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TECHNOLOGY, ISLAMABAD


# Digital Signature Based on Matrix Power Function over Galois Field

by

Muhammad Rafi

A thesis submitted in partial fulfillment for the
degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

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*To my parents, teachers and friends for their support and love.*


# CERTIFICATE OF APPROVAL

## Digital Signature Based on Matrix Power Function over Galois Field

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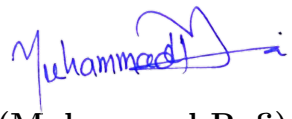

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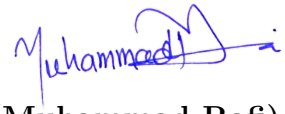

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# *Abstract*

Digital signatures ensure authenticity, integrity, and non-repudiation by linking the signer's identity to a document and creating a binding commitment. In the traditional process, the sender signs a message with their private key, and the receiver verifies the signature using the sender's public key. Proposed modification to an existing digital signature scheme by replacing matrices of integers entries with the matrices of polynomials entries from $GF(p^q)$. The matrix power function operates on matrices from $GL(n, GF(p^q))$, with the base matrix defined over a semigroup from $GF(p^q)$ and power matrices over a semiring of integer entries. The scheme's security is rooted in solving matrix based multivariate equations over a finite field, a problem that is computationally hard (NP-complete). This makes the matrix power function in $GF(p^q)$ strong candidate for digital signatures applications due to its one-way nature. Inverting the function is as challenging as solving these equations. This enhancement enables the creation of secure key exchange protocols and digital signatures using algebraic structures. Preliminary security analysis indicates the matrix power function potential as a robust one-way function since no polynomial time inversion algorithm exists. The matrix power function computational efficiency also makes it suitable for digital signatures protocols in resource constrained devices. To demonstrate it via example application, algorithms for matrix power function computation were implemented in the "Applied Computations in Commutative Algebra (ApCoCoA)".

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# Abbreviations

| | |
|---|---|
| **AES** | Advanced Encryption Standard |
| **DES** | Data Encryption Standard |
| **DLP** | Discrete Logarithm Problem |
| **DS** | Digital Signature |
| **DSA** | Digital Signature Algorithm |
| **DSMPF** | Digital Signature based on Matrix Power Function |
| **ECC** | Elliptic Curve Cryptosystem |
| **GF** | Galois Field |
| **GL** | General Linear Group |
| **MMDS** | Modular Matrix based Digital Signature |
| **MPF** | Matrix Power Function |
| **MQ** | Multivariate Quadratic |
| **NP** | Non-deterministic Polynomial |
| **RSA** | Rivest-Shamir-Adleman |

# Symbols

| | |
|---|---|
| $\alpha$ | Signature |
| $\beta$ | Signature |
| $M$ | Message |
| $\mathbb{Z}$ | Set Of Integers |
| $\mathbb{C}$ | Set of Complex Numbers |
| $\mathbb{N}$ | Set of Natural Numbers |
| $\mathbb{R}$ | Ring |
| $\mathbb{F}$ | Field |
| $\mathbb{Q}$ | Set of Rational Numbers |
| $\varphi$ | Euler's Totient Function |
| $\mathbb{M}_n R$ | Matrix Ring |
| $H$ | Hash Function |
| $\mathbb{H}$ | Group |

# Chapter 1

# Introduction

## 1.1 Cryptography

Cryptography has been essential for ensuring secure communication and protecting sensitive information in the presence of potential adversaries. Throughout history, various cryptographic methods have been developed to safeguard messages and enhance security. Over 2,000 years ago, shift ciphers based on alphabets were used, eventually evolving into more sophisticated techniques such as monoalphabetic ciphers, Playfair ciphers [1], and Hill ciphers [2] of different orders. As cryptographic systems have advanced, various challenges and vulnerabilities have emerged, necessitating the continuous development of more resilient cryptosystems [3].

Cryptography [4] provides the tools to conceal sensitive information and securely transmit it over vulnerable communication channels. A cryptosystem is a key component of modern cryptography and consists of five essential elements: plaintext, encryption algorithm, decryption algorithm, ciphertext, and key. While the primary goal of cryptography is to ensure data security, it also addresses other critical aspects, including data authenticity, availability, and integrity. Cryptography [5] is broadly classified into two main categories based on key management symmetric key cryptography and asymmetric key cryptography. In symmetric key cryptography, a single key is used for both encryption and decryption, shared between the sender and receiver.

However, a significant challenge is securely distributing the key, especially in protocols involving a large number of participants. If the key is compromised, the entire communication becomes vulnerable. Examples of symmetric key cryptography include DES (Data Encryption Standard) and AES (Advanced Encryption Standard).

To address key distribution issues, asymmetric key cryptography was introduced by Diffie-Hellman [6] in 1976. This approach uses a pair of keys public and private for encryption and decryption, ensuring that knowledge of one key does not compromise the security of the other. Examples of asymmetric cryptographic systems include RSA [7], ElGamal [8], and elliptic curve cryptosystems (ECC) [9].

In recent years, advancements in technologies such as the internet of things and quantum computing have influenced cryptographic development. Quantum cryptanalysis poses a significant challenge to conventional cryptographic systems like Diffie-Hellman, RSA, and ECC [10], making the resistance to quantum attacks crucial. One promising area of research is the use of One-Way functions (OWF), which rely on computationally hard problems that are difficult to solve efficiently, such as NP-complete problems. One such approach involves lattice-based cryptography, which leverage mathematical constructs resistant to quantum cryptanalysis. For instance, One-Way functions based on multivariate quadratic problems (MQ) have been proven to be NP-complete and NP-hard. Additionally, non-commutative and non-symmetric structures have been employed in cryptography, enhancing security further. For example, the use of matrix power functions and algebraic eraser concepts have been explored to create robust cryptographic primitives.

## 1.2 Digital Signature

Digital signatures [11] are a crucial component of modern cryptography, ensuring the authenticity, integrity, and non-repudiation of digital communications. They use asymmetric key cryptography, consisting of a pair of keys: a public key for verification and a private key for signing. A hash function creates a unique, fixed size value from the original data, which is then encrypted with the private key to generate the digital signature. Upon receiving the signed data, the recipient uses the corresponding public

key to decrypt and verify the signature. If the hash values match, the data is confirmed as authentic and unchanged.

Digital signatures [12] provide essential security features such as ensuring data integrity, authenticating the sender, and preventing repudiation of transactions. They are widely used in email security, software distribution, electronic commerce, and digital contracts to secure communications and verify the legitimacy of transactions. However, key management and computational efficiency remain important considerations for their adoption.

## 1.3 Literature Review

A digital signature is a mathematical scheme used for verifying the authenticity and integrity of digital messages or documents. It is similar to a handwritten signature or a stamped seal but offers a higher level of security. Digital signatures are widely used in electronic communication to prove the origin of messages and to ensure that the message has not been altered.

The concept of digital signatures emerged in the 1970s [13] as a response to the increasing need for secure communication in the growing world of computers and networks. The idea of using cryptographic techniques to verify identities and ensure message integrity was first explored by researchers like Whitfield Diffie and Martin Hellman, who introduced the concept of public key cryptography in their ground breaking 1976 paper. Public key cryptography allowed two parties to communicate securely over an insecure channel, using a pair of keys: one for encryption and the other for decryption. In the early 1980s, Ralph Merkle and Whitfield Diffie developed the first practical digital signature algorithm. Their work laid the foundation for modern digital signature schemes. In 1985, the RSA algorithm, named after its inventors Ron Rivest, Adi Shamir, and Leonard Adleman, was introduced and became the first widely accepted public key cryptosystem [14]. RSA allowed users to sign messages digitally, creating a secure way to verify the origin and integrity of digital data. This was a major breakthrough in ensuring secure communication over open networks like the internet. In the years that followed, various improvements and alternative

digital signature algorithms were developed. J. Deo and J. Smith develop signature on modular matrices [15]. A. Johnson and B. Smith worked of matrix power function [16]. The digital signature algorithm (DSA) [17], introduced by the U.S. National Security Agency (NSA) in 1991, became the basis for the digital signature standard (DSS), which was widely adopted in government and commercial systems.

## 1.4 Current Research

In this research focus is on the modular matrix based digital signature (MMDS) scheme introduced by S. K. Rososhek [18] followed by work on digital signature based on matrix power function (DSMPF) [19]. DSMPF scheme utilizes matrices from $GL(n, \mathbb{Z}_n)$, built upon finite field $\mathbb{Z}_n$ and incorporates the conjugacy search problem.

The primary focus is to modify the digital signature based on matrix poewer fuction(DSMPF) by taking the base matrix from $GL(n, GF(p^q))$. This modification enhances security, as it requires attackers to solve multivariate systems of equations and perform matrix decomposition i.e $(XY = Z)$ to gain access to the secret key. This computational challenge makes it infeasible for attackers to compromise the scheme. Constructed examples to demonstrate a digital signature scheme based on MPF over a galois field. Additionally, developed programs for computations using a Applied Computations in Commutative Algrbra (ApCoCoA) [20], illustrated few examples of the proposed scheme.

## 1.5 Thesis Structure

The rest of the thesis is structured as follows: Chapter 2 explores the foundational concepts and definitions of some basic terms used in cryptography and mathematical terms. It also covers hash functions and their key properties, concluding with an in-depth analysis of the matrix power function (MPF) and its associated properties.

Chapter 3 explores the digital signature scheme based on the MPF in $GL(n, \mathbb{Z}_N)$, proposed by Sundas Iqbal. This scheme draws inspiration from the article fast and

secure modular matrix based digital signature scheme by S.K. Rososhek. We review several well-known digital signature schemes and provide example to illustrate the workings of the modular matrix based digital signature scheme.

Chapter 4 examines a modified version of the digital signature scheme, referred as the digital signature scheme based on MPF over $GF(p)$ on $GL(n, GF(p^q))$ [21]. This modified scheme incorporates the matrix power function in $GF(p)$ to improve the security of the system. Structure of the algorithms for key generation, signing, and verification is presented. Furthermore, the chapter includes illustrative examples to explain the modified digital signature scheme.

# Chapter 2

# Preliminaries

This chapter covers fundamental definitions related to cryptography, key management, and a detailed explanation of digital signatures. Additionally, it explores algebraic concepts, polynomials, and operations performed on polynomials. These topics will serve as the foundation for developing a more comprehensive structure in the later sections of this thesis.

## 2.1 Cryptography

Cryptography is a structured and scientific approach to securing communication through encryption and decryption. It provides a reliable method for ensuring secure exchanges between two parties, with a primary focus on maintaining the confidentiality and protection of messages. The development of cryptographic protocols and algorithms is aimed at preventing unauthorized access and safeguarding sensitive information from hackers.

The term "cryptography" originates from two words: "Krypto," meaning hidden, and "Graphein," meaning writing [13]. It is a method used to transform a readable message into an unreadable format through encryption and then convert it back into a readable format through decryption. This process relies on mathematical, structural, scientific, and systematic techniques, utilizing appropriate keys and algorithms.

The process of transforming a plaintext message into a secure, encoded format known as ciphertext is achieved using an encryption algorithm along with a key. To retrieve the original message, the ciphertext is decrypted using a decryption algorithm and a corresponding key. The primary goal of this system is to ensure confidentiality, integrity, authenticity, and availability. This encryption system is composed of five essential components.

1. Plaintext
2. Cipertext
3. Encryption Algorithm
4. Decryption Algorithm
5. Keys

Initially, basic techniques were employed for this task, but over time, more advanced and reliable methods have been developed to meet evolving security needs.

### 2.1.1 Classification of Cryptography

There are two main branches of cryptography:

1. Symmetric Key Cryptography
2. Asymmetric Key Cryptography

#### 2.1.1.1 Symmetric Key Cryptography

Symmetric key cryptography is a cryptographic method that uses the same secret key for both encryption and decryption processes.

This means that the same key is applied to transform plaintext into ciphertext during encryption and to revert ciphertext back to plaintext during decryption. The security of this method relies on the secrecy of the key, which must be known only to the

communicating parties [14]. Some prominent examples of such scheme are DES [22] and AES [23].

Symmetric [24] cryptography is widely used due to its efficiency and speed, making it suitable for encrypting large amounts of data. However, managing and distributing keys securely remains a challenge.

This encryption method is sometimes mistakenly referred to as public key cryptography, but public key cryptography actually falls under asymmetric encryption, where different keys are used for encryption and decryption.

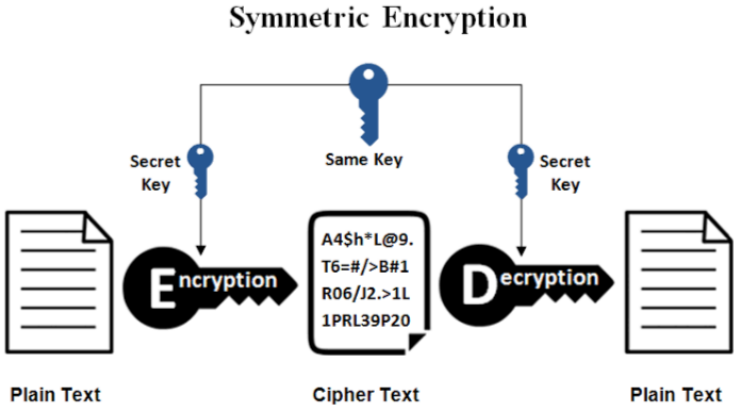


FIGURE 2.1: Symmetric Cryptography

#### 2.1.1.2 Asymmetric Key Cryptography

Asymmetric key cryptography, also known as public-key cryptography, is a cryptographic method that utilizes a pair of keys a public key and a private key for secure data exchange. In this system, one key is used to encrypt or digitally sign the data, while the other is used to decrypt the data or verify the digital signature. It hence the security of the data.

The public key is openly distributed, allowing anyone to encrypt messages or verify signatures, whereas the private key remains confidential to its owner, ensuring that only they can decrypt the messages or create valid digital signatures. This approach enhances security by eliminating the need to share secret keys between parties [25].

Examples included are RSA [7], elliptic curve cryptography (ECC) [9], and Diffie-Hellman [6]. The security of asymmetric cryptography is based on mathematical

algorithms that make it computationally infeasible to derive the private key from the public key [26].

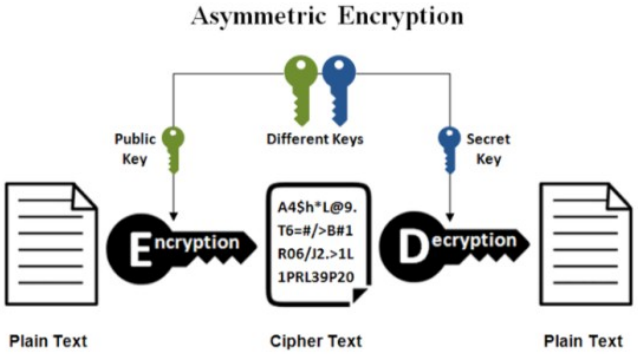


FIGURE 2.2: Asymmetric Cryptography

One key is public key and other key is a private key which is kept in between both the communication parties.

## 2.2 Digital Signature

Security is a core principle of cryptography, aiming to ensure that communication remains confidential, authentic, and tamper proof. One of the key elements used to achieve this goal is the digital signature [27]. Digital signatures are essential for ensuring the integrity of messages and verifying the identity of the sender. They are an integral part of various cryptographic techniques such as verification, authorization, and non-repudiation.

A digital signature allows an entity to authenticate its identity by creating a unique signature based on specific information or data. This process involves using a private key to generate the signature and a public key to verify it, ensuring that the message hasn't been altered and confirming the identity of the sender.

In the next chapter, more deep analysis and concept of digital signatures will be presented, both theoretical and mathematical explanation will be included. Explore different cryptographic protocols and algorithms such as RSA [7], DSA [17], and other relevant methods, to understand how they contribute to the creation and verification of digital signatures.

## 2.3 Hash Function

**Definition 2.3.1.** "A hash function is a mathematical algorithm that maps data of arbitrary size to a fixed-size output, called a hash value. It is commonly used for data integrity, verification, digital signatures, and cryptographic security [28]."

A hash function is a mathematical algorithm that takes an input (message) and returns a fixed size string of bytes, typically a hash value, which represents the input data. Hash functions are designed to be fast and efficient, providing a unique output for distinct inputs, though there is a possibility of collisions (two different inputs producing the same output). In cryptography, a hash function is considered a one-way function, meaning that it is computationally infeasible to reverse the hash value to obtain the original input. Hash functions are widely used in data structures such as hash tables and are essential in ensuring data integrity and authentication [28].

A hash function is deterministic, meaning that the same input will always generate the same output. It operates efficiently, regardless of the size of the input. Important characteristics of hash functions include pre-image resistance, which makes it difficult to reverse the hash and recover the original input; collision resistance, which ensures that two different inputs are unlikely to produce the same hash; and the avalanche effect, where small changes in the input lead to significant differences in the output.

Hash functions [29] are widely used in cryptography, for storing passwords securely, verifying data integrity, and in data structures such as hash tables. Common examples of hash functions include MD5 (which is no longer considered secure), SHA-256 [30] (commonly used for secure hashing), and SHA-3 [31] (the most recent cryptographic hash standard). For example, the SHA-256 hash of "Hello, World!" is

$$0535e4be2b79ffd93291305436bf889314e4a3faec05ecffcbb7df31f2e7e10$$

while the hash of "Hello, world!" is

$$315f5bdb76d078c43b8ac0064e4a0164612b1fce77c869345bfc94c75894edd3$$

demonstrating how even minor change in the input result in a vastly different hash value.

## 2.4 Mathematical Background

This section revisits some fundamental and essential definitions that are important and relevant for the thesis.

### 2.4.1 Groupoid

**Definition 2.4.1.** "A groupoid [32] is a category in which every morphism is an isomorphism. That is, for every morphism $f : x \to y$, there exists a morphism $f^{-1} : y \to x$ such that

$$f \circ f^{-1} = \mathrm{id}_y \quad \text{and} \quad f^{-1} \circ f = \mathrm{id}_x.$$"

For example the set of real numbers ($\mathbb{R}$) with the binary operation of addition ($+$) is a groupoid.

### 2.4.2 Semigroup

**Definition 2.4.2.** A non-empty set $\mathbb{H}$ along with a binary operation $\circ$ is termed a semigroup if the set $\mathbb{H}$ satisfies the following properties:

1. **Closure Property:** Let $a_1, a_2$ are the elements of $\mathbb{H}$ and $\circ$ is the binary operation. $\mathbb{H}$ is closed if

$$a_1 \circ a_2 \in \mathbb{H} \quad \text{for all} \quad a_1, a_2 \in \mathbb{H}.$$

2. **Associative Property:** Let $a_1, a_2, a_3$ are the elements $\mathbb{H}$,

$$(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3)$$

all $a_1, a_2, a_3 \in \mathbb{H}$.

### 2.4.3 Monoid

**Definition 2.4.3.** A non-empty set $\mathbb{H}$ endowed with a binary operation $\circ$ is called Monoid [33] if $\mathbb{H}$ show following properties over defined operation $\circ$.

1. **Closure Property:** Let $a_1, a_2$ are the elements of $\mathbb{H}$ and $\circ$ is the binary operation. $\mathbb{H}$ is closed if,

$$a_1 \circ a_2 \in \mathbb{H} \quad \text{for all} \quad a_1, a_2 \in \mathbb{H}.$$

2. **Associative Property:** Let $a_1, a_2, a_3$ are the elements of $\mathbb{H}$,

$$(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3) \quad \text{for all} \quad a_1, a_2, a_3 \in \mathbb{H}.$$

3. **Existence of Identity:** Suppose that $\varepsilon \in \mathbb{H}$ such that,

$$a_1 \circ \varepsilon = \varepsilon \circ a_1 = a_1 \in \mathbb{H}.$$

   Here $\varepsilon$ is the additive identity element.

For example set of integers ($\mathbb{Z}$) is monoid under mathematical operation $+$.

### 2.4.4 Group

**Definition 2.4.4.** A non-empty set $\mathbb{H}$ together with a binary operation $\circ$ is called group if it satisfies following properties over operation $\circ$ [34].

1. **Closure Property:** Binary operation $\circ$ is closed i.e

$$a_1 \circ a_2 \in \mathbb{H} \quad \text{for all} \quad a_1, a_2 \in \mathbb{H}.$$

2. **Associative Property:** Let $a_1, a_2, a_3$ are the elements of $\mathbb{H}$,

$$(a_1 \circ a_2) \circ a_3 = a_1 \circ (a_2 \circ a_3) \quad \text{for all} \quad a_1, a_2, a_3 \in \mathbb{H}.$$

3. **Existence of Identity:** Suppose that $\varepsilon \in \mathbb{H}$ such that,

$$a_1 \circ \varepsilon = \varepsilon \circ a_1 = a_1 \in \mathbb{H}.$$

Here $\varepsilon$ is the additive identity element.

4. **Existence of Inverses:** For each element $a_1 \in \mathbb{H}$, there exist $a_1^{'} \in \mathbb{H}$ such that,

$$a_1 \circ a_1^{'} = a_1^{'} \circ a_1 = \epsilon.$$

Here $\epsilon$ is the multiplicative identity element and $a_1^{'}$ is the multiplicative inverse of $a_1$.

**Example 2.4.5.** 1. With binary operation $+$, the set of integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$, the set of real numbers $\mathbb{R}$ and set of complex numbers $\mathbb{C}$ are groups.

2. The sets $\mathbb{Q} \setminus \{0\}$ , $\mathbb{R} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ also form group under multiplication as binary operation.

3. The general linear group $GL(n, \mathbb{R})$ is a group under matrix multiplication.

### 2.4.5 Abelian Group

A non-empty group $\mathbb{H}$ together with a binary operation $\circ$ is called Abelian Group if it satisfies the commutative law under the same binary operation that is,

$$a_1, a_2 \in \mathbb{H},$$

$$a_1 \circ a_2 = a_2 \circ a_1.$$

For example set of rational numbers ($\mathbb{Q}$), set of real numbers ($\mathbb{R}$).

### 2.4.6 Ring

The set denoted by $(R, +, \times)$ is called Ring [21] embedded with two binary operations addition $(+)$ and multiplication $(\times)$ if that satisfies the following features.
The following properties hold under addition.

1. **Closure Property:** Binary operation $(+)$is closed i.e

$$a_1 + a_2 \in R \quad \text{for all} \quad a_1, a_2 \in R.$$

2. **Associative Property:** Let $a_1, a_2, a_3$ are the elements of $R$,then

$$(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3) \quad \text{for all} \quad a_1, a_2, a_3 \in R.$$

3. **Existence of Identity:** Let $0 \in R$ Such that,

$$a_1 + 0 = 0 + a_1 = a_1 \in R.$$

   Here $0 \in R$ is additive identity.

4. **Existence of Inverses:** For each element $a_1 \in R$,there exist $a_1^{'} \in R$ such that,

$$a_1 + a_1^{'} = a_1^{'} + a_1 = 0.$$

   Here 0 is the identity element and $a_1^{'} \in R$ is additive inverse of $a \in R$.

5. **Commutative Law:** For all $a_1, a_2 \in R$,

$$a_1 + a_2 = a_2 + a_1.$$

The following properties hold under multiplication.

1. $(R, \times)$ is a semi-group or monoid.

2. **Closure Property:** Binary operation $\times$ is closed i.e

$$a_1.a_2 \in R \quad \text{for all} \quad a_1, a_2 \in R.$$

3. **Associative Property:** Let $a_1, a_2, a_3$ are the elements of $R$,then

$$(a_1 \times a_2) \times a_3 = a_1 \times (a_2 \times a_3) \quad \text{for all} \quad a_1, a_2, a_3 \in R.$$

4. **Existence of Identity:** Let $\epsilon \in R$ such that,

$$a_1 \times \epsilon = \epsilon \times a_1 = a_1 \in R.$$

Here $\epsilon \in R$ is the multiplicative identity.

5. **Commutative Law:** For all $a_1, a_2 \in R$,

$$a_1 \times a_2 = a_2 \times a_1.$$

**Example 2.4.6.**

1. The set of integers $\mathbb{Z}$ is Ring under addition(+) and multiplication($\times$).
2. Under usual addition (+) and multiplication($\times$) the set of rational numbers $\mathbb{Q}$ and set of complex numbers $\mathbb{C}$ are ring.
3. $M_2(R)$ is the ring of $2\times2$ matrices with co-efficient from $R$ under matrix addition (+) and multiplication ($\times$).

### 2.4.7 Field

The set ($\mathbb{F}$ ,+ , $\times$) with binary operations + and $\times$ is called Field $\mathbb{F}$ [35], if the following properties holds.

**F1:** $\mathbb{F}$ is closed under addition and multiplication. Let $a_1, a_2 \in F$ then,

$$a_1 + a_2 \in \mathbb{F}, \quad \text{and} \quad a_1 \times a_2 \in \mathbb{F}.$$

**F2:** $\mathbb{F}$ is associative under operation addition and multiplication. Let $a_1, a_2, a_3 \in \mathbb{F}$ , then

$$(a_1 + a_2) + a_3 = a_1 + (a_2 + a_3) \quad \text{and} \quad (a_1 \times a_2) \times a_3 = a_1 \times (a_2 \times a_3).$$

**F3:** $\mathbb{F}$ is commutative over addition and multiplication.Let $a_1, b_2 \in \mathbb{F}$ then,

$$a_1 + a_2 = a_2 + a_1 \quad \text{and} \quad a_1 \times a_2 = a_2 \times a_1.$$

**F4:** $\mathbb{F}$ has additive and multiplicative identities i.e

$$a_1 + \varepsilon = \varepsilon + a_1 = a_1 \quad \text{and} \quad a_1 \times \epsilon = \epsilon \times a_1 = a_1.$$

Here $\varepsilon$ is additive and $\epsilon$ is multiplicative identity.

**F5:** For each element $a_1 \in F$,there exist $a_1^{'} \in F$ such that,

$$a_1 + a_1^{'} = a_1^{'} + a_1 = \varepsilon.$$

Here $\varepsilon$ is the additive identity element and $a_1^{'} \in \mathbb{F}$ is additive inverse of $a_1 \in \mathbb{F}$.
For each element $a_1 \in \mathbb{F}$, there exist $a_1^{'} \in \mathbb{F}$ such that,

$$a_1 \times a_1^{'} = a_1^{'} \times a_1 = \epsilon.$$

Here $\epsilon$ is the multiplicative identity element and $a_1^{'} \in \mathbb{F}$ is multiplicative inverse of $a \in \mathbb{F}$.

**Example 2.4.7.** Set of real numbers $\mathbb{R}$ and set of rational numbers $\mathbb{Q}$ forms field under addition $(+)$ and multiplication $(\times)$.

### 2.4.8 Finite Group

A non-empty group $\mathbb{H}$ is called finite group if it contains finite elements. The number of elements in a finite group is called its order.

Order of finite group $\mathbb{H}$ is denoted by $\mid \mathbb{H} \mid$.

### 2.4.9 Galois Fields

**Definition 2.4.8.** "Let GF($p$) be the finite field with $p$ elements, where $p$ is a prime. Let $f(t)$ be an irreducible polynomial of degree $q$ over GF($p$). Then the quotient ring GF($p$)/$\langle f(t)\rangle$ forms a finite field with $p^q$ elements. This field is denoted by GF($p^q$) and is called the galois field of order $p^q$. The elements of GF($p^q$) are represented by polynomials of degree less than $q$ with coefficients in GF($p$), and arithmetic operations are performed under modulo $f(t)$ [36]".

1. GF($p$), $p$ is prime number which represent the numbers of element in GF. For example GF(7), there are seven elements in this finite field.

2. **GF($p^q$)**, here $p$ is also prime number and $q$ is degree of polynomials. This field is use to represent the polynomials elements with irreducible polynomial.
   For example GF($2^8$) is field with coefficient 0 and 1 and its degree of the polynomial $f(t)$ is 7.
   Consider a polynomial $B(t) \in$ GF($2^8$),

$$B(t) = b_7t^7 + b_6t^6 + b_5t^5 + b_4t^4 + b_3t^3 + b_2t^2 + b_1t^1 + b_0t^0.$$

   Every non-zero element in galois field has its inverse. As galois field shows all the properties of field [37].

### 2.4.10 Polynomials

"A polynomial in the variable $t$ with coefficients in a ring $R$ is an expression of the form:

$$f(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0,$$

where $n$ is a non-negative integer, $a_n, a_{n-1}, \ldots, a_0$ are elements of $R$, and $t$ is a formal symbol.

The set of all such polynomials is denoted by $R[t]$ [38] [39]."

### 2.4.11 Polynomial Modular Multiplication

Consider two polynomials from GF($p^2$):

$$P_1(t) = t + 1, \quad P_2(t) = t + 1,$$

with irreducible polynomial $t^2 + 1$.
The multiplication of these two polynomials is:

$$P_1(t) \times P_2(t) = (t + 1)(t + 1) = t^2 + t + t + 1.$$

Combining like terms gives:

$$t^2 + 2t + 1.$$

Now, reduce to modulo $t^2 + 1$:

$$t^2 \equiv -1 \pmod{t^2 + 1},$$
$$2t \equiv 2t \pmod{t^2 + 1},$$
$$1 \equiv 1 \pmod{t^2 + 1}.$$

Thus, the result is:

$$t^2 + 2t + 1 \equiv 2t \pmod{t^2 + 1}.$$

### 2.4.12 GCD Of Polynomials

"The greatest common divisor (GCD) of two polynomials $f(t)$ and $g(t)$ over a given field or ring is the highest-degree polynomial $d(t)$ that divides both $f(t)$ and $g(t)$ without remainder. Additionally, $d(t)$ is unique up to multiplication by a non-zero constant (a unit in the given field or ring) [40]".
GCD is unique polynomial of f(t) and g(t). If f(t) and g(t) are co-prime polynomial then their GCD is 1.
Euclidean algorithm can be used to find greatest common divisor for polynomials $f(t)$

and $g(t).d(t) = GCD(f(t), g(t))$ if $d(t)$ is the polynomial of greatest degree which divides both $f(t), g(t)$.

### 2.4.13 Euler's Totient Theorem

Let $N$ be a positive integer and $\varphi(N)$ denote the Euler's totient function, which counts the number of integers less than or equal to $N$ that are coprime with $N$. According to Euler's totient theorem [41], for any integer $a$ such that $\gcd(a, N) = 1$, the following relation holds:

$$a^{\varphi(N)} \equiv 1 \pmod{N}.$$

This theorem implies that the powers of any integer $a$ that are coprime with $N$ will eventually return to 1 modulo $N$, after raising $a$ to successive values of $\varphi(N)$.

### 2.4.14 Polynomial Inverse

"Let $f(t)$ be a polynomial over a field $F$. A polynomial $g(t)$ is called the *inverse* of $f(t)$ if their product equals 1 [39], i.e.,

$$f(t)g(t) = 1".$$

### 2.4.15 Modular Multiplicative Inverse

In modular arithmetic, compute the inverse $g(t)$ of $f(t)$ such that:

$$f(t)g(t) \equiv 1 \pmod{m(t)}.$$

where $m(t)$ is a given modulus polynomial. This is analogous to find the modular inverse of integers [42].

**Example 2.4.9.** Find inverse of 550 mod 1759.

Use the Euclidean algorithm to find the inverse of 550 mod 1759.

| **Q** | $\mathbf{T}_1$ | $\mathbf{T}_2$ | $\mathbf{T}_3$ | $\mathbf{S}_1$ | $\mathbf{S}_2$ | $\mathbf{S}_3$ |
|---|---|---|---|---|---|---|
| | 1 | 0 | 1759 | 0 | 1 | 550 |
| 3 | 0 | 1 | 550 | 1 | -3 | 109 |
| 5 | 1 | -3 | 109 | -5 | 16 | 5 |
| 21 | -5 | 16 | 5 | 106 | -339 | 4 |
| 1 | 106 | -339 | 4 | -111 | 355 | 1 |

TABLE 2.1: Inverse of 550 is 355.

**Example 2.4.10.** Find inverse of $t^3+t^2+t+1$ mod $t^8+t^4+t^3+t+1$.
By using the Euclidean algorithm to find the inverse of the polynomials $t^3+t^2+t+1$ mod $t^8+t^4+t^3+t+1$ follow the following steps identified in the table below:

| **Q(t)** | $\mathbf{T(t}_1\mathbf{)}$ | $\mathbf{T(t}_2\mathbf{)}$ | $\mathbf{T(t}_3\mathbf{)}$ | $\mathbf{S(t}_1\mathbf{)}$ | $\mathbf{S(t}_2\mathbf{)}$ | $\mathbf{S(t}_3\mathbf{)}$ |
|---|---|---|---|---|---|---|
| | 1 | 0 | $t^8+t^4+t^3$ $+t+1$ | 0 | 1 | $t^3+t^2+t+1$ |
| $t^5+t^4+1$ | 0 | 1 | $t^3+t^2+t+1$ | 1 | $t^5+t^4+1$ | $t^2$ |
| $t+1$ | 1 | $t^5+t^4+1$ | $t^2$ | $t+1$ | $t^6+t^4+t$ | $t+1$ |
| $t+1$ | $t+1$ | $t^6+t^4+t$ | $t+1$ | $t^2$ | $t^7+t^6+t^2$ $+t+1$ | 1 |

TABLE 2.2: Inverse of $t^3+t^2+t+1$ mod $t^8+t^4+t^3+t+1$ is $t^7+t^6+t^2+t+1$.

The final remainder is 1, which means that the GCD of $t^3+t^2+t+1$ mod $t^8+t^4+t^3+t+1$ and $t^7+t^6+t^2+t+1$ is 1 and inverse of $t^3+t^2+t+1$ mod $t^8+t^4+t^3+t+1$ is $t^7+t^6+t^2+t+1$.

### 2.4.16 Circulant Matrices

**Definition 2.4.11.** "A matrix $C \in \mathbb{C}^{n\times n}$ is called a **circulant matrix** if it has the following form:

$$C = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix},$$

where each row is a cyclic shift of the previous row. The entries of $C$ satisfy $C_{i,j} = c_{(j-i) \mod n}$ for all $i, j \in \{0, 1, \ldots, n-1\}$" [43].

**Example 2.4.12.** A $(3 \times 3)$ circulant matrix with first row $[1, 2, 3]$ is:

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}.$$

A $(4 \times 4)$ circulant matrix with first row $[4, 5, 6, 7]$ is:

$$C = \begin{bmatrix} 4 & 5 & 6 & 7 \\ 7 & 4 & 5 & 6 \\ 6 & 7 & 4 & 5 \\ 5 & 6 & 7 & 4 \end{bmatrix}.$$

### 2.4.17 Properties of Circulant Matrices

#### 2.4.17.1 Commutative Property

For circulant matrices $C_1$ and $C_2$ the commutative property holds:

$$C_1 \times C_2 = C_2 \times C_1.$$

#### 2.4.17.2 Fast Matrix-Vector Multiplication

Since multiplication by a circulant matrix is equivalent to a cyclic convolution, it can be computed efficiently using the fast fourier transform (FFT) in $O(n \log n)$ time.

### 2.4.18 Definition of Matrix Power Function (MPF)

**Definition 2.4.13.** "The left-sided MPF [44] [45] corresponding to matrix $T$ powered by matrix $W$ on the left, with MPF value equal to matrix $Q$ is given by:

$$^{W}T = Q,$$

here $W$ is circulant matrix."

Let supposed that, two $2 \times 2$ matrices, $T$ and $W$, defined as:

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}.$$

Define a transformation, denoted as $^{W}T$, which modifies $T$ based on the values in $W$. This transformation is given by:

$$^{W}T = \begin{bmatrix} t_{11}^{w_{11}} t_{21}^{w_{12}} & t_{12}^{w_{11}} t_{22}^{w_{12}} \\ t_{11}^{w_{21}} t_{21}^{w_{22}} & t_{12}^{w_{21}} t_{22}^{w_{22}} \end{bmatrix}.$$

Each element of the resulting matrix is obtained by taking elements of $T$, raising them to the powers specified by $W$, and multiplying accordingly.

This operation is not a standard matrix operation such as matrix multiplication but rather a custom transformation that applies exponentiation and multiplication element-wise based on the values in $W$.

**Example 2.4.14.** Consider two matrices $T$ and $W$:

$$T = \begin{bmatrix} x & 1 \\ x+1 & x \end{bmatrix}, W = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Applying the transformation ${}^{W}T$, define as:

$$ {}^{W}T = \begin{bmatrix} x^2 \cdot (x+1)^1 & 1^2 \cdot x^1 \\ x^1 \cdot (x+1)^2 & 1^1 \cdot x^2 \end{bmatrix}. $$

Simplifying each entry:

$$ {}^{W}T = \begin{bmatrix} x^3 + x^2 & x \\ x^3 + 2x^2 + x & x^2 \end{bmatrix}. $$

This transformation applies exponentiation and multiplication rather than standard matrix operations. Each entry in the transformed matrix is computed using the corresponding powers from $W$, leading to the final result.

**Definition 2.4.15.** "The right sided MPF corresponding to matrix $T$ powered by matrix $W$ on the right, with MPF value equal to matrix $D$, is given by [45]

$$ T^W = Q, $$

here W is circulant matrix."

Consider two matrices $T$ and $W$:

$$ T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \quad W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}. $$

Define a transformation $T^W$ as:

$$ T^W = \begin{bmatrix} t_{11}^{w_{11}} \cdot t_{21}^{w_{21}} & t_{11}^{w_{12}} \cdot t_{12}^{w_{22}} \\ t_{21}^{w_{11}} \cdot t_{22}^{w_{21}} & t_{21}^{w_{12}} \cdot t_{22}^{w_{22}} \end{bmatrix}. $$

In this transformation, each element of the resulting matrix is determined by raising specific entries of $T$ to the corresponding powers in $W$ and multiplying them accordingly.

**Example 2.4.16.** Consider two matrices $T$ and $W$:

$$ T = \begin{bmatrix} x & 1 \\ x+1 & x \end{bmatrix}, \quad W = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}. $$

The transformation $T^W$ is given by:

$$T^W = \begin{bmatrix} x & 1 \\ x+1 & x \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Applying the transformation:

$$T^W = \begin{bmatrix} x^2 \cdot 1^1 & x^1 \cdot 1^2 \\ (x+1)^2 \cdot x^1 & (x+1)^1 \cdot x^2 \end{bmatrix},$$

Simplifying:

$$T^W = \begin{bmatrix} x^2 & x \\ x^3 + 2x^2 + x & x^3 + x^2 \end{bmatrix}.$$

**Definition 2.4.17.** "The two-sided MPF [45] corresponds to matrix $T$ powered by matrix $W$ on the left and by matrix $X$ on the right, with MPF value equal to matrix $Q$, and is expressed as:

$$^W T^X = Q$$

W and X are circulant matrices [46]."

Consider the matrices:

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \quad W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

Define the transformation as:

$$^W T^X = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

Applying the transformation rule:

$$^W T^X = \begin{bmatrix} {t_{11}}^{w_{11}x_{11}} {t_{21}}^{w_{12}x_{11}} {t_{12}}^{w_{11}x_{21}} {t_{22}}^{w_{12}x_{21}} & {t_{11}}^{w_{11}x_{12}} {t_{21}}^{w_{12}x_{12}} {t_{12}}^{w_{11}x_{22}} {t_{22}}^{w_{12}x_{22}} \\ {t_{11}}^{w_{21}x_{11}} {t_{21}}^{w_{22}x_{11}} {t_{12}}^{w_{21}x_{21}} {t_{22}}^{w_{22}x_{21}} & {t_{11}}^{w_{21}x_{12}} {t_{21}}^{w_{22}x_{12}} {t_{12}}^{w_{21}x_{22}} {t_{22}}^{w_{22}x_{22}} \end{bmatrix}.$$

**Example 2.4.18.** Consider W and X two circulant matrices and T a base matrix:

$$T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

The operation ${}^{W}T^{X}$ is defined above: Step by step calculation:

First Element:

$$t_{11}{}^{w_{11}\cdot x_{11}} = 2^{1\cdot 2} = 4, \quad t_{21}{}^{w_{12}\cdot x_{11}} = 4^{2\cdot 2} = 256,$$
$$t_{12}{}^{w_{11}\cdot x_{21}} = 3^{1\cdot 3} = 27, \quad t_{22}{}^{w_{12}\cdot x_{21}} = 5^{2\cdot 3} = 15625.$$
$$\text{First element} = 4 \cdot 256 \cdot 27 \cdot 15625 = 431308800.$$

Second Element:

$$t_{11}{}^{w_{11}\cdot x_{12}} = 2^{1\cdot 1} = 2, \quad t_{21}{}^{w_{12}\cdot x_{12}} = 4^{2\cdot 1} = 16,$$
$$t_{12}{}^{w_{11}\cdot x_{22}} = 3^{1\cdot 2} = 9, \quad t_{22}{}^{w_{12}\cdot x_{22}} = 5^{2\cdot 2} = 625.$$
$$\text{Second element} = 2 \cdot 16 \cdot 9 \cdot 625 = 180000.$$

Third Element:

$$t_{11}{}^{w_{21}\cdot x_{11}} = 2^{3\cdot 2} = 64, \quad t_{21}{}^{w_{22}\cdot x_{11}} = 4^{4\cdot 2} = 4294967296,$$
$$t_{12}{}^{w_{21}\cdot x_{21}} = 3^{3\cdot 3} = 19683, \quad t_{22}{}^{w_{22}\cdot x_{21}} = 5^{4\cdot 3} = 95367431640625.$$
$$\text{Third element} = 64 \cdot 4294967296 \cdot 19683 \cdot 95367431640625 \approx 5.344 \times 10^{24}.$$

Fourth Element:

$$t_{11}{}^{w_{21}\cdot x_{12}} = 2^{3\cdot 1} = 8, \quad t_{21}{}^{w_{22}\cdot x_{12}} = 4^{4\cdot 1} = 256,$$
$$t_{12}{}^{w_{21}\cdot x_{22}} = 3^{3\cdot 2} = 729, \quad t_{22}{}^{w_{22}\cdot x_{22}} = 5^{4\cdot 2} = 152587890625.$$
$$\text{Fourth element} = 8 \cdot 256 \cdot 729 \cdot 152587890625 = 2.187 \times 10^{13}.$$

$${}^{W}T^{X} = \begin{bmatrix} 431308800 & 180000 \\ 5.344 \times 10^{24} & 2.187 \times 10^{13} \end{bmatrix}.$$

This is our final result.

### 2.4.19 Properties of ${}^{W}T^{X}$

#### 2.4.19.1 One-Sided Associativity

For matrices $T$, and $X$, the following one-sided associativity property holds:

$$(T^{X_1X_2}) = (T^{X_1})^{X_2} = T^{X_1X_2}.$$

Similarly, when the transformation involves $W$:

$$({}^{W_1W_2}T) = {}^{W_1}({}^{W_2}T) = {}^{W_1W_2}T.$$

#### 2.4.19.2 Two-Sided Associativity

When transformations involve both $W$ and $X$, two sided associativity holds:

$${}^{W_1}(T^{X_1}) = {}^{W_1W_2}(T^{X_1X_2}) = {}^{W_1W_2}T^{X_1X_2}.$$

#### 2.4.19.3 Compatibility with Scalar Powers

For scalar powers applied to $T$, if $T$ is raised to a scalar $k$:

$${}^{W}(T^{k})^{X} = \left({}^{W}T^{X}\right)^{k} = {}^{W}T^{kX}.$$

#### 2.4.19.4 Identity Behaviour

If $W = I$ (identity matrix) or $X = I$, the transformation simplifies:

$${}^{I}T^{X} = T^{X}, \quad {}^{W}T^{I} = {}^{W}T.$$

#### 2.4.19.5 Inverse Behaviour

If $W$ or $X$ are inverses, the operation satisfies:

$$^{W^{-1}}(T^{X^{-1}}) = \left(^{W}T^{X}\right)^{-1}.$$

### 2.4.20 One-Way Function

"A one-way function is a mathematical function that is easy to compute in one direction but significantly difficult to reverse. This means that given an input $t$, computing $f(t)$ is straightforward, but given $f(t)$, finding $t$ is computationally infeasible.[47]"

#### 2.4.20.1 Properties of One-Way Functions

1. **Efficient Computation**: For any given input $t$, the function $f(t)$ can be computed quickly using a standard algorithm.

2. **Hard to Invert**: Given an output $u = f(t)$, it is computationally difficult to determine the original input $t$.

**Example 2.4.19.** A common example is modular exponentiation in cryptography:

$$f(t) = g^t \mod p,$$

where, $g$ is a generator of a cyclic group, $p$ is a large prime number, $t$ is the input. While computing $f(t)$ is easy, reversing the process (finding $t$ from $g^t \mod p$) is difficult, which forms the basis of cryptographic protocols like Diffie-Hellman key exchange.

One-way functions are fundamental to cryptography, ensuring security in encryption, digital signatures, and password hashing.

### 2.4.21 Primitive Root in Polynomial Form

**Definition 2.4.20.** A primitive root [48] $g$ modulo $p$ is an integer such that the powers of $g$ modulo $p$ generate all the nonzero residues modulo $p$. In polynomial terms, this means that $g$ is a primitive root modulo $p$ if the following condition holds:

$$g^k \not\equiv 1 \mod p \quad \text{for all integers} \quad k < p-1,$$
$$g^{p-1} \equiv 1 \mod p.$$

This implies that the smallest exponent $k$ such that $g^k \equiv 1 \mod p$ is exactly $p-1$, which is the order of $g$ modulo $p$. Hence, $g$ must satisfy the polynomial equation:

$$g^k \equiv 1 \mod p, \quad \text{only when,} \quad k = p-1,$$
$$g^{p-1} - 1 \equiv 0 \mod p.$$

Thus, the polynomial $g^{p-1} - 1$ has a root at $g$ modulo $p$, and the order of $g$ (the smallest exponent $k$ for which $g^k \equiv 1 \mod p$) is exactly $p-1$.

# Chapter 3

# Modular Matrix Based Digital Signatures

Extensively studied various digital signature techniques and their associative algorithms. These techniques play a vital role in ensuring data authenticity, integrity, and non-repudiation in digital communication. Different approaches to digital signatures, such as digital signature algorithm (DSA), modular matrix based digital signature (MMDS), and digital signature scheme based on matrix power function (DSSMPF) are explored.

A extensive focus is on the digital signature scheme based on matrix power function, which offers enhanced security through matrix operations. This method involves the use of modular matrix transformations to generate and verify digital signatures. Example of this approach is demonstrated at the end.

## 3.1 El-Gamal Digital Signature

The ElGamal digital signature scheme [49] is a robust cryptanalysis method for generating secure and verified signatures in a simple and efficient manner. ElGamal digital signature scheme [50] is based on the hardness of the discrete logarithm problem (DLP) and provides a secure way to verify the sender's message.

### 3.1.1 Global Parameters

These parameters are publicly known and used in the signature scheme:

1. A large prime number $n_0$ used as the modulus in mathematical calculations. A large value of $n_0$ enhances security.

2. Let $a$ be a primitive root modulo $n_0$.

   This means that $a$ is an integer that generates a cyclic group under modular arithmetic modulo $n_0$.

   Specifically, the powers of $a$ will generate all integers less than $n_0$ that are co-prime to $n_0$,

   $$a < n_0.$$

3. $x_0$ is random integer known only to the signer. $x_0$ is a secret key such that,

   $$1 < x_0 < n_0 - 1.$$

4. Compute public key $z_0$ by using generator $a$ and secret key $x_0$.

   $$z_0 = a^{x_0} \mod n_0.$$

   So, public key is $(n_0, a, z_0)$.

### 3.1.2 Signature Generation

To generate a digital signature for a message $M$, the sender follows these steps:

1. On message $M$ cryptographic hash function is applied to converting it into a fixed length hash value to ensures its integrity.

2. Select a random integer $e$ such that $1 < e < n_0 - 1$. This value is unique for each signature, improving security by preventing repeated attacks.

3. The digital signature consists of two components, $\alpha$ and $\beta$, computed as follows:

$$\alpha = a^e \mod n_0,$$
$$\beta = \frac{H(M) - x_0 \cdot \alpha}{e} \mod (n_0 - 1).$$

The value $\alpha$ is generated from $a$, the random number $e$, and the modulus $n_0$.

It used during verification to confirm authenticity.

The computation of $\beta$ ensures that the signature is uniquely tied to the message $M$ and the sender's private key $x_0$, utilizing modular arithmetic for security.

### 3.1.3 Verification

To verify the ElGamal digital signature $(\alpha, \beta)$ follow these steps:

First compute $\Sigma_1$ as,

$$\Sigma_1 = z_0^{\alpha}\alpha^{\beta} \mod n_0.$$

Compute $\Sigma_2$

$$\Sigma_2 = a^{H(M)} \mod n_0.$$

If $\Sigma_1 = \Sigma_2$ then the signature are valid otherwise invalid.

### 3.1.4 Correctness

Correctness of this technique follows the following steps:

$$\beta = \frac{H(M) - x_0 \cdot \alpha}{e} \mod (n_0 - 1),$$
$$H(M) = x_0\alpha + e\beta \mod (n_0 - 1).$$

By Fermat's little theorem

$$a^{H(M)} = a^{x_0\alpha}a^{e\beta},$$

$$a^{H(M)} = (a^{x_0})^{\alpha}(a^{e})^{\beta},$$
$$a^{H(M)} = (z_0)^{\alpha}(\alpha)^{\beta}.$$

**Example 3.1.1.** For a message $M = 10$, generate digital signature by using ElGamal digital signature scheme and verify them. Secret key is 5 and generator is 2 along with prime number 19.

1. **Global Parameters**

   A prime number $n_0 = 19$.

   A generator $a = 2$ such that it is a primitive root of modulo $n_0$.

2. **Private key**

   $x_0 = 5$ such that $1 < 5 < 18$.

3. **Public key**

$$\begin{aligned} z_0 &= a^{x_0} \mod n_0, \\ &= 2^5 \mod 19, \\ &= 13. \end{aligned}$$

   So, public key is $(n_0, a, z_0) = (19, 2, 13)$.

4. **Message for Signature**

$$M = 10.$$

5. **Hash Function**

   To fixed the message length we use hash function.

$$\begin{aligned} H(M) &= H(10), \\ &= 14. \end{aligned}$$

6. **Signature Generation**

   (a) Choose random $e = 7$ such that $0 < 7 < 18$ and $gcd(7, 18) = 1$.

(b) For $\alpha = a^e \mod n_0 = 2^5 \mod 19 = 14$.

(c) For $\beta = \frac{H(M)-x_0\alpha}{e} \mod (n_0 - 1) = \frac{7-5\cdot 13}{4} \mod 16 = 10$.

7. **Signature**

$$(\alpha, \beta) = (13, 9).$$

8. **Verification**

First compute $\Sigma_1$ as,

$$\begin{aligned}
\Sigma_1 &= {z_0}^{\alpha}\alpha^{\beta} \mod n_0, \\
\Sigma_1 &= 13^{14}14^{10} \mod 19, \\
\Sigma_1 &= 6 \mod 19.
\end{aligned}$$

Compute $\Sigma_2$:

$$\begin{aligned}
\Sigma_2 &= a^{H(M)} \mod n_0, \\
\Sigma_2 &= 2^{14} \mod 19, \\
\Sigma_2 &= 6 \mod 19.
\end{aligned}$$

$\Sigma_1 = \Sigma_2$ so, signature are valid.

## 3.2 Digital Signature Algorithm (DSA)

Federal information processing standard (FIPS) used digital signature scheme for obtaining digital signatures algorithm (DSA).

*DSA* [51] is very secure, authentic and credible algorithm in signing a message. *DSA* is based asymmetric cryptography which work on bases of pair of keys.

Private or secure key for generation of signature and a public key for verifying these signatures [52]. DSA consist of following steps:

### 3.2.1 Global Parameters

DSA requires three globally shared parameters:

1. A large prime number $p_1$ is selected such that it lies within the range:

$$2^{L_0-1} < p_1 < 2^{L_0}, \quad \text{where } 512 \leq L_0 \leq 1024, \quad L_0 \text{ is a multiple of 64.}$$

2. A second prime number $p_2$, which is a divisor of $p_1 - 1$, is selected with a bit length of $N$ bits such that $2^{N-1} < p_2 < 2^N$.

3. A generator $a$ is determined using the formula:

$$a = e^{(p_1-1)/p_2} \mod p_1,$$

where $e$ is an integer such that $1 < e < (p_1 - 1)$ and $a > 1$.

### 3.2.2 Private Key ($x_0$)

Each user randomly selects a private key $x_0$, which must be an integer in the range $0 < x_0 < p_2$.

### 3.2.3 Public Key ($z_0$)

The corresponding public key is derived from the private key using the formula:

$$z_0 = a^{x_0} \mod p_1.$$

It is computationally infeasible to derive $x_0$ from $z_0$, ensuring security.

### 3.2.4 Signature

To generate a signature for a message $M$, the sender performs the following steps:

1. Compute the hash value of the message $M$ i.e $H(M)$.

2. Choose a random integer $k_0$ such that:

$$0 < k_0 < p_2.$$

3. Compute the first signature component $\alpha$:

$$\alpha = (a^{k_0} \mod p_1) \mod p_2.$$

4. Compute the second signature component $\beta$:

$$\beta = {k_0}^{-1}(H(M) + x_0\alpha) \mod p_2.$$

Here, $k_0^{-1}$ is the modular inverse of $k_0$ modulo $p_2$. Use Euclidean extended algorithm to compute the $k_0^{-1}$.

The digital signature are $(\alpha, \beta)$.

### 3.2.5 Signature Verification

Upon receiving a message $M$ and its signature $(\alpha, \beta)$, the recipient follows these steps to verify its authenticity:

1. Ensure that $\alpha$ and $\beta$ satisfy:

$$0 < \alpha < p_2, \quad 0 < \beta < p_2.$$

2. Compute $w$ the modular inverse of $\beta$. Use Euclidean extended algorithm for computing $w$.

$$w = \beta^{-1} \mod p_2.$$

3. Compute two intermediate values by using $H(M)$ and $w$.

$$\Sigma_1 = (H(M)w) \mod p_2,$$

$$\Sigma_2 = (\alpha w) \mod p_2.$$

4. Compute $\gamma$, which is the verification parameter:

$$\gamma = ((a^{\Sigma_1} z_0^{\Sigma_2}) \mod p_1) \mod p_2.$$

The signature is valid if:

$$\gamma = \alpha.$$

Since only the sender with knowledge of $x_0$ could have generated the original signature, a match confirms authenticity.

**Example 3.2.1.** Using the digital signature algorithm (DSA), generate a digital signature for the message $M = 5$ with the following parameters: a prime modulus $p_1 = 23$, a prime divisor $p_2 = 11$, and a private key $x_0 = 6$. Compute the public key $z_0$. Once the signature is generated, verify its validity using the public key $z_0$. Ensure all calculations adhere to the DSA specifications.

## Solution:

A prime modulus $p_1 = 23$.

A prime divisor that divides $(p_1 - 1)$ is $p_2 = 11$.

Produce a generator $a$ by using value of $e$:

$$a = e^{(p_1-1)/p_2} \mod p_1.$$

Where $e$ is an integer such that $1 < e < (p_1 - 1)$ and $a > 1$.

Let $e = 4$, then:

$$a = 4^{(22/11)} \mod 23 \quad = 4^2 \mod 23,$$

$$a = 16 \mod 23.$$

## Key Generation

A private key $x_0 = 6$.

Compute the public key $z_0$:

$$z_0 = a^{x_0} \mod p_1,$$
$$z_0 = 16^6 \mod 23 = 2.$$

## Signature Generation

Choose a random integer $k_0 = 3$.

Compute the first signature component $\alpha$:

$$\alpha = (a^{k_0} \mod p_1) \mod p_2,$$
$$\alpha = (16^3 \mod 23) \mod 11 = 9.$$

Compute the second signature component $\beta$ for $(M) = 5$, and $k^{-1} \mod p_2 = 4$,

$$\beta = k^{-1}(H(M) + x_0\alpha) \mod p_2,$$
$$\beta = 4(5 + 6.9) \mod 11,$$
$$\beta = 4.59 \mod 11 = 5.$$

The signatures are $(\alpha, \beta) = (9, 5)$.

## Signature Verification

Compute $w = \beta^{-1} \mod p_2$:

$$w = 5^{-1} \mod 11 = 9.$$

Extended euclidean algorithm used for calculating $5^{-1}$.

Compute $\Sigma_1$ and $\Sigma_2$:

$$\Sigma_1 = (H(M)w) \mod p_2 = (5.9) \mod 11 = 1,$$

$$\Sigma_2 = (\alpha w) \mod p_2 = (9.9) \mod 11 = 4.$$

Compute $\gamma$:

$$\gamma = ((a^{\Sigma_1} z_0^{\Sigma_2}) \mod p_1) \mod p_2,$$
$$\gamma = ((16^1 \times 2^4) \mod 23) \mod 11,$$
$$\gamma = (16 \times 16 \mod 23) \mod 11 = 9.$$

Since $\gamma = \alpha$, the signature is valid.

## 3.3 Modular Matrix Based Digital Signature

A modular matrix based digital signature (MMDS) [18] is a cryptographic technique that employs matrix functions and modular arithmetic to enhanced the authenticity and integrity of digital messages.

Conventional digital signature schemes such as RSA and ECC rely on prime factorization or elliptic curves. MMDS utilizes matrices over modular fields, providing potential advantages in security and computational efficiency.
Matrix operations can be parallelized, making MMDS potentially faster than traditional number-theoretic cryptographic schemes when optimized for hardware acceleration.

### 3.3.1 Key Generation

First step is to generate secret key and public key for generating digital signature based on modular matrix power function [53].

1. Consider two large and distinct prime number $p_1$ and $p_2$.

2. Compute the first part $N$ of public key as:

$$N = p_1 p_2$$

3. Let $X$ and $Y$ are two invertible $2\times2$ circulant matrices chosen from the collection of subgroup $H \subseteq GL(2, \mathbb{Z}_N)$, remember $GL(2, \mathbb{Z}_N)$ is collection of the group of invertible $2 \times 2$ matrices with entries in $\mathbb{Z}_N$.

$$H = \left\{ \mathbf{M} = \begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix} \;\middle|\; a_1, b_1 \in \mathbb{Z}_N \text{ and } (a_1^2 - b_1^2) \in \mathbb{Z}_N^* \right\}. \tag{3.1}$$

4. Compute $Z$, a part of public key.

$$Z = X^{-1}Y,$$

where $X^{-1}$ is the modular inverse of $X$ under $\mathbb{Z}_N$. Euclidean extended algorithm is used for computing $X^{-1}$.

5. The public and private Keys are:

$$\text{Public Key:} \quad (N, Z) \quad , \quad \text{Private Key:} \quad (X, Y).$$

### 3.3.2 Digital Signature Generation

For generating digital signature perform following steps:

1. Choose two random matrices $R$ and $Q$, $R$ is circulant matrix and $Q$ is base matrix of order 2 along with two random numbers $\eta$ and $\zeta$.
   - $R \in H$.
   - $Q \in GL(2, \mathbb{Z})$.
   - $\eta, \zeta \in \mathbb{Z}_N$.
2. Session private key is defined as:

$$(\eta, \zeta, R, Q).$$

3. Automorphisms of matrix $Q$: There are three automorphisms define on a matrix $Q \in M_2(\mathbb{Z}_N)$:

$$\phi_Z(Q) = Z^{-1}QZ.$$
$$\phi_{YR}(Q) = (YR)^{-1}Q(YR).$$
$$\phi_{XR}(Q) = (XR)^{-1}Q(XR).$$

First part of digital signature $\alpha$ is computed as:

$$\alpha = \zeta\phi_{XR}(Q).$$
$$\Sigma_1 = \phi_{YR}(Q).$$
$$\eta\Sigma_1 = \eta\phi_{YR}(Q).$$
$$\tau = \eta + \zeta.$$

4. Now, first convert the message $M$ into binary numbers $(M)_2$ and then do following calculation for generating second part of digital signature $\beta$.

$$\beta = \mathcal{H}\left((M)_2 \| (\tau\Sigma_1)_2\right),$$

where $\mathcal{H}$ is a cryptographic hash function.

The session public key and signature are:

$$\text{Session Public Key:} \quad \eta\Sigma_1.$$
$$\text{Signature:} \quad (\alpha, \beta).$$

### 3.3.3 Verification

Now, verify computed digital signatures $(\alpha, \beta)$ for the message $M$ by following steps:

1. The public key is $(N, Z)$ and the session public key is $\eta\Sigma_1$.

2. Compute $\gamma$:

$$\gamma = \eta\Sigma_1 + \phi_Z(\alpha).$$

Where $\phi_Z$ is the automorphism.

$$\phi_Z(Q) = Z^{-1}QZ.$$

Recompute the hash value:

$$\beta' = \mathcal{H}\left((M)_2 \| (\gamma)_2\right).$$

$(M)_2$ and $(\gamma)_2$ are in base 2.

If:

$$\beta' = \beta,$$

then the signature are valid otherwise, signature are invalid.

### 3.3.4 Correctness

The correctness of a signature can be verified by checking that the signature matches the computed value. For this purpose performed the following steps :

Let,

$$\gamma = \eta\Sigma_1 + \phi_Z(\alpha),$$
$$\gamma = \eta\Sigma_1 + \zeta(X^{-1}Y)^1(XR)(X^{-1}Q),$$

$$\gamma = \eta\Sigma_1 + \zeta\Sigma_1,$$
$$\gamma = (\eta + \zeta)\Sigma_1 = \tau\Sigma_1.$$

From above equations:

$$\beta' = \mathcal{H}\left((M)_2 \| (\gamma)_2\right),$$
$$\beta' = \mathcal{H}\left((M)_2 \| (\tau\Sigma_1)_2\right),$$

As,

$$\beta' = \beta.$$

**Example 3.3.1.** To illustrate the MMDS scheme consider a matrix of order 2 from the general linear group $GL(2, \mathbb{Z}_N)$ with two circulant matrices of order 2.

All the calculation are performed under modular arithmetic with mod $N$.

## Key Generation

1. Choose two distinct prime numbers $p_1$ and $p_2$ such that,

$$p_1 = 7, \quad p_2 = 11.$$

2. Compute $N$ a part of public key:

$$N = p_1 p_2 = (7)(11) = 77.$$

3. Choose $X$ and $Y$ are two invertible $2 \times 2$ circulant matrices.

$$X = \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix}$$

$$Y = \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}.$$

4. Compute the matrix $Z$ other part of public key.

   For Z first compute $X^{-1}$.

$$\begin{aligned} X^{-1} &= \frac{1}{\det(X)} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix} \bmod 77, \\ &= 72^{-1} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix} \bmod 77. \end{aligned}$$

| $Q$ | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|---|---|
| | 1 | 0 | 77 | 0 | 1 | 72 |
| 1 | 0 | 1 | 72 | 1 | -1 | 5 |
| 14 | 1 | -1 | 5 | -14 | 15 | 2 |
| 2 | -14 | 15 | 2 | 29 | -31 | 1 |

TABLE 3.1: Extended Euclidean Algorithm $72^{-1} \mod 77$

The modular inverse of 72 mod 77 is $72^{-1} = -31 \bmod 77$. However, express $-31$ as a positive value:

$$-31 + 77 = 46.$$

So,

$72^{-1} \equiv 46 \bmod 77$.

$$X^{-1} = 46 \begin{bmatrix} 11 & 70 \\ 70 & 11 \end{bmatrix} \bmod 77,$$

$$X^{-1} = \begin{bmatrix} 44 & 63 \\ 63 & 44 \end{bmatrix} \bmod 77.$$

$$Z = X^{-1}Y = \begin{bmatrix} 44 & 63 \\ 63 & 44 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix} \bmod 77,$$

$$Z = \begin{bmatrix} 15 & 54 \\ 54 & 15 \end{bmatrix} \bmod 77.$$

5. Public Key:

$$\text{Public Key} = (N, Z) = (77, \begin{bmatrix} 15 & 54 \\ 54 & 15 \end{bmatrix}).$$

6. Private Key:

$$\text{Private Key} = (X, Y) = \left( \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix}, \begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}. \right)$$

**Signature Generation**

Private key is used for generating a digital signature for a message $M$.

1. Choose two random matrices $R$ and $Q$.$R$ is circulant matrix and $Q$ is matrix of order 2 along with two random values $\eta$ and $\zeta$.

$$R = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}, \quad \eta = 3, \quad \zeta = 4.$$

2. Session private key:

$$(\eta, \zeta, R, Q) = (3, 4, \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix}).$$

3. Compute automorphisms matrices:

$$\phi_Z(Q) = Z^{-1} Q Z.$$

4. Compute $Z^{-1}$

$$Z^{-1} = 4^{-1} \begin{bmatrix} 15 & -54 \\ -54 & 15 \end{bmatrix} \bmod 77.$$

| $Q$ | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|---|---|
| | 1 | 0 | 77 | 0 | 1 | 4 |
| 19 | 0 | 1 | 4 | 1 | -19 | 1 |

TABLE 3.2: Extended Euclidean Algorithm $4^{-1} \mod 77$

$4^{-1}$ is $-19$, for positive value add 77.

$$4^{-1} = 58 \bmod 77.$$
$$Z^{-1} = 58 \begin{bmatrix} 15 & 23 \\ 23 & 15 \end{bmatrix} \bmod 77,$$
$$Z^{-1} = \begin{bmatrix} 23 & 25 \\ 25 & 23 \end{bmatrix} \bmod 77.$$

Now for automorphisms:

$$\phi_Z(Q) = \begin{bmatrix} 23 & 25 \\ 25 & 23 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} \begin{bmatrix} 15 & 54 \\ 54 & 15 \end{bmatrix},$$
$$\phi_Z(Q) = \begin{bmatrix} 45 & 21 \\ 17 & 62 \end{bmatrix} \bmod 77.$$
$$\phi_{YR}(Q) = (YR)^{-1} Q (YR).$$

$$YR = \begin{bmatrix} 38 & 42 \\ 42 & 38 \end{bmatrix} \bmod 77.$$
$$(YR)^{-1} = 65^{-1} \begin{bmatrix} 38 & 35 \\ 35 & 38 \end{bmatrix} \bmod 77.$$

| $Q$ | $A_1$ | $A_2$ | $A_3$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|---|---|
| | 1 | 0 | 77 | 0 | 1 | 65 |
| 1 | 0 | 1 | 65 | 1 | -1 | 12 |
| 5 | 1 | -1 | 12 | -5 | 6 | 5 |
| 2 | -5 | 6 | 5 | 11 | -13 | 2 |
| 2 | 11 | -13 | 2 | -27 | 32 | 1 |

TABLE 3.3: Extended Euclidean Algorithm $65^{-1} \bmod 77$

$$65^{-1} = 32 \bmod 77.$$

$$(YR)^{-1} = 32 \begin{bmatrix} 38 & 35 \\ 35 & 38 \end{bmatrix},$$

$$(YR)^{-1} = \begin{bmatrix} 61 & 42 \\ 42 & 61 \end{bmatrix}.$$

$$\phi_{YR}(Q) = \begin{bmatrix} 61 & 42 \\ 42 & 61 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} \begin{bmatrix} 38 & 42 \\ 42 & 38 \end{bmatrix} \bmod 77,$$

$$\phi_{YR}(Q) = \begin{bmatrix} 44 & 17 \\ 4 & 54 \end{bmatrix} \bmod 77$$

For computing this $\phi_{XR}(Q) = (XR)^{-1}Q(XR)$,

First we compute $(XR)^{-1}$,

$$(XR) = \begin{bmatrix} 76 & 68 \\ 68 & 76 \end{bmatrix} \bmod 77.$$

$$(XR)^{-1} = 74^{-1} \begin{bmatrix} 76 & 9 \\ 9 & 76 \end{bmatrix} \bmod 77.$$

Use extended euclidean algorithm for calculating $74^{-1}$.

| $Q$ | $\mathbf{A}_1$ | $\mathbf{A}_2$ | $\mathbf{A}_3$ | $\mathbf{B}_1$ | $\mathbf{B}_2$ | $\mathbf{B}_3$ |
|---|---|---|---|---|---|---|
| | 1 | 0 | 77 | 0 | 1 | 74 |
| 1 | 0 | 1 | 74 | 1 | -1 | 3 |
| 24 | 1 | -1 | 3 | -24 | 25 | 2 |
| 1 | -24 | 25 | 2 | 25 | -26 | 1 |

TABLE 3.4: Extended Euclidean Algorithm $74^{-1} \bmod 77$

$74^{-1}$ is $-26$, add 77 to get a positive value.

$$74^{-1} = -26 + 77 = 51.$$

$$(XR)^{-1} = 51 \begin{bmatrix} 76 & 9 \\ 9 & 76 \end{bmatrix} \bmod 77,$$

$$(XR)^{-1} = \begin{bmatrix} 26 & 74 \\ 74 & 27 \end{bmatrix} \bmod 77.$$

$$\phi_{XR}(Q) = \begin{bmatrix} 26 & 74 \\ 74 & 26 \end{bmatrix} \begin{bmatrix} 9 & 10 \\ 11 & 12 \end{bmatrix} \begin{bmatrix} 76 & 68 \\ 68 & 74 \end{bmatrix} \bmod 77,$$

$$\phi_{XR}(Q) = \begin{bmatrix} 16 & 46 \\ 52 & 5 \end{bmatrix} \bmod 77.$$

5. Computing Signature:

$$\alpha = \zeta\phi_{XR}(Q),$$

$$\alpha = 4 \begin{bmatrix} 16 & 46 \\ 52 & 5 \end{bmatrix} \bmod 77,$$

$$\alpha = \begin{bmatrix} 64 & 30 \\ 54 & 20 \end{bmatrix} \bmod 77.$$

Suppose

$$\Sigma_1 = \phi_{YR}(Q),$$

Compute

$$\eta\Sigma_1 = \eta \begin{bmatrix} 44 & 17 \\ 4 & 54 \end{bmatrix} \bmod 77,$$

$$\eta\Sigma_1 = 3\begin{bmatrix} 44 & 17 \\ 4 & 54 \end{bmatrix} \bmod 77,$$
$$\eta\Sigma_1 = \begin{bmatrix} 55 & 51 \\ 12 & 8 \end{bmatrix} \bmod 77.$$

Let,

$$\gamma = \eta + \zeta = 3 + 4 = 7$$
$$\gamma\Sigma_1 = 7\begin{bmatrix} 44 & 17 \\ 4 & 54 \end{bmatrix} \bmod 77$$
$$\gamma\Sigma_1 = \begin{bmatrix} 0 & 42 \\ 28 & 70 \end{bmatrix} \bmod 77$$

For $\beta$:

$$\beta = \mathcal{H}\left((M)_2 \| (\gamma\Sigma_1)_2\right),$$

Convert $M$ and $\gamma\Sigma_1$ into binary.

$$0\|42\|28\|70 \rightarrow 00000000\|00101010\|00011100\|01000110$$

**Signature Verification**

Digital signature $(\alpha, \beta)$ for the message $M$ verified by using following steps.

1. The master public key

$$N = 77 \quad Z = \begin{bmatrix} 15 & 54 \\ 54 & 15 \end{bmatrix}.$$

and session public key is

$$\eta\Sigma_1 = \begin{bmatrix} 36 & 11 \\ 61 & 35 \end{bmatrix} \bmod 77.$$

2. Compute $\lambda$:

$$\begin{aligned}
\lambda &= \eta\Sigma_1 + \phi_z\alpha, \\
\lambda &= \eta\Sigma_1 + Z^{-1}\alpha Z, \\
\lambda &= \begin{bmatrix} 55 & 51 \\ 12 & 8 \end{bmatrix} + \begin{bmatrix} 23 & 25 \\ 25 & 23 \end{bmatrix}^{-1} \begin{bmatrix} 64 & 30 \\ 55 & 20 \end{bmatrix} \begin{bmatrix} 15 & 54 \\ 54 & 15 \end{bmatrix}, \\
\lambda &= \begin{bmatrix} 0 & 42 \\ 28 & 70 \end{bmatrix} \bmod 77. \\
\beta' &= H\left((M)_2 \parallel (\gamma\Sigma_1)_2\right). \\
\beta' &= H\left((M)_2 \parallel (\lambda)_2\right).
\end{aligned}$$

3. $\beta' = \beta$, confirms that digital signatures are valid.

## 3.4 Digital Signature Scheme Based on Matrix Power Function

The work of S. K. Rososhek [18] is extended and presented in MPhil thesis [19] by introducing digital signature scheme based on matrix power function.

The scheme utilizes base matrix over $\mathbb{Z}_N$ and power matrix with integers modulo $\varphi(N)$. This explores the applications of matrix power functions in digital signatures.

### 3.4.1 Key Generation

Choose two large prime numbers $p_1$ and $p_2$ where $p_1 \neq p_2$ such that their product is $N$.

$$N = p_1 p_2.$$

Compute Euler's Totient function:

$$\varphi(N) = \varphi(p_1 p_2). \tag{3.2}$$

Choose two random matrices $X$ and $Y$ from general linear group $GL(n, \mathbb{Z}_N)$.

Compute public key matrix $Z$ by using $X$ and $Y$ as:

$$Z = XY \qquad \text{mod } \varphi(N). \tag{3.3}$$

Publish the master public key:

$$K_{pub} = \{N, Z\}.$$

The secret key $K_{pr}$ is:

$$K_{pr} = \{X, Y\}.$$

### 3.4.2 Signature Generation

For generating signature for a message $M$ we will pass though following steps:
Choose a random matrix circulant matrix $R$ from $GL(m, \mathbb{Z}_N)$
Choose another random matrix $Q$ from the general group $GL(n, \mathbb{Z}_N)$.
Choose a random element $\eta$ from $Z_{91}$.
The session private key is

$$K_{ses} = \{\eta, R, Q\}.$$

Now calculate $\phi_Z, \phi_{X^2YR}$ and $\phi_{XR}$ as follows:

$$\phi_Z(Q) = Q^Z \qquad \text{mod } N.$$
$$\phi_{XR^{YR}}(Q) = Q^{X^2RY} \qquad \text{mod } N.$$
$$\phi_{XR} = Q^{XR} \qquad \text{mod } N.$$

From the above calculation compute the signature $\alpha$ and $\beta$ with the help of following steps:

$$\alpha = Q^{XR} \mod N. \tag{3.4}$$

$$\Sigma = Q^{X^2RY} \mod N, \tag{3.5}$$

$$\eta\Sigma = \eta Q^{X^2RY} \mod N, \tag{3.6}$$

$$\mu = \eta + 1, \tag{3.7}$$

$$\beta = H\left((M)_2 \| (\mu)_2\right). \tag{3.8}$$

$(M)_2$ and $(\mu)_2$ are in binary representation.

$\eta\Sigma$ is the session public key for the verification of signatures $(\alpha, \beta)$ of the message $M$.

### 3.4.3 Signature Verification

For verification the receiving person will perform following steps:

The receiving person will use master public key and session public key for verification. For this purpose compute:

$$\lambda = \eta\Sigma + \alpha^Z,$$
$$\lambda = \eta\Sigma + \alpha^{XY},$$
$$\lambda = \eta\Sigma + Q^{XRXY},$$
$$\lambda = \eta\Sigma + Q^{X^2RY}.$$

Compute $\beta'$:

$$\beta' = H((M)_2 \| (\lambda)_2). \tag{3.9}$$

Since,

$$\beta = \beta'.$$

So, signature of message $M$ are valid.

### 3.4.4 Correctness Algorithm

The correctness of the signatures is verified by following the steps outlined below:

From Eq. (3.4)

$$\lambda = \eta\Sigma + (\alpha^{Z}),$$

From Eq. (3.3)

$$\lambda = \eta\Sigma + (\alpha^{XY}),$$

$$\lambda = \eta\Sigma + Q^{XRXY},$$

From Eq. (3.5)

$$\lambda = \eta\Sigma + \Sigma,$$

Taking $\Sigma$ common

$$\lambda = (\eta + 1)\Sigma,$$

Now, from equation (3.6)

$$\lambda = \mu\Sigma.$$

From equation (3.9):

$$\beta' = H((M)_2 \| (\mu\Sigma)_2),$$

$$\Rightarrow \beta' = \beta.$$

**Example 3.4.1.** $\mathrm{GL}(3, \mathbb{Z}_N)$ will be used to solve an example to illustrate the previously defined method of the modular matrix-based algorithm. For the calculation of modulo $N$, $p_1 = 13$ and $p_2 = 7$ are chosen. All steps of the power matrix are carried out modulo $\varphi(N)$, while the base matrix calculations are done modulo $N$.

**Key Generation**

Two prime numbers $p_1 = 13$ and $p_2 = 7$ are given in the statement.

Computed $N$:

$$N = p_1 p_2 = 13(7) = 91.$$

Compute Euler's totient function:

$$\varphi(N) = \varphi(91)$$

$$\varphi(N) = \varphi(13.7) = (13-1)(7_1) = 72.$$

Two circulant matrices $X$ and $Y$ are chosen from $\mathrm{GL}(3, \mathbb{Z}_{91})$:

$$X = \begin{pmatrix} 15 & 14 & 3 \\ 3 & 15 & 14 \\ 14 & 3 & 15 \end{pmatrix}, \quad Y = \begin{pmatrix} 11 & 24 & 20 \\ 20 & 11 & 24 \\ 24 & 20 & 11 \end{pmatrix}.$$

Their product is computed as follows:

$$\begin{aligned} &Z = XY, \\ &Z = \begin{pmatrix} 15 & 14 & 3 \\ 3 & 15 & 14 \\ 14 & 3 & 15 \end{pmatrix} \begin{pmatrix} 11 & 24 & 20 \\ 20 & 11 & 24 \\ 24 & 20 & 11 \end{pmatrix} \mod 72, \\ &Z = \begin{pmatrix} 517 & 574 & 669 \\ 669 & 517 & 574 \\ 574 & 669 & 517 \end{pmatrix} \mod 72, \\ &Z = \begin{pmatrix} 13 & 70 & 21 \\ 21 & 13 & 70 \\ 70 & 21 & 13 \end{pmatrix} \mod 72. \end{aligned}$$

The public key consists of $n = 91$ and the matrix:

$$Z = \begin{pmatrix} 13 & 70 & 21 \\ 21 & 13 & 70 \\ 70 & 21 & 13 \end{pmatrix}.$$

The private key includes the chosen matrices:

$$X = \begin{pmatrix} 15 & 14 & 3 \\ 3 & 15 & 14 \\ 14 & 3 & 15 \end{pmatrix}, \quad Y = \begin{pmatrix} 11 & 24 & 20 \\ 20 & 11 & 24 \\ 24 & 20 & 11 \end{pmatrix}.$$

**Signature Generation**

Performs following steps to generate the signature:

1. Choose the right circulant matrix $R \in \mathrm{GL}(3, \mathbb{Z}_{91})$ and a random matrix $Q \in \mathrm{GL}(3, \mathbb{Z}_{91})$.

$$R = \begin{pmatrix} 31 & 53 & 7 \\ 7 & 31 & 53 \\ 53 & 7 & 31 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Z}_{91}),$$

$$Q = \begin{pmatrix} 25 & 11 & 20 \\ 30 & 32 & 44 \\ 17 & 21 & 15 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Z}_{91}).$$

2. Let the random element $\eta$ be:

$$\eta = 47 \in \mathbb{Z}_{91}.$$

3. Session private key:

$$\eta = 47, \quad R = \begin{pmatrix} 31 & 53 & 7 \\ 7 & 31 & 53 \\ 53 & 7 & 31 \end{pmatrix}, \quad Q = \begin{pmatrix} 25 & 11 & 20 \\ 30 & 32 & 44 \\ 17 & 21 & 15 \end{pmatrix}.$$

4. Compute the product $XR \bmod 72$:

$$X = \begin{pmatrix} 15 & 14 & 3 \\ 3 & 15 & 14 \\ 14 & 3 & 15 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} 31 & 53 & 7 \\ 7 & 31 & 53 \\ 53 & 7 & 31 \end{pmatrix},$$

$$XR = \begin{pmatrix} 15 & 14 & 3 \\ 3 & 15 & 14 \\ 14 & 3 & 15 \end{pmatrix} \begin{pmatrix} 31 & 53 & 7 \\ 7 & 31 & 53 \\ 53 & 7 & 31 \end{pmatrix} \bmod 72,$$

$$XR = \begin{pmatrix} 2 & 26 & 4 \\ 4 & 2 & 26 \\ 26 & 4 & 2 \end{pmatrix} \bmod 72.$$

5. Compute $\phi_{XR}(Q)$:

$$\phi_{XR}(Q) = Q^{XR}.$$

$$\phi_{XR}(Q) = \begin{pmatrix} 25 & 11 & 20 \\ 30 & 32 & 44 \\ 17 & 21 & 15 \end{pmatrix}^{\begin{pmatrix} 2 & 26 & 4 \\ 4 & 2 & 26 \\ 26 & 4 & 2 \end{pmatrix}} \bmod 91,$$

$$\phi_{XR}(Q) = \begin{pmatrix} (25)^2(11)^4(20)^{26} & (25)^2 6(11)^2(20)^4 & ((25)^4(11)^{26}(20)^2 \\ (30)^2(32)^4(44)^{26} & (30)^{26}(32)^2(44)^4 & (30)^4(32)^{26}(44)^2 \\ (17)^2(21)^4(15)^{26} & (17)^{26}(21)^2(15)^4 & (17)^4(21)^{26}(15)^2 \end{pmatrix} \bmod 91,$$

After computing by using $\bmod 91$:

$$Q^{XR} = \begin{pmatrix} 53 & 14 & 74 \\ 74 & 35 & 81 \\ 1 & 42 & 79 \end{pmatrix} \bmod 91.$$

Now compute $X^2YR \mod \varphi(N)$:

$$X^2YR = \begin{pmatrix} 15 & 14 & 3 \\ 3 & 15 & 14 \\ 14 & 3 & 15 \end{pmatrix}^2 \begin{pmatrix} 11 & 24 & 20 \\ 20 & 11 & 24 \\ 24 & 20 & 11 \end{pmatrix} \begin{pmatrix} 31 & 53 & 7 \\ 7 & 31 & 53 \\ 53 & 7 & 31 \end{pmatrix} \bmod 72,$$

$$X^2YR = \begin{pmatrix} 60 & 58 & 42 \\ 42 & 60 & 58 \\ 58 & 42 & 60 \end{pmatrix} \mod 72$$

6. Now compute $\phi_{X^2YR}(Q) \mod \varphi(N)$:

$$\phi_{X^2YR}(Q) = Q^{X^2YR},$$

$$\phi_{X^2YR}(Q) = \begin{pmatrix} 25 & 11 & 20 \\ 30 & 32 & 44 \\ 17 & 21 & 15 \end{pmatrix} \begin{pmatrix} 60 & 58 & 42 \\ 42 & 60 & 58 \\ 58 & 42 & 60 \end{pmatrix},$$

$$\phi_{X^2YR}(Q) = \begin{pmatrix} (25)^{60}(11)^{42}(20)^{58} & (25)^{58}(11)^{60}(20)^{42} & ((25)^{42}(11)^{58}(20)^{60} \\ (30)^{60}(32)^{42}(44)^{58} & (30)^{58}(32)^{60}(44)^{42} & (30)^{42}(32)^{58}(44)^{60} \\ (17)^{60}(21)^{42}(15)^{58} & (17)^{58}(21)^{60}(15)^{42} & (17)^{42}(21)^{58}(15)^{60} \end{pmatrix} \mod 91.$$

$$\phi_{X^2YR}(Q) = \begin{pmatrix} 9 & 35 & 79 \\ 74 & 14 & 29 \\ 53 & 21 & 15 \end{pmatrix}.$$

7. Now compute signatures $(\alpha, \beta)$:

$$\alpha = \phi_{XR}(Q),$$

$$\alpha = Q^{XR},$$

$$\alpha = (Q)^{XR} = \begin{pmatrix} 53 & 14 & 74 \\ 74 & 35 & 81 \\ 1 & 42 & 79 \end{pmatrix} \mod 91.$$

$$\Sigma = \phi_{X^2YR}(Q),$$

$$\Sigma = Q^{X^2YR},$$

$$\Sigma = \begin{pmatrix} 9 & 35 & 79 \\ 74 & 14 & 29 \\ 53 & 21 & 15 \end{pmatrix} \mod 91.$$

Since,

$$\eta = 47.$$

Hence,

$$\eta\Sigma = 47 \begin{pmatrix} 9 & 35 & 79 \\ 74 & 14 & 29 \\ 53 & 21 & 15 \end{pmatrix} \mod 91,$$

$$\eta\Sigma = \begin{pmatrix} 59 & 7 & 79 \\ 20 & 21 & 89 \\ 34 & 63 & 33 \end{pmatrix} \mod 91.$$

Let,

$$\mu = \eta + 1,$$

$$\mu = 47 + 1.$$

Compute $\mu\Sigma$:

$$\mu\Sigma = (48) \begin{pmatrix} 9 & 35 & 79 \\ 74 & 14 & 29 \\ 53 & 21 & 15 \end{pmatrix} \mod 91,$$

$$\mu\Sigma = \begin{pmatrix} 68 & 42 & 61 \\ 3 & 35 & 27 \\ 87 & 14 & 55 \end{pmatrix} \mod 91.$$

**Verification**

$$\lambda = \eta\Sigma + \alpha^{Z},$$

$$\lambda = \eta\Sigma + \alpha^{XY},$$

$$\lambda = \begin{pmatrix} 59 & 7 & 79 \\ 20 & 21 & 89 \\ 34 & 63 & 33 \end{pmatrix} + \begin{pmatrix} 9 & 35 & 79 \\ 74 & 14 & 29 \\ 53 & 21 & 15 \end{pmatrix} \mod 91,$$

$$\lambda = \begin{pmatrix} 68 & 42 & 61 \\ 3 & 35 & 27 \\ 87 & 14 & 55 \end{pmatrix} \mod 91,$$

$$\lambda = \mu\Sigma.$$

As

$$\lambda = \mu\Sigma.$$

$\Rightarrow$ This verify the signatures.

# Chapter 4

# Digital Signature Based on MPF over Galois Field

A digital signature scheme based on finite fields like galois filed $GF(p)$ is constructed and extended the digital siganture based on $MPF$ on $\mathrm{GL}(n, \mathbb{Z}_N)$ proposed by Sundas Iqbal [19] inspired by S. K. Rososhak's work [18]. The key generation, signature generation, and verification algorithms are outlined in the next section. The scheme is implemented using ApCoCoA [54], with examples provided for illustration.

## 4.1 DS Scheme Based on MPF over $GF(p)$

The idea is extended to a digital signature scheme based on matrix power function (MPF) [55] in $GF(p^q)$. A public platform group defined over the galois field $GF(p^q)$ is chosen. The general linear group over $GF(p^q)$ is represented as $GL(n, GF(p^q))$. When working in this platform, an irreducible polynomial is required for mathematical operations. Let $\mathbf{m}(\mathbf{x})$ be the chosen irreducible polynomial, and all calculations on the base matrix are performed modulo $\mathbf{m}(\mathbf{x})$. The powering matrices are chosen from the general linear group $GL(n, \mathbb{Z}_N)$.

The complete working scheme (algorithm) for constructing private and public keys, session keys, signature generation, and signature verification is explained. This scheme

operates in the general linear group $GL(n, GF(p^q))$ and utilizes power matrices [56] from a subgroup of $GL(n, \mathbb{Z}_N)$ consisting of circulant matrices [43].

### 4.1.1 Key Generation

For key generation perfomed following steps [17].

Select two large prime number $p_1$ and $p_2$ where $p_1 \neq p_2$ and compute $N$ as

$$N = p_1 p_2.$$

Compute Euler's totient function:

$$\varphi(N) = \varphi(p_1 p_2) = (p_1 - 1)(p_2 - 1). \tag{4.1}$$

Choose two random circulant matrices $X$ and $Y$ from $GL(n, \mathbb{Z}_N)$.
Now compute public key matrix $Z$ by using matrices $X$ and $Y$.

$$Z = XY \qquad \text{mod } \varphi(N). \tag{4.2}$$

Publish the master public key:

$$K_{pub} = \{N, Z\}.$$

The secret key $K_{pr}$ is:

$$K_{pr} = \{X, Y\}.$$

### 4.1.2 Signature Generation

For generating signature for a message $M$ perform the following steps:
Choose a random circulant matrix $R$ from $GL(n, \mathbb{Z}_N)$
Choose another random matrix $Q$ from the general group $GL(n, GF(p^q))$.

Choose a random polynomial $\eta$.

The session private key is

$$K_{ses} = \{\eta, R, Q\}.$$

Now compute $\phi_Z, \phi_{(XR)Z}$ and $\phi_{XR}$ as follows:

$$\begin{aligned} \phi_Z(Q) &= Q^Z \mod m(x). \\ \phi_{(Z)^{XR}}(Q) &= Q^{Z^{XR}} \mod m(x). \\ \phi_{XR} &= Q^{XR} \mod m(x). \end{aligned}$$

From the above calculation compute the signature $\alpha$ and $\beta$ with the help of following steps:

$$\alpha = Q^{XR} \mod m(x). \tag{4.3}$$

$$\Sigma = (Q^Z)^{XR} \mod m(x). \tag{4.4}$$

$$\eta\Sigma = \eta(Q^Z)^{XR} \mod m(x). \tag{4.5}$$

$$\mu = \eta + 1. \tag{4.6}$$

$$\beta = H((M)_2 \| (\mu\Sigma)_2). \tag{4.7}$$

$(M)_2$ and $(\mu)_2$ are in binary representation (base 2). $\eta\Sigma$ is the session public key for the verification of signatures $(\alpha, \beta)$ of the message $M$.

### 4.1.3 Verification

For verification of signature the receiving person will perform following steps: The receiving person will use master public key and session public key for verification. For this purpose compute $\lambda$:

$$\lambda = \eta\Sigma + \alpha^Z,$$

$$\begin{aligned} \lambda &= \eta\Sigma + \alpha^{XY}, \\ \lambda &= \eta\Sigma + (Q^{XR})^{XY}. \end{aligned}$$

Compute $\beta'$:

$$\beta' = H((M)_2 \| (\lambda)_2). \tag{4.8}$$

If $\beta = \beta'$ then signature of message $M$ will valid.

From Eq. (4.3)

$$\lambda = \eta\Sigma + (\alpha)^Z,$$

From Eq. (4.2)

$$\lambda = \eta\Sigma + (Q^{XR})^Z.$$

As,

$$(Q^{XR})^Z = (Q^Z)^{XR}.$$

From Eq. (4.4)

$$\lambda = \eta\Sigma + \Sigma,$$

Taking $\Sigma$ common,

$$\lambda = (\eta + 1)\Sigma,$$

Now, from equation (4.6)

$$\lambda = \mu\Sigma.$$

Equation can be written as:

$$\beta = H((M)_2 \| (\mu\Sigma)_2),$$

$$\beta' = \beta.$$

## 4.2 Illustrated Examples

To illustrate digital signature on MPF over galois field let solve computational examples based on general linear group of $GF(p^q)$ with power matrices from the subgroup of general linear group which contain circulant matrices.

**Example 4.2.1.** A public platform is defined over the extended galois field $GF(p^q)$. Let $m(x) = x^8 + x^4 + x^3 + x + 1$ be the chosen irreducible polynomial. The order of the matrix is 2. The general linear group is defined as $GL(2, \mathbb{Z}_{91})$, with $p_1 = 7$

and $p_2 = 13$. All calculations for power matrices are performed modulo $\varphi(N)$, while calculations for base matrices are performed modulo $m(x)$.

## Key Generation

1. Two prime numbers $p_1 = 13$ and $p_2 = 7$ are given in the statement.

2. Compute $N$:

$$N = p_1 p_2,$$
$$N = 7(13) = 91.$$

3. Compute Euler's totient function:

$$\varphi(N) = \varphi(p_1 p_2),$$
$$\varphi(N) = \varphi(7.13) = (7-1)(13-1) = 72.$$

4. Two circulant matrices $X$ and $Y$ are chosen from $\mathrm{GL}(2, \mathbb{Z}_{91})$:

$$X, Y \in GL(2, \mathbb{Z}_{91}).$$
$$X = \begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix} \quad Y = \begin{bmatrix} 42 & 19 \\ 19 & 42 \end{bmatrix} \in GL(2, \mathbb{Z}_{91}).$$

5. Compute $Z$ from $X$ and $Y$:

$$Z = XY \bmod \varphi(N).$$
$$Z = \begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix} \begin{bmatrix} 42 & 19 \\ 42 & 19 \end{bmatrix} \quad \bmod 72.$$

$$Z = \begin{bmatrix} 34 & 63 \\ 63 & 34 \end{bmatrix} \quad \bmod 72.$$

6. Master public key is:

$$N = 91, Z = \begin{bmatrix} 34 & 63 \\ 63 & 34 \end{bmatrix}.$$

7. Master secret key is:

$$X = \begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix} \quad Y = \begin{bmatrix} 42 & 19 \\ 19 & 42 \end{bmatrix}.$$

## Signature Generation

For generating signature perform following comprehensive steps:

1. Choose a circulant matrix:

$$R = \begin{bmatrix} 13 & 25 \\ 25 & 13 \end{bmatrix} \in GL(2, \mathbb{Z}_{91}).$$

2. Choose a base matrix:

$$Q = \begin{bmatrix} x^3 + x^2 + 1 & x^5 + x^2 + 1 \\ x^7 + x^5 + x^3 + 1 & x^4 + x^2 + 1 \end{bmatrix} \bmod m(x) \in GF(2, 2^8).$$

3. Let the random polynomial $\eta$ be

$$\eta = x^2.$$

4. Session private key is:

$$\eta = x^2, \quad R = \begin{bmatrix} 13 & 25 \\ 25 & 13 \end{bmatrix}.$$

$$Q = \begin{bmatrix} x^3 + x^2 + 1 & x^5 + x^2 + 1 \\ x^7 + x^5 + x^3 + 1 & x^4 + x^2 + 1 \end{bmatrix} \bmod m(x) \in GF(2, 2^8).$$

5. For computing $\phi_Z(Q)$:

$$\phi_Z(Q) = Q^Z \bmod m(x),$$

$$\phi_Z(Q) = \begin{bmatrix} x^3+x^2+1 & x^5+x^2+1 \\ x^7+x^5+x^3+1 & x^4+x^2+1 \end{bmatrix} \begin{bmatrix} 34 & 63 \\ 63 & 34 \end{bmatrix} \bmod m(x),$$

$$\phi_Z(Q) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \bmod m(x),$$

$$\begin{aligned} Q_{11} &= (x^3+x^2+1)^{34}(x^5+x^2+1)^{63} \bmod m(x), \\ Q_{12} &= (x^3+x^2+1)^{63}(x^5+x^2+1)^{34} \bmod m(x), \\ Q_{21} &= (x^7+x^5+x^3+1)^{34}(x^4+x^2+1)^{63} \bmod m(x), \\ Q_{22} &= (x^7+x^5+x^3+1)^{63}(x^4+x^2+1)^{34} \bmod m(x), \end{aligned}$$

After solving it on ApCoCoA [20], we get

$$\phi_Z(Q) = \begin{bmatrix} x^7+x^6+x^5+x^4+x^3+x^2+x & x^7+x^6+x^5+x^3+x^2 \\ x^7+x^5+x^2+1 & x^7+x^5+x^4+x+1 \end{bmatrix} \bmod m(x).$$

6. Compute $XR$:

$$XR = \begin{bmatrix} 21 & 16 \\ 16 & 21 \end{bmatrix} \begin{bmatrix} 13 & 25 \\ 25 & 13 \end{bmatrix} \bmod 72,$$

$$XR = \begin{bmatrix} 25 & 13 \\ 13 & 25 \end{bmatrix} \bmod 72.$$

7. Compute $\phi_{XR}(Q)$:

$$\phi_{XR}(Q) = Q^{XR},$$

$$\phi_{XR}(Q) = \begin{bmatrix} x^3+x^2+1 & x^5+x^2+1 \\ x^7+x^5+x^3+1 & x^4+x^2+1 \end{bmatrix} \begin{bmatrix} 25 & 13 \\ 13 & 25 \end{bmatrix} \bmod m(x),$$

$$\phi_{XR}(Q) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \bmod m(x),$$

$$\begin{aligned} Q_{11} &= (x^3+x^2+1)^{25}(x^5+x^2+1)^{13} \bmod m(x), \\ Q_{12} &= (x^3+x^2+1)^{13}(x^5+x^2+1)^{25} \bmod m(x), \\ Q_{21} &= (x^7+x^5+x^3+1)^{25}(x^4+x^2+1)^{13} \bmod m(x), \\ Q_{22} &= (x^7+x^5+x^3+1)^{13}(x^4+x^2+1)^{25} \bmod m(x), \end{aligned}$$

After sloving it on ApCoCoA [20], we get:

$$\phi_{XR}(Q) = \begin{bmatrix} x^6+x^5+x+1 & x^5+x^4+x^3+x+1 \\ x^4+x^3+x^2+x+1 & x^6+x^5+x^3+x^2+x \end{bmatrix} \bmod m(x).$$

8. Compute $\phi_{Z^{XR}}(Q)$:

$$\phi_{Z^{XR}}(Q) = (Q^Z)^{XR},$$

$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} x^7+x^6+x^5+x^4+x^3+x^2+x & x^7+x^6+x^5+x^3+x^2 \\ x^7+x^5+x^2+1 & x^7+x^5+x^4+x+1 \end{bmatrix} \begin{bmatrix} 25 & 13 \\ 13 & 25 \end{bmatrix} \bmod m(x),$$

$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \bmod m(x),$$

$$\begin{aligned} Q_{11} &= (x^7+x^6+x^5+x^4+x^3+x^2+x)^{25}(x^7+x^6+x^5+x^3+x^2)^{13} \bmod m(x), \\ Q_{12} &= x^7+x^6+x^5+x^4+x^3+x^2+x)^{13}(x^7+x^6+x^5+x^3+x^2)^{25} \bmod m(x), \\ Q_{21} &= (x^7+x^5+x^2+1)^{13}(x^7+x^5+x^4+x+1)^{25} \bmod m(x), \\ Q_{22} &= (x^7+x^5+x^2+1)^{25}(x^7+x^5+x^4+x+1)^{13} \bmod m(x), \end{aligned}$$

After solving it on ApCoCoA [20], we get:

$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} x^7+x^5+1 & x^7+x^5+x^4+x^2+1 \\ x^7+x^6+x^3+x+1 & x^6+x^2+1 \end{bmatrix} \bmod m(x).$$

9. Compute signatures $(\alpha, \beta)$:

$$\alpha = \phi_{XR}(Q),$$
$$\alpha = Q^{XR},$$
$$\alpha = \begin{bmatrix} x^6+x^5+x+1 & x^5+x^4+x^3+x+1 \\ x^4+x^3+x^2+x+1 & x^6+x^5+x^3+x^2+x \end{bmatrix} \bmod m(x).$$

and,

$$\Sigma = \phi_{Z^{XR}}(Q),$$
$$\Sigma = (Q^Z)^{XR},$$
$$\Sigma = \begin{bmatrix} x^7+x^5+1 & x^7+x^5+x^4+x^2+1 \\ x^7+x^6+x^3+x+1 & x^6+x^2+1 \end{bmatrix} \bmod m(x).$$

Since,

$$\eta = x^2.$$

Hence,

$$\eta\Sigma = x^2 \begin{bmatrix} x^7+x^5+1 & x^7+x^5+x^4+x^2+1 \\ x^7+x^6+x^3+x+1 & x^6+x^2+1 \end{bmatrix} \bmod m(x),$$
$$\eta\Sigma = \begin{bmatrix} x^9+x^7+x^2 & x^9+x^7+x^6+x^4+x^2 \\ x^9+x^8+x^5+x^3+x^2 & x^8+x^4+x^2 \end{bmatrix} \bmod m(x),$$
$$\eta\Sigma = \begin{bmatrix} x^7+x^5+x^4+x & x^7+x^6+x^5+x \\ 1 & x^3+x^2+x+1 \end{bmatrix} \bmod m(x).$$

As, $\mu = \eta + 1,$

$$\mu = x^2 + 1.$$

Compute $\mu\Sigma$:

$$\mu\Sigma = (x^2+1)\begin{bmatrix} x^7+x^5+1 & x^7+x^5+x^4+x^2+1 \\ x^7+x^6+x^3+x+1 & x^6+x^2+1 \end{bmatrix} \bmod m(x),$$

$$\mu\Sigma = \begin{bmatrix} x^9+x^5+x^2+1 & x^9+x^6+x^5+1 \\ x^9+x^8+x^7+x^6+x^5+x^2+x+1 & x^8+x^6+x^4+1 \end{bmatrix} \bmod m(x),$$

$$\mu\Sigma = \begin{bmatrix} x^4+x+1 & x^6+x^4+x^2+x+1 \\ x^7+x^6+x^3+x & x^6+x^3+x \end{bmatrix} \bmod m(x),$$

$$\beta = H((M)_2 \| (\mu.\Sigma)_2).$$

## Verification

$$\lambda = \eta\Sigma + (Q^{XR})^Z,$$

$$(Q^{XR})^Z = \begin{bmatrix} x^7+x^5+1 & x^7+x^5+x^4+x^2+1 \\ x^7+x^6+x^3+x+1 & x^6+x^2+1 \end{bmatrix} \begin{bmatrix} 34 & 63 \\ 63 & 34 \end{bmatrix} \bmod m(x),$$

$$(Q^{XR})^Z = \begin{bmatrix} x^7+x^5+1 & x^7+x^5+x^4+x^2+1 \\ x^7+x^6+x^3+x+1 & x^6+x^2+1 \end{bmatrix} \bmod m(x),$$

$$\lambda = \begin{bmatrix} x^7+x^5+x^4+x & x^7+x^6+x^5+x \\ 1 & x^3+x^2+x+1 \end{bmatrix} + (Q^{XR})^Z \bmod m(x),$$

$$\lambda = \begin{bmatrix} x^4+x+1 & x^6+x^4+x^2+x+1 \\ x^7+x^6+x^3+x & x^6+x^3+x \end{bmatrix},$$

$$\lambda = \mu\Sigma,$$

$$\beta' = H((M)_2 \| (\lambda)_2).$$

Hence,

$$\beta = \beta'.$$

This verifies the signatures.

**Example 4.2.2.** A public platform is defined over the extended Galois Field $GF(p^q)$. Let $m(x) = x^8 + x^4 + x^3 + x + 1$ be the chosen irreducible polynomial. The order of the matrix is 3. The general linear group is defined as $GL(3, \mathbb{Z}_{91})$, with $p_1 = 7$ and $p_2 = 13$. All calculations for power matrices are performed modulo $\varphi(N)$, while calculations for base matrices are performed modulo $m(x)$.

## Key Generation

1. Two prime numbers $p_1 = 13$ and $p_2 = 7$ are given in the statement.

2. Compute $N$:

$$N = p_1 p_2 = 7(13) = 91.$$

3. Compute Euler's totient function:

$$\varphi(N) = \varphi(p_1 p_2),$$
$$\varphi(N) = \varphi(7.13) = (7-1)(13-1) = 72.$$

4. Two circulant matrices $X$ and $Y$ are chosen from $\mathrm{GL}(3, \mathbb{Z}_{91})$:

$$X = \begin{bmatrix} 11 & 16 & 13 \\ 13 & 11 & 16 \\ 16 & 13 & 11 \end{bmatrix} \quad Y = \begin{bmatrix} 5 & 7 & 13 \\ 13 & 5 & 7 \\ 7 & 13 & 5 \end{bmatrix} \in GL(3, \mathbb{Z}_{91}).$$

5. Compute $Z$ from $X$ and $Y$

$$Z = \begin{bmatrix} 11 & 16 & 13 \\ 13 & 11 & 16 \\ 16 & 13 & 11 \end{bmatrix} \begin{bmatrix} 5 & 7 & 13 \\ 13 & 5 & 7 \\ 7 & 13 & 5 \end{bmatrix} \mod 72,$$

$$Z = \begin{bmatrix} 354 & 326 & 320 \\ 320 & 354 & 326 \\ 326 & 320 & 354 \end{bmatrix} \mod 72,$$

$$Z = \begin{bmatrix} 66 & 38 & 32 \\ 32 & 66 & 38 \\ 38 & 32 & 66 \end{bmatrix} \mod 72.$$

6. Master public key is:

$$N = 91, Z = \begin{bmatrix} 66 & 38 & 32 \\ 32 & 66 & 38 \\ 38 & 32 & 66 \end{bmatrix}.$$

7. Master secret key is:

$$X = \begin{bmatrix} 11 & 16 & 13 \\ 13 & 11 & 16 \\ 16 & 13 & 11 \end{bmatrix} \quad Y = \begin{bmatrix} 5 & 7 & 13 \\ 13 & 5 & 7 \\ 7 & 13 & 5 \end{bmatrix}.$$

## Signature Generation

For Generating Signature we will perform following comprehensive steps:

(a) Choose a circulant matrix $R$:

$$R = \begin{bmatrix} 5 & 11 & 17 \\ 17 & 5 & 11 \\ 11 & 17 & 5 \end{bmatrix} \in GL(3, \mathbb{Z}_{91}).$$

(b) Choose a base matrix $Q$:

$$Q = \begin{bmatrix} x^2+x+1 & x^3+x^2 & x^2+1 \\ x^4+x^3+x & x^2+1 & x^3+x+1 \\ x^4+x^2+1 & x^3+1 & x^4+x+1 \end{bmatrix} \mod m(x) \in GF(2, 2^5).$$

(c) Let a random polynomial $\eta$ be,

$$\eta = x^2.$$

(d) Session private key is:

$$\eta = x^2, \quad R = \begin{bmatrix} 5 & 11 & 17 \\ 17 & 5 & 11 \\ 11 & 17 & 5 \end{bmatrix}$$

$$Q = \begin{bmatrix} x^2+x+1 & x^3+x^2 & x^2+1 \\ x^4+x^3+x & x^2+1 & x^3+x+1 \\ x^4+x^2+1 & x^3+1 & x^4+x+1 \end{bmatrix} \bmod m(x) \in GF(2,2^5).$$

(e) Compute $\phi_Z(Q)$:

$$\phi_Z(Q) = Q^Z \bmod x^5+x^4+x^3+x+1,$$

$$\phi_Z(Q) = \begin{bmatrix} x^2+x+1 & x^3+x^2 & x^2+1 \\ x^4+x^3+x & x^2+1 & x^3+x+1 \\ x^4+x^2+1 & x^3+1 & x^4+x+1 \end{bmatrix} \begin{bmatrix} 66 & 38 & 32 \\ 32 & 66 & 38 \\ 38 & 32 & 66 \end{bmatrix} \bmod m(x),$$

$$\phi_Z(Q) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \bmod m(x),$$

$$Q_{11} = (x^2+x+1)^{66}(x^3+x^2)^{32}(x^2+1)^{38} \bmod m(x),$$

$$Q_{12} = (x^4+x^3+x)^{38}(x^2+1)^{66}(x^3+x+1)^{32} \bmod m(x),$$

$$Q_{13} = (x^4+x^2+1)^{32}(x^3+1)^{38}(x^4+x+1)^{66} \bmod m(x),$$

$$Q_{21} = (x^4+x^3+x)^{66}(x^2+1)^{32}(x^3+x+1)^{38} \bmod m(x),$$

$$Q_{22} = (x^4+x^3+x)^{38}(x^2+1)^{66}(x^3+x+1)^{32} \bmod m(x),$$

$$Q_{23} = x^4+x^3+x)^{32}(x^2+1)^{38}(x^3+x+1)^{66} \bmod m(x),$$

$$Q_{31} = x^2+x+1)^{66}(x^3+x^2)^{32}(x^2+1)^{38} \bmod m(x),$$

$$Q_{32} = (x^4+x^3+x)^{38}(x^2+1)^{66}(x^3+x+1)^{32} \bmod m(x),$$

$$Q_{33} = (x^4+x^2+1)^{32}(x^3+1)^{38}(x^4+x+1)^{66} \bmod m(x),$$

Solving it on ApCoCoA [20], we get:

$$\phi_Z(Q) = \begin{bmatrix} x^4+x^2+1 & x & x^2 \\ x^4+x^2+x & x^3+x^2 & x^4+x^3+x^2+1 \\ x^3+x & x+1 & x^4+x^2+x \end{bmatrix} \bmod m(x).$$

(f) Compute $XR$:

$$XR = \begin{bmatrix} 11 & 16 & 13 \\ 13 & 11 & 16 \\ 16 & 13 & 11 \end{bmatrix} \begin{bmatrix} 5 & 11 & 17 \\ 17 & 5 & 11 \\ 11 & 17 & 5 \end{bmatrix} \bmod 72,$$

$$XR = \begin{bmatrix} 38 & 62 & 68 \\ 68 & 38 & 62 \\ 62 & 68 & 38 \end{bmatrix} \bmod 72.$$

(g) Compute $\phi_{XR}(Q)$:

$$\phi_{XR}(Q) = Q^{XR},$$

$$\phi_{XR}(Q) = \begin{bmatrix} x^2+x+1 & x^3+x^2 & x^2+1 \\ x^4+x^3+x & x^2+1 & x^3+x+1 \\ x^4+x^2+1 & x^3+1 & x^4+x+1 \end{bmatrix} \begin{bmatrix} 38 & 62 & 68 \\ 68 & 38 & 62 \\ 62 & 68 & 38 \end{bmatrix} \bmod m(x),$$

$$\phi_{XR}(Q) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \bmod m(x),$$

$$Q_{11} = (x^2+x+1)^{38}(x^3+x^2)^{68}(x^2+1)^{62} \bmod m(x),$$

$$Q_{12} = (x^4+x^3+x)^{62}(x^2+1)^{38}(x^3+x+1)^{68} \bmod m(x),$$

$$Q_{13} = (x^4+x^2+1)^{68}(x^3+1)^{62}(x^4+x+1)^{38} \bmod m(x),$$

$$Q_{21} = (x^4+x^3+x)^{38}(x^2+1)^{68}(x^3+x+1)^{62} \bmod m(x),$$

$$Q_{22} = (x^4+x^3+x)^{62}(x^2+1)^{38}(x^3+x+1)^{68} \bmod m(x),$$

$$Q_{23} = x^4+x^3+x)^{68}(x^2+1)^{62}(x^3+x+1)^{38} \bmod m(x),$$

$$Q_{31} = x^2 + x + 1)^{38}(x^3 + x^2)^{68}(x^2 + 1)^{62} \bmod m(x),$$
$$Q_{32} = (x^4 + x^3 + x)^{62}(x^2 + 1)^{38}(x^3 + x + 1)^{66} \bmod m(x),$$
$$Q_{33} = (x^4 + x^2 + 1)^{66}(x^3 + 1)^{62}(x^4 + x + 1)^{38} \bmod m(x),$$

$$\phi_{XR}(Q) = \begin{bmatrix} x^4 + x^2 + 1 & x & x^2 \\ x^4 + x^2 + x & x^3 + x^2 & x^4 + x^3 + x^2 + x \\ x^3 + x & x + 1 & x^4 + x^2 + x \end{bmatrix} \bmod m(x).$$

(h) Compute $\phi_{X^2YR}(Q)$:

$$\phi_{Z^{XR}}(Q) = Q^{Z^{XR}},$$
$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} x^4 + x^2 + 1 & x & x^2 \\ x^4 + x^2 + x & x^3 + x^2 & x^4 + x^3 + x^2 + x \\ x^3 + x & x + 1 & x^4 + x^2 + x \end{bmatrix}^{\begin{bmatrix} 66 & 38 & 32 \\ 32 & 66 & 38 \\ 38 & 32 & 66 \end{bmatrix}},$$
$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \bmod x^5 + x^4 + x^3 + x + 1,$$

$$Q_{11} = (x^4 + x^2 + 1)^{66}(x)^{32}(x^2)^{38} \bmod m(x),$$
$$Q_{12} = (x^4 + x^2 + 1)^{38}(x)^{66}(x^2)^{32} \bmod m(x),$$
$$Q_{13} = (x^4 + x^2 + 10^{32}(x)^{38}(x^2)^{66} \bmod m(x),$$
$$Q_{21} = (x^4 + x^2 + x)^{66}(x^3 + x^2)^{32}(x^4 + x^3 + x^2 + x)^{38} \bmod m(x),$$
$$Q_{22} = (x^4 + x^2 + x)^{38}(x^3 + x^2)^{66}(x^4 + x^3 + x^2 + x)^{32} \bmod m(x),$$
$$Q_{23} = (x^4 + x^2 + x)^{32}(x^3 + x^2)^{38}(x^4 + x^3 + x^2 + x)^{66} \bmod m(x),$$
$$Q_{31} = (x^3 + x)^{66}(x + 1)^{32}(x^4 + x^2 + x)^{38} \bmod m(x),$$
$$Q_{32} = (x^3 + x)^{38}(x + 1)^{66}(x^4 + x^2 + x)^{32} \bmod m(x),$$
$$Q_{33} = x^3 + x)^{32}(x + 1)^{38}(x^4 + x^2 + x)^{66} \bmod m(x),$$

$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} x^2 & x^4+x^3+1 & x^4 \\ x^2+1 & x^4+x^3+x^2+x & x^4 \\ x^4+x^3+x^2+1 & x^3+x^2+x & x^4 \end{bmatrix} \bmod m(x).$$

(i) Compute signatures $(\alpha, \beta)$:

$$\alpha = \phi_{XR}(Q),$$
$$\alpha = Q^{XR},$$
$$\alpha = \begin{bmatrix} x^4+x^2+1 & x & x^2 \\ x^4+x^2+x & x^3+x^2 & x^4+x^3+x^2+x \\ x^3+x & x+1 & x^4+x^2+x \end{bmatrix} \bmod m(x).$$

and,

$$\Sigma = \phi_{Z^{XR}}(Q),$$
$$\Sigma = (Q^Z)^{XR},$$
$$\Sigma = \begin{bmatrix} x^2 & x^4+x^3+1 & x^4 \\ x^2+1 & x^4+x^3+x^2+x & x^4 \\ x^4+x^3+x^2+1 & x^3+x^2+x & x^4 \end{bmatrix} \bmod m(x).$$

Hence,

$$\eta\Sigma = x^2 \begin{bmatrix} x^2 & x^4+x^3+1 & x^4 \\ x^2+1 & x^4+x^3+x^2+x & x^4 \\ x^4+x^3+x^2+1 & x^3+x^2+x & x^4 \end{bmatrix} \bmod m(x),$$
$$\eta\Sigma = \begin{bmatrix} x^4 & x^6+x^5+x^2 & x^6 \\ x^4+x^2 & x^6+x^5+x^4+x^3 & x^6 \\ x^6+x^5+x^4+x^2 & x^5+x^4+x^3 & x^6 \end{bmatrix} \bmod m(x),$$
$$\eta\Sigma = \begin{bmatrix} x^4 & x^4+1 & x^3+x^2+1 \\ x^4+x^2 & x^3+x^2+x & x^3+x^2+1 \\ x & x+1 & x^3+x^2+1 \end{bmatrix} \bmod m(x),$$

Since $\eta = x^2$,

$$\mu = \eta + 1,$$

$$\mu = x^2 + 1.$$

Compute $\mu\Sigma$:

$$\mu\Sigma = (x^2+1)\begin{bmatrix} x^2 & x^4+x^3+1 & x^4 \\ x^2+1 & x^4+x^3+x^2+x & x^4 \\ x^4+x^3+x^2+1 & x^3+x^2+x & x^4 \end{bmatrix} \bmod m(x),$$

$$\mu\Sigma = \begin{bmatrix} x^4+x^2 & x^6+x^5+x^4+x^3+x^2+1 & x^6+x^4 \\ x^4+1 & x^6+x^5+x^2+x & x^6+x^4 \\ x^6+x^5+x^3+1 & x^5+x^4+x^2+x & x^6+x^4 \end{bmatrix} \bmod m(x),$$

$$\mu\Sigma = \begin{bmatrix} x^4+x^2 & x^3 & x^4+x^3+x^2+1 \\ x^4+1 & x^4 & x^4+x^3+x^2+1 \\ x^4+x^3+x^2+x+1 & x^3+x^2+1 & x^3+x^2+1 \end{bmatrix} \bmod m(x),$$

$$\beta = H((M)_2 \| (\mu\Sigma)_2).$$

## Verification

$$\lambda = \eta\Sigma + \alpha^Z,$$

$$\lambda = \eta\Sigma + (Q^{XR})^Z,$$

$$Q^{XR^Z} = \begin{bmatrix} x^4+x^2+1 & x & x^2 \\ x^4+x^2+x & x^3+x^2 & x^4+x^3+x^2+x \\ x^3+x & x+1 & x^4+x^2+x \end{bmatrix} \begin{bmatrix} 66 & 38 & 32 \\ 32 & 66 & 38 \\ 38 & 32 & 66 \end{bmatrix},$$

$$Q^{XR^Z} = \begin{bmatrix} x^2 & x^4+x^3+1 & x^4 \\ x^2+1 & x^4+x^3+x^2+x & x^4 \\ x^4+x^3+x^2+1 & x^3+x^2+x & x^4 \end{bmatrix} \bmod m(x),$$

$$\lambda = \begin{bmatrix} x^4 & x^4+1 & x^3+x^2+1 \\ x^4+x^2 & x^3+x^2+x & x^3+x^2+1 \\ x & x+1 & x^3+x^2+1 \end{bmatrix} + Q^{XR^Z},$$

$$\lambda = \begin{bmatrix} x^4+x^2 & x^3 & x^4+x^3+x^2+1 \\ x^4+1 & x^4 & x^4+x^3+x^2+1 \\ x^4+x^3+x^2+x+1 & x^3+x^2+1 & x^3+x^2+1 \end{bmatrix} \bmod m(x),$$

$$\lambda = \mu\Sigma,$$

$$\beta' = H((M)_2 \| (\lambda)_2).$$

As,

$$\beta = \beta'.$$

This verifies the signatures.

**Example 4.2.3.** A public platform is defined over the extended *Galois Field* $GF(p^q)$. Let $m(x) = x^4 + x^2 + x + 1$ be the chosen irreducible polynomial. The order of the matrix is 4. The general linear group is defined as $GL(4, \mathbb{Z}_{33})$, with prime numbers $p_1 = 3$ and $p_2 = 11$. All calculations for power matrices are performed.

## Key Generation

1. Two prime numbers $p_1 = 3$ and $p_2 = 11$ are given in the statement.

2. Compute $N$

$$N = p_1 p_2 = 3(11) = 33.$$

3. Compute Euler's totient function:

$$\varphi(N) = \varphi(p_1 p_2),$$
$$\varphi(N) = \varphi(3.13) = (3-1)(11-1) = 20.$$

4. Two circulant matrices $X$ and $Y$ are chosen from $\mathrm{GL}(4, \mathbb{Z}_{33})$:

$$X = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 7 & 2 & 3 & 5 \\ 5 & 7 & 2 & 3 \\ 3 & 5 & 7 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 4 & 3 & 5 & 11 \\ 11 & 4 & 3 & 5 \\ 5 & 11 & 4 & 3 \\ 3 & 5 & 11 & 4 \end{bmatrix} \in GL(4, \mathbb{Z}_{33}).$$

5. Compute $Z$ from $X$ and $Y$:

$$Z = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 7 & 2 & 3 & 5 \\ 5 & 7 & 2 & 3 \\ 3 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 & 5 & 11 \\ 11 & 4 & 3 & 5 \\ 5 & 11 & 4 & 3 \\ 3 & 5 & 11 & 4 \end{bmatrix} \mod 20,$$

$$Z = \begin{bmatrix} 7 & 8 & 16 & 0 \\ 0 & 7 & 8 & 16 \\ 16 & 0 & 7 & 8 \\ 8 & 16 & 0 & 7 \end{bmatrix} \mod 20.$$

6. Master public key is:

$$N = 33, Z = \begin{bmatrix} 7 & 8 & 16 & 0 \\ 0 & 7 & 8 & 16 \\ 16 & 0 & 7 & 8 \\ 8 & 16 & 0 & 7 \end{bmatrix} \mod 20.$$

7. Master secret key is:

$$X = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 7 & 2 & 3 & 5 \\ 5 & 7 & 2 & 3 \\ 3 & 5 & 7 & 2 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & 5 & 11 \\ 11 & 4 & 3 & 5 \\ 5 & 11 & 4 & 3 \\ 3 & 5 & 11 & 4 \end{bmatrix}.$$

## Signature Generation

For generating signature perform following comprehensive steps:

1. Choose a circulant matrix $R \in GL(4, \mathbb{Z}_{33})$:

$$R = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 4 & 5 & 3 & 7 \\ 7 & 4 & 5 & 3 \\ 3 & 7 & 4 & 5 \end{bmatrix} \in GL(4, \mathbb{Z}_{33}).$$

2. Choose a base matrix $Q$ from $GL(4, 2^4)$:

$$Q = \begin{bmatrix} x^2 & x & x+1 & 1 \\ x^2+1 & x^3 & x^2+1 & x+1 \\ x & x^2+1 & x^2+x+1 & x^3+1 \\ x^3+1 & x^2+x+1 & x^3+x & x^2+1 \end{bmatrix} \bmod m(x) \in GL(4, 2^4).$$

3. Let a random polynomial $\eta$ be,

$$\eta = x^2 + x.$$

4. Session private key is:

$$K_{ses} = (\eta, \quad R \quad Q)$$
$$\eta = x^2 + x,$$
$$R = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 4 & 5 & 3 & 7 \\ 7 & 4 & 5 & 3 \\ 3 & 7 & 4 & 5 \end{bmatrix},$$
$$Q = \begin{bmatrix} x^2 & x & x+1 & 1 \\ x^2+1 & x^3 & x^2+1 & x+1 \\ x & x^2+1 & x^2+x+1 & x^3+1 \\ x^3+1 & x^2+x+1 & x^3+x & x^2+1 \end{bmatrix} \bmod m(x) \in GF(4, 2^4).$$

5. Compute $\phi_Z(Q)$:

$$\phi_Z(Q) = Q^Z \bmod x^4 + x^2 + x + 1,$$

$$\phi_Z(Q) = \begin{bmatrix} x^2 & x & x+1 & 1 \\ x^2+1 & x^3 & x^2+1 & x+1 \\ x & x^2+1 & x^2+x+1 & x^3+1 \\ x^3+1 & x^2+x+1 & x^3+x & x^2+1 \end{bmatrix} \begin{bmatrix} 7 & 8 & 16 & 0 \\ 0 & 7 & 8 & 16 \\ 16 & 0 & 7 & 8 \\ 8 & 16 & 0 & 7 \end{bmatrix} \bmod m(x),$$

After solving it on ApCoCoA [20], we get:

$$\phi_Z(Q) = \begin{bmatrix} x^2+1 & x^2 & x+1 & x^3+x^2 \\ x^3+x & x^2+x & x^3+1 & x^3+1 \\ x^2+1 & x^2+x & x^3+x^2 & x^2+1 \\ x^3+x & x^3+x^2+x+1 & x^3+x^2+x+1 & x+1 \end{bmatrix} \bmod m(x).$$

6. Compute $XR$:

$$XR = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 7 & 2 & 3 & 5 \\ 5 & 7 & 2 & 3 \\ 3 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & 7 & 4 \\ 4 & 5 & 3 & 7 \\ 7 & 4 & 5 & 3 \\ 3 & 7 & 4 & 5 \end{bmatrix} \bmod 20,$$

$$XR = \begin{bmatrix} 11 & 8 & 13 & 14 \\ 14 & 11 & 8 & 13 \\ 13 & 14 & 11 & 8 \\ 8 & 13 & 14 & 11 \end{bmatrix} \bmod 20.$$

7. Compute $\phi_{XR}(Q)$:

$$\phi_{XR}(Q) = Q^{XR},$$

$$\phi_{XR}(Q) = \begin{bmatrix} x^2 & x & x+1 & 1 \\ x^2+1 & x^3 & x^2+1 & x+1 \\ x & x^2+1 & x^2+x+1 & x^3+1 \\ x^3+1 & x^2+x+1 & x^3+x & x^2+1 \end{bmatrix} \begin{bmatrix} 11 & 8 & 13 & 14 \\ 14 & 11 & 8 & 13 \\ 13 & 14 & 11 & 8 \\ 8 & 13 & 14 & 11 \end{bmatrix} \bmod m(x),$$

After solving it on ApCoCoA [20], we get:

$$\phi_{XR}(Q) = \begin{bmatrix} x^2+1 & x^2+x & x+1 & x^3+x \\ x^3+x^2 & x^2+1 & x+1 & x^2+x \\ x^2+x & x+1 & x^2+1 & x^3+1 \\ x^3+x^2 & x^3+x^2+x+1 & x^3+x & x+1 \end{bmatrix} \bmod m(x),$$

8. Compute $\phi_{Z^{XR}}(Q)$:

$$\phi_{Z^{XR}}(Q) = (Q^Z)^{XR},$$

$$XR = \begin{bmatrix} 11 & 8 & 13 & 14 \\ 14 & 11 & 8 & 13 \\ 13 & 14 & 11 & 8 \\ 8 & 13 & 14 & 11 \end{bmatrix},$$

$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} x^2+1 & x^2 & x+1 & x^3+x^2 \\ x^3+x & x^2+x & x^3+1 & x^3+1 \\ x^2+1 & x^2+x & x^3+x^2 & x^2+1 \\ x^3+x & x^3+x^2+x+1 & x^3+x^2+x+1 & x+1 \end{bmatrix}^{XR} \bmod m(x),$$

After solving it on ApCoCoA [20], we get:

$$\phi_{Z^{XR}}(Q) = \begin{bmatrix} x^3+x^2 & x^3+x & x+1 & x^2+1 \\ x^3+1 & x+1 & x^2+1 & x^3+x \\ x^3+x^2+x+1 & x^2+1 & x^2+1 & x^2+x \\ x^2+x & x^2+1 & x^3+x^2+x+1 & x^2+x \end{bmatrix} \bmod m(x).$$

9. Compute signatures $(\alpha, \beta)$:

$$\alpha = \phi_{XR}(Q),$$

$$\alpha = Q^{XR},$$

$$\alpha = \begin{bmatrix} x^2+1 & x^2+x & x+1 & x^3+x \\ x^3+x^2 & x^2+1 & x+1 & x^2+x \\ x^2+x & x+1 & x^2+1 & x^3+1 \\ x^3+x^2 & x^3+x^2+x+1 & x^3+x & x+1 \end{bmatrix} \mod m(x).$$

and,

$$\Sigma = \phi_{Z^{XR}}(Q),$$
$$\Sigma = (Q^Z)^{XR},$$
$$\Sigma = \begin{bmatrix} x^3+x^2 & x^3+x & x+1 & x^2+1 \\ x^3+1 & x+1 & x^2+1 & x^3+x \\ x^3+x^2+x+1 & x^2+1 & x^2+1 & x^2+x \\ x^2+x & x^2+1 & x^3+x^2+x+1 & x^2+x \end{bmatrix} \mod m(x).$$

Since,

$$\eta = x^2 + x,$$

$$\eta\Sigma = (x^2+x) \begin{bmatrix} x^3+x^2 & x^3+x & x+1 & x^2+1 \\ x^3+1 & x+1 & x^2+1 & x^3+x \\ x^3+x^2+x+1 & x^2+1 & x^2+1 & x^2+x \\ x^2+x & x^2+1 & x^3+x^2+x+1 & x^2+x \end{bmatrix} \mod m(x),$$

$$\eta\Sigma = \begin{bmatrix} x^5+x^3 & x^5+x^4+x^3+x^2 & x^3+x & x^4+x^3+x^2+x \\ x+^5 x^4+x^2+x & x^3+x & x^4+x^3+x^2+x & x^5+x^4+x^3+x^2 \\ x^5+x & x^4+x^3+x^2+x & x^4+x^3+x^2+x & x^4+x^2 \\ x^4+x^2 & x^4+x^3+x^2+x & x^5+x & x^4+x^2 \end{bmatrix},$$

$$\eta\Sigma = \begin{bmatrix} x^2+x & x^2+1 & x^3+x & x^3+1 \\ x^3+x^2+x+1 & x^3+x & x^3+1 & x^2+1 \\ x^3+x^2 & x^3+1 & x^3+1 & x+1 \\ x+1 & x^3+1 & x^3+x^2 & x+1 \end{bmatrix} \mod m(x).$$

$$\mu = \eta + 1,$$

$$\mu = x^2 + x + 1,$$

10. Compute $\mu\Sigma$:

$$\mu\Sigma = \mu \begin{bmatrix} x^3+x^2 & x^3+x & x+1 & x^2+1 \\ x^3+1 & x+1 & x^2+1 & x^3+x \\ x^3+x^2+x+1 & x^2+1 & x^2+1 & x^2+x \\ x^2+x & x^2+1 & x^3+x^2+x+1 & x^2+x \end{bmatrix} \bmod m(x),$$

$$\mu\Sigma = \begin{bmatrix} x^3+x & x^3+x^2+x+1 & x^3+1 & x^3+x^2 \\ x^2+x & x^3+1 & x^3+x^2 & x^3+x^2+x+1 \\ x+1 & x^3+x^2 & x^3+x^2 & x^2+1 \\ x^2+1 & x^3+x^2 & x+1 & x^2+1 \end{bmatrix} \bmod m(x).$$

$$\beta = H((M)_2 \| (\mu.\Sigma)_2).$$

## Verification

$$\lambda = \eta\Sigma + \alpha^Z,$$
$$\lambda = \eta\Sigma + (Q^{XR})^Z,$$

$$\phi_{XR}(Q) = \begin{bmatrix} x^2 & x & x+1 & 1 \\ x^2+1 & x^3 & x^2+1 & x+1 \\ x & x^2+1 & x^2+x+1 & x^3+1 \\ x^3+1 & x^2+x+1 & x^3+x & x^2+1 \end{bmatrix} \begin{bmatrix} 11 & 8 & 13 & 14 \\ 14 & 11 & 8 & 13 \\ 13 & 14 & 11 & 8 \\ 8 & 13 & 14 & 11 \end{bmatrix} \bmod m(x),$$

$$(Q^{XR})^Z = \begin{bmatrix} x^2+1 & x^2+x & x+1 & x^3+x \\ x^3+x^2 & x^2+1 & x+1 & x^2+x \\ x^2+x & x+1 & x^2+1 & x^3+1 \\ x^3+x^2 & x^3+x^2+x+1 & x^3+x & x+1 \end{bmatrix} \begin{bmatrix} 7 & 8 & 16 & 0 \\ 0 & 7 & 8 & 16 \\ 16 & 0 & 7 & 8 \\ 8 & 16 & 0 & 7 \end{bmatrix} \bmod m(x),$$

$$(Q^{XR})^Z = \begin{bmatrix} x^3+x^2 & x^3+x & x+1 & x^2+1 \\ x^3+1 & x+1 & x^2+1 & x^3+x \\ x^3+x^2+x+1 & x^2+1 & x^2+1 & x^2+x \\ x^2+x & x^2+1 & x^3+x^2+x+1 & x^2+x \end{bmatrix} \bmod m(x).$$

$$\lambda = \begin{bmatrix} x^2+x & x^2+1 & x^3+x & x^3+1 \\ x^3+x^2+x+1 & x^3+x & x^3+1 & x^2+1 \\ x^3+x^2 & x^3+1 & x^3+1 & x+1 \\ x+1 & x^3+1 & x^3+x^2 & x+1 \end{bmatrix} + (Q^{XR})^Z,$$

$$\lambda = \begin{bmatrix} x^3+x & x^3+x^2+x+1 & x^3+1 & x^3+x^2 \\ x^2+x & x^3+1 & x^3+x^2 & x^3+x^2+x+1 \\ x+1 & x^3+x^2 & x^3+x^2 & x^2+1 \\ x^2+1 & x^3+x^2 & x+1 & x^2+1 \end{bmatrix} \bmod m(x),$$

$$\lambda = \mu.\Sigma.$$

$$\beta' = H((M)_2 \| (\lambda)_2).$$

As,

$$\beta = \beta'.$$

This verifies the signatures.

## 4.3 Types of Attacks and Countermeasures

### 4.3.1 Algebraic Attacks

MPF schemes often involve modular matrix exponentiation, which can be converted into polynomial systems. Attackers may attempt to solve these systems using algebraic techniques [57] like Gröbner bases. As base matrix $Q$ has polynomial entries. Attacker may use different Algebraic techniques to solve the system of equations.

Matrix $Q^Z$ where $Z$ is product of X and Y which make the underlying equations computationally infeasible to solve. First attacker suppose that $\alpha$ is this matrix.

$$\alpha = \begin{bmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{bmatrix}, \quad Z = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix},$$

$$\alpha^Z = \begin{bmatrix} f_1(x)^{w_{11}} f_2(x)^{w_{21}} & f_1(x)^{w_{12}} f_2(x)^{w_{22}} \\ f_3(x)^{w_{11}} f_4(x)^{w_{21}} & f_3(x)^{w_{12}} f_4(x)^{w_{22}} \end{bmatrix}.$$

and

$$\eta\Sigma = \begin{bmatrix} g_1(x) & g_2(x) \\ g_3(x) & g_4(x) \end{bmatrix},$$

$$\eta\Sigma + \alpha^Z = \begin{bmatrix} g_1(x) & g_2(x) \\ g_3(x) & g_4(x) \end{bmatrix} + \begin{bmatrix} f_1(x)^{w_{11}} f_2(x)^{w_{21}} & f_1(x)^{w_{12}} f_2(x)^{w_{22}} \\ f_3(x)^{w_{11}} f_4(x)^{w_{21}} & f_3(x)^{w_{12}} f_4(x)^{w_{22}} \end{bmatrix}.$$

Here $\eta\Sigma$ is private session key , which is unknown to attacker. After adding four algebraic equation will form with eight unknown values. So, computationally is very hard to find all eight value which are in power (0ne-way Function) after solving fours equations.

#### 4.3.1.1 Countermeasures:

1. Used large primes $p$ to make the underlying equations computationally infeasible to solve. Matrix $Q$ power is matrix $Z$ and that matrix is a product of $X$ and $Y$.

2. Introduced random padding or non-linear terms into MPF operations to disrupt algebraic structures as $GF(2^8)$ mean we are using higher degree polynomials.

### 4.3.2 Rank-Based Attacks

A rank attack [58] on a matrix over a finite field GF($p$) is a cryptographic attack that exploits the properties of matrix and the rank to compromise the security of a cryptographic scheme. This type of attack is relevant in schemes that operate with code-based cryptography, lattice-based cryptography, or multivariate quadratic systems. The rank of a matrix is defined as the maximum number of linearly independent

rows or columns in the matrix. In the case of matrices over a finite field GF($p$), the rank is computed modulo $p$. A finite field GF($p$) consists of $p$ elements, where $p$ is a prime number. Arithmetic operations such as addition, multiplication, and inversion are performed modulo $p$.

$$\mathrm{GF}(p) = \{0, 1, 2, \ldots, p-1\}$$

The attacker exploits the rank properties of matrices to recover secret information or break the cryptographic scheme. For example, if the secret key or a critical component of the scheme is represented as a low-rank matrix, the attacker might use rank-based techniques to deduce the secret information. Specifically, an attacker might compute the rank of a given matrix over GF($p$), and if the matrix has a rank lower than expected, it could reveal structural weaknesses in the system. The rank of this matrix can be computed using standard row reduction techniques. In GF($p$), we perform row operations modulo $p$. If the rank of matrix is low, it may indicate a vulnerability in the cryptographic scheme.

#### 4.3.2.1 Countermeasures

There are several countermeasures to defend against rank attacks:

1. Ensured that matrices used in the cryptographic scheme have sufficiently high rank to resist rank attacks as
2. Introduced randomness in matrix generation to make it harder for attackers to exploit rank properties
3. Worked over larger finite fields GF($p^q$), where $m > 1$, to increase the complexity of rank computations and make the attack more difficult like GF($2^8$).

### 4.3.3 Known-Plaintext Attacks

In this type of attack, the attacker has knowledge of some plaintext. Using this information, the attacker attempts to recover the secret key or devise an algorithm

to decode subsequent signatures. Assume the attacker knows a pair of corresponding plaintext $M_y$ from previous communication. With this knowledge, the attacker aims to reveal the secret communication key to decode future messages, such as the next plaintext $M_{y+1}$. In our scheme over $GF(p)$, the attacker's first goal is to find the unknown matrices $X$ and $Y$, given that $Z$ is publicly available. The attacker would attempt to derive the key $Z = XY$. However, this type of attack is infeasible in our scheme for several reasons. Therefore, in every session, the key keeps changing, making it impossible for the attacker to derive a future key based on previous keys. This dynamic feature ensures that the scheme is secure against known-plaintext attacks, as the adversary cannot predict or access new keys based on previous communication.

#### 4.3.3.1 Countermeasures

1. $X$ and $Y$ are randomly generated matrices for both participants, and changing these matrices will directly affect the key.

2. The session key is different for each communication session, preventing reuse of keys.

3. The calculation of $R$ and $Q$ relies on randomly generated polynomials from both participants. Since these polynomials change each time, the key changes with each session.

### 4.3.4 Forgery Attacks

Forgery attacks [59] are a significant threat to the security of digital signature schemes. The goal of a forgery attack is to create a valid digital signature for a message without having access to the private key of the legitimate signer.

In this context, we explore forgery attacks against digital signatures, particularly in the context of MPF (Multivariate Polynomial Functions) over GF($p$), where $p$ is a prime number. The primary objective of a forgery attack is to create a valid signature for a message without access to the private key. A attackers can use different methods for this purpose.

#### 4.3.4.1 How Forgery Attacks work against MPF over GF($p$)

In the context of digital signatures based on MPF over GF($p$), forgery attacks may exploit weaknesses in several areas:

1. If the MPF has certain algebraic properties or patterns like in matrix $Q$ or random polynomial $\eta$, attackers may find ways to generate valid signatures without knowing the private key.
2. The way MPF is incorporated into the signature generation and verification process can introduce vulnerabilities.
3. An inappropriate choice of the prime $p$ or the size of the field could facilitate forgery attempts by simplifying the attacker's computations.

Attackers might use advanced mathematical techniques to analyze the MPF and exploit relationships between the matrix and the field elements. This could involve finding collisions in the MPF or solving related algebraic problems.

#### 4.3.4.2 Countermeasures

To prevent forgery attacks, it's essential to adopt several countermeasures:

1. The MPF should be designed to resist known mathematical attacks, such as collision resistance, and have strong cryptographic properties as by using session private key and using random matrices like $R$ and $Q$.
2. The integration of MPF into the signature generation and verification process must be carefully analyzed to avoid introducing vulnerabilities.
3. The size of GF($p$) should be large enough to make computations infeasible.

### 4.3.5 Brute Force and Exhaustive Search

To construct a secure protocol based on the properties of the Galois field $GF(p^q)$, we can choose a prime $p$ with 60 decimal digits and a polynomial of degree greater than

7 (seven). This ensures that the protocol is resistant to brute-force attacks [60] due to the enormous size of the field and the complexity introduced by the high-degree polynomial.

Below, I will explain the role of matrices in this context and how they function within $GF(p)$.

In such a field, matrices can be used to represent elements, and operations such as matrix multiplication are performed within the field $GF(p)$. The security of the protocol comes from the fact that finding the inverse or solving for the unknown elements in large matrices over $GF(p^q)$ is computationally difficult. The use of high-degree polynomials further complicates the computation, adding an additional layer of security.

The structure of the field and the properties of the matrices ensure that adversaries cannot easily break the system, even with access to known plaintext-ciphertext pairs.

#### 4.3.5.1 Countermeasures

1. Use large primes $p$ (e.g., $p > 2^{256}$) to increase the key space.
2. Ensure matrix entries are chosen uniformly at random to prevent predictability.

### 4.3.6 Symmetry-Based Attacks

**Description:** Symmetric matrices ($\alpha = (Q^Z)^{XR}$) simplify MPF-related computations and may expose vulnerabilities. In matrix-based digital signatures, symmetry-based attacks exploit symmetric properties to derive private keys or forge signatures.

#### 4.3.6.1 Countermeasures

1. Used non-symmetric matrices like $R$, $X$, and $Y$ matrices for key generation to eliminate symmetry-based vulnerabilities.
2. Ensured matrix entries are independently randomized like matrices $N$ and $Z$ to prevent symmetry.

### 4.3.7 Modular Reduction Attacks

**Description:** Small primes $p$ or predictable modular reductions can expose matrix vulnerabilities. In matrix-based digital signatures, modular reduction attacks exploit weaknesses in modular arithmetic to derive private keys or forge signatures.

#### 4.3.7.1 Countermeasures

1. Select large primes $p$ to increase the complexity of modular reductions.
2. Introduce randomization in operations involving modular arithmetic to prevent predictability.

### 4.3.8 Fault Injection Attacks

**Description:** Deliberate faults during MPF computations (e.g., bit flips) can reveal sensitive information. In matrix-based digital signatures, fault injection attacks aim to corrupt computations to derive private keys or forge signatures.

#### 4.3.8.1 Countermeasures

1. Implemented fault detection mechanisms to identify and reject tampered computations.
2. Use redundancy in computations to verify correctness and prevent exploitation of faults like private key $(X, Y)$ , session private key $(R, Q)$ , public key $(Z)$ and master public key $(N, Z)$.

# Chapter 5

# Conclusion

In this research, we analyze a digital signature scheme based on the matrix power function (MPF) and propose a new algorithm utilizing MPF in the Galois field $GF(p^q)$. Additionally, we review the article Fast and Secure Modular Matrix Based Digital Signature Scheme by S.K. Rososhak [18] and modify the work of Sundas Iqbal [19], which is based on the general linear group $GL(n, \mathbb{Z}_{\mathbb{N}})$. Our approach leverages matrices from $GL(n, \mathbb{Z}_{\mathbb{N}})$ to develop key generation algorithms for both public and private keys.

To enhance security, we introduce session-based public keys by performing additional multiplications with existing public keys. Our methodology involves selecting two random matrices from $GL(n, \mathbb{Z}_{\mathbb{N}})$ and a base matrix $Q$ from $GL(n, GF(p^q))$. A randomly chosen polynomial $\eta$ is used in intermediate computations, ensuring added complexity and making the system resistant to inversion attacks. The signature generation process relies on extensive mathematical operations, particularly matrix power multiplications, which differ from conventional multiplication techniques.

Furthermore, we compute a security parameter $\mu$ from $\eta$ to further strengthen the robustness of our scheme. The proposed digital signature mechanism, named Digital Signature based on Matrix Power Function over Galois Field of Polynomials, employs the computational power of ApCoCoA for executing matrix power functions efficiently. Our framework offers flexibility, allowing future research to extend the model using other commutative or non-commutative algebraic structures. By integrating these

refinements, we provide a more secure and computationally efficient cryptographic signature scheme suitable for modern cryptographic applications.

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