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(α, β)-$F\zeta$-Contraction Mapping and Fixed Points

by

Qurat-ul-ain

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Dedicated to my family

CERTIFICATE OF APPROVAL

(α, β)-$F\zeta$-Contraction Mapping and Fixed Points

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Abstract

This research study aims to propose some significant fixed-point results within the framework of two intricate extensions of metric spaces: b-metric spaces and b-like metric spaces. By integrating the concept of admissibility into F-contractions and leveraging the properties of the ζ-function, we develop a novel and more generalized rational contraction termed the (α, β)-$F\zeta$- contraction. This research delves into the thorough construction of this new contraction, demonstrating its validity and applicability through illustrative examples. Moreover, the fixed-point results obtained in this study significantly expand and generalize many established results from the existing literature. This work opens up multiple possibilities for further analysis, providing a fertile ground for researchers interested in exploring the dynamic interplay between admissibility, F-contractions, and the ζ-function in the metric fixed point theory context.

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Abbreviations

(α, β)-$F\zeta$-CM	(α, β)-$F\zeta$- Contraction Mapping
BCP	Banach Contraction Principle
b**-ms**	b-metric space
cms	Complete metric space
Cb**-ms**	Complete b-metric space
Cb**-mls**	Complete b-metric like space
dms	Discrete metric space
$F\zeta$-CM	$F\zeta$- Contraction Mapping
fp	Fixed Points
ms	Metric Space

Symbols

$\mathbb{N}$	Natural Number
$\mathbb{R}$	Real Number
$\mathfrak{D}$	Metric Function
$\mathfrak{H}$	Non-empty space
$\mathfrak{M}$	Family of F-Mappings
$\exists$	There exist
$\forall$	For all

Chapter 1

Introduction

Mathematics, widely recognized as the most powerful tool in the history of science and technology, has played a crucial role in solving everyday problems, establishing itself as the cornerstone of all scientific fields. Mathematical concepts offer indispensable instruments for finding solutions to complex problems. Notably, functional analysis, a pivotal branch of mathematics that emerged in the early 19th century, has proven essential in this endeavour. Functional analysis is a crucial branch of mathematics that not only underpins numerous mathematical disciplines but also has significant applications in the applied sciences, driving innovation and progress. Functional analysis, a highly abstract and specialized field of mathematics, evolved from the foundations of classical analysis. The driving force behind its development was the need to address a wide range of applied problems, including those related to ordinary and partial differential equations, numerical analysis, the calculus of variations, approximation theory, and integral equations, among others. Functional analysis was first used in behaviour analysis by B. F. Skinner [1] in 1948. Functional analysis relies on several fundamental concepts, including:

i. The Hahn-Banach theorem

ii. The uniform boundedness principle

iii. The open mapping principle

These three principles form the foundation of functional analysis, providing crucial tools for establishing many of the fields core results and applications. Metric spaces play a crucial role in mathematics as they provide a framework for studying spatial relationships, distances, and limits. Metric spaces extend the concept of Euclidean space, allowing for the study of more general and abstract spaces. The concept of a metric space was first introduced by Maurice Frechet in 1906 [2]. However, the term metric space was later coined by Felix Hausdorff. In Functional Analysis [3], metric fixed point theory, a crucial branch of the field, seamlessly integrates topology, geometry, and analysis. This strong framework plays a vital role in establishing the existence and uniqueness of solutions to differential and integral equations, thereby providing a foundational for various branches of mathematics and applications in diverse fields. Poincaré initiated this theory in 1866 [4], introducing his first fixed point theorem without a proof, thereby establishing himself as a pioneer in the field. In his 1912 paper [5], Brouwer introduced a fixed point theorem for the unit sphere, offering critical evidence of fixed point existence and making a substantial contribution to the field. This early work laid the foundation for Kakutani [6] subsequent generalizations on set-valued mappings. The fixed point theory, employed as a successive approximation technique, offers three significant benefits:

i. Existence guarantee: Ensures the presence of solutions for nonlinear problems.

ii. Uniqueness establishment: Confirms the sole existence of a solution.

iii. Iterative approximation: Facilitates the approximation of solutions through successive iterations.

The fixed-point theory, which originated in the late 19th century, was utilized for iterative approximation to explore the existence and uniqueness of solutions to differential equations, paving the way for significant advancements in mathematical analysis.

This approach is deeply rooted in the works of prominent mathematicians such as Fredholm, Volterra, Liouville, Cauchy, and notably Picard, who made significant contributions to the search for unique solutions to differential equations. Banach fixed point theorem, also known as the contraction mapping theorem (BCP) [7], stands as a landmark achievement in the history of mathematics, providing a crucial milestone for researchers exploring metric fixed point theory and significantly impacting the development of the field.

Banach Contraction Principle (BCP) ensures the existence and uniqueness of fixed points in metric spaces, providing an effective method for finding them. Stefan Banach, a prominent mathematician (1892-1945), introduced the Banach Fixed Point Theorem in 1922, a milestone in mathematical history. The Banach Contraction Principle (BCP) states If a complete metric space $(\mathfrak{H}, \mathfrak{D})$ is mapped to itself by a contraction mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$, then $\mathcal{T}$ has a unique fixed point. A contraction mapping is a function that satisfies the condition

$$\mathfrak{D}\left(\mathcal{T}\xi_1, \mathcal{T}\xi_2\right) \leq \eta\mathfrak{D}\left(\xi_1, \xi_2\right) \text{ for all } \xi_1, \xi_2 \in \mathfrak{H},$$

where η is a constant in the range (0, 1). The Banach Contraction Principle has significant applications in ensuring the existence and uniqueness of solutions to specific ordinary differential equations. The Banach Contraction Principle (BCP) [7], which pairs a complete space with an appropriate integral operator, ensures the existence and uniqueness of solutions to integral equations. The Banach Contraction Principle has undergone significant generalizations and extensions since its inception. Edelstein [8] gave the first generalization of the Banach contraction condition in 1962 by taking constant $\eta = 1$ and using distinct points from a compact space $\mathfrak{H}$. In the same year, Rakotch [9] established a contractive condition by replacing the constant η of the contraction with a monotonic decreasing function $\eta : [0, \infty) \longrightarrow [0, 1]$. Presic [10], Kannan [11], Keeler *et al.* [12] worked on BCP by altering the contraction condition. Fomin [13] and Gupta [14] introduced a rational expression and generalized the Banach Contraction Principle (BCP).

The concepts of b-metric space were introduced by Bakhtin [15] and Bourbaki [16]. Furthermore, Czerwik [17] introduced the concept of a b-metric space, which was designed to generalize the Banach contraction theorem. This axiom relaxes the traditional triangular inequality, providing a weaker condition that still enables the extension of the contraction theorem.

In the triangle inequality, he added a constant $\mathfrak{s}$. In the scenario where $\mathfrak{s} = 1$, b-metric space is a metric space. In general, this idea is not as strong as metric spaces. Czerwik was eager to investigate these spaces topological features in further detail. Matthews [18](1992) introduced partial metric spaces as a generalization of metric spaces, allowing for non-zero self-distances. Later, in 2012, A.A. Harandi [19] further extended this concept by introducing metric-like spaces, where the self-distance of a point can exceed the distance to any other point, providing a more relaxed notion of distance. Subsequently, in 2014, S. Shukla [20] proposed the notion of partial b-metrics offers a further refinement of both partial metrics and b -metrics. Meanwhile, Alghamdi [21] et al. (2013) introduced b-metric-like spaces, which integrate the concepts of partial b-metric spaces and metric-like spaces, thereby creating a more comprehensive framework. It is important to note that b-metric-like spaces include all the abstract spaces discussed in this research.

In 2012, Wardowski [22] presented another well-known contraction, F-contraction. Sagroi *et al.*[23] proved fixed point results on F-contraction in 2013 with some applications on integral equations. Alot of work is done in this area; see for examples,[24–31].
The concept of F-contraction has been extensively generalized in multiple directions. One notable generalization is the (α, F)-contractive mapping, which was first introduced by Kamran [32] *et al.* in 2016 . The F-contraction generalization was made for single-valued mappings in b-metric spaces. The research in this thesis is in continuation of this studies. This thesis generalizes the results of Jain et al. [33] by introducing generalized (α, β)-$F\zeta$-contractions in b-metric space.

The organization of the thesis is given below:

Chapter 2 includes every basic idea in functional analysis that is required for the discussion that follows. Examples are provided to clarify the key ideas surrounding metric spaces. A foundation for the main results is provided by a few significant fixed-point results that are presented.

Chapter 3 gives a comprehensive review of the article ζ-Contractive mappings on the platform of b-metric spaces and b-metric-like spaces by Jain *et al.* [33]. The theorems are well elaborated, along with examples.

In Chapter 4, the findings of Jain *et al.* are further expanded by utilizing a new contraction condition, the generalized (α, β)-$F\zeta$-Contraction Mapping, to explore fixed points. It should be noted that the findings of Jain *et al.* serve as an exception to the gener-alization of the results presented in this study. The established theorems are supported by one non-trivial example. Many of the current results are the exception to the general findings presented in Chapter 4. Some corollaries authenticate this fact.
Chapter 5 summarizes all the extensions of our work and offers additional recommendations for future research.

Chapter 2

Literature Review

This chapter offers a thorough and accessible introduction to the fundamental principles of functional analysis, starting with the basics of metric, b-metric spaces and b-metric-like spaces, including definitions, explanations, and illustrative examples to ensure a solid grasp of the concepts. The chapter goes on to examine different types of mapping, providing clear examples to illustrate each concept. The chapter concludes by laying a strong groundwork for the main theorem, introducing key fixed point results.

2.1 Metric Space

The idea of metric spaces, which Maurice Rene Frechet [2] presented in 1906, expanded the definition of distance to a broader context. Later on these spaces acted as bridge between Topological spaces and Real analysis, and provided an establishment to Metric Fixed point theory. Metric spaces helped to solve several problems in several branches of mathematics by offering a framework for dealing with numerous mathematical problems.

Definition 2.1.1. Metric Space
"A metric space is a pair $(\mathfrak{H}, \mathfrak{D})$, where $\mathfrak{H}$ is a set and $\mathfrak{D}$ is a metric on $\mathfrak{H}$ (or distance function on $\mathfrak{H}$), that is, a function defined on $\mathfrak{H} \times \mathfrak{H}$ such that for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{H}$ we have:

(M1) $\mathfrak{D}(\xi_1, \xi_2) \geq 0.$

(M2) $\mathfrak{D}(\xi_1, \xi_2) = 0 \iff \xi_1 = \xi_2$.

(M3) $\mathfrak{D}(\xi_1, \xi_2) = \mathfrak{D}(\xi_2, \xi_1)$ (Symmetry).

(M4) $\mathfrak{D}(\xi_1, \xi_2) \leq \mathfrak{D}(\xi_1, \xi_3) + \mathfrak{D}(\xi_3, \xi_2)$ (Triangle inequality).

The symbol × denotes the Cartesian product of sets. $\mathfrak{A} \times \mathfrak{B}$ is the set of all ordered pairs $(\mathtt{a}, \mathtt{b})$, where $\mathtt{a} \in \mathfrak{A}$ and $\mathtt{b} \in \mathfrak{B}$. Hence $\mathfrak{H} \times \mathfrak{H}$ is the set of all ordered pairs of elements of $\mathfrak{H}$." [34]

Example 2.1.2.

A function $\mathfrak{D} : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\mathfrak{D}(\xi_1, \xi_2) = \sqrt{|\xi_1 - \xi_2|}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$ is a metric, known as usual metric on $\mathbb{R}$. [34]

Example 2.1.3. Although trivial and somewhat pointless, it's worth noting that the only metric on the empty set is the empty function, which has no domain and, therefore, no effect.

Similarly, the only metric on a singleton set, though not much more interesting, is simply the zero function.

Example 2.1.4.

Let $\mathfrak{H}$ be the set of all real valued functions $(\xi_1, \xi_2, ...)$ which are functions of an independent real variable ϑ and are defined and continuous on a given closed interval $\mathbb{I} = [\alpha, \beta]$.

Define $\mathfrak{D} : \mathfrak{H} \times \mathfrak{H} \dashrightarrow \mathbb{R}$ by

$$\mathfrak{D}\,(\xi_1, \xi_2) = \max_{\vartheta \in \mathbb{I}} |\xi_1(\vartheta) - \xi_2(\vartheta)|,$$

where max denotes the maximum.

$\mathfrak{H}$ is a metric space denoted by $\mathfrak{C}[\alpha, \beta]$ the function space.

Clearly $(M1), (M2)$ and $(M3)$ are satisfied.

$(M4)$, For $\xi_1, \xi_2, \xi_3 \in \mathfrak{H}$ consider

$$\begin{aligned}
\mathfrak{D}\left(\xi_1, \xi_3\right) &= \max_{\vartheta \in \mathbb{I}} \left|\xi_1(\vartheta) - \xi_3(\vartheta)\right| \\
&= \max_{\vartheta \in \mathbb{I}} \left|\xi_1(\vartheta) - \xi_2(\vartheta) + \xi_2(\vartheta) - \xi_3(\vartheta)\right| \\
&\leq \max_{\vartheta \in \mathbb{I}} \left\{\left|\xi_1(\vartheta) - \xi_2(\vartheta)\right| + \left|\xi_2(\vartheta) - \xi_3(\vartheta)\right|\right\} \\
&= \max_{\vartheta \in \mathbb{I}} \left|\xi_1(\vartheta) - \xi_2(\vartheta)\right| + \max_{\vartheta \in \mathbb{I}} \left|\xi_2(\vartheta) - \xi_3(\vartheta)\right| \\
&= \mathfrak{D}\left(\xi_1, \xi_2\right) + \mathfrak{D}\left(\xi_2, \xi_3\right)
\end{aligned}$$

for all ξ_1, ξ_2 and ξ_3 in $\mathfrak{H}$.

Example 2.1.5. Discrete Metric Space

Let $\mathfrak{D}$ be the metric is defined on a non-empty set $\mathfrak{H}$ as follows

$$\mathfrak{D}\left(\xi_1, \xi_2\right) = \begin{cases} 0 & \text{if} \quad \xi_1 = \xi_2, \\ 1 & \text{if} \quad \xi_1 \neq \xi_2. \end{cases}$$

This space $(\mathfrak{H}, \mathfrak{D})$ is called a discrete metric space.

Definition 2.1. Open ball and Closed ball

“Let $\mathfrak{H}$ be a non-empty set for a given point $\xi_1 \in \mathfrak{H}$ and a real number $r > 0$, we define three types of sets:

(i) $B(\xi_1, r) = \{\xi_2 \in \mathfrak{H} | \mathfrak{D}(\xi_1, \xi_2) < r\}$ (Open ball)

(ii) $\overline{B(\xi_1, r)} = \{\xi_2 \in \mathfrak{H} | \mathfrak{D}(\xi_1, \xi_2) \leq r\}$ (Closed ball)

(iii) $S(\xi_1, r) = \{\xi_2 \in \mathfrak{H} | \mathfrak{D}(\xi_1, \xi_2) = r\}$ (Sphere).” [34]

Definition 2.2.

“A subset $\mathcal{M}$ of a metric space $\mathfrak{H}$ is said to be an open set if it contains a ball about each of its points.” [34]

Definition 2.3.

“A subset $\mathcal{M}$ of $\mathfrak{H}$ is said to be closed if its complement (in $\mathfrak{H}$) is open, that is, $\mathcal{M}_\mathsf{c} = \mathfrak{H} - \mathcal{M}$ is open.” [34]

Definition 2.4. Topology Induced by a Metric

"The metric topology in a metric space $(\mathfrak{H}, \mathfrak{D})$. For $\xi_1 \in \mathfrak{H}$ and $r > 0$ let

$$B(\xi_1, r) = \{\xi_2 \in \mathfrak{H} | \mathfrak{D}(\xi_1, \xi_2) < r\}.$$

$B(\xi_1, r)$ is called the open ball centered at ξ_1 of radius r. The metric topology on a metric space $\mathfrak{H}$ is the topology obtained by taking as open sets the collection of all sets $\mathcal{P}$ in $\mathfrak{H}$ which have the property $S \in \mathcal{P}$ provided each point $\xi_1 \in S$ is the center of some open ball $B(\xi_1, r)$, which also lies in S." [35]

Definition 2.5. Convergence of a Sequence, Limit

"A sequence $\left\{\xi_{1_{m_2}}\right\}$ in a metric space $\mathfrak{H} = (\mathfrak{H}, \mathfrak{D})$ is said to converge or to be convergent if there is an $\xi_1 \in \mathfrak{H}$ such that

$$\lim_{m_2 \to \infty} \mathfrak{D}(\xi_{1_{m_2}}, \xi_1) = 0,$$

ξ_1 is called the limit of $(\xi_{1_{m_2}})$ and we write

$$\lim_{m_2 \to \infty} \xi_{1_{m_2}} = \xi_1,$$

simply,

$$\xi_{1m_2} \longrightarrow \xi_1.$$

We say that $(\xi_{1_{m_2}})$ converges to ξ_1 or has the limit ξ_1". [7]

Definition 2.6. Cauchy Sequence

" A sequence $\left\{\xi_{1_{m_2}}\right\}$ in a metric space $\mathfrak{H} = (\mathfrak{H}, \mathfrak{D})$ is said to Cauchy or fundamental if for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that

$$\mathfrak{D}(\xi_{1_{m_1}}, \xi_{1_{m_2}}) < \epsilon$$

for every $m_1, m_2 > N$." [7]

Example 2.1.6.

The sequence $\{\xi_{1_l}\} = \left\{\frac{1}{l} : l \in N\right\}$ is a Cauchy sequence in metric space $[0, 1]$ with usual metric.

Definition 2.7. Complete Metric Space

"A metric space $(\mathfrak{H}, \mathfrak{D})$ is said to be complete if every Cauchy sequence in $\mathfrak{H}$ converges to an element in $\mathfrak{H}$."[36]

Example 2.1.7.

Suppose l^{∞} be the space of all bounded sequence of complex numbers, with the metric defined by

$$\mathfrak{D}\left(\xi_1, \xi_2\right) = \sup_{r \in \mathbb{N}} |\xi_{1_r} - \xi_{2_r}|,$$

where $\xi_1 = \{\xi_{1_r}\}$ and $\xi_2 = \{\xi_{2_r}\}$, then $(l^{\infty}, \mathfrak{D})$ is a complete metric space.

Example 2.1.8.

Let $\mathfrak{H} = \mathbb{R}$, define a metric $\mathfrak{D}$ on $\mathfrak{H}$ by

$$\mathfrak{D}(\xi_1, \xi_2) = |\tan^{-1}(\xi_1) - \tan^{-1}(\xi_2)|,$$

then, $\mathfrak{D}$ is not a complete metric on $\mathfrak{H}$.

2.2 Some Generalizations of Metric Space

Stefan Czerwik [17] and Bakhtin [15] developed b-metric spaces as an extension of metric spaces, modifying the triangle inequality.

Definition 2.8. b-Metric Space

"Let $\mathfrak{H}$ be a non-empty set and let $\mathfrak{s} \geq 1$ be a given real number.
A function
$\mathfrak{D}_{\mathrm{b}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, \infty)$, is said to be b-metric if for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{H}$ the following conditions are satisfied;

(B1) $\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) = 0 \Leftrightarrow \xi_1 = \xi_2$;

(B2) $\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) = \mathfrak{D}_{\mathrm{b}}(\xi_2, \xi_1)$;

(B3) $\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_3) \leq \mathfrak{s}[\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) + \mathfrak{D}_{\mathrm{b}}(\xi_2, \xi_3)]$.

The triple $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s})$ is called a b-metric space." [37]

Definition 2.9.

"Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s})$ be a b-metric space, $\{\xi_{1_{m_2}}\}$ be a sequence in $\mathfrak{H}$ and $\xi_1 \in \mathfrak{H}$. Then,

(i) the sequence $\{\xi_{1_{m_2}}\}$ is said to be Cauchy sequence, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that: for all $m_2 \geq \mathbb{N}$ and for all $p \in \mathbb{N}$, we have

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+p}}) < \epsilon.$$

(ii) the sequence $\{\xi_{1_{m_2}}\}$ is said to be convergent to ξ_1, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that: for all $m_2 \geq \mathbb{N}$, we have

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_1) < \epsilon.$$

In this case, we write:

$$\lim_{m_2 \to \infty} \xi_{1_{m_2}} = \xi_1$$

(iii) the space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s})$ is called a complete b-metric space if every Cauchy sequence is convergent in it."

Example 2.2.1. Let $\mathfrak{H} := l_p[0, 1]$ be the space of all real function $\xi_1(\vartheta)$, $\vartheta \in [0, 1]$ such that $\int_0^1 |\xi_1(\vartheta)|^p < \infty$ with $0 < p < 1$.

Define $\mathfrak{D}_{\mathrm{b}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}^+$ as :

$$\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) = \left(\int_0^1 |\xi_{1_{m_2}} - \xi_{2_{m_2}}|^p \right)^{\frac{1}{p}},$$

Then $\mathfrak{D}$ is a b-metric space with coefficients $\mathfrak{s} = 2^{\frac{1}{p}}$.

Example 2.2.2.

Let $\mathfrak{H} := l_p(\mathbb{R})$ with $0 < p < 1$, where $l_p(\mathbb{R}) := \left\{ \{\xi_{1_{m_2}}\} \subset \mathbb{R} : \sum_{m_2=1}^{\infty} |\xi_{1_{m_2}}|^p < \infty \right\}$.

Define $\mathfrak{D}_{\mathrm{b}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}^+$as :

$$\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) = \left(\sum_{m_2=1}^{\infty} |\xi_{1_{m_2}} - \xi_{2_{m_2}}|^p \right)^{\frac{1}{p}},$$

where $\xi_1 = \{\xi_{1_{m_2}}\}$ and $\xi_2 = \{\xi_{2_{m_2}}\}$. Then $\mathfrak{D}$ is a b-metric space with coefficients $\mathfrak{s} = 2^{\frac{1}{p}}$.

Definition 2.10. Partial Metric Space

"Let $\mathfrak{H}$ be a non-empty set. Then, a mapping $\mathfrak{D}_p : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, +\infty)$ is called a partial metric if for all $\xi_1, \xi_2 \in \mathfrak{H}$,

$$\text{P1} : \mathfrak{D}_p(\xi_1, \xi_2) = 0 \iff \mathfrak{D}_p(\xi_1, \xi_1) = \mathfrak{D}_p(\xi_1, \xi_2) = \mathfrak{D}_p(\xi_2, \xi_2);$$

$$\text{P2} : \mathfrak{D}_p(\xi_1, \xi_1) \leq \mathfrak{D}_p(\xi_1, \xi_2);$$

$$\text{P3} : \mathfrak{D}_p(\xi_1, \xi_2) = \mathfrak{D}_p(\xi_2, \xi_1);$$

$$\text{P4} : \mathfrak{D}_p(\xi_1, \xi_3) \leq \mathfrak{D}_p(\xi_1, \xi_2) + \mathfrak{D}_p(\xi_2, \xi_3) - \mathfrak{D}_p(\xi_2, \xi_2),$$

The pair $(\mathfrak{H}, \mathfrak{D}_p)$ is called a partial metric space." [38]

Example 2.2.3. Let $(\mathfrak{H}, \mathfrak{D}_p)$ be a partial metric space. Then

i: $\xi_{1_{m_2}}$ is a Cauchy sequence in $(\mathfrak{H}, \mathfrak{D}_p)$ if and only if it is a Cauchy sequence in the metric space $(\mathfrak{H}, \mathfrak{D}_p^{\mathfrak{s}})$.

ii: A partial metric space $(\mathfrak{H}, \mathfrak{D}_p)$ is complete if and only if the metric space $(\mathfrak{H}, \mathfrak{D}_p^{\mathfrak{s}})$ is complete.

Furthermore,

$$\lim_{m_2 \longrightarrow \infty} p^{\mathfrak{s}}(\xi_{1_{m_2}}, \xi_1) = 0$$

if and only if

$$p(\xi_1, \xi_1) = \lim_{m_2 \longrightarrow \infty} p^{\mathfrak{s}}(\xi_{1_{m_2}}, \xi_1) = \lim_{m_1, m_2 \longrightarrow \infty} p^{\mathfrak{s}}(\xi_{1_{m_2}}, \xi_{1_{m_1}}).$$

Example 2.2.4.

A mapping $\mathfrak{D}_p : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}^+$; for all $(\xi_1, \xi_2) \in \mathbb{R}^+$, defined by

$$\mathfrak{D}_p(\xi_1, \xi_2) = \max\{\xi_1, \xi_2\},$$

is partial metric space $(\mathfrak{H}, \mathfrak{D}_p)$.

Example 2.2.5.

Let $\mathbb{I}$ be the collection of non-empty closed bounded intervals in $\mathbb{R}$

$\mathbb{I} = \{[\mathtt{a}, \mathtt{b}] : \mathtt{a} \leq \mathtt{b}\}$. For $[\mathtt{a}, \mathtt{b}], [\mathtt{c}, \mathtt{d}] \in \mathbb{I}$,

define $\mathfrak{D}_p([\mathtt{a}, \mathtt{b}], [\mathtt{c}, \mathtt{d}]) = \max(\mathtt{b}, \mathtt{d}) - \min(\mathtt{a}, \mathtt{c})$.

Then, it can be shown that $\mathfrak{D}_p$ is a partial metric over $\mathbb{I}$.

Definition 2.11. Hausdorff Metric Space

"Let $(\mathfrak{H}, \mathfrak{D})$ be a metric space and $CB(\mathfrak{H})$ denotes the collection of all non-empty closed and bounded subsets of $\mathfrak{H}$. For $\mathfrak{A}, \mathfrak{B} \in \mathfrak{H}$, define

$$H(\mathfrak{A}, \mathfrak{B}) = \max\left\{\sup_{\mathtt{a}\in\mathfrak{A}} \mathfrak{D}(\mathtt{a}, \mathfrak{B}), \sup_{\mathtt{b}\in\mathfrak{B}} \mathfrak{D}(\mathtt{b}, \mathfrak{A})\right\},$$

where $\mathfrak{D}(\xi_1, \mathfrak{A}) = \inf\{\mathfrak{D}(\xi_1, \mathtt{a}) : \mathtt{a} \in \mathfrak{A}\}$ is the distance of point ξ_1 to the set $\mathfrak{A}$. It is known that H is a metric on $CB(\mathfrak{H})$, called the Hausdorff metric induced by the metric $\mathfrak{D}$.". [7]

Example 2.2.6.

Let $P = \{0, 1, 2, 3, ..., 10\}$ we select two non-empty set $\mathfrak{A} = \{1, 5\}$ and $\mathfrak{B} = \{6, 9\}$. Now $H(\mathfrak{A}, \mathfrak{B})$ can be calculated as follows

$$\begin{aligned}
H(\mathfrak{A}, \mathfrak{B}) &= \max\left\{\sup_{\mathtt{a}\in\mathfrak{A}} \mathfrak{D}(\mathtt{a}, \mathfrak{B}), \sup_{\mathtt{b}\in\mathfrak{B}} \mathfrak{D}(\mathtt{b}, \mathfrak{A})\right\} \\
&= \max\{\{5, 1\}, \{1, 4\}\} \\
&= \max\{5, 4\} \\
&=5.
\end{aligned}$$

2.3 Results on b-Metric-Like Spaces

A b-metric-like space is a mathematical construct that generalizes the notion of a metric space, where the distance between points is measured using a metric function. In the literature, b-metric-like spaces have been studied extensively in the context of functional analysis and topology.

They are defined as spaces where the metric function satisfies a relaxed version of the triangle inequality, known as the relaxed triangle inequality or b-metric inequality. This relaxation allows for a wider range of distance functions, enabling the study of more general spaces. Research on b-metric-like spaces has led to important results in areas such as fixed point theory, operator theory, and geometric analysis, and has applications in fields like computer science, physics, and engineering. The study of b-metric-like spaces continues to be an active area of research, with new results and applications emerging regularly.

Definition 2.12. Metric-Like Space

"Let $\mathfrak{H}$ be a non-empty set. Then, a mapping $\mathfrak{D}_{\mathrm{ml}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, +\infty)$ is called a metric-like space if for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{H}$,

(ml1) $\mathfrak{D}_{\mathrm{ml}}(\xi_1, \xi_2) = 0 \Longrightarrow \xi_1 = \xi_2$;

(ml2) $\mathfrak{D}_{\mathrm{ml}}(\xi_1, \xi_2) = \mathfrak{D}_{\mathrm{ml}}(\xi_2, \xi_1)$;

(ml3) $\mathfrak{D}_{\mathrm{ml}}(\xi_1, \xi_3) \leq \mathfrak{D}_{\mathrm{ml}}(\xi_1, \xi_2) + \mathfrak{D}_{\mathrm{ml}}(\xi_2, \xi_3)$.

The pair $(\mathfrak{H}, \mathfrak{D}_{\mathrm{ml}})$ is called a metric-like space." [39]

Definition 2.13. **b-Metric-Like Space**

"Let $\mathfrak{H}$ be a non-empty set. Then, a mapping $\mathfrak{D}_{\mathrm{bl}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, +\infty)$ is called a b-metric-like if there exists a number $\mathfrak{s} \geq 1$ such that for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{H}$,

(bml1) $\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) = 0 \Longrightarrow \xi_1 = \xi_2$;

(bml2) $\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) = \mathfrak{D}_{\mathrm{bl}}(\xi_2, \xi_1)$;

(bml3) $\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_3) \leq \mathfrak{s} \left(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) + \mathfrak{D}_{\mathrm{bl}}(\xi_2, \xi_3) \right)$.

Then, the triplet $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s})$ is called a b-metric-like space." [21]

Definition 2.14.

"Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ be a b-metric-like space and let $\left\{\xi_{1_{m_2}}\right\}$ be a sequence of points of $\mathfrak{H}$. A point $\xi_1 \in \mathfrak{H}$ is said to be the limit of sequence $\left\{\xi_{1_{m_2}}\right\}$ is

$$\lim_{n \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_{1_{m_2}}) = \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1),$$

and we say that the sequence $\left\{\xi_{1_{m_2}}\right\}$ is convergent to ξ_1 and denote it by $\xi_{1_{m_2}} \longrightarrow \xi_1$ as $n \longrightarrow +\infty$." [21]

Definition 2.15. Cauchy Sequence, Completeness

“Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ be a b-metric-like space.

(i) A sequence $\left\{\xi_{1_{m_2}}\right\}$ in $\mathfrak{H}$ is called Cauchy sequence if $\lim\limits_{m_1, m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_1}}, \xi_{1_{m_2}})$ exists and is finite.

(ii) $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ is said to be complete if every Cauchy sequence $\left\{\xi_{1_{m_2}}\right\}$ in $\mathfrak{H}$ converges to $\xi_1 \in \mathfrak{H}$ so that

$$\lim_{m_1, m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_1}}) = \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1) = \lim_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_1).\text{”}[21]$$

Proposition 2.3.1.

“Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ be a b-metric-like space and $\left\{\xi_{1_{m_2}}\right\}$ be a sequence in $\mathfrak{H}$ such that for some $\xi_1 \in \mathfrak{H}$, $\lim\limits_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_1) = 0$. Then,

(i) ξ_1 is unique.

(ii) $\frac{1}{\mathfrak{s}}\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1) \leq \lim\limits_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_1) \leq \mathfrak{s}\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) \qquad$ for all $\xi_1, \xi_2 \in \mathfrak{H}$.”[21]

Lemma 2.3.2.

“Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ be a b-metric-like space and $\left\{\xi_{1_{m_2}}\right\}$ be a sequence in $\mathfrak{H}$ such that

$$\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) \leq \mathfrak{s}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})$$

for some $\mathfrak{s} \in [0, 1)$ and for each $m_2 \in \mathbb{N}$. Then, $\left\{\xi_{1_{m_2}}\right\}$ is a Cauchy sequence $\lim\limits_{m_1, m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_1}}) = 0$.”

Remark 2.16.

The class of b-metric-like spaces is a generalization of both metric-like spaces and b-metric spaces.

While every b-metric space is a b-metric-like space with the same parameter, and every b-metric-like space with parameter 1 is a metric-like space, not every metric-like space is a b-metric-like space, and not every b-metric-like space is a b-metric space.

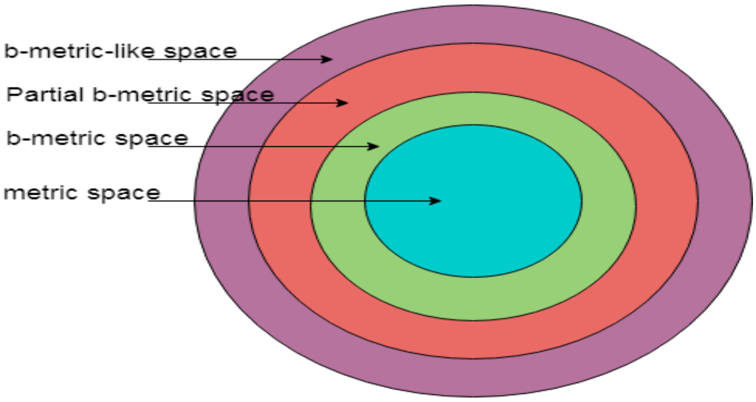

FIGURE 2.1: Relation between different metric spaces

Example 2.3.3.

Let $C_{\mathfrak{b}}(\mathfrak{H}) = \left\{ f : \mathfrak{H} \longrightarrow \mathbb{R} : \sup\limits_{\xi_1 \in \mathfrak{H}} |f(\xi_1)| < +\infty \right\}$. The function $\mathfrak{D}_{\mathrm{bl}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}_+$, defined by

$$\mathfrak{D}_{\mathrm{bl}}(f, g) = \sqrt[3]{\sup_{\xi_1 \in \mathfrak{H}} \left(|f(\xi_1)| + |g(\xi_1)| \right)^3}$$

for all $f, g \in C_{\mathfrak{b}}(\mathfrak{H})$, is a b-metric-like with constant $\mathfrak{s} = \sqrt[3]{4}$, and so, the triplet $\left(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \sqrt[3]{4}\right)$ forms a b-metric-like space.

Example 2.3.4.

Let $\mathfrak{H} = \{0, 1, 2\}$, and let

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) = \begin{cases} 2, & \text{if } \xi_1 = \xi_2 = 0, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Then $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}})$ is a b-metric-like space with the constant $\mathfrak{s} = 2$.

Example 2.3.5.

Let $\mathfrak{H} = [0, \infty)$. Define the function $\mathfrak{D}_{\mathrm{bl}} : \mathfrak{H}^2 \longrightarrow [0, \infty)$ by

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) = (\xi_1 + \xi_2)^2 .$$

Then, $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}})$ is a b-metric-like space with constant $\mathfrak{s} = 2$. Clearly, $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}})$ is not a b-metric space. Indeed, for all $\xi_1, \xi_2, \xi_3 \in \mathfrak{H}$,

$$\begin{aligned} \mathfrak{D}_{\mathrm{bl}}(x, y) = (\xi_1 + \xi_2)^2 \leq (\xi_1 + \xi_3 + \xi_3 + \xi_2)^2 &= (\xi_1 + \xi_3)^2 + (\xi_3 + \xi_2)^2 + 2(\xi_1 + \xi_3)(\xi_3 + \xi_2) \\ &\leq 2\left[(\xi_1 + \xi_3)^2 + (\xi_3 + \xi_2)^2\right] \\ &= 2\left(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_3) + \mathfrak{D}_{\mathrm{bl}}(\xi_3, \xi_2)\right) \end{aligned}$$

and so $(\mathfrak{D}_{\mathrm{bl}}1)$ holds. Clearly, $(\mathfrak{D}_{\mathrm{bl}}2)$ and $(\mathfrak{D}_{\mathrm{bl}}3)$ hold.

Example 2.3.6.

Let $\mathfrak{H} = [0, \infty)$. Define the function $\mathfrak{D}_{\mathrm{bl}} : \mathfrak{H}^2 \longrightarrow [0, \infty)$ by

$$\mathfrak{D}_{\mathrm{bl}}\,(\xi_1, \xi_2) = (\max\{\xi_1 + \xi_2\})^2 .$$

Then, $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}})$ is a b-metric-like space with constant $\mathfrak{s} = 2$.

Clearly, $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}})$ is not a b-metric space.

2.4 Mappings on Metric Spaces

This section explores various types of mappings defined on metric spaces, providing a comprehensive overview of their properties and behavior.

Definition 2.17. Lipschitzian Mapping

“Consider a metric space $(\mathfrak{H}, \mathfrak{D})$. A self mapping $\mathcal{T}\colon \mathfrak{H} \dashrightarrow \mathfrak{H}$ is called Lipschitzian map, if for small number $\epsilon \geq 0$ the mapping $\mathcal{T}$satisfies the following,

$$\mathfrak{D}\,(\mathcal{T}(\xi_1), \mathcal{T}(\xi_2)) \leq \ \epsilon\, \mathfrak{D}(\xi_1, \xi_2) \qquad \text{for all} \quad \xi_1, \xi_2 \in \mathfrak{H}.\text{”}[40]$$

Definition 2.18. Contraction Mapping

“Let $(\mathfrak{H}, \mathfrak{D})$ is a metric space and $\mathcal{T}\colon \mathfrak{H} \dashrightarrow \mathfrak{H}$ is called contraction mapping. Thus, there exists a constant $\epsilon < 1$ such that

$$\mathfrak{D}\,(\mathcal{T}(\xi_1), \mathcal{T}(\xi_2)) \leq \ \epsilon\, \mathfrak{D}(\xi_1, \xi_2) \qquad \text{for all} \quad \xi_1, \xi_2 \in \mathfrak{H}.\text{”}[40]$$

Example 2.4.1.

Let $\mathfrak{H} = [0, 1]$ is equipped with the usual metric, we define a map $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ as

$$\mathcal{T}(\xi_1) = \frac{1}{2 + \xi_1},$$

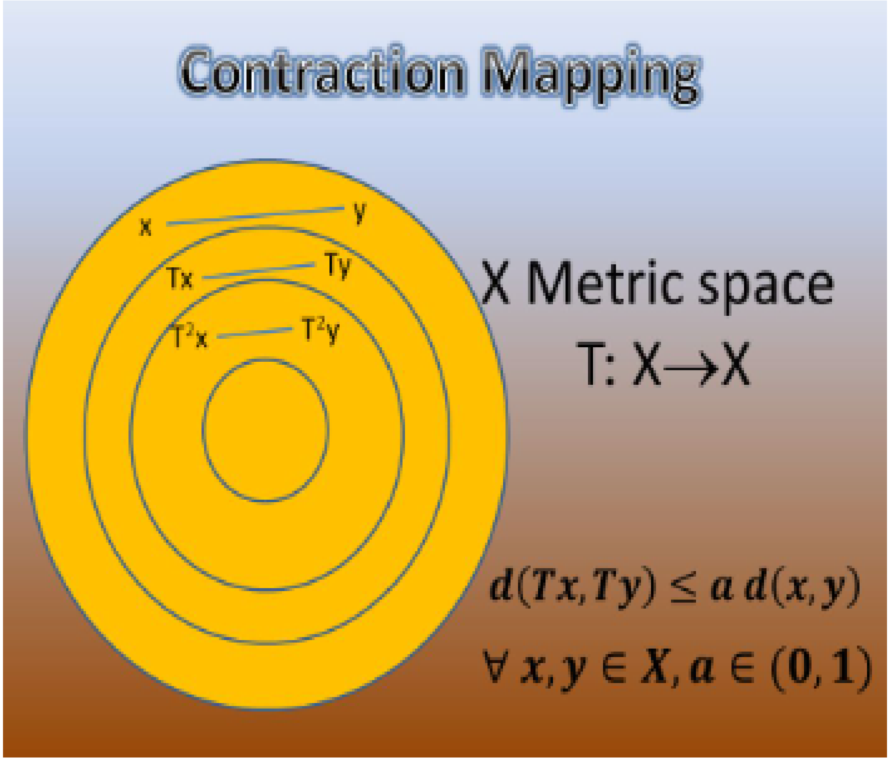

FIGURE 2.2: Contraction Mapping

is a contraction mapping with $\epsilon = 0.25 \in (0, 1)$.

Definition 2.19. Contractive Mapping

"Let $(\mathfrak{H}, \mathfrak{D})$ is a metric space and $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ is called contractive mapping, if

$$\mathfrak{D}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) < \mathfrak{D}(\xi_1, \xi_2) \qquad \text{for all} \quad \xi_1, \xi_2 \in \mathfrak{H} \quad \text{with} \quad \xi_1 \neq \xi_2.\text{"}[40]$$

Note that Contraction ⇒ Contractive ⇒ Lipschitzian.

Definition 2.20. Non-expansion Mapping

"Let $(\mathfrak{H}, \mathfrak{D})$ is a metric space and a function $\varrho : \mathfrak{N} \longrightarrow \mathfrak{N}$ is non-expansion, if

$$\mathfrak{D}(\varrho\xi_1, \varrho\xi_2) \le \mathfrak{D}(\xi_1, \xi_2) \qquad \text{for all} \quad \xi_1, \xi_2 \in \mathfrak{N}.\text{"} \quad [40]$$

Definition 2.21. Continuous Mapping

"Let $\mathfrak{H}_1 = (\mathfrak{H}_1, \mathfrak{D}_1)$ and $\mathfrak{H}_2 = (\mathfrak{H}_2, \mathfrak{D}_2)$ be metric spaces. A mapping $\mathcal{T}: \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ is said to be Continuous mapping at a point $\xi_{1_0} \in \mathfrak{H}_1$ if for every $\epsilon > 0$ there is a $\mu > 0$, such that

$$\mathfrak{D}_2(\mathcal{T}\xi_1, \mathcal{T}\xi_{1_0}) < \epsilon \quad \text{for all} \quad \mathfrak{D}_1(\xi_1, \xi_{1_0}) < \mu.\text{"} \quad [34]$$

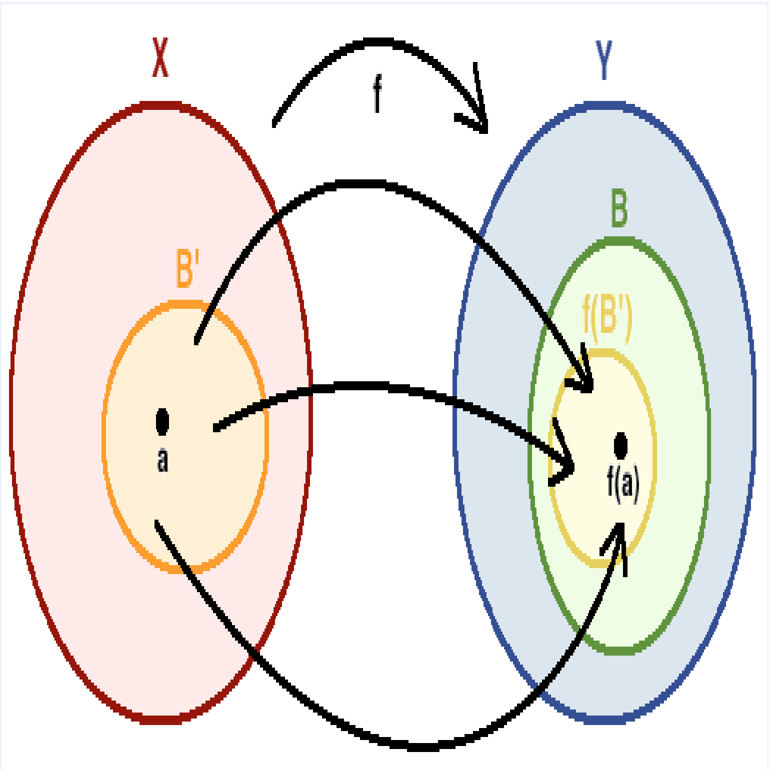

FIGURE 2.3: Continuous Mapping

Theorem 2.4.2.

"Let $\mathfrak{H}_1 = (\mathfrak{H}_1, \mathfrak{D}_1)$ and $\mathfrak{H}_2 = (\mathfrak{H}_2, \mathfrak{D}_2)$ be two metric spaces and $\mathcal{T}\colon \mathfrak{H}_1 \longrightarrow \mathfrak{H}_2$ be a mapping.

Then $\mathcal{T}$ is continuous at a point $\mathsf{c} \in \mathfrak{H}_1$ if and only if, for all sequences $\{\mathsf{a}_{m_2}\}$ in $\mathfrak{H}_1$ which converge to a point c, the sequence $\mathcal{T}(\mathsf{a}_{m_2})$ converges to $\mathcal{T}(\mathsf{c})$." [34]

2.4.1 Fixed Point and Some Classical Results

In 1880 Henri Poincare [41] showed that solution to certain important analytical problems could be studied by defining a set K and a function $\mathcal{T} : M \longrightarrow M$ in such a way that solution corresponds to the fixed point of the function $\mathcal{T}$, with this achievement fixed points become more important for getting the solution of problems arising in various areas of Mathematical analysis such as existence of solution of system of linear equations, eigen value problems, integral equations and differential equations.

Definition 2.22. Fixed Point

"let $(\mathfrak{H}, \mathfrak{D})$ be a metric space, and let $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be a mapping. A point ξ_1 in $\mathfrak{H}$ is called a fixed point of $\mathcal{T}$ if $\mathcal{T}(\xi_1) = \xi_1$."

A fixed point is defined geometrically as the intersection of the line $\xi_2 = \xi_1$ and a real valued function $\xi_2 = \mathcal{T}(\xi_1)$. There could be a fixed point or not for a function. Additionally, in the event that it has, the fixed point could not be unique. [34]

Example 2.4.3.

$\mathcal{T}(\xi_1) = \tan \xi_1$: This function has an infinite number of fixed points, one in each interval $\left(\frac{m_2\pi}{2}, \frac{(m_2+2)\pi}{2}\right)$ along the positive x-axis and one in each interval $\left(\frac{-(m_2+2)\pi}{2}, \frac{-m_2\pi}{2}\right)$ along the negative x-axis.

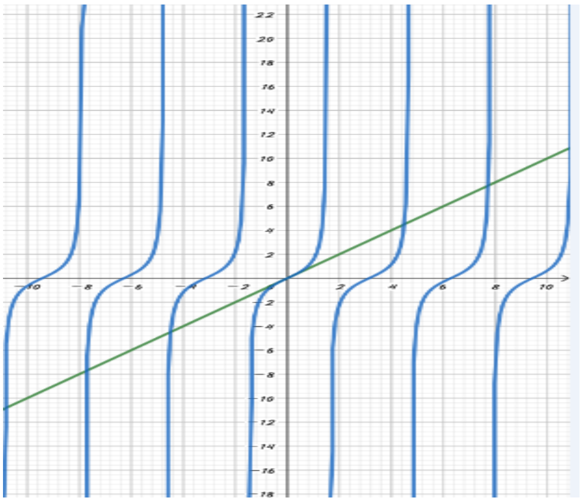

Figure 2.4: Fixed Points of Tangent Function

Example 2.4.4.

Define a mapping $\mathcal{T}$ on $\mathbb{R}$, as:

$$\mathcal{T}(\xi_1) = \xi_1^2 - 3\xi_1 + 4,$$

where $\xi_1 = 2$ is a fixed point of mapping.

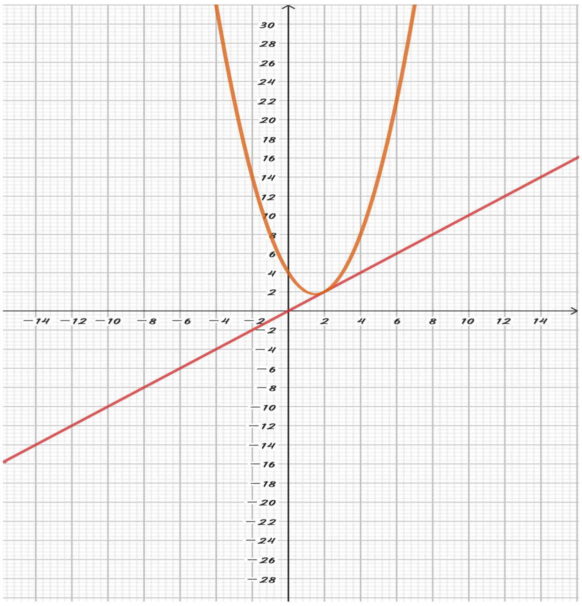

Figure 2.5: One Fixed Point

Example 2.4.5.

Define translation mapping $\mathcal{T} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as

$$\mathcal{T}(\xi_1) = \xi_1 + 1$$

then, the mapping $\mathcal{T}$ has no fixed point.

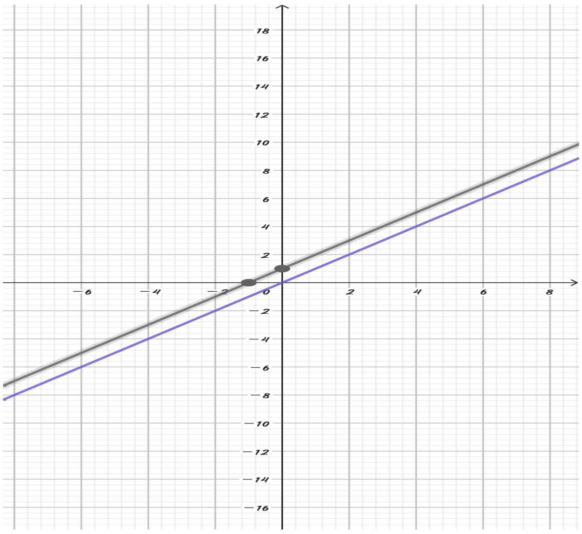

Figure 2.6: No Fixed Point

Example 2.4.6.

Define a mapping $\mathcal{T}$ on $\mathbb{R}$, as:

$$\mathcal{T}(\xi_1) = \xi_1^2 - 4\xi_1 + 3,$$

the mapping have two fixed points.

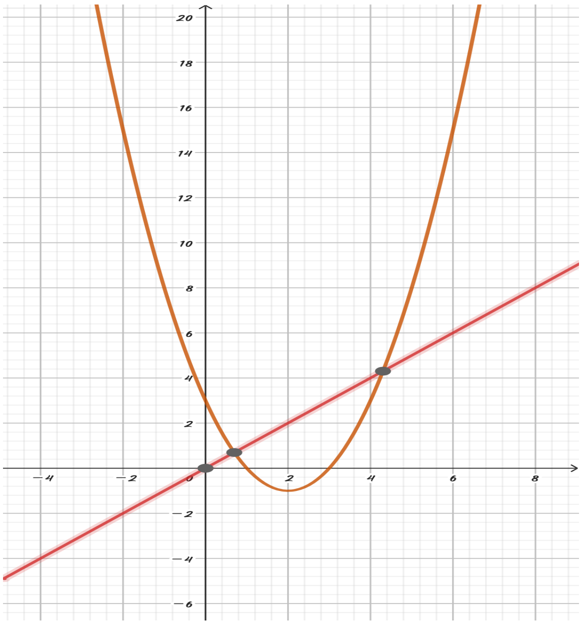

Figure 2.7: Two Fixed Points

Example 2.4.7.

Define a mapping $\mathcal{T} : \mathbb{R} \longrightarrow \mathbb{R}$, as:

$$\mathcal{T}(\xi_1) = \xi_1^2 - 2,$$

where $\xi_1 = -1, 2$ are fixed points of mapping.

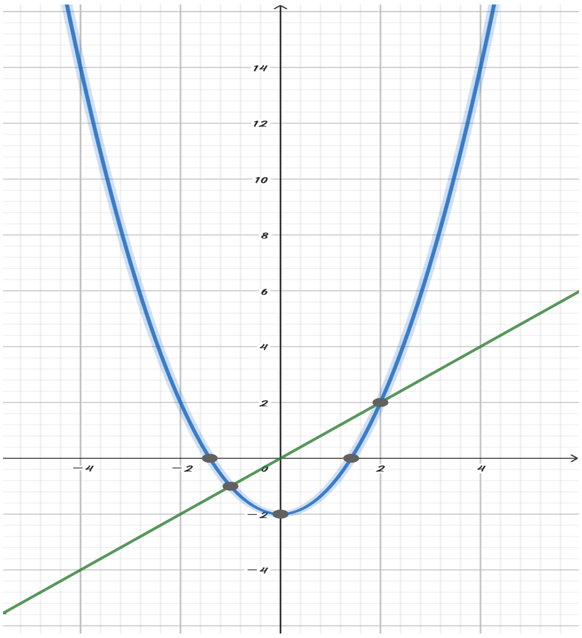

Figure 2.8: Two Fixed Points

Next we will present some crucial fixed point theorems that form the foundation of fixed point theory and its applications.

In 1910, famous Brouwer Theorem was given, although this theorem didn't give any information about location of fixed points but it gave a direction towards existence of fixed point.

Theorem 2.4.8. Brouwer's Theorem

"Every continuous mapping from a closed ball of Euclidean space into itself has a fixed point." [42]

In 1930 an improved version of above result was presented by Schauder, with addition of compactness condition.

Theorem 2.4.9. Schauder Theorem

"Every continuous function from a convex compact subset of Euclidean space to itself has a fixed point." [43]

In 1922 Stefan Banach presented a theorem which is known as Banach Contraction Principle(BCP)/(Contraction mapping theorem), It ensures the existence and uniqueness of a fixed point.

Theorem 2.4.10. Banach Contraction Principle

“Let $(\mathfrak{H}, \mathfrak{D})$ be a complete metric space and let $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be a contraction mapping. Then $\mathcal{T}$ has a unique fixed point ξ_{1_0}, and for each $\xi_1 \in \mathfrak{H}$,

$$\lim_{m_2 \longrightarrow \infty} \mathcal{T}^{m_2} (\xi_1) = \xi_{1_0}.$$

Moreover,

$$\mathfrak{D} \left(\mathcal{T}^{m_2} (\xi_1), \xi_{1_0} \right) \leq \frac{\eta^{m_2}}{\eta - 1} \mathfrak{D} \left(\xi_1, \mathcal{T} (\xi_1) \right) . \text{”[7]}$$

The first generalization of Banach theorem was proposed by Edelstein in which he replaced the complete space by compact space.

Theorem 2.4.11. Edelstein Theorem

“Let $(\mathfrak{H}, \mathfrak{D})$ be a compact metric space and let $\mathcal{T}$ be a mapping on $\mathfrak{D}$. Assume

$$\mathfrak{D}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) < \mathfrak{D}(\xi_1, \xi_2)$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$ with $\xi_1 \neq \xi_2$. Then, $\mathcal{T}$ has a unique fixed point.” [44]

Czerwik extended the renowned Banach Contraction Principle (BCP) by substituting traditional metric spaces with more general b-metric spaces, thereby proposing a significant generalization of the classic theorem.

Theorem 2.4.12. Banach Fixed Point Theorem in b-metric space

“Let $(\mathfrak{H}, \mathfrak{D})$ be a complete b-metric space with constant $\mathfrak{s} \geq 1$ and define the sequence $\left\{\xi_{1_{m_2}}\right\}_{\xi_2=1}^{\infty} \subset \mathfrak{H}$. Let $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be a contraction with the restrictions $\eta \in [0.1)$ and $\eta\mathfrak{s} < 1$. Then, there exists $\xi_1^* \in \mathfrak{H}$ such that $\xi_{1_{m_2}} \longrightarrow \xi_1^*$ is unique fixed point of $\mathcal{T}$.” [45]

Chapter 3

ζ-Contractive Mappings on the Platform of b-metric Spaces and b-metric-like Spaces

This chapter reviews Jain and Kaur et al.[33] fixed point theorems for ζ-contractive mappings in bms and bmls. Having already discussed b-metric spaces, we will go ahead and discuss the concept of ζ-contractive mappings. The concept of b-metric spaces was first proposed by Bakhtin in 1989 and later formally defined by Czerwik in 1993, with the aim of extending the Banach Contraction Principle (BCP).

3.1 Preliminaries

The definitions and results which follow below are necessary for the main results presented in [33].

Theorem 3.1.1.

Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}})$ be a Cbms with constant $\mathfrak{s} \geq 1$. Suppose $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ obeys the inequality:

$$\begin{aligned}\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq \lambda_1^* \mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) + \lambda_2^* \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) \\ + \lambda_3^* \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2) + \lambda_4^* \left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)\right),\end{aligned}$$

where all λ_i^* are non-negative reals. Also $\lambda_1^* + \lambda_2^* + \lambda_3^* + 2\lambda_4^* < 1$ for $\mathfrak{s} \in [1,2]$ and $\frac{2}{\mathfrak{s}} < \lambda_1^* + \lambda_2^* + \lambda_3^* + 2\lambda_4^* < 1$ for $\mathfrak{s} > 2$ the mapping $\mathcal{T}$ has a unique fp. [46]

Lemma 3.1.2.

Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ be a b-metric space with convergent sequences $\xi_{1_{m_2}} \longrightarrow \xi_1$ and $\xi_{2_{m_2}} \longrightarrow \xi_2 \in \mathfrak{H}$. Then,

$$\frac{1}{\mathfrak{s}^2}\mathfrak{D}_{\mathrm{b}}\left(\xi_1, \xi_2\right) \leq \liminf_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2}}, \xi_{2_{m_2}}\right) \leq \limsup_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2}}, \xi_{2_{m_2}}\right) \leq \mathfrak{s}^2 \mathfrak{D}_{\mathrm{b}}\left(\xi_1, \xi_2\right).$$

In particular, if $\xi_1 = \xi_2$, then $\liminf_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2}}, \xi_{2_{m_2}}\right) = 0$.
Furthermore, for each $\xi_3 \in \mathfrak{H}$, there exists

$$\frac{1}{\mathfrak{s}}\mathfrak{D}_{\mathrm{b}}\left(\xi_1, \xi_3\right) \leq \liminf_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2}}, \xi_3\right) \leq \limsup_{m_2 \to +\infty} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2}}, \xi_3\right) \leq \mathfrak{s}\mathfrak{D}_{\mathrm{b}}\left(\xi_1, \xi_3\right). [47]$$

Lemma 3.1.3.

Each sequence $\left(\xi_{1_{m_2}}\right)_{m_2 \in \mathbb{N}}$ of elements from a bms $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}})$ of constant $\mathfrak{s}$, the inequality

$$\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_0}, \xi_{1_\eta}\right) \leq \sum_{i=0}^{\eta-1} \mathfrak{s}^{m_2} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_i}, \xi_{1_{i+1}}\right)$$

is true $\forall$ $m_2 \in \mathbb{N}$ and every $\eta \in \{1, 2, ..., 2^{m_2}\}$. [48]

Lemma 3.1.4.

A sequence $\left\{\xi_{1_{m_2}}\right\}$ in a b-metric space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$, is Cauchy, if it satisfies: for some $\lambda^* \in [0,1)$ such that $\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}\right) \leq \lambda^* \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}\right)$ and all $m_2 \in \mathbb{N}$.

Proof.

First note that

$$\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_2+1}}, \xi_{1_{m_2}}\right) \leq {\lambda^*}^{m_2} \mathfrak{D}_{\mathrm{b}}\left(\xi_1, \xi_{1_0}\right), \tag{3.1}$$

for every $m_2 \in \mathbb{N}$. For all $m_1, \eta \in \mathbb{N}$ with the notation $p = \log_2 \eta$, we have

$$\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+1}}, \xi_{1_{m_1+\eta}}\right) \leq \mathfrak{s}\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+1}}, \xi_{1_{m_1+2}}\right) + \mathfrak{s}\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2}}, \xi_{1_{m_1+\eta}}\right)$$

$$
\begin{aligned}
&\leq \mathfrak{s}\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+1}}, \xi_{1_{m_1+2}}\right) + \mathfrak{s}^2\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2}}, \xi_{1_{m_1+2^2}}\right) + \\
&\quad \mathfrak{s}^2\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2^2}}, \xi_{1_{m_1+\eta}}\right) \\
&\leq \mathfrak{s}\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+1}}, \xi_{1_{m_1+2}}\right) + \mathfrak{s}^2\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2}}, \xi_{1_{m_1+2^2}}\right) + \\
&\quad \mathfrak{s}^3\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2^2}}, \xi_{1_{m_1+2^3}}\right) \\
&+ \mathfrak{s}^3\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2^3}}, \xi_{1_{m_1+\eta}}\right) \\
&\ldots \\
&\leq \sum_{m_2=1}^{p} \mathfrak{s}^{m_2}\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2^{m_2-1}}}, \xi_{1_{m_1+2^{m_2}}}\right) + \mathfrak{s}^{p+1}\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+2^p}}, \xi_{1_{m_1+\eta}}\right).
\end{aligned}
$$

Using Lemma 3.1.3 and equation (3.1), we obtain

$$
\begin{aligned}
\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+1}}, \xi_{1_{m_1+\eta}}\right) &\leq \sum_{m_2=1}^{p} \mathfrak{s}^{2m_2}\left(\sum_{i=m_1}^{m_1+2^{m_2-1}-1} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{2^{m_2-1}+i}}, \xi_{1_{2^{m_2-1}+i+1}}\right)\right) + \\
&\quad \mathfrak{s}^{2(p+1)}\left(\sum_{i=m_1}^{m_1+k-2^p-1} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{2^p+i}}, \xi_{1_{2^p+i+1}}\right)\right) \\
&\leq \sum_{m_2=1}^{p+1} \mathfrak{s}^{2m_2}\left(\sum_{i=m_1}^{m_1+2^{m_2-1}-1} \mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{2^{m_2-1}+i}}, \xi_{1_{2^{m_2-1}+i+1}}\right)\right) \\
&\leq \mathfrak{D}\left(\xi_{1_0}, \xi_1\right) \sum_{m_2=1}^{p+1} \mathfrak{s}^{2m_2}\left(\sum_{i=0}^{2^{m_2-1}-1} {\lambda^*}^{m_1+2^{n-1}+i}\right) \\
&\leq \frac{\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_0}, \xi_1\right){\lambda^*}^{m_1}}{1-\lambda^*} \sum_{m_2=1}^{p+1} \mathfrak{s}^{2m_2}{\lambda^*}^{2^{m_2-1}} \\
&= {\lambda^*}^{m_1}\frac{\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_0}, \xi_1\right)}{1-\lambda^*} \sum_{m_2=1}^{p+1} {\lambda^*}^{2m_2\log_{\lambda^*}\mathfrak{s}+2^{m_2-1}}.
\end{aligned}
$$

Since $\lim\limits_{m_2\longrightarrow\infty} 2m_2\log_{\lambda^*}\mathfrak{s} + 2^{m_2-1} - m_2 = \infty$, for a fixed $M > 0$, there exists $m_{2_0} \in \mathbb{N}$ such that $2m_2\log_{\lambda^*}\mathfrak{s} + 2^{m_2-1} - m_2 \geq M$, that is $2m_2\log_{\lambda^*}\mathfrak{s} + 2^{m_2-1} - m_2 \leq {\lambda^*}^M{\lambda^*}^{m_2}$ for all $m_2 \in \mathbb{N}, m_2 \geq m_{2_0}$, ensuring the convergence of the series $\sum_{m_2=1}^{\infty} {\lambda^*}^{2m_2\log_{\lambda^*}\mathfrak{s}+2^{m_2-1}}$ with sum S which leads to the conclusion that

$$
\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_{m_1+1}}, \xi_{1_{m_1+k}}\right) \leq {\lambda^*}^{m_1}\frac{\mathfrak{D}_{\mathrm{b}}\left(\xi_{1_0}, \xi_1\right)S}{1-\lambda^*},
$$

for all $m_1, \eta \in \mathbb{N}$. As $\lambda^{*^{m_2}} \longrightarrow 0$ as $m_2 \longrightarrow \infty$, we conclude that the sequence $\left(\xi_{1_{m_2}}\right)_{m_2 \in \mathbb{N}}$ is Cauchy.[48] ☐

3.2 ζ-Contractive Mapping of Type-I

In this section, a new contractive mapping is defined in the context of b-metric space as detailed below, after first defining a new class of functions.

Definition 3.1.

All of the functions combined $\zeta : [0, +\infty)^{m_1} \dashrightarrow [0, +\infty)$ for any $m_1 \in \mathbb{N}$ is defined as ε_{m_1} in a way that

(i) $\zeta\left(\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{m_1}\right) < \max \vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{m_1}$ if $\left(\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{m_1}\right) \neq (0, 0, 0, ..., 0)$,

(ii) Consider m_1 sequences $\left\{\vartheta_i^{(m_2)}\right\}_{m_2 \in \mathbb{N}}$, in $[0, +\infty)$, where $1 \leq i \leq m_1$. Suppose that each sequence converges in the sense that:
$\limsup\limits_{m_2 \to +\infty} \vartheta_i^{(m_2)} = \vartheta_i < +\infty$ for all $i = 1$ to m_1.
Then, the following inequality holds:
$\liminf\limits_{m_2 \to +\infty} \zeta\left(\vartheta_1^{(m_2)}, \vartheta_2^{(m_2)}, \vartheta_3^{(m_2)}, ..., \vartheta_{m_1^{(m_2)}}\right) \leq \zeta\left(\vartheta_1, \vartheta_2, \vartheta_3, ..., \vartheta_{m_1}\right).$

Definition 3.2.

Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ be a b-metric space. The mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ is called a ζ-contractive mapping of type-I if there exists $\zeta \in \varepsilon_4$ such that

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq \frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)}{2\mathfrak{s}}\Big\}, \tag{3.2}$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$.

Theorem 3.2.1.

Given a complete b-metric space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ and a type-I ζ-contraction mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$, there exists a unique fixed point for $\mathcal{T}$.

Proof.

Suppose $\xi_{1_0} \in \mathfrak{H}$. Define a sequence $\left\{\xi_{1_{m_2}}\right\}$ in $\mathfrak{H}$ recursively by

$$\xi_{1_{m_2}} = \mathcal{T}\xi_{1_{m_2-1}}$$

$\forall\ m_2 \geq 1$. Assuming that

$$\xi_{1_{m_2}} \neq \xi_{1_{m_2+1}}$$

for $m_2 \in \mathbb{N}$ otherwise, $\mathcal{T}$ has a fp, we aim to show that the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy. In order to prove this, let $m_2 \in \mathbb{N}$ and consider

$$\begin{aligned}\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) &\leq \frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\} \qquad (3.3)\\ &< \frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\}\\ &= \frac{1}{\mathfrak{s}}\max\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\}\\ &\leq \frac{1}{\mathfrak{s}}\max\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2}\Big\},\end{aligned}$$

which implies that

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) < \frac{1}{\mathfrak{s}}\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) \qquad \text{for all } m_2 \geq 1. \qquad (3.4)$$

Case-I:

According to Lemma 3.1.3 and equation (3.4), the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy, whenever $\mathfrak{s} > 1$.

Case-II:

When $\mathfrak{s} = 1$, equation (3.4) implies that the sequence $\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\}$ decreases monotonically and remains bounded below. As a result, the distance $\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})$ converges to a non-negative value η. Assuming η is positive, and taking the $\liminf\limits_{m_2\to+\infty}$ in equation (3.2), we get

$$\eta \leq \zeta(\eta, \eta, \eta, \eta'),$$

where

$$\eta' = \limsup_{m_2\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2} \leq \limsup_{m_2\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2} = \eta.$$

At this point,

$$\eta \leq \zeta(\eta, \eta, \eta, \eta') < \max(\eta, \eta, \eta, \eta') = \eta$$

which leads to a contradictory; thus,

$$\lim_{m_2 \longrightarrow +\infty} \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) = 0. \tag{3.5}$$

Assume that the sequence $\left\{\xi_{1_{m_2}}\right\}$ is not Cauchy, then $\exists$ a positive real numberϵ such that for every natural number r, there exists integers m_{1_r} and m_{2_r} satisfying $m_{1_r} > m_{2_r} \geq r$

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \geq \epsilon. \tag{3.6}$$

Moreover, suppose that $m_{1_r} > m_{2_r}$ is the least natural number such that equation (3.6) is satisfied.

Then,

$$\begin{aligned}
\epsilon \leq & \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \\
\leq & \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r-1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r-1}}}}, \xi_{1_{m_{2_r}}}) \\
\leq & \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r-1}}}}) + \epsilon \\
< & \mathfrak{D}_{\mathrm{b}}(\xi_{1_r}, \xi_{1_{r-1}}) + \epsilon,
\end{aligned}$$

thus, using (3.5) and taking $\liminf\limits_{n \to +\infty}$, we get

$$\lim_{r \longrightarrow +\infty} \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) = \epsilon. \tag{3.7}$$

Now, consider

$$\begin{aligned}
\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_{r+1}}}}) \leq \zeta \Bigg(& \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r+1}}}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_r}}}, \xi_{1_{m_{2_{r+1}}}}), \\
& \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_r}}})}{2} \right) \Bigg)
\end{aligned}$$

Therefore, we have

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \leq \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_{r+1}}}}, \xi_{1_{m_{2_r}}})$$

$$\begin{aligned}
&\le \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_{r+1}}}}, \xi_{1_{m_{2_r}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_{r+1}}}}) \\
&\le \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_{r+1}}}}, \xi_{1_{m_{2_r}}}) + \zeta\Big(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}), \\
&\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r+1}}}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_r}}}, \xi_{1_{m_{2_{r+1}}}}), \\
&\left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_r}}})}{2}\right)\Bigg).
\end{aligned}$$

Therefore, applying $\liminf\limits_{m_2\to+\infty}$ to both sides, and combining equations (3.5) and (3.7), we obtain: $\epsilon \le 0 + 0 + \zeta(\epsilon, 0, 0, \epsilon')$, where

$$\begin{aligned}
\epsilon' &= \limsup_{r\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_r}}})}{2} \\
&\le \limsup_{r\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_r}}}, \xi_{1_{m_{2_{r+1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{1_r}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}})}{2} \\
&= \frac{\epsilon + 0 + 0 + \epsilon}{2} = \epsilon.
\end{aligned}$$

Consequently, $\epsilon \le \xi(\epsilon, 0, 0, \epsilon') < \max(\epsilon, 0, 0, \epsilon') = \epsilon$, which is contradictory. Hence, $\{\xi_{1_{m_2}}\}$ is a Cauchy sequence in the metric space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}})$, whenever $\mathfrak{s} \ge 1$.

Given that $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \ge 1)$ is a complete b-metric space, we can conclude that the sequence $\xi_{1_{m_2}}$converges to a point $\xi_1 \in \mathfrak{H}$. That is, $\xi_{1_{m_2}} \longrightarrow \xi_1$.

Now, consider

$$\begin{aligned}
\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_{1_{m_2}}, \mathcal{T}\xi_1) \le \frac{1}{\mathfrak{s}}\zeta\Big(&\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \mathcal{T}\xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \\
&\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_{1_{m_2}})}{2\mathfrak{s}}\Bigg),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2+1}}, \mathcal{T}\xi_1) \le \frac{1}{\mathfrak{s}}\zeta\Big(&\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \\
&\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\Bigg).
\end{aligned}$$

Using Lemma 3.1.3 and taking $\liminf\limits_{m_2\to+\infty}$ on both sides, we get

$$\frac{1}{\mathfrak{s}}(\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) \le \frac{1}{\mathfrak{s}}\zeta\left(0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l\right),$$

$$\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) \leq \zeta\left(0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l\right),$$

$$\begin{aligned} l =& \limsup_{n \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}} \\ \leq& \lim_{m_2 \longrightarrow +\infty} \frac{\mathfrak{s}\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) + 0}{2\mathfrak{s}} \\ =& \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1)}{2}. \end{aligned}$$

$$\begin{aligned} \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) \leq& \zeta(0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l) \\ &< \max\{0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l\} = \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \end{aligned}$$

which is contradictory. Therefore $\mathcal{T}\xi_1 = \xi_1$. Let $\mathcal{T}\xi_2 = \xi_2$ for some $\xi_2 \in \mathfrak{H}$ and suppose that $\xi_1 \neq \xi_2$; then, consider

$$\begin{aligned} \mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) &= \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \\ &\leq \frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_1, \xi_1)}{2\mathfrak{s}}\right) \\ &\leq \frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), 0, 0, \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2)}{\mathfrak{s}}\right) \\ &\leq \frac{1}{\mathfrak{s}}\max\left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), 0, 0, \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2)}{\mathfrak{s}}\right) \\ &= \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2)}{\mathfrak{s}}, \end{aligned}$$

which is contradictory. Therefore, $\xi_1 = \xi_2$. □

Remark 3.3.

Theorem 3.2.1 remains valid if we replace the term $\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)}{2\mathfrak{s}}$ in (3.2) is replaced

by $\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)}{\mu\mathfrak{s}}$, where μ is defined as a $\mathbb{R}$ such that

$$\mu\mathfrak{s} = \begin{cases} 2, & \text{if } \mathfrak{s} = 1, \\ 3, & \text{if } 1 < \mathfrak{s} \leq 2, \\ 1, & \text{if } \mathfrak{s} > 2, \end{cases}$$

where μ' is any $\mathbb{R}$ in the interval $\left(\frac{2}{\mathfrak{s}}, 1 + \frac{1}{\mathfrak{s}}\right)$.

Corollary 3.2.2.

Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}})$ be a complete b-metric space with parameter $\mathfrak{s} \geq 1$, and let $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ be a mapping that satisfies the following property: $\exists$ a constant $q \in \left[0, \frac{1}{\mathfrak{s}}\right)$ so that

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq q \max\left\{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)}{2\mathfrak{s}}\right\}, \tag{3.8}$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$. Thus, $\mathcal{T}$ has a unique fixed point.

Proof.

Consider a function $\zeta \in \varepsilon_4$ defined as: $\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = q\mathfrak{s} \max\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$. According to Theorem 3.2.1, which implies that $\mathcal{T}$ has a unique fixed point. $\square$

Example 3.2.3.

Consider the set $\mathfrak{H} = \left\{\frac{1}{\sqrt{m_2}} : m_2 \in \mathbb{N}\right\} \cup \{0\}$. Define a function $\mathfrak{D}_{\mathrm{b}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, \infty)$ as follows: $\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) = |\xi_1 - \xi_2|^2$ for all $\xi_1, \xi_2 \in \mathfrak{H}$. This defines a b-metric $\mathfrak{D}_{\mathrm{b}}$ on $\mathfrak{H}$ with parameter $\mathfrak{s} = 2$.

Define a mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ as follows: for each $n \in \mathbb{N}$, set $\mathcal{T}(\frac{1}{\sqrt{m_2}}) = \frac{1}{\sqrt{2(m_2+1)}}$, and define $\mathcal{T}(0) = 0$. Then, define

$$\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \begin{cases} \frac{\max\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}}{1+\vartheta_1}, & \vartheta_1 > 0, \\ \frac{1}{2}\max\{\vartheta_2, \vartheta_3, \vartheta_4\}, & \text{otherwise.}, \end{cases}$$

Then, considering all ξ_1, $\xi_2 \in \mathfrak{H}$, (3.2) holds, thereby the conditions of theorem 3.2.1 are fulfilled.

Moreover, we notice that if equation (3.8) holds for all elements ξ_1, $\xi_2 \in \mathfrak{H}$, then we observe

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq qN(\xi_1, \xi_2),$$

$\forall\, \xi_1, \xi_2 \in \mathfrak{H}$, where

$N(\xi_1, \xi_2) = \max\left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)}{2\mathfrak{s}}\right).$

Consequently, we observe that

$$\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{\sqrt{2(m_2+1)}}, \frac{1}{\sqrt{2(m_1+1)}}\right) \leq qN\left(\frac{1}{\sqrt{(m_2)}}, \frac{1}{\sqrt{(m_1)}}\right) \qquad \forall\, m_1, m_2 \in \mathbb{N}, m_1 \neq m_2.$$

that is

$$\frac{\left|\frac{1}{\sqrt{(m_2+1)}} - \frac{1}{\sqrt{(m_1+1)}}\right|^2}{N\left(\frac{1}{\sqrt{(m_2)}}, \frac{1}{\sqrt{(m_1)}}\right)} \leq 2q \qquad \text{for all } m_1, m_2 \in \mathbb{N}, m_1 \neq m_2.$$

Taking $\lim\limits_{m_1,m_2 \to +\infty}$, leads to a contradiction, namely $2q \geq 1$.

Remark 3.4.

Considering remark 3.3, Corollary 3.2.2 is also valid, if the term $\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1,\xi_2)}{2\mathfrak{s}}$ is replaced by $\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1,\xi_2)}{\mu\mathfrak{s}}$, where μ is defined as in remark 3.3.

Corollary 3.2.4.

Suppose $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ is a Cbms, and $\mathcal{T}\colon \mathfrak{H} \longrightarrow \mathfrak{H}$ is a mapping that satisfies the following condition:

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq \lambda_1^*\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) + \lambda_2^*\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) + \lambda_3^*\mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2) + \lambda_4^*\left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)\right), \tag{3.9}$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$, in which $\lambda_1^* + \lambda_2^* + \lambda_3^* + \mu s \lambda_4^* < \frac{1}{\mathfrak{s}}$
and $\lambda_i^* \geq 0\ \forall;\ i = \{1, 2, 3, 4\}$. Then, the mapping $\mathcal{T}$ has a unique fixed point.

Proof.

Let $\zeta \in \varepsilon_4$ be defined as $\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \mathfrak{s}(\lambda^*\vartheta_1 +_1 \lambda^*\vartheta_2 +_2 \lambda^*\vartheta_3 + {}_3\mu\mathfrak{s}\lambda^*\vartheta_4). \quad {}_4$

□

Then, by Theorem 3.2.1 and remark 3.4, $\mathcal{T}$ has exactly one fixed point.

3.2.1 ζ-Contractive Mapping of Type-II

Next, we introduce a contractive mapping in the b-metric space.

Definition 3.5.

Consider a b-metric space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$. A mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ is called ζ-contractive if ∃ a constant $\zeta \in \varepsilon_5$ so that

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq \frac{1}{\mathfrak{s}}\zeta\left\{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2)}{2\mathfrak{s}}, \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)\right\}, \tag{3.10}$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$.

Theorem 3.2.5.

Consider a complete b-metric space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$. If a mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ is ζ-contractive of type-II, then $\mathcal{T}$ has a single fixed point.

Remark 3.6.

In addition, Theorem 3.2.5 holds true if $\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2)}{2\mathfrak{s}}$ in (3.10) is changed to $\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2)}{\mu\mathfrak{s}}$, where remark 3.3 defines μ.

Corollary 3.2.6.

Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ be a Cbms and $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ act as a mapping that satisfies the subsequently property: $\exists$ a constant $q \in \left[0, \frac{1}{\mathfrak{s}}\right)$ so that

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq q \max\left\{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2)}{\mu\mathfrak{s}}, \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)\right\}, \tag{3.11}$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$. Consequently, $\mathcal{T}$ possesses a unique fixed point.

Proof.

Define $\zeta \in \varepsilon_5$ as follows: $\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) = q\mathfrak{s} \max(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5)$. Consequently, the mapping $\mathcal{T}$ possesses a unique fixed point. □

Corollary 3.2.7.

Let $\mathcal{T}$: $\mathfrak{H} \dashrightarrow \mathfrak{H}$ be a mapping such that $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ is a complete b-metric space.

$$\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \leq \lambda_1^*\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) + \lambda_2^*\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) + \lambda_3^*\mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2) + \lambda_4^*\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \lambda_5^*\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2), \tag{3.12}$$

$\forall\ \xi_1, \xi_2 \in \mathfrak{H}$, as well as $\lambda_1^* + \lambda_2^* + \lambda_3^* + \mu\mathfrak{s}\lambda_4^* + \lambda_5^* < \frac{1}{\mathfrak{s}}$ and $\lambda_i^* \geq 0\ \forall\ i = \{1, 2, 3, 4, 5\}$. If $\mathcal{T}$ satisfies (3.12) then $\mathcal{T}$ has a single fp.

Proof.

Let $\zeta \in \varepsilon_5$ be defined as $\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5) = \mathfrak{s}(\lambda_1^*\vartheta_1 + \lambda_2^*\vartheta_2 + \lambda_3^*\vartheta_3 + \mu\mathfrak{s}\lambda_4^*\vartheta_4 + \lambda_5^*\vartheta_5)$. Then, $\mathcal{T}$ has a single fixed point, as per Theorem 3.2.5. □

3.3 Existence Results in b-metric-like Spaces

Theorem 3.3.1.

Consider a complete b-metric-like space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$. Let $\left\{\xi_{1_{m_2}}\right\}$ be a sequence of mappings, subject to the condition that there exists a constant $\zeta \in \varepsilon_4$ such that

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_1,\mathcal{T}\xi_2) \le \Bigg(\frac{1}{\mathfrak{s}}\zeta\Big\{&\mathfrak{D}_{\mathrm{bl}}(\xi_1,\xi_2),\mathfrak{D}_{\mathrm{bl}}(\xi_1,\mathcal{T}\xi_1),\mathfrak{D}_{\mathrm{bl}}(\xi_2,\mathcal{T}\xi_2),\\ &\frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_1,\xi_2)-\mathfrak{D}_{\mathrm{bl}}(\xi_2,\xi_2)}{2\mathfrak{s}}\Big\}\Bigg),\end{aligned} \tag{3.13}$$

$\forall\ \xi_1,\xi_2 \in \mathfrak{H}$ with $\mathfrak{D}_{\mathrm{bl}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_1,\xi_2)-\mathfrak{D}_{\mathrm{bl}}(\xi_2,\xi_2)$. As such, there is only one fixed point in the mapping $\mathcal{T}$.

Proof.

Suppose $\xi_{1_0} \in \mathfrak{H}$. Define a sequence $\left\{\xi_{1_{m_2}}\right\}$ in $\mathfrak{H}$ recursively by

$$\xi_{1_{m_2}} = \mathcal{T}\xi_{1_{m_2-1}}$$

$\forall\ m_2 \ge 1$. Assuming that

$$\xi_{1_{m_2}} \neq \xi_{1_{m_2+1}}$$

for $m_2 \in \mathbb{N}$ otherwise, $\mathcal{T}$ has a fp, we aim to show that the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy. In order to prove this, let $m_2 \in \mathbb{N}$ and take into

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\mathcal{T}\xi_{1_{m_2}})+\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_{1_{m_2-1}},\xi_{1_{m_2}}) &= \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2+1}})+\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2}})\\ &\ge \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2}});\end{aligned}$$

therefore, using (3.13), we have

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}}) &\le \frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}}),\\ &\qquad\frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2+1}})+\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2}})-\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2}})}{2\mathfrak{s}}\Big\}\\ &< \frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}}),\frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\}\\ &= \frac{1}{\mathfrak{s}}\max\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}),\frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\}\\ &\le \frac{1}{s}\max\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}),\frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}})+\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}})}{2}\Big\},\end{aligned} \tag{3.14}$$

which implies that

$$\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}}) < \frac{1}{\mathfrak{s}}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}},\xi_{1_{m_2}}) \qquad \text{for all } m_2 \ge 1. \tag{3.15}$$

Case-I:

According to lemma 3.1.3 and equation (3.15), the sequence $\{\xi_{1_n}\}$ is Cauchy, whenever $\mathfrak{s} > 1$.

Case-II:

When $\mathfrak{s} = 1$, equation (3.15) implies that the sequence $\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\}$ decreases monotonically and remains bounded below. As a result, the distance $\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})$ converges to a non-negative value η. Assuming η is positive, and taking the $\liminf\limits_{m_2\to+\infty}$ in equation (3.14), we get

$$\eta' = \limsup_{m_2\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}}}{2} \leq \limsup_{m_2\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2} = \eta.$$

At this point,

$$\eta \leq \zeta(\eta, \eta, \eta, \eta') < \max(\eta, \eta, \eta, \eta') = \eta,$$

which leads to a contradiction; thus,

$$\lim_{m_2\longrightarrow+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) = 0. \tag{3.16}$$

Moreover,

$$\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}}) \leq \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2+1}}, \xi_{1_{m_2}}),$$

Using equation (3.16) and considering $\limsup\limits_{m_2\to+\infty}$, we get

$$\lim_{m_2\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}}) = 0, \tag{3.17}$$

Assume that the sequence $\{\xi_{1_{m_2}}\}$ is not Cauchy, then $\exists$ a positive real numberϵ such that for every natural number r, there exists there exist integers m_{1_r} and m_{2_r} satisfying $m_{1_r} > m_{2_r} \geq r$

$$\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \geq \epsilon. \tag{3.18}$$

Moreover, suppose that $m_{1_r} > m_{2_r}$ is the least natural number such that equation (3.18) is satisfied. Then,

$$\begin{aligned}\epsilon \leq &\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}})\\ \leq &\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r-1}}}}) + \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_{r-1}}}}, \xi_{1_{m_{2_r}}})\\ \leq &\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r-1}}}}) + \epsilon\\ < &\mathfrak{D}_{\text{bl}}(\xi_{1_r}, \xi_{1_{r-1}}) + \epsilon.\end{aligned}$$

thus, using (3.16) and taking $\liminf\limits_{m_2 \to +\infty}$, we get

$$\lim_{r \longrightarrow +\infty} \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) = \epsilon. \tag{3.19}$$

Let us now assume that there are an infinite number of r such that

$$\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \mathcal{T}\xi_{1_{m_{2_r}}}) + \mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) < \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{2_r}}}, \xi_{1_{m_{2_r}}}).$$

Using equation (3.17), and considering $\limsup\limits_{m_2 \to +\infty}$, we obtain

$$\limsup_{r \longrightarrow +\infty} \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \mathcal{T}\xi_{1_{m_{2_r}}}) + \limsup_{r \longrightarrow +\infty} \mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) < \limsup_{r \longrightarrow +\infty} \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{2_r}}}, \xi_{1_{m_{2_r}}})$$

$$\limsup_{r \longrightarrow +\infty} \{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \mathcal{T}\xi_{1_{m_{2_r}}}) + \mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}})\} = 0,$$

$$\lim_{r \longrightarrow +\infty} \{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_{r+1}}}})\} = \lim_{r \longrightarrow +\infty} \{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_{r+1}}}}, \xi_{1_{m_{2_r}}})\} = 0.$$

$$\begin{aligned}\epsilon = \lim_{r \longrightarrow +\infty} \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \leq \limsup_{r \longrightarrow +\infty} \{&\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_{r+1}}}}) +\\ &\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{2_{r+1}}}}, \xi_{1_{m_{2_r}}})\} = 0.\end{aligned}$$

Which is contradictory.

Consequently, there exists a natural number r_0 such that for all integers r greater than or equal to r_0,
$\mathfrak{D}_{\text{bl}}(\xi_{1_{m_{1_r}}}, \mathcal{T}\xi_{1_{m_{2_r}}}) + \mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \geq \mathfrak{D}_{\text{bl}}(\xi_{1_{m_{2_r}}}, \xi_{1_{m_{2_r}}})$, Thus, for all $r \geq r_0$, using

(3.13),

$$\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}}) \leq \zeta\left\{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \right.$$
$$\left. \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}}) - \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_r}})}{2}\right\}.$$

Now,

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_r}}) &\leq \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_{r+1}}}, \xi 1_{m_{2_r}}) \\ &\leq \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_{r+1}}}, \xi 1_{m_{2_r}}) + \zeta\left\{(\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \right. \\ &\quad \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \\ &\quad \left. \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}}) - \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_r}})}{2}\right\}.\end{aligned}$$

Consequently, applying $\liminf\limits_{n\to+\infty}$ to both sides and using equations (3.16) and (3.19), we have: $\epsilon \leq 0+0+\max(\epsilon, 0, 0, \epsilon')$

$$\begin{aligned}\epsilon' &= \limsup_{r\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}}) - \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_r}})}{2} \\ &\leq \limsup_{r\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{1_r}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) - 0}{2} \\ &= \frac{\epsilon+0+0+\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore, the following inequalities hold: $\epsilon \leq \zeta(\epsilon, 0, 0, \epsilon') < \max(\epsilon, 0, 0, \epsilon')=\epsilon$ leading to a contradiction. As a result, the sequence $\left\{\xi 1_{m_{2_r}}\right\}$ is Cauchy in the space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ with

$$\lim_{m_1, m_2\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi 1_{m_1}) = 0$$

.

Since, $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ is a Cbmls, it follows that $\exists$ an element $\xi_1 \in \mathfrak{H}$ such that $\xi 1_{m_2} \longrightarrow \xi_1$,

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1) = \lim_{n\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1) = \lim_{m_1, m_2\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi 1_{m_1}) = 0.$$

Moreover, according to proposition 2.3.1, ξ_1 is unique.

Assuming that $\mathcal{T}\xi_1 \neq \xi_1$, let us consider

$$\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_2}, \mathcal{T}\xi_1) \leq \frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi 1_{m_2}), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), \right.$$
$$\left. \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi 1_{m_2}) - \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1)}{2\mathfrak{s}}\right),$$

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2+1}}, \mathcal{T}\xi_1) \leq \frac{1}{\mathfrak{s}}\zeta\Big(&\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1),\\ &\frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big).\end{aligned}$$

Applying $\liminf\limits_{m_2 \to +\infty}$ to both sides and utilizing Proposition 2.3.1, we obtain

$$\frac{1}{\mathfrak{s}}(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1) \leq \frac{1}{\mathfrak{s}}\zeta\left(0, 0, \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), l\right),$$

that is

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1) \leq \zeta\left(0, 0, \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), l\right),$$

where

$$\begin{aligned} l &= \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\\ &\leq \limsup_{n \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1)}{2\mathfrak{s}} + \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\\ &\leq \lim_{m_2 \longrightarrow +\infty} \frac{\mathfrak{s}\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1) + 0}{2\mathfrak{s}}\\ &= \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1)}{2} \end{aligned},$$

Thus,

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1) \leq& \zeta(0, 0, \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), l)\\ &< \max\{0, 0, \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), l\} = \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1),\end{aligned}$$

which is a contradictory.

Therefore, $\mathcal{T}\xi_1 = \xi_1$. Let $\mathcal{T}\xi_2 = \xi_2$ for some $\xi_2 \in \mathfrak{H}$, then, by (3.13), $\mathfrak{D}_{\mathrm{bl}}(\xi_2, \xi_2) = 0$.

Now, suppose that $\xi_1 \neq \xi_2$; and consider

$$\begin{aligned}&\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) = \mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_1, \mathcal{T}\xi_2)\\ &\leq \frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{bl}}(\xi_2, \mathcal{T}\xi_1) - \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1)}{2\mathfrak{s}}\right),\\ &= \frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{bl}}(\xi_2, \mathcal{T}\xi_1)}{2\mathfrak{s}}\right),\\ &\leq \frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2), 0, 0, \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2)}{\mathfrak{s}}\right)\\ &< \frac{1}{\mathfrak{s}}\max\left(\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2), 0, 0, \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2)}{\mathfrak{s}}\right),\end{aligned}$$

$$= \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2)}{\mathfrak{s}},$$

which is a contradiction. Therefore, $\xi_1 = \xi_2$. $\square$

Example 3.3.2.

Let $\mathfrak{H} = [0, +\infty)$. Define $\mathfrak{D}_{\mathrm{bl}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, +\infty)$ by

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_2) = (\xi_1 + \xi_2)^2$$

for all $\xi_1, \xi_2 \in \mathfrak{H}$. Consequently, $\mathfrak{D}_{\mathrm{bl}}$ defines a b-metric-like structure on $\mathfrak{H}$ with parameter $\mathfrak{s} = 2$, , but it does not constitute a b-metric on $\mathfrak{H}$.

Let us define a mapping $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ by setting $\mathcal{T}(\xi_1) = \frac{\xi_1}{2}$.

Furthermore, we define

$$\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \frac{1}{2} \max\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}.$$

For all elements $\xi_1, \xi_2 \in \mathfrak{H}$, we have

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_1, \xi_2) \geq \mathfrak{D}_{\mathrm{bl}}(\xi_2, \xi_2),$$

equation (3.13) in Theorem 3.3.1 holds true and the fixed point of $\mathcal{T}$ is unique and equals 0.

Theorem 3.3.3.

Consider the following system of linear equations

$$\mathfrak{A}\xi_1 = \mathtt{b}, \tag{3.20}$$

in which the matrix $\mathfrak{A}$ is a column vector of constants, denoted by $\mathfrak{A} = [\mathtt{b}_i]_{l \times m_2}$, and the column matrix ξ_1 is composed of n unknown variables, denoted by $\xi_1 = [\xi_{1_i}]_{l \times n}$.

For every $\xi_1 = [\xi_{1_i}]_{l \times m_2}$ and $\xi_2 = [\xi_{2_i}]_{l \times n}$ and for values of i from 1 to m_2,

$$|(\mathtt{a}_{ii} + 1)(\xi_{1_i} - \xi_{2_i}) + \sum_{j=1, j\neq 1}^{n} \mathtt{a}_{jj}(\xi_{1_j} - \xi_{2_j})| \left(1 + \max_{\eta=1}^{n} |\xi_{1_\eta} - \xi_{2_\eta}|\right) \leq |\xi_{1_i} - \xi_{2_i}|; \tag{3.21}$$

Therefore, a unique solution exists for the system.

Proof.

Let $\mathfrak{H} = \left\{ [\xi_{1_i}]_{1\times m_2} \mid \xi_{1_i}$ is real for all values of $i =$ from 1 to m_2, with m_2 fixed$\right\}$ and $\mathfrak{D}_\mathrm{b} : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, +\infty)$ is defined to be

$$\mathfrak{D}_\mathrm{b}(\xi_1, \xi_2) = \max_{i=1}^{m_2} |\xi_{1_i} - \xi_{2_i}|$$

for every $\xi_1 = [\xi_{1_i}]_{l\times m_2}, \xi_2 = [\xi_{2_i}]_{l\times m_2} \in \mathfrak{H}$. So, obviously $(\mathfrak{H}, \mathfrak{D}_\mathrm{b})$ is a Cbms with constant $\mathfrak{s} = 1$. Let's build a square matrix $\mathfrak{C}$ of size $m_2 \times m_2$ now, with $\mathfrak{C} = [\mathtt{c}_{ij}]$, through

$$\mathtt{c}_{ij} = \begin{cases} \mathtt{a}_{ij} + 1, & \text{if } i = j, \mathtt{a}_{ij}, \\ \text{if } i \neq j. \end{cases}$$

Consequently, equation (3.20) simplifies to

$$\xi_1 = \mathfrak{C}\xi_1 - \mathtt{b}. \tag{3.22}$$

Equation (3.21) simplifies to

$$|\sum_{j=1}^{m_2} \mathtt{c}_{jj}\left(\xi_{1_j} - \xi_{2_j}\right)| \left(1 + \max_{\eta=1}^{m_2} |\xi_{1_\eta} - \xi_{2_\eta}|\right) \leq |\xi_{1_i} - \xi_{2_i}| \quad \text{for all values of } i \text{ from 1 to } m_2. \tag{3.23}$$

Next, define a mapping $\mathcal{T}$ from $\mathfrak{H}$ into itself, given by

$$\mathcal{T}\xi_1 = \mathfrak{C}\xi_1 - \mathtt{b}, \qquad \text{where } \xi_1 \in \mathfrak{H}.$$

Consider two matrices $\xi_1 = [\xi_{1_i}]_{l\times m_2}, \xi_2 = [\xi_{2_i}]_{l\times m_2}$. Assume that applying the mapping $\mathcal{T}$ to ξ_1 yields $\mathcal{T}\xi_1 = \mathtt{u} = [\mathtt{u}_j]_{l\times m_2}$ and applying $\mathcal{T}$ to ξ_2 yields $\mathcal{T}\xi_2 = \hbar = [\hbar_i]_{l\times m_2}$.

Consequently,

$$\mathtt{u}_i = \sum_{j=1}^{m_2} \mathtt{c}_{ij}\xi_{1_j} - \mathtt{b}_i \qquad \text{for all values of } i \text{ from 1 to } m_2.$$

Moreover,

$$\hbar_i = \sum_{j=1}^{m_2} \mathtt{c}_{ij}\xi_{2_j} - \mathtt{b}_i \qquad \text{for all values of } i \text{ from 1 to } m_2.$$

Introduce

$$\zeta(\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4) = \begin{cases} \frac{\max\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}}{1+\vartheta_1}, & \vartheta_1 > 0, \\ \frac{1}{2}\max\{\vartheta_2, \vartheta_3, \vartheta_4\}, & \text{otherwise.}, \end{cases}$$

Next, utilizing (3.23),

$$\begin{aligned} \mathfrak{D}_{\mathrm{b}}\left(\mathcal{T}\xi_1, \mathcal{T}\xi_2\right) &= \max_{i=1}^{m_2} |u_i - \hbar_i| \\ &= \max_{i=1}^{m_2} |\sum_{i=1}^{m_2} \mathsf{c}_{ij}\left(\xi_{1_j} - \xi_{2_j}\right)| \\ &\leq \max_{i=1}^{m_2}\left(\frac{|\xi_{1_i} - \xi_{2_i}|}{1 + \max_{\eta=1}^{m_2} |\xi_{1_\eta} - \xi_{2_\eta}|}\right) \\ &\leq \zeta\left(\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2), \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \mathfrak{D}_{\mathrm{b}}(\xi_2, \mathcal{T}\xi_2), \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_2) + \mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1, \xi_2)}{2\mathfrak{s}}\right). \end{aligned}$$

Thus, the mapping $\mathcal{T}$ has a unique fixed point, implying that system (3.20) has a unique solution. □

Chapter 4

(α, β)-$F\zeta$-Contraction Mapping and Fixed Points

Jain and Kaur [33] developed a new class of functions, thereby defining a new contractive mapping in the framework of b-metric space. In this chapter we extended their ideas by involving (α, β)-admissible mapping and using the idea of F-contraction introduced by Wardowski [49]. For this purpose the following section provides some basic definition.

4.1 Basic Definitions

Definition 4.1. α-admissible mapping

Let $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ and $\alpha : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}_+$, we say that $\mathcal{T}$ is an α-admissible mapping if $\alpha(\xi_1, \xi_2) \geq 1$ implies $\alpha(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \geq 1$ for $\xi_1, \xi_2 \in \mathfrak{H}$.[50]

Example 4.1.1. α-admissible mapping

Let $\mathfrak{H} = \mathbb{R}$. Define $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$ and $\alpha : \mathfrak{H} \times \mathfrak{H} \longrightarrow [0, \infty)$ by

$$\mathcal{T}(\xi_1) = \begin{cases} \ln|\xi_1|, & \text{if } \xi_1 \neq 0 \\ 3, & \text{otherwise} \end{cases}$$

$$\alpha(\xi_1, \xi_2) = \begin{cases} 3, & \text{if } \xi_1 \geq \xi_2 \\ 0, & \text{otherwise} \end{cases}$$

One can easily verify that $\mathcal{T}$ is α-admissible .

Definition 4.2. (α, β)**-admissible mapping**

Let $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}}, \mathfrak{s} \geq 1)$ be a b-metric space and $(\alpha, \beta) : \mathfrak{H} \times \mathfrak{H} \longrightarrow (0, \infty)$, be two functions then $\mathcal{T}$: $\mathfrak{H} \times \mathfrak{H}$ is called (α, β)-admissible if
$\alpha(\xi_1, \xi_2) \geq 1 \Longrightarrow \alpha(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \geq 1$ and $\beta(\xi_1, \xi_2) \geq 1 \Longrightarrow \beta(\mathcal{T}\xi_1, \mathcal{T}\xi_2) \geq 1$
for all $\xi_1, \xi_2 \in \mathfrak{H}$.[51]

Definition 4.3. F-mappings

Let $\mathfrak{M}$ denote the family of all function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties

(F_1) It is strictly increasing.

(F_2) For each sequence $\{\xi_{1_{m_2}}\}$ of positive numbers, we have,

$$\lim_{m_2 \to \infty} \xi_{1_{m_2}} = 0 \Leftrightarrow \lim_{m_2 \to \infty} F(\xi_{1_{m_2}}) = -\infty.[49]$$

Example 4.1.2.

The following mappings from $\mathbb{R}^+ \to \mathbb{R}$ are the examples of F- mapping:

(i) : $F(\xi_1) = -\dfrac{1}{\sqrt{\xi_1}}$; where $\xi_1 > 0$.

(ii) : $F(\xi_1) = \ln(\xi_1)$ where $\xi_1 > 0$.

(iii) : $F(\xi_1) = \ln(\xi_1^2 + \xi_1)$ where $\xi_1 > 0$.

Definition 4.4. F-Contraction Mapping

Let $F \in \mathcal{F}$ and let $\mathcal{T}$ be a mapping from a metric space $(\mathfrak{H}, \mathfrak{D})$ in to itself. If there is a positive number τ such that $\forall\, \xi_1, \xi_2 \in \mathfrak{H}$ for which

$$\mathfrak{D}(\mathcal{T}\xi_1, \mathcal{T}\xi_2) > 0, \Longrightarrow \tau + F\left(\mathfrak{D}(\mathcal{T}\xi_1, \mathcal{T}\xi_2)\right) \leq F\mathfrak{D}\left(\xi_1, \xi_2\right), \tag{4.1}$$

holds, then the mapping $\mathcal{T}$ is called an F-contraction. [49]

Remark 4.5.

Obviously, by choosing $F(\xi_1)=\ln(\xi_1)$ and $\tau=\ln(\frac{1}{\eta})$, $\eta\in[0,1]$ the conditions of (4.1) becomes the Banach contraction.

Example 4.1.3.

Let $\mathfrak{H}=\{\xi_{1_{m_2}}: m_2\in\mathbb{N}\}$ where $\xi_{1_{m_2}}=\sum_{\eta=1}^{m_2}\eta=\frac{1}{2}m_2(m_2+1)$ equipped with usual metric, $\mathfrak{D}(\xi_1,\xi_2)=|\xi_1-\xi_2| \quad \forall\, \xi_1,\xi_2\in\mathfrak{H}$. Then $(\mathfrak{H},\mathfrak{D})$ is complete metric space.

Let $\mathcal{T}:\mathfrak{H}\to\mathfrak{H}$ be defined as

$$\mathcal{T}\xi_{1_{m_2}}=\begin{cases}\xi_{1_{m_2}} & \text{for } m_2=1,\\ \xi_{1_{m_2-1}} & \text{for otherwise.}\end{cases}$$

Obviously $\mathcal{T}$ is F-contraction for $\tau=1$ and $F(\xi_1)=\xi_1+\ln\xi_1$ but it is not Banach contraction.

Definition 4.6. (α,β)-**$F\zeta$-Contraction**

Let $(\mathfrak{H},\mathfrak{D}_{\mathrm{b}},\mathfrak{s}\geq 1)$ be a b-metric space. An (α,β)-admissible mapping $\mathcal{T}$ is called (α,β)-$F\zeta$-contraction, if $\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1,\mathcal{T}\xi_2)>0$

$$\tau+F(\alpha(\xi_1,\xi_2)\beta(\xi_1,\xi_2)\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1,\mathcal{T}\xi_2))\leq F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_1,\xi_2),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_2,\mathcal{T}\xi_2),\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\mathrm{b}}(\xi_2,\mathcal{T}\xi_1)}{2\mathfrak{s}}\Big\}\right), \tag{4.2}$$

for all $\xi_1,\xi_2\in\mathfrak{H}$.

Theorem 4.1.4.

Let $(\mathfrak{H},\mathfrak{D}_{\mathrm{b}},\mathfrak{s}\geq 1)$ be a b-metric space.

Suppose that $\mathcal{T}:\mathfrak{H}\to\mathfrak{H}$ is called (α,β)-$F\zeta$-contraction mapping with the following property

(i) $\exists \quad \xi_{1_0}\in\mathfrak{H}$ so that $\alpha(\xi_{1_0},\mathcal{T}\xi_{1_0})\geq 1$ and $\beta(\xi_{1_0},\mathcal{T}\xi_{1_0})\geq 1$

(ii) If there exist a sequence $\{\xi_{1_{m_2}}\}\in\mathfrak{H}$ such that $\xi_{1_{m_2}}\longrightarrow\xi_1$ then $\alpha\left(\xi_{1_{m_2}},\xi_1\right)\geq 1$ and $\beta\left(\xi_{1_{m_2}},\xi_1\right)\geq 1$.

Therefore, the mapping $\mathcal{T}$ admits a fixed point.

Proof.

Suppose $\xi_{1_0} \in \mathfrak{H}$. Define a sequence $\left\{\xi_{1_{m_2}}\right\}$ in $\mathfrak{H}$ recursively by

$$\xi_{1_{m_2}} = \mathcal{T}\xi_{1_{m_2-1}}$$

$\forall\ n \geq 1$. Assuming that

$$\xi_{1_{m_2}} \neq \xi_{1_{m_2+1}}$$

for $m_2 \in \mathbb{N}$ otherwise, $\mathcal{T}$ has a fp, we aim to show that the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy. In order to prove this, let $m_2 \in \mathbb{N}$ and take into

$$\begin{aligned}
\tau + F\left(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\right) &\leq \tau + F\left(\alpha(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\beta(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\right)\\
&\leq F\Bigg(\frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}),\right.\\
&\qquad \left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\right)\Bigg) \qquad (4.3)\\
&< F\Bigg(\frac{1}{\mathfrak{s}}\zeta\left(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}),\right.\\
&\qquad \left.\frac{\mathfrak{s}\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2\mathfrak{s}}\right)\Bigg)\\
&= F\Bigg(\frac{1}{\mathfrak{s}}\max\left(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}),\right.\\
&\qquad \left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2}\right)\Bigg)\\
&\leq F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\tau + F(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right)\\
\Longrightarrow F\left(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\right) &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right) - \tau\\
\Longrightarrow F(\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right)
\end{aligned}$$

Since, F is increasing function.

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) < \left(\frac{1}{\mathfrak{s}}\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\right) \quad \forall\ m_2 \geqslant 1.$$

Now, consider the following two cases,

Case-I:

According to lemma 3.1.3 and equation (4.4), the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy, whenever $\mathfrak{s} > 1$.

Case-II:

When $\mathfrak{s} = 1$, equation (4.4) implies that the sequence $\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\}$ decreases monotonically and remains bounded below. As a result, the distance $\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})$ converges to a non-negative value η. Assuming η is positive, and taking the $\liminf\limits_{m_2\rightarrow+\infty}$ in equation (4.3), we get

$\eta \leq \zeta(\eta, \eta, \eta, \eta')$,

in which

$$\eta' = \limsup_{n\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2} \leq \limsup_{m_2\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2} = \eta.$$

At this point,

$$\eta \leq \zeta(\eta, \eta, \eta, \eta') < \max\left(\eta, \eta, \eta, \eta'\right) = \eta,$$

which leads to a contradiction; thus,

$$\lim_{n\longrightarrow+\infty} \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) = 0. \tag{4.4}$$

Assume that the sequence $\left\{\xi_{1_{m_2}}\right\}$ does not converge to a finite limit, i.e., it is not a Cauchy sequence, then $\exists$ a positive real number ϵ such that for every natural number r, there exists integers m_{1_r} and m_{2_r} satisfying $m_{1_r} > m_{2_r} \geq r$

$$\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \geq \epsilon. \tag{4.5}$$

Moreover, suppose that $m_{1_r} > m_{2_r}$ is the least natural number such that equation (4.5) is satisfied.

Then,

$$\begin{aligned}
\epsilon \leq & \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{2_r}}}) \\
\leq & \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r-1}}}}) + \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r-1}}}}, \xi_{1_{m_{2_r}}}) \\
\leq & \mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}}, \xi_{1_{m_{1_{r-1}}}}) + \epsilon \\
< & \mathfrak{D}_{\mathrm{b}}(\xi_{1_r}, \xi_{1_{r-1}}) + \epsilon,
\end{aligned}$$

thus, using (4.4) and taking $\lim\limits_{n \to +\infty}$, we get

$$\lim_{r \longrightarrow +\infty} \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) = \epsilon. \tag{4.6}$$

Now, consider

$$\begin{aligned}
\tau + F(\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}})) \leq{}& \tau + F\Big(\alpha(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}})\beta(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) \\
& \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}})\Big) \\
\leq{}& F\Big(\zeta \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \\
& \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{1_r}})}{2}\right)\Big),
\end{aligned}$$

$$\begin{aligned}
\tau + F(\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}})) \leq{}& F\Big(\zeta \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \\
& \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}})}{2}\right)\Big),
\end{aligned}$$

$$\begin{aligned}
F(\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}})) \leq{}& F\Big(\zeta \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \\
& \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}})}{2}\right)\Big) - \tau,
\end{aligned}$$

$$\begin{aligned}
F(\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}})) \leq{}& F\Big(\max \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \\
& \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}})}{2}\right)\Big).
\end{aligned}$$

F is increasing function.

$$\begin{aligned}
\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}}) \leq{}& \Big(\max \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}} \xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \\
& \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}} \xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}})}{2}\right)\Big),
\end{aligned}$$

$$\Longrightarrow \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) \leq \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_{r+1}}}, \xi 1_{m_{2_r}}),$$

$$\leq \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_{r+1}}}, \xi 1_{m_{2_r}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_{r+1}}}),$$

$$\leq \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_{r+1}}}, \xi 1_{m_{2_r}}) + \Big((\max \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}),$$

$$\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r+1}}}), \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}), \left(\frac{\mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{b}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}})}{2}\right)\Big).$$

Therefore, Applying $\liminf\limits_{n\to+\infty}$ to both sides and applying equations (4.4) and (4.6) yields $\epsilon \le 0+0+\zeta(\epsilon,0,0,\epsilon')$

$$\begin{aligned}
\epsilon' =& \limsup_{r\longrightarrow+\infty}\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}},\xi_{1_{m_{2_{r+1}}}})+\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}},\xi_{1_{m_{2_r}}})}{2}\\
\le& \limsup_{r\longrightarrow+\infty}\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}},\xi_{1_{m_{2_r}}})+\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{2_r}}},\xi_{1_{m_{2_{r+1}}}})+\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_{r+1}}}},\xi_{1_{m_{1_r}}})+\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_{1_r}}},\xi_{1_{m_{2_r}}})}{2}\\
=&\frac{\epsilon+0+0+\epsilon}{2}=\epsilon.
\end{aligned}$$

Thus, we have $\epsilon \le \zeta(\epsilon,0,0,\epsilon') < \max(\epsilon,0,0,\epsilon')$=$\epsilon$ which results in a contradictory. Hence, the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy in the space $(\mathfrak{H},\mathfrak{D}_{\mathrm{b}},\mathfrak{s}\ge 1)$.
Since, $(\mathfrak{H},\mathfrak{D}_{\mathrm{b}},\mathfrak{s}\ge 1)$ is a Cbms, $\exists$ a limit $\xi_1\in\mathfrak{H}$ such that the sequence $\xi_{1_{m_2}}$ converges to ξ_1. Now, let us consider

$$\begin{aligned}
\tau+F\left(\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_{1_{m_2}},\mathcal{T}\xi_1)\right) &\le \tau+F\left(\alpha(\xi_{1_{m_2}},\xi_1)\beta(\xi_{1_{m_2}},\xi_1)\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_{1_{m_2}},\mathcal{T}\xi_1)\right)\\
&\le F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\right.\\
&\qquad\left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_{1_{m_2}})}{2\mathfrak{s}}\Big\}\right),\\
F\left(\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_{1_{m_2}},\mathcal{T}\xi_1)\right) &\le F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\right.\\
&\qquad\left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_{1_{m_2}})}{2\mathfrak{s}}\Big\}\right)-\tau,\\
F\left(\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_{1_{m_2}},\mathcal{T}\xi_1)\right) &\le F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\right.\\
&\qquad\left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_{1_{m_2}})}{2\mathfrak{s}}\Big\}\right),
\end{aligned}$$

F is increasing function.

$$\begin{aligned}
\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_{1_{m_2}},\mathcal{T}\xi_1) \le& \left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_{1_{m_2}}),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\right.\\
&\left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_{1_{m_2}})}{2\mathfrak{s}}\Big\}\right)
\end{aligned}$$

Which implies that

$$\begin{aligned}
\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2+1}},\mathcal{T}\xi_1) \le& \left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}}),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\right.\\
&\left.\frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\mathrm{b}}(\xi_1,\xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\}\right).
\end{aligned}$$

Using lemma 3.1.2 and applying $\liminf\limits_{m_2\to+\infty}$ to both sides, we get

$$\frac{1}{\mathfrak{s}}(\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1) \le \frac{1}{\mathfrak{s}}\zeta\Big\{0,0,\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),l\Big\},$$

that is

$$\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) \le \zeta\Big\{0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l\Big\},$$

where

$$\begin{aligned} l &= \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}} \\ &\le \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_{1_{m_2}}, \mathcal{T}\xi_1)}{2\mathfrak{s}} + \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_{1_{m_2+1}})}{2\mathfrak{s}} \\ &\le \lim_{m_2 \longrightarrow +\infty} \frac{\mathfrak{s}\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) + 0}{2\mathfrak{s}} \\ &= \frac{\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1)}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) &\le \zeta(0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l) \\ &< \max\{0, 0, \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), l\} = \mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1), \end{aligned}$$

which is contradiction.

$$\mathfrak{D}_{\mathrm{b}}(\xi_1, \mathcal{T}\xi_1) = 0.$$

Therefore $\mathcal{T}\xi_1 = \xi_1$. □

Example 4.1.5.

Let $\mathfrak{H} = \left\{\frac{1}{2^{m_2}} \in \mathbb{N}\right\} \cup \{0\}$. Define $\mathfrak{D}_{\mathrm{b}} : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{R}$ by

$$\mathfrak{D}_{\mathrm{b}}(\xi_1, \xi_2) = |\xi_1 - \xi_2|^2 \text{ for all } \xi_1, \xi_2 \in \mathfrak{H}.$$

Then, the metric space $(\mathfrak{H}, \mathfrak{D}_{\mathrm{b}})$ has a b-metric structure , where the metric $\mathfrak{D}_{\mathrm{b}}$ has a scale parameter $\mathfrak{s}$ of 2.

Define $\mathcal{T}$: $\mathfrak{H} \dashrightarrow \mathfrak{H}$ by

$\mathcal{T}\left(\frac{1}{2^{m_2}}\right)=\frac{1}{2^{m_2+1}}\ \forall\ m_2\in\mathbb{N}$ and $\mathcal{T}(0)=0, F(\xi_1)=\ln(\xi_1)$. Define

$$\alpha(\xi_1,\xi_2)=\begin{cases}1 & \xi_1,\xi_2\in\{\frac{1}{2^{m_2}}:m_2\in\mathbb{N}\}\\ 0 & \text{otherwise}\end{cases}$$

$$\beta(\xi_1,\xi_2)=\begin{cases}1 & \xi_1,\xi_2\in\{\frac{1}{2^{m_2}}:m_2\in\mathbb{N}\}\\ 0 & \text{otherwise}\end{cases}$$

Choose

$$\xi_1=\frac{1}{2^{m_2}};\xi_2=\frac{1}{2^{m_1}}$$

then, Consider,

$$\begin{aligned}\ln\left(\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{2^{m_2+1}},\frac{1}{2^{m_1+1}}\right)\right)-\ln\left(\frac{1}{2}\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{2^{m_2}},\frac{1}{2^{m_1}}\right)\right)&=\ln\left|\frac{1}{2^{m_2+1}}-\frac{1}{2^{m_1+1}}\right|^2-\ln\frac{1}{2}\left|\frac{1}{2^{m_2}}-\frac{1}{2^{m_1}}\right|^2\\ &=\ln\frac{\left|\frac{1}{2^{m_2+1}}-\frac{1}{2^{m_1+1}}\right|^2}{\frac{1}{2}\left|\frac{1}{2^{m_2}}-\frac{1}{2^{m_1}}\right|^2}\\ &=\ln\frac{\frac{1}{4}\left|\frac{1}{2^{m_2}}-\frac{1}{2^{m_1}}\right|^2}{\frac{1}{2}\left|\frac{1}{2^{m_2}}-\frac{1}{2^{m_1}}\right|^2}\\ &=\ln\frac{1}{2}<-\frac{1}{2}\end{aligned}$$

$$\ln\left\{\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{2^{m_2+1}},\frac{1}{2^{m_1+1}}\right)-\ln\left(\frac{1}{2}\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{2^{m_2}},\frac{1}{2^{m_1}}\right)\right)\right\}<-\frac{1}{2},$$

$$\frac{1}{2}+\ln\left(\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{2^{m_2+1}},\frac{1}{2^{m_1+1}}\right)\right)<\ln\left(\frac{1}{2}\mathfrak{D}_{\mathrm{b}}\left(\frac{1}{2^{m_2}},\frac{1}{2^{m_1}}\right)\right).$$

$$\begin{aligned}\implies\tau+F(\alpha(\xi_1,\xi_2)\beta(\xi_1,\xi_2)\mathfrak{D}_{\mathrm{b}}(\mathcal{T}\xi_1,\mathcal{T}\xi_2))\leq F\Bigg(\frac{1}{\mathfrak{s}}\zeta\Big\{&\mathfrak{D}_{\mathrm{b}}(\xi_1,\xi_2),\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_1),\mathfrak{D}_{\mathrm{b}}(\xi_2,\mathcal{T}\xi_2),\\ &\frac{\mathfrak{D}_{\mathrm{b}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\mathrm{b}}(\xi_2,\mathcal{T}\xi_1)}{2\mathfrak{s}}\Big\}\Bigg),\end{aligned}$$

where $\tau=\frac{1}{2},\ F(\xi_1)=\ln(\xi_1),\quad \xi_1,\xi_2\in\mathfrak{H}=\{\frac{1}{2^{m_2}}:m_2\in\mathbb{N}\}\ \alpha(\xi_1,\xi_2)=1,\beta(\xi_1,\xi_2)=1.$

Hence contraction is satisfied.

$\Rightarrow \mathcal{T}$ has a fp.

4.2 Results on b-Metric Like Spaces

Definition 4.7.

Let $(\mathfrak{H},\mathfrak{D}_{\text{bl}},\mathfrak{s}\geq 1)$ is bmls along with $\mathcal{T}:\mathfrak{H}\times\mathfrak{H}\longrightarrow[0,\infty)$ be an (α,β)-admissible mapping. $\mathcal{T}$ is called (α,β)-ζF_{bl}-Contraction, if it satisfies the following condition

$$\mathfrak{D}_{\text{bl}}\left(\mathcal{T}\xi_1,\mathcal{T}\xi_2\right)>0$$

$$\tau+F\left(\alpha(\xi_1,\xi_2)\beta(\xi_1,\xi_2)\mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_1,\mathcal{T}\xi_2)\right)\leq F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\text{bl}}(\xi_1,\xi_2),\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1),\mathfrak{D}_{\text{bl}}(\xi_2,\mathcal{T}\xi_2),\frac{\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_1,\xi_2)-\mathfrak{D}_{\text{bl}}(\xi_2,\xi_2)}{2\mathfrak{s}}\Big\}\right), \tag{4.7}$$

for all $\xi_1,\xi_2\in\mathfrak{H}$.

Theorem 4.2.1.

Let $(\mathfrak{H},\mathfrak{D}_{\text{bl}},\mathfrak{s}\geq 1)$ be a Cb-mls and $\mathcal{T}:\mathfrak{H}\rightarrow\mathfrak{H}$ be (α,β)-$F_{\text{bl}}\zeta$-contraction mappings with the following properties:

(i) $\exists\quad \xi_{1_0}\in\mathfrak{H}$ so that $\alpha(\xi_{1_0},\mathcal{T}\xi_{1_0})\geq 1$ and $\beta(\xi_{1_0},\mathcal{T}\xi_{1_0})\geq 1$

(ii) if there exist a sequence $\{\xi_{1_{m_2}}\}\in\mathfrak{H}$ so that $\xi_{1_{m_2}}\longrightarrow\xi_1$ the $\alpha\left(\xi_{1_{m_2}},\xi_1\right)\geq 1$ and $\beta\left(\xi_{1_{m_2}},\xi_1\right)\geq 1$

(iii) $\mathfrak{D}\text{bl}(\xi_1,\mathcal{T}\xi_2)+\mathfrak{D}_{\text{bl}}(\mathcal{T}\xi_1,\xi_2)\geq\mathfrak{D}_{\text{bl}}(\xi_2,\xi_2)$

Therefore,

The mapping $\mathcal{T}$ admits a fixed point.

Proof.

Suppose $\xi_{1_0}\in\mathfrak{H}$. Define a sequence $\left\{\xi_{1_{m_2}}\right\}$ in $\mathfrak{H}$ recursively by

$$\xi_{1_{m_2}}=\mathcal{T}\xi_{1_{m_2-1}}$$

$\forall\, m_2 \geq 1$. Assuming that

$$\xi_{1_{m_2}} \neq \xi_{1_{m_2+1}}$$

for $m_2 \in \mathbb{N}$ otherwise, $\mathcal{T}$ has a fp, we aim to show that the sequence $\left\{\xi_{1_{m_2}}\right\}$ is Cauchy. In order to prove this, let $m_2 \in \mathbb{N}$ and take into

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \mathcal{T}\xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) &= \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}}) \\ &\geq \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}});\end{aligned}$$

therefore, using (4.7), we have

$$\begin{aligned}\tau + F(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) &\leq \tau + F(\alpha(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\beta(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) \\ &\leq F\Bigg(\frac{1}{\mathfrak{s}}\zeta\,\Big(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \\ &\qquad \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}}) - \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}})}{2\mathfrak{s}}\Big)\Bigg), \qquad (4.8) \\ &\leq F\Bigg(\frac{1}{\mathfrak{s}}\zeta\,\Big(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \\ &\qquad \frac{\mathfrak{s}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}})}{2\mathfrak{s}}\Big)\Bigg), \\ &\leq F\Bigg(\frac{1}{\mathfrak{s}}\zeta\,\Big(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}), \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}), \\ &\qquad \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}})}{2}\Big)\Bigg),\end{aligned}$$

$$\begin{aligned}\tau + F(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) &\leq \tau + F(\alpha(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\beta(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) \\ &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right), \\ \tau + F(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right), \\ F(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right) - \tau, \\ F(\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})) &< F\left(\frac{1}{\mathfrak{s}}\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}})\}\right),\end{aligned}$$

F is increasing function.

$$\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) < \frac{1}{\mathfrak{s}}\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) \quad \text{for all } m_2 \geq 1. \tag{4.9}$$

Case-I:

When $\mathfrak{s} > 1$, lemma 3.1.3 and equation (4.9), imply that $\left\{\xi_{1_{m_2}}\right\}$ is a sequence that Cauchy in $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$, having the property that

$$\lim_{m_1, m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_1}}) = 0.$$

Case-II:

When $\mathfrak{s} = 1$ equation (4.9) implies that the sequence $\{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})\}$ is monotonically decreasing. Considering $\liminf\limits_{m_2 \to +\infty}$ in (4.8), we get

$\eta \leq \zeta(\eta, \eta, \eta, \eta'),$

where

$$\eta' = \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2+1}})}{2} \leq \limsup_{m_2 \longrightarrow +\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2-1}}, \xi_{1_{m_2}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}})}{2} = \eta.$$

At this point,

$$\eta \leq \zeta(\eta, \eta, \eta, \eta') < \max(\eta, \eta, \eta, \eta') = \eta,$$

this is a contradictory statement.

Consequently,

$$\lim_{n \longrightarrow +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) = 0. \tag{4.10}$$

Furthermore,

$$\mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}}) \leq \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2+1}}) + \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2+1}}, \xi_{1_{m_2}}),$$

considering $\limsup\limits_{m_2 \to +\infty}$, and applying equation (4.10) we get

$$\lim_{m_2 \longrightarrow +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_2}}) = 0, \tag{4.11}$$

Suppose that

$$\lim_{m_1, m_2 \to +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi_{1_{m_2}}, \xi_{1_{m_1}}) \neq 0$$

then there exists a positive real number ϵ such that for every natural number r, there exists there exist integers m_{1_r} and m_{2_r} satisfying $m_{1_r} > m_{2_r} \geq r$

$$\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) \geq \epsilon. \tag{4.12}$$

Moreover, suppose that $m_{1_r} > m_{2_r}$ is the least natural number so that the equation (4.12) is satisfied.

Then,

$$\begin{aligned}
\epsilon &\leq \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) \\
&\leq \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r-1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r-1}}}, \xi 1_{m_{2_r}}) \\
&\leq \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{1_{r-1}}}) + \epsilon \\
&< \mathfrak{D}_{\mathrm{bl}}(\xi 1_{r}, \xi 1_{r-1}) + \epsilon.
\end{aligned}$$

Hence, applying equation (4.10) and taking $\lim\limits_{r \to +\infty}$, we obtain

$$\lim_{r \longrightarrow +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) = \epsilon. \tag{4.13}$$

Assume that there exists an infinite sequence of r values such that

$$\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \mathcal{T}\xi 1_{m_{2_r}}) + \mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) < \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_r}}).$$

Considering $\limsup\limits_{r \to +\infty}$, and using (4.11), we obtain

$$\limsup_{r \longrightarrow +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \mathcal{T}\xi 1_{m_{2_r}}) + \limsup_{r \longrightarrow +\infty} \mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) < \limsup_{r \longrightarrow +\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_r}}).$$

$$\limsup_{r \longrightarrow +\infty} \{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \mathcal{T}\xi 1_{m_{2_r}}) + \mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}})\} = 0,$$

which means that

$$\lim_{r \longrightarrow +\infty} \{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}})\} = \lim_{r \longrightarrow +\infty} \{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}})\} = 0.$$

Now,

$$\epsilon=\lim_{r\longrightarrow+\infty}\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}})\leq\limsup_{r\longrightarrow+\infty}\{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_{r+1}}},\xi1_{m_{2_r}})\}=0,$$

This is a contradictory statement.

Hence, we can conclude that there exists a natural number r_0 such that for all $r\geq r_0$,

$\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\mathcal{T}\xi1_{m_{2_r}})+\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi1_{m_{1_r}},\xi1_{m_{2_r}})\geq\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})$, Thus, for all $r\geq r_0$, using (4.7),

$$\begin{aligned}\tau+F(\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}}))&\leq\tau+F(\alpha(\xi1_{m_{1_r}},\xi1_{m_{2_r}})\beta(\xi1_{m_{1_r}},\xi1_{m_{2_r}})\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}}))\\&\leq F\Bigg(\zeta\{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_{r+1}}}),\\&\qquad\left(\frac{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_r}})-\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})}{2}\}\right)\Bigg),\end{aligned}$$

$$\begin{aligned}\tau+F(\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}}))\leq F\Bigg(&\{\zeta\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_{r+1}}}),\\&\frac{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_r}})-\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})}{2}\}\Bigg),\end{aligned}$$

$$\begin{aligned}F(\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}}))\leq F\Bigg(&\{\zeta(\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_{r+1}}}),\\&\frac{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_r}})-\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})}{2}\}\Bigg)-\tau,\end{aligned}$$

$$\begin{aligned}F(\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}}))\leq F\{\zeta\Bigg(&\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_{r+1}}}),\\&\frac{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_r}})-\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})}{2}\}\Bigg),\end{aligned}$$

F is increasing function.

$$\begin{aligned}\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}})\leq\{\max\Bigg(&\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_{r+1}}}),\\&\frac{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_r}})-\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})}{2}\}\Bigg)\end{aligned}$$

$$\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}})\leq\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_{r+1}}},\xi1_{m_{2_r}}),$$

$$\begin{aligned}&\leq\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_{r+1}}},\xi1_{m_{2_r}})+\zeta\Big\{(\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_r}}),\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{1_{r+1}}}),\\&\qquad\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_{r+1}}}),\frac{\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_r}},\xi1_{m_{2_{r+1}}})+\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{1_{r+1}}},\xi1_{m_{2_r}})-\mathfrak{D}_{\mathrm{bl}}(\xi1_{m_{2_r}},\xi1_{m_{2_r}})}{2}.\Big\}\end{aligned}$$

Hence, applying $\liminf_{r\to+\infty}$ to each side and applying equations (4.10) and (4.13), we obtain $\epsilon \leq 0 + 0 + \max(\epsilon, 0, 0, \epsilon')$

$$\begin{aligned}
\epsilon' =& \limsup_{r\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{2_r}}) - \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_r}})}{2} \\
\leq& \limsup_{r\longrightarrow+\infty} \frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{2_r}}, \xi 1_{m_{2_{r+1}}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_{r+1}}}, \xi 1_{m_{1_r}}) + \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_{1_r}}, \xi 1_{m_{2_r}}) - 0}{2} \\
=& \frac{\epsilon + 0 + 0 + \epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, $\epsilon \leq \zeta(\epsilon, 0, 0, \epsilon') < \max(\epsilon, 0, 0, \epsilon') = \epsilon$ which is conflicting. Thus $\left\{\xi 1_{m_2}\right\}$ is a Cauchy sequence in $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ with

$$\lim_{m_1, m_2\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi 1_{m_1}) = 0$$

.

Now, $(\mathfrak{H}, \mathfrak{D}_{\mathrm{bl}}, \mathfrak{s} \geq 1)$ is a Completebmls. Consequently, $\exists \xi_1 \in \mathfrak{H}$ so that $\xi 1_{m_2} \longrightarrow \xi_1$,

$$\mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1) = \lim_{m_2\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1) = \lim_{m_1, m_2\to+\infty} \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi 1_{m_1}) = 0. \tag{4.14}$$

Additionally, based on proposition 2.3.1, ξ_1 is unique.

Consider that $\mathcal{T}\xi_1 \neq \xi_1$. At this point, take into consideration

$$\begin{aligned}
\tau + F(\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_2}, \mathcal{T}\xi_1)) \leq& \tau + F\left(\alpha(\xi 1_{m_2}, \xi_1)\beta(\xi 1_{m_2}, \xi_1)\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_2}, \mathcal{T}\xi_1)\right) \\
\leq& F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi 1_{m_2}), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1),\right. \\
&\left.\frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi 1_{m_2}) - \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi_1)}{2\mathfrak{s}}\Big\}\right),
\end{aligned}$$

$$\begin{aligned}
F(\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2+1}, \mathcal{T}\xi_1)) \leq& F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi 1_{m_2+1}), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1),\right. \\
&\left.\frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi 1_{m_2+1})}{2\mathfrak{s}}\Big\}\right) - \tau,
\end{aligned}$$

$$\begin{aligned}
F(\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2+1}, \mathcal{T}\xi_1)) \leq& F\left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi 1_{m_2+1}), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1),\right. \\
&\left.\frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \xi 1_{m_2+1})}{2\mathfrak{s}}\Big\}\right),
\end{aligned}$$

F is increasing function.

$$\begin{aligned}
(\mathfrak{D}_{\mathrm{bl}}(\mathcal{T}\xi 1_{m_2}, \mathcal{T}\xi_1)) \leq& \left(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \xi_1), \mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi 1_{m_2}), \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi_1),\right. \\
&\left.\frac{\mathfrak{D}_{\mathrm{bl}}(\xi 1_{m_2}, \mathcal{T}\xi_1) + \mathfrak{D}_{\mathrm{bl}}(\xi_1, \mathcal{T}\xi 1_{m_2})}{2\mathfrak{s}}\Big\}\right),
\end{aligned}$$

Which implies that

$$(\mathfrak{D}_{\text{bl}}(\xi_{1_{m_2+1}},\mathcal{T}\xi_1)) \leq \Bigg(\frac{1}{\mathfrak{s}}\zeta\Big\{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_2}}\xi_{1_{m_2}},\xi_1),\mathfrak{D}_{\text{bl}}(\xi_{1_{m_2}},\xi_{1_{m_2+1}}),\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1), \frac{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\text{bl}}(\xi_1,\xi_{1_{m_2+1}})}{2\mathfrak{s}}\Big\}\Bigg),$$

Considering $\liminf\limits_{m_2\to+\infty}$ on both sides and applying Proposition 2.3.1, we obtain

$$\frac{1}{\mathfrak{s}}(\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1) \leq \frac{1}{\mathfrak{s}}\zeta\Big\{0,0,\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1),l\Big\},$$

that is

$$\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1) \leq \zeta\Big\{0,0,\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1),l\Big\},$$

where

$$\begin{aligned} l &= \limsup_{m_2\longrightarrow+\infty}\frac{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)+\mathfrak{D}_{\text{bl}}(\xi_1,\xi_{1_{m_2+1}})}{2\mathfrak{s}} \\ &\leq \limsup_{m_2\longrightarrow+\infty}\frac{\mathfrak{D}_{\text{bl}}(\xi_{1_{m_2}},\mathcal{T}\xi_1)}{2\mathfrak{s}} + \limsup_{m_2\longrightarrow+\infty}\frac{\mathfrak{D}_{\text{bl}}(\xi_1,\xi_{1_{n+1}})}{2\mathfrak{s}} \\ &\leq \lim_{m_2\longrightarrow+\infty}\frac{\mathfrak{s}\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1)+0}{2\mathfrak{s}} \\ &= \frac{\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1)}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1) \leq{}& \zeta(0,0,\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1),l) \\ &< \max\{0,0,\mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1),l\} = \mathfrak{D}_{\text{bl}}(\xi_1,\mathcal{T}\xi_1), \end{aligned}$$

which is a contradiction. Therefore, $\mathcal{T}\xi_1 = \xi_1$. $\square$

Chapter 5

Conclusion

We summarize our research in the following manners:

- A brief historical review provides a foundation for a compact exploration of fixed point theory.
- The concept of contractions is illuminated through the detailed analysis of particular mapping instances.
- Particular attention is devoted to F-contraction mappings, accompanied by illustrative examples.
- A dedicated section explores various crucial concepts pertinent to metric fixed point theory.
- The work of Jain et al. [1] is subjected to a thorough and exhaustive review. For this task two types of contractive mapping are established. Existence of fixed point on platform b-metric and b-metric-like spaces are discussed.
- For establishing the existence of fixed point following steps are adopted.

(i) Construction of an iterative sequence.

(ii) Proof of Cauchyness of this sequence.

(iii) Existence of fixed point.

- Through the incorporation of the admissibility notion into F-contractions and the utilization of the ζ-function's features, we create a new and expanded rational contraction known as the (α, β)-$F\zeta$- contraction.
- This study thoroughly constructs this new contraction and uses illustrative examples to show its validity and application.
- Several well-established findings from the body of current literature are greatly expanded and generalized by the fixed-point results acquired in this work.
- A non-trivial illustrative example is offered to demonstrate the validity of the established theorem.
- Several corollaries are presented, showing that our main result generalizes a wide range of existing fixed point results, which can be seen as special cases.
- The theorems presented in this work generalize the results of Jain et al. [33], which can be viewed as a special case of the more general results presented here.

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