsilon < 1$, (4.1) holds. It follows the conditions of Theorem 4.1.1. From above it is clear that S is continuous. By choosing $\mu = -3$ we have

$$\alpha(-3, \mathsf{S}(-3)) = \alpha(-3, 1)$$
$$= 1.$$

implies

$$\alpha(\mathsf{S}(\mathsf{S}(-3), \mathsf{S}(-3)) = \alpha(\mathsf{S}(1), \mathsf{S}(-3))$$

$$\begin{aligned} &= \alpha(\frac{-1}{3}, 1) \\ &= 1. \end{aligned}$$

It shows that α is orbital admissible. For condition (iii) there exist $\mu_0 = -2 \in \mathfrak{M}$ such that

$$\begin{aligned} \alpha(\mu_0, \mathsf{S}(\mu_0)) &= \alpha(-2, \mathsf{S}(-2)) \\ &= \alpha(-2, \frac{2}{3}) \\ &= 1 \end{aligned}$$

Example 4.3.2. Consider $\mathfrak{M} = (-\infty, 0], \mathfrak{N} = [0, \infty)$

Define $\rho : (-\infty, 0] \times [0, \infty) \to [0, \infty)$ as $\rho(\mathsf{u}, \mathsf{v}) = (\mu - \nu)^2$. Then, $(\mathfrak{M}, \mathfrak{N}, \rho)$ is a complete BbMS.

Define $\mathsf{S} : (-\infty, 0] \cup [0, \infty) \rightleftarrows (-\infty, 0] \cup [0, \infty)$ by $\mathsf{S}\mu = \frac{-\mu}{2}$ $\forall$ $\mu \in (-\infty, 0] \cup [0, \infty)$, and $\theta(\mathsf{t}) = \frac{t^2}{1+t}$, $\alpha(\mu, \nu) = 1$, $\mathcal{R}(\mu, \nu) = max\{\rho(\mu, \nu), \rho(\mu, \mathsf{S}\mu), \rho(\mathsf{S}\nu, \nu), \frac{\rho(\mu,\mathsf{S}\mu)+\rho(\mathsf{S}\nu,\nu)}{2}\}$ for all $(\mu, \nu) \in \mathfrak{M} \times \mathfrak{N}$,

As $\mathsf{S}((-\infty, 0]) \subset [0, \infty)$ and $\mathsf{S}([0, \infty)) \subset (-\infty, 0]$.

Let $\mu \in (-\infty, 0]$, $\nu \in [0, \infty)$. and $(\mathcal{R}(\mu, \nu)) = 2$ Now,

$$\begin{aligned} \theta(\mathcal{R}(\mu, \nu)) &= \theta(2) = \frac{4}{3} = 1.33, \\ \theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) &= \theta(\rho(\frac{-\nu}{2}, \frac{-\mu}{2})) = \theta((\frac{\nu}{2} - \frac{\mu}{2})^2) = \frac{(\mu - \nu)^4}{2\mu - 2\nu + 4}. \end{aligned}$$

By taking $\delta = 2\epsilon$, $0 < \epsilon < 1$, and carefully choosing the values of μ, ν that satisfies (4.15). It follows the conditions of Theorem 4.2.1. From above, condition (i) and (ii) are obvious. For condition (iii) there exist $\mu_0 = -2 \in \mathfrak{M}$ such that

$$\begin{aligned} \alpha(\mu_0, \mathsf{S}(\mu_0)) &= \alpha(-2, \mathsf{S}(-2)) \\ &= \alpha(-2, \frac{2}{3}) \\ &= 1 \end{aligned}$$

Hence, all the condition are satisfied and S has a FP.

Clearly 0 is the FP of S.

Corollary 4.3.1. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BbMS with $b \geq 1$ and $\theta \in \Theta$. Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ be a cont mapping. Then, for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \theta(\rho(\mu, \nu)) < \epsilon + \delta \Rightarrow \theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \frac{\epsilon}{\mathsf{b}A} \tag{4.42}$$

here $\theta \in \Theta$ and $A \geq 1$. Then, S has a FP.

Proof. It follows by choosing $\alpha(\mu, \nu) = \sqrt{A}$ in Theorem 4.1.1. □

Corollary 4.3.2. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BbMS with $b \geq 1$ and $\theta \in \Theta$. Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightleftarrows (\mathfrak{M}, \mathfrak{N})$ be a cont mapping. Then, for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \mathcal{R}(\rho(\mu, \nu)) < \epsilon + \delta \Rightarrow \theta(\rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \frac{\epsilon}{\mathsf{b}A} \tag{4.43}$$

here $\theta \in \Theta$ and $A \geq 1$. Then, S has a FP.

Proof. It follows by choosing $\alpha(\mu, \nu) = \sqrt{A}$ in Theorem 4.1.1. □

Corollary 4.3.3. Let $(\mathfrak{M}, \mathfrak{N}, \rho)$ be a BbMS with $b \geq 1$ and $\theta \in \Theta$. Let $\mathsf{S} : (\mathfrak{M}, \mathfrak{N}) \rightrightarrows (\mathfrak{M}, \mathfrak{N})$ be a cov mapping, Then, for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\epsilon \leq \rho(\mu, \nu)) < \epsilon + \delta \Rightarrow \rho(\mathsf{S}\nu, \mathsf{S}\mu)) < \frac{\epsilon}{\mathsf{b}A} \tag{4.44}$$

here $A \geq 1$. Then, S has a FP.

Proof. It follows by taking $\alpha(\mu, \nu) = \sqrt{A}$ in Theorem 4.3.1. □

4.4 Application

Theorem 4.4.1. Let us consider the following integral equation

$$\mathfrak{p}(\varphi) = p(\varphi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \pi, \mathfrak{p}(\pi))\, d\pi + \lambda_2 \int \mathfrak{B}_2(\varphi, \eta, \mathfrak{p}(\pi))\, d\pi \tag{4.45}$$

$$\varphi \in \mathsf{F}_1 \cup \mathsf{F}_2, \mathsf{F}_1 \cup \mathsf{F}_2$$

is a Lebesgue measurable set with finite measures and λ_1, λ_2 are constants.

(1) Suppose that

$$\mathfrak{B} : \mathsf{F}_1^2 \cup \mathsf{F}_2^2 \times [0,\infty) \to [0,\infty)$$

and

$$\mathfrak{B}_2 : \mathsf{F}_1^2 \cup \mathsf{F}_2^2 \times [0,\infty) \to [0,\infty).$$

(2) There is a CF

$$\sigma : \mathsf{F}_1^2 \cup \mathsf{F}_2^2 \times [0,\infty)$$

and

$$k \in (0,1)$$

such that for all

$$(\varphi,\pi) \in \mathsf{F}_1^2 \cup \mathsf{F}_2^2$$

and

$$p(\varphi) \in \mathsf{L}^\infty(F_1) \cup \mathsf{L}^\infty(F_2).$$

$$\|\lambda_i(\mathfrak{B}_i(\varphi,\pi,\mathfrak{p}(\pi))) - \lambda_i(\mathfrak{B}_i(\varphi,\pi,\xi(\pi)))\|^2 \leq \frac{k}{4}\sigma(\varphi,\pi)|\mathfrak{p}(\pi) - \xi(\pi)|^2,$$

for all $\mathsf{i} = 1,2$ and $\|\sigma(\varphi,\pi)d\pi\|^2 \leq 1$ that is $\sup_{\varphi\in\mathsf{F}_1\cup\mathsf{F}_2} \int |\sigma(\varphi,\pi)d\pi| \leq 1$.

Then, (4.45) gives a unique solution in $\mathsf{L}^\infty(\mathsf{F}_1) \cup \mathsf{L}^\infty(\mathsf{F}_2)$.

Proof. Assume $\mathfrak{M} = \mathsf{L}^\infty(\mathsf{F}_1)$ and $\mathfrak{N} = \mathsf{L}^\infty(\mathsf{F}_2)$. be two normed linear spaces, where F_1 and F_2 are two Lebesgue measurable sets with $p(\mathsf{F}_1 \cup \mathsf{F}_2) < \infty$.
Consider $\rho : \mathfrak{M} \times \mathfrak{N} \to [0,\infty)$ as

$$\rho(\mu,\nu) = \|\mu - \nu\|_\infty^2.$$

Since $(\mathfrak{M},\mathfrak{N},\rho)$ is a complete BbMS with $\mathsf{b} \geq 1$. Define a cov mapping as

$$\mathsf{S}(\mathfrak{p}(\varphi)) = p(\varphi) + \lambda_1 \int \mathfrak{B}_1(\varphi,\pi,\mathfrak{p}(\pi))d\pi + \lambda_2 \int \mathfrak{B}_2(\varphi,\pi,\mathfrak{p}(\pi))d\pi.$$

Now, for each $\epsilon > 0$, there exist $\delta > 0$ such that $\epsilon \leq \rho(\mathfrak{p}(\pi), \xi(\pi)) < \epsilon + \delta$.

$$\begin{aligned}
\rho(\mathsf{S}\mathfrak{p}(\pi), \mathsf{S}\xi(\pi)) &= \|\mathsf{S}\mathfrak{p}(\pi), \mathsf{S}\xi(\pi)\|^2 \\
&= \|p(\pi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \pi, \mathfrak{p}(\pi))\, d\pi + \lambda_2 \int \mathfrak{B}_2(\varphi, \pi, \mathfrak{p}(\pi))\, d\pi \\
&\quad - \xi(\pi) + \lambda_1 \int \mathfrak{B}_1(\varphi, \eta, \mathfrak{p}(\pi))\, d\pi - \lambda_2 \int \mathfrak{B}_2(\varphi, \pi, \mathfrak{p}(\pi))\, d\pi\|^2 \\
&\leq \lambda_1 (\frac{k}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|^2) + \lambda_2 (\frac{k}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|^2) \\
&\leq \frac{k}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|^2 + \frac{k}{4}\sigma(\varphi, \pi)|\mathfrak{p}(\pi) - \xi(\pi)|^2 \\
&= \frac{k}{2}\rho(\mathfrak{p}(\pi), \xi(\pi)) \\
&< \frac{1}{2}(\epsilon + \delta) \\
&< \frac{\epsilon}{b}.
\end{aligned}$$

Hence, condition of Theorems 4.3.1 proved. So, S has a unique FP, and (4.45) gives unique solution. □

Chapter 5

Conclusion

This thesis reviews and expands upon the work of M.Kumar [21] on "New FP theorems for $(\alpha - \theta)$ M-K contraction in BMS." The main goal of this research is to extend their findings within the framework of bMS. To achieve this,

- (i) We introduce the cont $(\alpha - \theta)$ M-K contraction mapping in BbMS and prove certain results for this type of contraction.
- (ii) The existence and uniqueness of FP has been proved by using generalized $(\alpha - \theta)$ M-K contraction mapping in BbMS.
- (iii) To introduce the $(\alpha - \theta)$ M-K contraction mapping in the framework of BbMS and prove the FP result for these mapping.

These findings are supported by meaningful examples that demonstrate their validity. Additionally, these results are applied to obtain analytical solutions for integral equations. Our results presented in this study could be useful for determining FP in the context of bMS.

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