CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Fixed Point Theorem for Mapping Contracting Perimeter of Triangles Embedded with F-Contraction in b-Metric Spaces

by

Hira Ulfat

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in the

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*Dedicated to my **Father** and my (Late)**Mother***

CERTIFICATE OF APPROVAL

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Abstract

In this thesis, the concept of a mapping contracting perimeter of triangles embedded with F-contraction in the framework of b-metric space is introduced. Some fixed point results are established using this mappings, also Banach Contraction Principle is derived as corollary of main result. Additionally, we construct examples of mapping contracting perimeters of triangles embedded with F-contraction which are not contraction mappings in the framework of b-metric space.

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Abbreviations

BCP	Banach Contraction Principle
b-MS	b-metric space
FP	Fixed Point
MS	Metric Space
MCPT	Mapping contracting perimeters of triangles

Symbols

$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
$\forall$	for all
$\Rightarrow$	Implies
∞	Infinity
$\exists$	there exists

Chapter 1

Introduction

1.1 Historical Background

Mathematics is the foundation of science and technology. It plays a vital role in every aspect of our lives, from simple calculations to complex problem-solving. One of the key branches of mathematics is fixed point (FP) theory. The FP theory is a crucial area of mathematics that has numerous practical applications. The FP theory has a vast array of applications in different areas of science including, mathematical economics, theory of optimization and the theory of approximation. FP theory has emerged as one of the fastest-growing field of mathematical research in the past 5-7 decades.

FP theory is also essential in other scientific fields that rely heavily on mathematics, including physics, engineering and computer science. The history of FP theory dates back to 1886 when the French mathematician Henri Poincare [1] first began working on it. A FP theorem states that, under certain conditions, a function $\Upsilon : \mathcal{U} \to \mathcal{U}$ has at least one point η where $\Upsilon(\eta) = \eta$. A point η does not change its value even when a particular transformation is applied. Brouwer [2] investigated the FP problem and developed FP theorems to solve $\Upsilon(\eta) = \eta$. Additionally, he presented several FP results across different dimensions. The concept of metric spaces (MS) has a rich history that spans over a century, with its roots in the early 20th century. Pioneering mathematician Maurice Frechet [3] laid the foundation for this field. He defined a MS as a set of points together with a distance function that obeys specific fundamental properties. Later, in 1922, Stefan Banach [4] from Poland, made a discovery about FPs. He presented a crucial result known as the "Banach Contraction Principle" (BCP). This theorem asserts that every contraction mapping in a complete MS possesses a unique FP. Mathematically, the BCP can be expressed as follows:

"Let $(\mathcal{U}, \sigma)$ be a complete MS, and let $\Upsilon : \mathcal{U} \to \mathcal{U}$ be a self-map on $\mathcal{U}$. If there exists a constant $\alpha \in [0, 1)$ such that

$$\sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha\sigma(\eta, \xi)$$

for all $\eta, \xi \in \mathcal{U}$, then Υ has a unique FP".

The BCP has two significant consequences. Firstly, it ensures the existence and uniqueness of a FP for contraction mappings. Secondly, it provides a method to determine the FP of such mappings, which is a remarkable achievement. Researchers have extended the BCP by either modifying the contraction condition or considering different types of spaces.

Edelstein [5] first extended the idea of BCP in 1961 by taking into account the globally contractive mapping on compact space. Afterward, Presic [6] extended the BCP to operators defined on product spaces in 1965. Kannan [7] changed the BCP mapping from contraction mapping to Kannan mapping in 1968. Kannan introduced the mapping in following way;

"A function $\Upsilon : \mathcal{U} \to \mathcal{U}$ on a MS $(\mathcal{U}, \sigma)$ is called Kannan if there exists a constant $\alpha \in [0, \frac{1}{2})$ then $\forall\ \eta, \xi \in \mathcal{U}$,

$$\sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha[\sigma(\eta, \Upsilon\eta) + \sigma(\xi, \Upsilon\xi)]".$$

Later on, he proved that if $(\mathcal{U}, \sigma)$ be a complete MS then, Υ has a unique FP in $\mathcal{U}$. In 1969, Keeler and Meir [8] expanded the BCP, which is expressed as follows:

"Let $(\mathcal{U}, \sigma)$ be a complete MS, such that $\Upsilon : \mathcal{U} \to \mathcal{U}$ be an operator. Suppose that for every $\epsilon > 0\ \exists\ \delta(\epsilon) > 0$, such that $\forall\ \eta, \xi \in \mathcal{U}$

$$\epsilon \leq \sigma(\eta, \xi) < \epsilon + \delta(\epsilon) \Rightarrow \sigma(\Upsilon\eta, \Upsilon\xi) < \epsilon.$$

Then, Υ has unique FP".

Nadler [9] further extended BCP, using set valued contraction mapping in place of single valued contraction mapping. Set valued contractions play an important role in FP theory, optimization and differential inclusions. The BCP was further developed by Chatterjea [10] in 1972, who demonstrated the FP findings using Chatterjea type contraction mapping rather than contraction mapping. Chatterjea introduced mapping in following way;

"A function $\Upsilon : \mathcal{U} \to \mathcal{U}$ on a MS $(\mathcal{U}, \sigma)$ is called Chatterjea-type contraction if there exists a constant $\alpha \in [0, 1)$ then $\forall\ \eta, \xi \in \mathcal{U}$,

$$\sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha[\sigma(\eta, \Upsilon\xi) + \sigma(\Upsilon\eta, \xi)]".$$

In 1974, a theorem for FP on complete MS a generalized form of BCP was established by Ciric [11]. Ciric introduced a novel class of contractive mappings in the context of MS. A Ciric type mapping does not necessarily have to be continuous, but it must be continuous at the FP. Dass and Gupta [12] further expanded BCP in 1975 by demonstrating FP findings using rational contraction mapping.

A MS is a vast concept, and even small modifications to its axioms can lead to the creation of different structures, such as 2-MS [13], cone MS [14], and others. The notion of b-MS was pioneered by Bakhtin [15] in 1989, and later refined by Czerwik [16] in 1993. This innovation introduced a new coefficient in the triangular inequality of MS, laying the groundwork for the development of b-MS. Researchers have developed a wide range of FP results utilizing the framework of b-MS.

Ma et al. [17] introduced the concept of C*-algebra-valued contraction mappings. Building on this, Batul et al. [18] generalized the idea by relaxing the contraction condition initially proposed by Ma et al. In another development, Shehwar et al. [19] extended Caristi's FP theorem to mappings defined on C*-algebra-valued MS. They demonstrated the existence of FPs by employing the concept of minimal elements within these spaces and introduced a partial order on the set $\mathcal{U}$.

Motivated by the work of Berinde [20, 21], the notion of F-contraction was first introduced by Wardowski [22], a Polish mathematician, in his 2012 paper titled

"FPs of a new type of contractive mappings in complete MS". This new concept allowed for the study of mappings that do not satisfy the traditional contraction conditions, but still exhibit some of the desirable properties of contraction mappings. Later on, FP for F-contraction were proved by Secelean [23] using iterated function. Abbas et al. [24] extended the work of Wardowski and established various results of FPs using F-contraction mapping. Since then, the study of F-contractions has undergone significant development, with contributions from numerous mathematicians. A significant extension of Wardowski's research was achieved by Batul et al. [25] in 2022, who established innovative FP results for fuzzy mappings in b-MS using (α_*, F)-contractions. Researchers have explored various aspects of F-contractions, including their existence, uniqueness and stability, as well as their applications in different fields.

Evgenity petrov [26], obtain some FP theorems for "mapping contracting perimeters of triangles" (MCPT) in the framework of MS. In this paper author introduced a new type of mappings in MS which can be characterized as MCPT.
Influenced by the work of Evgenity Petrov, we bring to light some FP theorems for MCPT embedded with F-cntraction in the framework of b-MS. Moreover, relevant examples are provided to support our findings.

The thesis is further divided into four chapters, which are organized in the following manner:

Chapter 2: This chapter lays the foundation for the subsequent chapters by introducing the essential definitions and concepts that form the backbone of the thesis. This chapter investigate the fundamental notions of MS, b-MS, FP, BCP and F-contraction. Some examples are also presented to illustrate these concepts.

Chapter 3: It provides a comprehensive review of article "FP theorems for mapping contracting perimeters of triangle" [26]. Specifically, it focuses on FP results for MCPT in MS. This chapter presents a detailed review of these results, accompanied by illustrative examples.

Chapter 4: In this chapter, the concept of CPT F-mappings in the platform of b-MS has been introduced and some FP results has been established using these mappings, also BCP is derived as corollary of main result. Furthermore, interesting examples are provided to support our results.

Chapter 5: provides the conclusion of the thesis.

Chapter 2

Preliminaries

This chapter establishes the fundamental concepts, definitions and examples that will be crucial for understanding the concept presented in the next chapters. This chapter's primary objective is to give a solid foundation of key findings, explanations and examples that will be applied and expanded upon in subsequent chapters.

2.1 Metric Space (MS)

Maurice Frechet [3], a French mathematician, presented the idea of MS in his 1906 article "Sur quelques points du calcul fonctionnel" (On certain aspects of functional calculus). A MS is a fundamental mathematical concept that enables the measurement of distances and exploration of geometric relationships between objects. MS offer a powerful tool for analyzing complex structures.

Definition 2.1.1.

"A pair $(\mathcal{U}, \sigma)$ is a MS when $\mathcal{U}$ is a set and σ is a metric on $\mathcal{U}$ (or a distance function on $\mathcal{U}$), that is, a function defined on $\mathcal{U} \times \mathcal{U}$ such that for all $\eta, \xi, \zeta \in \mathcal{U}$ we have :

(**M1**): σ maps distances to real, finite and non-negative numbers.

(**M2**): The distance $\sigma(\eta, \xi)$ is zero if and only if η and ξ coincide.

(**M3**): $\sigma(\eta, \xi) = \sigma(\xi, \eta)$ (Symmetry).

(**M4**): $\sigma(\eta,\, \xi) \leq \sigma(\eta,\, \zeta) + \sigma(\zeta,\, \xi)$ (Triangular inequality).

The symbol $\times$ is used to indicate the Cartesian product operation, which generates a new set containing all ordered pairs from the original sets. Hence, $\mathcal{U} \times \mathcal{U}$ is set the of all ordered pairs of elements of $\mathcal{U}$." [27]

Here are some illustrations of MS:

Example 2.1.2.

Consider $\mathcal{U} = \mathbb{R}$, the distance function $\sigma : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be defined as $\forall\ \ \eta,\, \xi \in \mathcal{U}$

$$\sigma(\eta, \xi) = |\eta - \xi|.$$

Then, σ satisfies all axioms of metric. Hence, $(\mathcal{U}\ ,\ \sigma)$ is a MS known as usual metric.

Example 2.1.3.

Let $\mathcal{U} = \mathsf{C}[\mathsf{a}, \mathsf{b}]$ be the set of all real valued continuous functions on closed interval [a,b] and $\sigma : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be the distance function defined by:

$$\sigma(\mathsf{f},\ \mathsf{g}) = \max_{\mathsf{s} \in [a,b]} |\mathsf{f}(\mathsf{s}) - \mathsf{g}(\mathsf{s})|, \qquad \forall\ \ \mathsf{f}, \mathsf{g} \in \mathcal{U}.$$

Then, σ satisfies all of the properties of metric. Thus, the pair $(\mathcal{U}\ ,\ \sigma)$ is a MS.

Example 2.1.4.

Consider $\mathcal{U} = \ell^{\infty}$ is the set of all bounded sequences of complex numbers, and the distance function $\sigma : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be defined as:

$$\sigma(\eta, \xi) = \sup_{n \in \mathbb{N}} |\eta_{\mathsf{n}} - \xi_{\mathsf{n}}| \quad \text{where} \ \ \forall\ \ \eta, \xi \in \mathcal{U}.$$

Here, σ satisfies (M1)- (M4). Thus, $(\mathcal{U}, \sigma)$ is a MS.

2.1.1 Convergence, Cauchyness and Completeness in Metric Space

In the realm of MS, understanding the concepts of convergence, Cauchy sequences, and completeness is crucial for analyzing and manipulating mathematical structures. Convergence, the foundation of these concepts, describes the behavior of sequences approaching a limit point. Cauchy sequences, characterized by terms getting arbitrarily close, serve as a gateway to studying convergence. Completeness, a fundamental property of MS. Exploring convergence, Cauchy sequences, and completeness in MS provides a deeper understanding of the underlying structure of mathematical objects, enabling the development of powerful theorems and uses in the fields such as calculus, functional analysis and fixed point (FP) theory.

Definition 2.1.5.

"A sequence $\{\eta_{\mathsf{n}}\}$ in a MS $\mathcal{U} = (\mathcal{U}, \sigma)$ is said to converge or to be convergent if there is an $\eta \in \mathcal{U}$ such that

$$\lim_{\mathsf{n}\to\infty} \sigma(\eta_{\mathsf{n}}, \eta) = 0.$$

η is called the limit of (η_{n}) and write as

$$\lim_{\mathsf{n}\to\infty} \eta_{\mathsf{n}} = \eta$$

or, simply,

$$\eta_{\mathsf{n}} \longrightarrow \eta.$$

Then, (η_{n}) converges to η or has the limit η. If (η_{n}) is not convergent, it is said to be divergent."[27]

Example 2.1.6.

Suppose that $\mathcal{U} = \mathbb{R}$ with σ be the usual metric on $\mathbb{R}$. Consider a sequence $\{\eta_{\mathsf{n}}\} = \dfrac{1}{n}$. Then

$$\lim_{\mathsf{n}\to\infty} \eta_{\mathsf{n}} = \lim_{\mathsf{n}\to\infty} \frac{1}{n} = 0.$$

$\Rightarrow \{\eta_{\mathsf{n}}\}$ is a convergent sequence in $\mathcal{U}$.

Let $\mathcal{U} = (0, 1)$ be an open interval on $\mathbb{R}$ with the usual metric $\sigma(\eta, \xi) = |\eta - \xi|$, then the sequence $\{\eta_n\} = \frac{1}{n}$ is not convergent since '0' is not in $\mathcal{U}$.

Definition 2.1.7.

"A sequence $\{\eta_n\}$ in a MS $\mathcal{U} = (\mathcal{U}, \sigma)$ is said to be Cauchy (or fundamental) if for every $\epsilon > 0$ there is an integer $\mathcal{N} = \mathcal{N}(\epsilon)$ such that

$$\sigma(\eta_m, \eta_n) < \epsilon, \qquad \text{for every } \ m, n > \mathcal{N}.\text{"}[27]$$

Example 2.1.8.

Consider the sequence $\{\eta_n\}$ in $\mathcal{U} = \mathbb{R}$ with usual metric. Let $0 < \alpha < 1$, $\{\eta_n\}$ satisfies the condition $|\eta_{n+1} - \eta_n| < \alpha^n$, $\forall$ $n \in \mathbb{N}$. Then, $\{\eta_n\}$ form a Cauchy sequence in $\mathbb{R}$. As, $m, n \in \mathbb{N}$ and $m > n$.

Then,

$$\begin{aligned} |\eta_m - \eta_n| &\leq |\eta_n - \eta_{n+1}| + |\eta_{n+1} - \eta_{n+2}| + \cdots + |\eta_{m-1} - \eta_m| \\ &\leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1} \\ &= \frac{\alpha^n(1 - \alpha^{m-n})}{1 - \alpha} \\ &< \frac{\alpha^n}{1 - \alpha}. \end{aligned}$$

As, $0 < \alpha < 1$, $\alpha^n \longrightarrow 0$ and for any $\epsilon > 0$, choose $\mathcal{N} \in \mathbb{N}$ such that $\dfrac{\alpha^n}{1 - \alpha} < \epsilon$, Thus, $\forall$ $m, n > \mathcal{N}$, we conclude that

$$|\eta_m - \eta_n| \leq \frac{\alpha^n}{1 - \alpha} < \epsilon.$$

Thus, $\{\eta_n\}$ is a Cauchy sequence in $\mathcal{U}$.

Definition 2.1.9.

"A space $(\mathcal{U}, \sigma)$ be a complete MS if every Cauchy sequence in $\mathcal{U}$ converges (that is, has a limit which is an element in $\mathcal{U}$)."[27]

The following are examples of complete MS:

Example 2.1.10.

$\mathbb{R}$ and $\mathbb{C}$ are complete MS with usual metric.

Example 2.1.11.

Under the following metric define on Euclidean space $\mathbb{R}^n$ and Unitary space $\mathbb{C}^n$ are complete MS

$$\sigma(\eta, \xi) = \left(\sum_{j=1}^{n} (\eta_{\mathsf{j}} - \xi_{\mathsf{j}})^2 \right)^{\frac{1}{2}},$$

where $\eta = \{\eta_{\mathsf{j}}\}$ and $\eta = \{\xi_{\mathsf{j}}\} \in \mathbb{R}^n$ or $\mathbb{C}^n$.
Consider $\{\eta_{\mathsf{m}}\}$ be the Cauchy sequence in $\mathbb{R}^n$ (or $\mathbb{C}^n$), and $\eta_{\mathsf{m}} = ($ $\eta_1^{(\mathsf{m})}$, $\eta_2^{(\mathsf{m})}$, $\cdots$, $\eta_{\mathsf{n}}^{(\mathsf{m})}$ $)$.
So, $\forall$ $\epsilon > 0$, $\exists$ $\mathcal{N}(\epsilon)$ s.t

$$\sigma(\eta_{\mathsf{m}}, \xi_{\mathsf{k}}) = (\sum_{j=1}^{n} (\eta_{\mathsf{j}}^{\mathsf{m}} - \xi_{\mathsf{j}}^{\mathsf{k}})^2)^{\frac{1}{2}} < \epsilon, \qquad \mathsf{m}, \mathsf{k} > \mathcal{N}(\epsilon),$$

$$\Rightarrow ((\eta_{\mathsf{j}}^{\mathsf{m}} - \xi_{\mathsf{j}}^{\mathsf{k}})^2)^{\frac{1}{2}} < \epsilon, \qquad \mathsf{m}, \mathsf{k} > \mathcal{N}(\epsilon).$$

Taking square on both side, yields:

$$(\eta_{\mathsf{j}}^{\mathsf{m}} - \xi_{\mathsf{j}}^{\mathsf{k}})^2 < \epsilon^2, \qquad \text{where} \quad \mathsf{m}, \mathsf{k} > \mathcal{N}(\epsilon) \text{ and } \quad \mathsf{j} = 1, 2, \cdots, n,$$

which implies:

$$|\eta_{\mathsf{j}}^{\mathsf{m}} - \xi_{\mathsf{j}}^{\mathsf{k}}| < \epsilon, \qquad \text{where} \quad \mathsf{m}, \mathsf{k} > \mathcal{N}(\epsilon) \text{ and } \quad \mathsf{j} = 1, 2, \cdots, n.$$

Given that each sequence $(\eta_1^{(\mathsf{m})}, \eta_2^{(\mathsf{m})}, \cdots)$ is Cauchy in $\mathbb{R}$ (or $\mathbb{C}$) and $\mathbb{R}$ (or $\mathbb{C}$) is complete, we have $\eta_{\mathsf{j}}^{(\mathsf{m})} \longrightarrow \eta_{\mathsf{j}}$ as $m \longrightarrow \infty$ for all j. Using these limits, we define $\eta_{\mathsf{m}} = ($ $\eta_1^{(\mathsf{m})}, \eta_2^{(\mathsf{m})}, \cdots, \eta_{\mathsf{n}}^{(\mathsf{m})}$ $) \in \mathbb{R}^n$ (or $\mathbb{C}^n$). Then, for any $\epsilon > 0$, $\exists$ $\mathcal{N}(\epsilon)$ such that $\sigma(\eta_{\mathsf{m}}, \eta) < \epsilon$, whenever $\mathsf{m} > \mathcal{N}(\epsilon)$, as $\mathsf{k} \longrightarrow \infty$.
This demonstrates the completeness of $\mathbb{R}^n$ (or $\mathbb{C}^n$).

Definition 2.1.12.

"Let $\mathcal{U} = (\mathcal{U}, \sigma_1)$ and $\mathcal{V} = (\mathcal{V}, \sigma_2)$ be MS. A mapping Υ: $\mathcal{U} \longrightarrow \mathcal{V}$ is said to be continuous at a point $\eta_0 \in \mathcal{U}$ if for every $\epsilon > 0$ $\exists$ $\delta > 0$, such that

$$\sigma_2(\Upsilon(\eta), \Upsilon(\eta_0)) < \epsilon,$$

for all η satisfying

$$\sigma_1(\eta, \eta_0) < \delta.\text{''}[27]$$

Example 2.1.13.

Consider a MS $\mathcal{U} = [0,\ 1]$, with usual metric. A mapping $\Upsilon\colon \mathcal{U} \longrightarrow \mathcal{U}$ is defined by:

$$\Upsilon(\eta) = 2\eta + 1$$

for all η in $[0,\ 1]$. Then, Υ is continuous on $\mathcal{U}$.

2.2 Banach Contraction Principle (BCP)

The Banach contraction principle (BCP), or Banach FP theorem, is a basic principle in mathematical analysis, particularly functional analysis. By formalizing the concept of contraction mappings in MS, this principle provides a rigorous and reliable approach to illustrating the convergence of iterative processes to a FP. The principle's significance stems from its ability to guarantee the existence of a unique FP for a contraction mapping, providing a solid foundation for solving equations, modeling dynamic systems, and informing iterative numerical methods. With far-reaching implications in various fields, including nonlinear analysis, numerical analysis, dynamical systems and optimization, the BCP is a fundamental concept in mathematics. It's broad applicability, simplicity and elegance make it a cornerstone of modern mathematics.

Definition 2.2.1.

"A FP of a mapping $\Upsilon\colon \mathcal{U} \longrightarrow \mathcal{U}$ of a set $\mathcal{U}$ into itself is an $\eta \in \mathcal{U}$ which is mapped onto itself (is "kept fixed" by Υ), that is,

$$\Upsilon\eta = \eta$$

the image $\Upsilon\eta$ coincides with η."[27]

For real valued functions, FPs are the points of intersection of the line $\xi = \eta$ and the function $\xi = \Upsilon(\eta)$. A mapping may have one, multiple, or no FP.

Some examples of FP are illustrated below:

Example 2.2.2.

Suppose Υ: $\mathbb{R} \longrightarrow \mathbb{R}$ be a mapping and defined as

$$\Upsilon(\eta) = \frac{\eta}{6} + 5,$$

then Υ possesses a unique FP $\eta = 6$. The graphical representation is given in Figure 2.1.

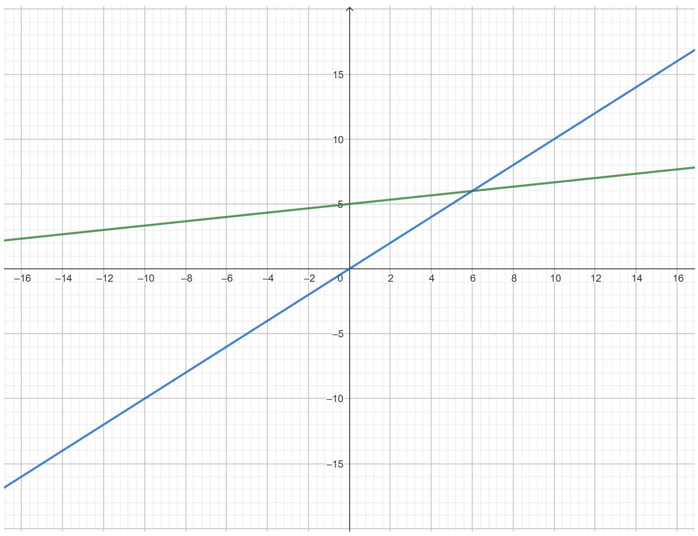

FIGURE 2.1: Graph of $\Upsilon(\eta) = \frac{\eta}{6} + 5$

Example 2.2.3.

Suppose a mapping Υ: $\mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$\Upsilon(\eta) = \eta + 5,$$

then Υ doesnot possesses any FP. The graphical representation is given in Figure 4.2.

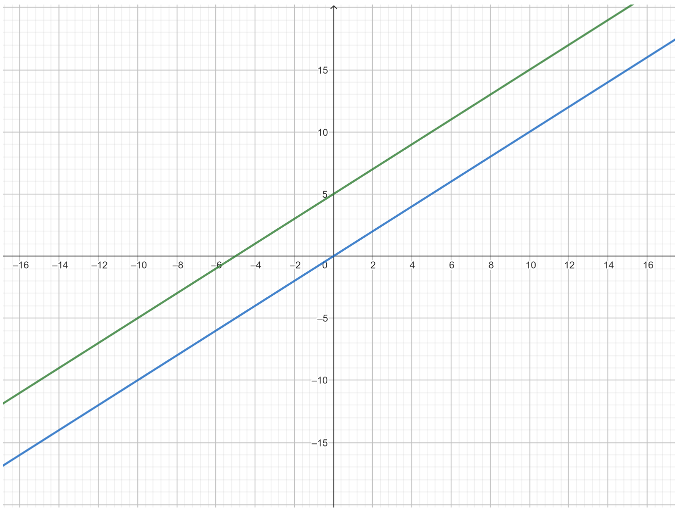

FIGURE 2.2: Graph of $\Upsilon(\eta) = \eta + 5$

Example 2.2.4.

Suppose a mapping $\Upsilon\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$\Upsilon(\eta) = \eta^2 - 4\eta + 4,$$

then Υ has two FPs, $\eta = 1, 4$. The graphical representation is given in Figure 2.3.

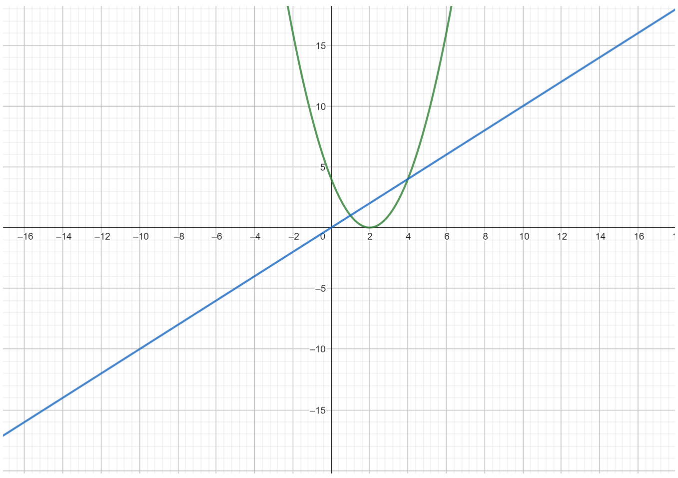

FIGURE 2.3: Graph of $\Upsilon(\eta) = \eta^2 - 4\eta + 4$

Example 2.2.5.

Suppose a mapping $\Upsilon\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$\Upsilon(\eta) = \eta + \sin\eta,$$

then Υ possesses infinite many FPs. The graphical representation is given in Figure 2.4.

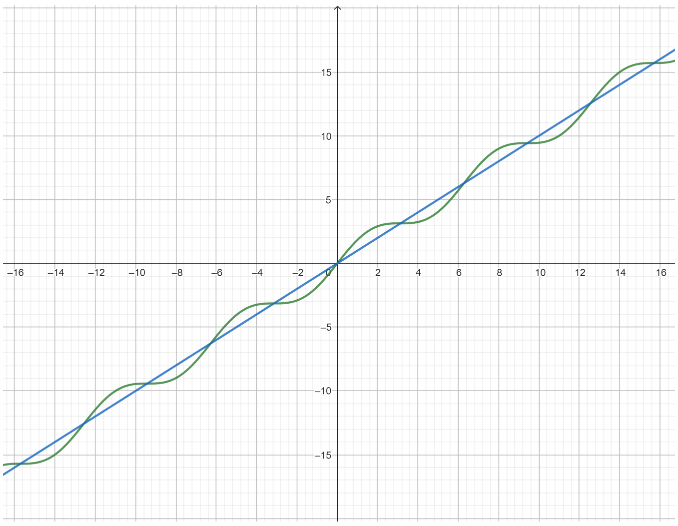

FIGURE 2.4: Graph of $\Upsilon(\eta) = \eta + \sin\eta$

Definition 2.2.6.

"Let $(\mathcal{U}, \sigma)$ be a MS. Then a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is called a contraction mapping on $\mathcal{U}$ if there exists a positive real number $\alpha \in [0,1)$ such that

$$\sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha\sigma(\eta, \xi), \qquad \forall\ \eta, \xi \in \mathcal{U}. \tag{2.1}$$

Geometrically, this means that any points η and ξ have images that are closer together than those points η and ξ; more precisely, the ratio $\frac{\sigma(\Upsilon\eta,\Upsilon\xi)}{\sigma(\eta,\xi)}$ does not exceed a constant α which is strictly less than 1." [27]

Example 2.2.7.

Consider the MS $\mathcal{U} = \mathbb{R}$, with metric $\sigma(\eta, \xi) = |\eta - \xi|$. Define a function Υ: $\mathcal{U} \longrightarrow \mathcal{U}$ as follows:

$$\Upsilon(\eta) = \frac{2\eta}{3} + 1 \qquad \forall\ \eta \in \mathcal{U}.$$

Then

$$\sigma(\Upsilon\eta, \Upsilon\xi) = \left|\frac{2\eta}{3} + 1 - (\frac{2\xi}{3} + 1)\right|,$$

$$
\begin{aligned}
&= \left| \frac{2\eta}{3} + 1 - \frac{2\xi}{3} - 1 \right|, \\
&= \left| \frac{2\eta}{3} - \frac{2\xi}{3} \right|, \\
&= \frac{2}{3}|\eta - \xi|, \\
&= \frac{2}{3}\sigma(\eta, \xi).
\end{aligned}
$$

Thus, Υ is a contraction mapping with $\alpha = \frac{2}{3}$.

Example 2.2.8.

Consider a MS $\mathcal{U} = [0,\ 1]$, equipped with usual metric. Define a function Υ: $\mathcal{U} \longrightarrow \mathcal{U}$ as follows:

$$\Upsilon(\eta) = \frac{\eta + 1}{4} \qquad\qquad \forall\ \eta \in \mathsf{X}.$$

Then, Υ is a contraction mapping with $\alpha = \frac{1}{4}$.

Remark:

All contraction mapping posses continuity.

Example 2.2.9.

Consider the function $\Upsilon(\eta) = \frac{1}{2}\eta$ defined on the real numbers $\mathbb{R}$ with the standard Euclidean distance metric σ. Then,

$$\sigma(\Upsilon(\eta), \Upsilon(\xi)) = \left| \frac{1}{2}\eta - \frac{1}{2}\xi \right| = \frac{1}{2}|\eta - \xi| = \frac{1}{2}\sigma(\eta, \xi)$$

This shows that Υ satisfies the contraction mapping condition with $\alpha = \frac{1}{2}$. Since α is between 0 and 1, Υ is indeed a contraction mapping. Moreover, as a linear function, Υ is also continuous.

In 1922, Banach's revolutionary work introduced the concept of a FP theorem, ensuring uniqueness and existence of FP for contraction mappings.

Theorem 2.2.10.

"Consider a MS $\mathcal{U} = (\mathcal{U}, \sigma)$, where $\mathcal{U} \neq \emptyset$. Suppose that $\mathcal{U}$ is complete and let Υ: $\mathcal{U} \longrightarrow \mathcal{U}$ be a contraction on $\mathcal{U}$. Then Υ has precisely one FP." [27]

Example 2.2.11.

Consider $\mathcal{U} = \mathbb{R}$, and $(\mathcal{U}, \sigma)$ be a usual MS. Now, define a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ as:

$$\Upsilon(\eta) = \frac{\eta}{5} + 2 \qquad\qquad \text{where,} \quad \eta \in \mathcal{U}.$$

Here, Υ is a contraction mapping, where $\alpha = \frac{1}{5}$. Then, Υ possess only one FP i.e $\eta = \frac{5}{2}$.

2.3 b-Metric Space (b-MS)

The b-MS that represent a significant extension of classical MS introduced by Bakhtin [15]. By relaxing the traditional triangle inequality, these spaces accommodate more flexible distance measurements, governed by a constant $\mathsf{s} \geq 1$. This generalized framework enables the analysis of complex systems and non-traditional distances, with profound implications for FP theory.

Definition 2.3.1.

"Let $\mathcal{U}$ be a nonempty set and let $\mathsf{s} \geq 1$ be a given real number. A function $\sigma_{\mathsf{b}} : \mathcal{U} \times \mathcal{U} \to \mathbb{R}^+$ is called a b-metric provided that, for all $\eta, \xi, \zeta \in \mathcal{U}$,

$(\mathsf{M}_{\mathsf{b}_1})$: $\sigma_{\mathsf{b}}(\eta, \xi) \geq 0$,

$(\mathsf{M}_{\mathsf{b}_2})$: $\sigma_{\mathsf{b}}(\eta, \xi) = 0$ if and only if $\eta = \xi$,

$(\mathsf{M}_{\mathsf{b}_3})$: $\sigma_{\mathsf{b}}(\eta, \xi) = \sigma_{\mathsf{b}}(\xi, \eta)$,

$(\mathsf{M}_{\mathsf{b}_4})$: $\sigma_{\mathsf{b}}(\eta, \zeta) \leq \mathsf{s}\,[\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta)]$.

The pair $(\mathcal{U}, \sigma_{\mathsf{b}})$ is called a b-MS."[15]

In general, b-metric is not continuous function. However, throughout the thesis, we will assume that the b-MS is continuous.

Remark:

The definition of b-MS is an expansion of the standard MS. Using $\mathsf{s} = 1$ in Definition (2.3.1) yields the definition of the MS.

Some examples of b-MS as follows:

Example 2.3.2.

Consider $\mathcal{U} = \mathbb{R}$, the mapping $\Upsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined $\forall\ \eta_1, \eta_2 \in \mathcal{U}$ as:

$$\Upsilon(\eta_1, \eta_2) = |\eta_1 - \eta_2|^2.$$

Then,

$(\mathsf{M_{b_1}})$: $\Upsilon(\eta_1, \eta_2) = |\eta_1 - \eta_2|^2 \geq 0$, $\forall\ \eta_1, \eta_2 \in \mathcal{U}$.

As, absolute value is always positive. so, $\Upsilon(\eta_1, \eta_2) \geq 0$.

$(\mathsf{M_{b_2}})$: $\Upsilon(\eta_1, \eta_2) = 0 \Leftrightarrow |\eta_1 - \eta_2|^2 = 0 \Leftrightarrow \eta_1 - \eta_2 = 0 \Leftrightarrow \eta_1 = \eta_2$.

$(\mathsf{M_{b_3}})$: Now,

$$\begin{aligned}\Upsilon(\eta_1, \eta_2) &= |\eta_1 - \eta_2|^2,\\ &= |-(\eta_2 - \eta_1)|^2,\\ &= |\eta_2 - \eta_1|^2,\\ &= \Upsilon(\eta_2, \eta_1).\end{aligned}$$

$\Rightarrow$ Symmetric property holds.

$(\mathsf{M_{b_4}})$: $\forall\ \eta_1,\ \eta_2,\ \eta_3 \in \mathcal{U}$

$$\begin{aligned}\Upsilon(\eta_1, \eta_3) &= |\eta_1 - \eta_3|^2,\\ &= |\eta_1 - \eta_2 + \eta_2 - \eta_3|^2.\end{aligned}$$

By using the inequality: $(\mathsf{a} + \mathsf{b})^{\mathsf{p}} \leq 2^{\mathsf{p}-1}(\mathsf{a}^{\mathsf{p}} + \mathsf{b}^{\mathsf{p}}), \quad where\ \ \mathsf{p} \geq 1.$

$$\begin{aligned}\Upsilon(\eta_1, \eta_3) &\leq 2(|\eta_1 - \eta_2|^2 + |\eta_2 - \eta_3|^2),\\ &\leq 2(\Upsilon(\eta_1, \eta_2) + \Upsilon(\eta_2, \eta_3)).\end{aligned}$$

$\Rightarrow$ $(\mathcal{U}, \Upsilon)$ is a b-MS, with $\mathsf{s} = 2$.

Example 2.3.3.

Consider a set $\mathcal{U} = l_{\mathsf{p}}(\mathbb{R})$, where $0 < \mathsf{p} < 1$, and

$$l_{\mathsf{p}}(\mathbb{R}) = \{\{\eta_{\mathsf{n}}\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |\eta_{\mathsf{n}}|^{\mathsf{p}} < \infty\}.$$

The distance function σ_{b}: $\mathcal{U} \times \mathcal{U} \to \mathbb{R}^{+}$ is defined as follows:

$$\sigma_{\mathsf{b}}(\eta, \xi) = \left(\sum_{n=1}^{\infty} |\eta_{\mathsf{n}} - \xi_{\mathsf{n}}|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}},$$

where $\eta = \{\eta_{\mathsf{n}}\}$, $\xi = \{\xi_{\mathsf{n}}\}$. Then,

$(\mathsf{M}_{\mathsf{b}_1})$:

$$\sigma_{\mathsf{b}}(\eta, \xi) = \left(\sum_{n=1}^{\infty} |\eta_{\mathsf{n}} - \xi_{\mathsf{n}}|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}} \geq 0, \text{ where } 0 < \mathsf{p} < 1.$$

Absolute value is always non-negative.

$(\mathsf{M}_{\mathsf{b}_2})$: Consider,

$$\begin{aligned} \sigma_{\mathsf{b}}(\eta, \xi) &= 0, \\ \Leftrightarrow \left(\sum_{n=1}^{\infty} |\eta_{\mathsf{n}} - \xi_{\mathsf{n}}|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}} &= 0, \\ \Leftrightarrow |\eta_{\mathsf{n}} - \xi_{\mathsf{n}}|^{\mathsf{p}} &= 0, \\ \Leftrightarrow \eta_{\mathsf{n}} - \xi_{\mathsf{n}} &= 0, \\ \Leftrightarrow \eta_{\mathsf{n}} &= \xi_{\mathsf{n}}. \end{aligned}$$

$(\mathsf{M}_{\mathsf{b}_3})$: To prove symmetric property, consider,

$$\begin{aligned} \sigma_{\mathsf{b}}(\eta, \xi) &= \left(\sum_{n=1}^{\infty} |\eta_{\mathsf{n}} - \xi_{\mathsf{n}}|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}}, \\ &= \left(\sum_{n=1}^{\infty} |-(\xi_{\mathsf{n}} - \eta_{\mathsf{n}})|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}}, \\ &= \left(\sum_{n=1}^{\infty} |\xi_{\mathsf{n}} - \eta_{\mathsf{n}}|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}}, \\ &= \sigma_{\mathsf{b}}(\xi, \eta). \end{aligned}$$

$\Rightarrow$ Symmetric property holds.

$(\mathsf{M}_{\mathsf{b}_4})$: Now, consider,

$$\sigma_{\mathsf{b}}(\eta, \zeta) = \left(\sum_{n=1}^{\infty} |\eta_{\mathsf{n}} - \zeta_{\mathsf{n}}|^{\mathsf{p}} \right)^{\frac{1}{\mathsf{p}}},$$

$$= \left(\sum_{n=1}^{\infty} |\eta_n - \xi_n + \xi_n - \zeta_n|^p \right)^{\frac{1}{p}}.$$

Using inequality: $(a + b)^p \leq (a^p + b^p)$, *where* $0 < p < 1$.

$$\leq \left(\sum_{n=1}^{\infty} |\eta_n - \xi_n|^p + \sum_{n=1}^{\infty} |\xi_n - \zeta_n|^p \right)^{\frac{1}{p}}.$$

Now by using the inequality: $(a + b)^{\frac{1}{p}} \leq 2^{\frac{1}{p}-1}(a^{\frac{1}{p}} + b^{\frac{1}{p}})$, *where* $0 < p < 1$.

$$\leq 2^{\frac{1}{p}-1} \left[\left(\sum_{n=1}^{\infty} |\eta_n - \xi_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |\xi_n - \zeta_n|^p \right)^{\frac{1}{p}} \right],$$

$$\leq 2^{\frac{1}{p}} \left[\left(\sum_{n=1}^{\infty} |\eta_n - \xi_n|^p \right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |\xi_n - \zeta_n|^p \right)^{\frac{1}{p}} \right],$$

$$\sigma_b(\eta, \zeta) \leq 2^{\frac{1}{p}}(\sigma_b(\eta, \xi) + \sigma_b(\xi, \zeta)).$$

σ_b is a b-MS, with coefficient $s = 2^{\frac{1}{p}}$.

Definition 2.3.4.

"A sequence $\{\eta_n\}$ in a b-MS $\mathcal{U} = (\mathcal{U}, \sigma_b)$ is said to converge or to be convergent if there is an $\eta \in \mathcal{U}$ such that:

$$\lim_{n\to\infty} \sigma_b(\eta_n, \eta) = 0.$$

η is called the limit of $\{\eta_n\}$ and we write

$$\lim_{n\to\infty} \eta_n = \eta,$$

or, simply,

$$\eta_n \longrightarrow \eta.$$

Then, $\{\eta_n\}$ converges to η or has the limit η." [15]

Definition 2.3.5.

"A sequence $\{\eta_n\}$ in a b-MS $\mathcal{U} = (\mathcal{U}, \sigma_b)$ is said to be-Cauchy (or fundamental) if for every $\epsilon > 0$ there is an integer $\mathcal{N} = \mathcal{N}(\epsilon)$ such that:

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{m}}, \eta_{\mathsf{n}}) < \epsilon, \quad \text{for every} \quad \mathsf{m}, \mathsf{n} > \mathcal{N}.\text{"}[15]$$

Definition 2.3.6.

"A b-MS $\mathcal{U} = (\mathcal{U}, \sigma_{\mathsf{b}})$ is said to be complete b-MS if every Cauchy sequence in $\mathcal{U}$ converges (that is, has a limit which is an element in X)."[15]

2.4 F-Contraction mappings

In 2012, D. Wardowski [22] introduced F-contraction mappings, a significant generalization of the BCP, facilitating efficient solutions to FP problems in MS.

Definition 2.4.1.

"Suppose, $\mathsf{F} : \mathbb{R}^+ \to \mathbb{R}$ be a function that satisfies the following:

(**F**-1):F is increasing, i.e., $\forall\ \eta, \xi \in \mathbb{R}^+$ such that $\eta < \xi, \ \Rightarrow\ \mathsf{F}(\eta) < \mathsf{F}(\xi)$.

(**F**-2):For any sequence $\{\eta_n\}_{n=1}^{\infty}$ of positive real numbers, $\lim\limits_{n\to\infty} \eta_n = 0 \Leftrightarrow$ $\lim\limits_{n\to\infty} \mathsf{F}(\eta_n) = -\infty$.

(**F**-3): There exist $k \in (0,1)$ such that $\lim\limits_{\alpha\to 0^+} (\alpha)^k \mathsf{F}(\alpha) = 0$." [22]

Definition 2.4.2.

"Let $(\mathcal{U}, \sigma)$ be a MS. A mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ is said to be a Wardowski F-contraction if there exists $\tau > 0$ such that

$$\sigma(\Upsilon\eta, \Upsilon\xi) > 0 \ \Rightarrow\ \tau + \mathsf{F}(\sigma(\Upsilon\eta, \Upsilon\xi)) \leq \mathsf{F}(\sigma(\eta, \xi)) \quad \forall\ \eta, \xi \in \mathcal{U} \text{ and } \mathsf{F} \in \mathcal{F}.\text{"}[22]$$

In 2015, Cosentine et al. [28] introduced a condition in Definition (2.4.1) to derive certain fixed-point results in b-MS. In this chapter, we further extend this definition by incorporating an additional condition into Definition (2.4.1).

(**F**-4):"Let $\mathsf{s} \geq 1$, be a real number. For each sequence $\{\beta_n\}_{n\in\mathbb{N}}$ of positive real numbers such that;

$$\tau + \mathsf{F}(\mathsf{s}^2\beta_n) \leq \mathsf{F}(\beta_{n-1}) \tag{2.2}$$

$\forall$ n $\in \mathbb{N}$ and some $\tau > 0$, then,

$$\tau + \mathsf{F}(\mathsf{s}^{\mathsf{n}}\beta_n) \leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-2}\beta_{n-1}).\text{''} \tag{2.3}$$

Throughout the thesis, $\mathcal{F}$ denotes the collection of mappings that satisfy (F-1) to (F-4).

Example 2.4.3.

Define a mapping $\mathsf{F} : \mathbb{R}^+ \to \mathbb{R}$ as $\mathsf{F}(\eta) = \ln(\eta)$ for $\eta > 0$, $\Upsilon : \mathcal{U} \to \mathcal{U}$ be defined as $\Upsilon(\eta) = \dfrac{\eta}{2}$ and σ be a usual metric. Then:

$$\sigma(\Upsilon\eta, \Upsilon\xi) > 0 \Rightarrow \tau + \mathsf{F}(\sigma(\Upsilon\eta, \Upsilon\xi)) \leq \mathsf{F}(\sigma(\eta, \xi)), \forall\ \eta, \xi \in \mathbb{R} \text{ and } \mathsf{F} \in \mathcal{F},$$

$$\tau + \ln\left|\frac{\eta}{2} - \frac{\xi}{2}\right| \leq \ln|\eta - \xi|,$$

by taking exponential bothsides, we obtain,

$$e^{\tau}.e^{\ln\left|\frac{\eta-\xi}{2}\right|} \leq e^{\ln|\eta-\xi|},$$

$$\Rightarrow \frac{1}{2}|\eta - \xi| \leq e^{-\tau}|\eta - \xi|.$$

$\Rightarrow \Upsilon$ is a F-contraction.

Wardowski's key result generalized the BCP as follows:

Theorem 2.4.4.

"Assume that $(\mathcal{U}, \sigma)$ is a complete MS and $\Upsilon : \mathcal{U} \to \mathcal{U}$ is an F-contraction. Then, Υ possesses a unique FP".[22]

Chapter 3

Fixed Point Theorem for Mappings Contracting Perimeters of Triangles

3.1 Introduction

In this chapter, a new type of mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ that is a mapping contracting perimeters of triangle (MCPT) is reviewed. The concept of MCPT is presented in this chapter and a fixed point (FP) theorem for these mappings is proved, using ideas from Banach's classical theorem. Furthermore, some examples are given to validate results.

3.2 Mapping Contracting Perimeters of Triangle (MCPT)

MCPT uses three points, unlike other contraction mappings that use two, and a condition is applied to the mapping Υ to prevent the occurrence of periodic points with prime period 2.

Definition 3.2.1.

Consider a metric space (MS) $(\mathcal{U}, \sigma)$ with at least three elements, i.e., $|\mathcal{U}| \geq 3$. A mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is said to contracting perimeters of triangles on $\mathcal{U}$ if there exists a constant $\alpha \in [0,1)$ such that the following inequality

$$\sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\xi, \Upsilon\zeta) + \sigma(\Upsilon\eta, \Upsilon\zeta) \leq \alpha(\sigma(\eta, \xi) + \sigma(\xi, \zeta) + \sigma(\eta, \zeta)) \tag{3.1}$$

holds for all possible combinations of three pairwise distinct points η, ξ, ζ in $\mathcal{U}$.

Remark:

It is essential for $\eta, \xi, \zeta \in \mathcal{U}$ is to be pairwise distinct. If they are not pairwise distinct, this definition reduces to the definition of a contraction mapping.

Proposition 3.2.2.

Let $(\mathcal{U}, \sigma)$ be a complete MS and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a MCPT. Then Υ is continuous.

Proof. Suppose that $(\mathcal{U}, \sigma)$ be a MS with $|\mathcal{U}| \geq 3$, $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a MCPT on $\mathcal{U}$ and let η_0 be an isolated point in $\mathcal{U}$. Then, clearly, Υ is continuous at η_0.

Now suppose that η_0 be a limit point. Let us show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $\sigma(\Upsilon\eta_0, \Upsilon\eta) < \epsilon$ whenever $\sigma(\eta_0, \eta) < \delta$. Since η_0 is a limit point, for every $\delta > 0$ there exists $\xi \in \mathcal{U}$ such that $\sigma(\eta_0, \xi) < \delta$. By using (3.1)

$$\begin{aligned} \sigma(\Upsilon\eta_0, \Upsilon\eta) &\leq \sigma(\Upsilon\eta_0, \Upsilon\eta) + \sigma(\Upsilon\eta_0, \Upsilon\xi) + \sigma(\Upsilon\eta, \Upsilon\xi) \\ &\leq \alpha(\sigma(\eta_0, \eta) + \sigma(\eta_0, \xi) + \sigma(\eta, \xi)). \end{aligned}$$

Using the triangular inequality $\sigma(\eta, \xi) \leq \sigma(\eta_0, \eta) + \sigma(\eta_0, \xi)$,

$$\begin{aligned} \sigma(\Upsilon\eta_0, \Upsilon\eta) &\leq \alpha(\sigma(\eta_0, \eta) + \sigma(\eta_0, \xi) + \sigma(\eta_0, \eta) + \sigma(\eta_0, \xi)) \\ &\leq 2\alpha(\sigma(\eta_0, \eta) + \sigma(\eta_0, \xi)) \\ &< 2\alpha(\delta + \delta) \\ &= 4\alpha\delta. \end{aligned}$$

Setting $\quad \delta = \epsilon/(4\alpha)$,

$$\sigma(\Upsilon\eta_0, \Upsilon\eta) < \epsilon.$$

Hence, Υ is continuous. $\square$

Definition 3.2.3.

Consider a mapping Υ on the MS $\mathcal{U}$. A point $\eta \in \mathcal{U}$ is called a periodic point of period n if:

$$\Upsilon^{\mathsf{n}}(\eta) = \eta,$$

where n is the smallest positive integer satisfying this, is called the prime period of η.

Theorem 3.2.4.

Consider a complete MS $(\mathcal{U}, \sigma)$ with at least three elements, i.e., $|\mathcal{U}| \geq 3$ and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a MCPT on $\mathcal{U}$. Then Υ has a FP if and only if Υ does not have periodic points with a prime period 2. Moreover, Υ possesses at most two FPs.

Proof. Suppose that no point is periodic with prime period 2 under the mapping Υ. Our objective is to show that Υ has a FP. Let $\eta_0 \in \mathcal{U}$ and,
$\Upsilon\eta_0 = \eta_1, \Upsilon\eta_1 = \eta_2, \cdots, \Upsilon\eta_{\mathsf{n}} = \eta_{\mathsf{n}+1}, \cdots$.
Assume that, for all $\mathsf{i} = 0, 1, 2, \cdots$, there are no FP of the mapping Υ among the points η_{i}. Our goal is to demonstrate the distinctness of every point η_{i}. We have $\eta_{\mathsf{i}} \neq \eta_{\mathsf{i}+1} = \Upsilon\eta_{\mathsf{i}}$ because η_{i} is not a FP of Υ. We also know that $\eta_{\mathsf{i}+2} = \Upsilon(\Upsilon(\eta_{\mathsf{i}})) \neq \eta_{\mathsf{i}}$ since Υ lacks any periodic points of prime period 2. Moreover, $\eta_{\mathsf{i}+1} \neq \eta_{\mathsf{i}+2} = \Upsilon\eta_{\mathsf{i}+1}$, since $\eta_{\mathsf{i}+1}$ is not a FP of Υ. As a result, pairwise distinct points are η_{i}, $\eta_{\mathsf{i}+1}$, and $\eta_{\mathsf{i}+2}$. Furthermore, suppose

$$\begin{aligned}
\wp_0 &= \sigma(\eta_0, \eta_1) + \sigma(\eta_1, \eta_2) + \sigma(\eta_2, \eta_0),\\
\wp_1 &= \sigma(\eta_1, \eta_2) + \sigma(\eta_2, \eta_3) + \sigma(\eta_3, \eta_1),\\
\wp_2 &= \sigma(\eta_2, \eta_3) + \sigma(\eta_3, \eta_4) + \sigma(\eta_4, \eta_2),\\
&\vdots\\
\wp_{\mathsf{n}} &= \sigma(\eta_{\mathsf{n}}, \eta_{\mathsf{n}+1}) + \sigma(\eta_{\mathsf{n}+1}, \eta_{\mathsf{n}+2}) + \sigma(\eta_{\mathsf{n}+2}, \eta_{\mathsf{n}}),\\
&\vdots
\end{aligned}$$

applying the contraction condition to the pairwise distinct points η_{i}, $\eta_{\mathsf{i}+1}$, and $\eta_{\mathsf{i}+2}$, we obtain

$$\begin{aligned}\sigma(\eta_1,\eta_2)+\sigma(\eta_2,\eta_3)+\sigma(\eta_1,\eta_3) &= \sigma(\Upsilon\eta_0,\Upsilon\eta_1)+\sigma(\Upsilon\eta_1,\Upsilon\eta_2)+\sigma(\Upsilon\eta_0,\Upsilon\eta_2),\\ &\leq \alpha(\sigma(\eta_0,\eta_1)+\sigma(\eta_1,\eta_2)+\sigma(\eta_0,\eta_2)\\ \Rightarrow \wp_1 &\leq \alpha\wp_0.\end{aligned}$$

Similarly,

$$\begin{aligned}\wp_2 &\leq \alpha\wp_1,\\ \wp_3 &\leq \alpha\wp_2,\\ &\vdots\\ \wp_n &\leq \alpha\wp_{n-1}.\\ &\vdots\end{aligned}$$

Since, $\alpha \in [0,1)$, Therefore

$$\wp_0 > \wp_1 > \cdots > \wp_n > \cdots . \tag{3.2}$$

Assume that $j \geq 3$ is the smallest natural number such that $\eta_j = \eta_i$ for some i satisfying $0 \leq i < j-2$. Then, we have $\eta_{j+1} = \eta_{i+1}$ and $\eta_{j+2} = \eta_{i+2}$. Consequently, $\wp_i = \wp_j$ which contradicts (3.2). This shows that all η_i's are distinct.
Further, we have to show that $\{\eta_n\}$ is a Cauchy sequence. It is clear that

$$\begin{aligned}\sigma(\eta_1,\eta_2) &\leq \sigma(\eta_0,\eta_1)+\sigma(\eta_1,\eta_2)+\sigma(\eta_2,\eta_0),\\ \sigma(\eta_1,\eta_2) &\leq \wp_0.\end{aligned}$$

Likewise, one can obtain,

$$\begin{aligned}\sigma(\eta_2,\eta_3) &\leq \wp_1 \leq \alpha\wp_0,\\ \sigma(\eta_3,\eta_4) &\leq \wp_2 \leq \alpha\wp_1 \leq \alpha^2\wp_0,\\ &\vdots\\ \sigma(\eta_n,\eta_{n+1}) &\leq \wp_{n-1} \leq \alpha^{n-1}\wp_0,\\ \sigma(\eta_{n+1},\eta_{n+2}) &\leq \wp_n \leq \alpha^n\wp_0,\\ &\vdots\end{aligned}$$

Now, by the triangular inequality,

$$\sigma(\eta_n, \eta_{n+p}) \leq \sigma(\eta_n, \eta_{n+1}) + \sigma(\eta_{n+1}, \eta_{n+p}),$$
$$\sigma(\eta_n, \eta_{n+p}) \leq \sigma(\eta_n, \eta_{n+1}) + \sigma(\eta_{n+1}, \eta_{n+2}) + \sigma(\eta_{n+2}, \eta_{n+p}).$$

Continuing in this way,

$$\begin{aligned}
\sigma(\eta_n, \eta_{n+p}) \leq& \sigma(\eta_n, \eta_{n+1}) + \sigma(\eta_{n+1}, \eta_{n+2}) + \cdots + \sigma(\eta_{n+p-1}, \eta_{n+p}) \\
\leq& \alpha^{n-1}\wp_0 + \alpha^n \wp_0 + \cdots + \alpha^{n+p-2}\wp_0 \\
=& \alpha^{n-1}(1 + \alpha + \cdots + \alpha^{p-1})\wp_0 \\
=& \alpha^{n-1}\frac{1-\alpha^p}{1-\alpha}\wp_0. \\
\Rightarrow \sigma(\eta_n, \eta_{n+p}) <& \alpha^{n-1}\frac{1}{1-\alpha}\wp_0, \quad \text{since } \alpha \in [0, 1).
\end{aligned}$$

Taking limit $n \longrightarrow \infty$, we get $\sigma(\eta_n, \eta_{n+p}) \longrightarrow 0$ for every $p > 0$.
It follows that $\{\eta_n\}$ is a Cauchy sequence. As $(\mathcal{U}, \sigma)$ is a complete MS, $\{\eta_n\}$ has a limit η^* in $\mathcal{U}$.

To show that $\Upsilon\eta^* = \eta^*$, we apply the triangular inequality and inequality (3.1)

$$\begin{aligned}
\sigma(\eta^*, \Upsilon\eta^*) \leq& \sigma(\eta^*, \eta_n) + \sigma(\eta_n, \Upsilon\eta^*) \\
=& \sigma(\eta^*, \eta_n) + \sigma(\Upsilon\eta_{n-1}, \Upsilon\eta^*) \\
\leq& \sigma(\eta^*, \eta_n) + \sigma(\Upsilon\eta_{n-1}, \Upsilon\eta^*) + \sigma(\Upsilon\eta_{n-1}, \Upsilon\eta_n) + \sigma(\Upsilon\eta_n, \Upsilon\eta^*) \\
\leq& \sigma(\eta^*, \eta_n) + \alpha(\sigma(\eta_{n-1}, \eta^*) + \sigma(\eta_{n-1}, \eta_n) + \sigma(\eta_n, \eta^*)).
\end{aligned}$$

Taking the limit as $n \to \infty$, we note that each term in the preceding sum vanishes, yielding:
$\sigma(\eta^*, \Upsilon\eta^*) = 0$.
Therefore, we conclude that $\Upsilon\eta^* = \eta^*$.

In order to prove there exists at most two FPs. Assume by contradiction that Υ has at least three pairwise distinct FPs, say η, ξ, and ζ. That is:
$\Upsilon\eta = \eta$, $\Upsilon\xi = \xi$ and $\Upsilon\zeta = \zeta$.
Then by contraction condition,

$$\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta)=\sigma(\Upsilon\eta,\Upsilon\xi)+\sigma(\Upsilon\xi,\Upsilon\zeta)+\sigma(\Upsilon\eta,\Upsilon\zeta),$$
$$\leq \alpha(\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta)),$$

which is a contradiction. Thus, we conclude that Υ possesses at most two FPs.

Conversely, Suppose that Υ possesses a FP η^*. We have to prove that there are no periodic points in Υ with a prime period 2. For this, suppose by contradiction that Υ has a periodic point η of prime period 2, that is $\Upsilon(\Upsilon\eta)=\eta$. Define $\xi=\Upsilon\eta$ and $\eta=\Upsilon\xi$. Then

$$\sigma(\Upsilon\eta,\Upsilon\xi)+\sigma(\Upsilon\xi,\Upsilon\eta^*)+\sigma(\Upsilon\eta,\Upsilon\eta^*)=\sigma(\xi,\eta)+\sigma(\eta,\eta^*)+\sigma(\xi,\eta^*),$$

which contradicts to (3.1). Thus, our supposition is wrong which implies that Υ does not have periodic points with a prime period 2. □

Remark:

Assume that the mapping Υ has a FP η^*, which is a limit of some iterative sequence, following the assumptions of Theorem (3.4). In order to ensure that $\eta_n \neq \eta^*$ for all $n = 1,2,\cdots$, for this let η_0 be an initial point, and iterative sequence will be $x_1=\Upsilon\eta_0, \eta_2=\Upsilon\eta_1,\cdots$. In that case, there is only one FP, η^*. Assume, in fact, that Υ has another FP η^{**}. For every $n = 1,2,\cdots$, it is evident that $\eta_n \neq \eta^{**}$. Thus, for every $n = 1,2,\cdots$, we have that the points η^*, η^{**}, and η_n are pairwise distinct.

Now, by contraction condition

$$\sigma(\eta^*,\eta^{**})+\sigma(\eta^*,\eta_{n+1})+\sigma(\eta^{**},\eta_{n+1})=\sigma(\Upsilon\eta^*,\Upsilon\eta^{**})+\sigma(\Upsilon\eta^*,\Upsilon\eta_n)+\sigma(\Upsilon\eta^{**},\Upsilon\eta_n)$$
$$\leq \alpha(\sigma(\eta^*,\eta^{**})+\sigma(\eta^*,\eta_n)+\sigma(\eta^{**},\eta_n)).$$

As $n \longrightarrow 0$ we get $\sigma(\eta^*,\eta_{n+1}) \longrightarrow 0$, $\sigma(\eta^*,\eta_n) \longrightarrow 0$, $\sigma(\eta^{**},\eta_{n+1}) \longrightarrow \sigma(\eta^{**},\eta^*)$ and $\sigma(\eta^{**},\eta_n) \longrightarrow \sigma(\eta^{**},\eta^*)$. Hence,

$$2\sigma(\eta^*,\eta^{**}) \leq \alpha 2\sigma(\eta^*,\eta^{**})$$
$$1 \leq \alpha,$$

which contradicts to condition (3.1) i.e. $0 \leq \alpha < 1$. This shows that $\eta^* = \eta^{**}$. Therefore, Υ has a unique FP.

The following is an example of a MCPT with exactly two FPs.

Example 3.2.5.

Let $\mathcal{U} = \{\eta, \xi, \zeta\}$, Suppose $\sigma : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}$ be defined as follows:

$$\sigma(\eta, \xi) = 1, \ \sigma(\xi, \zeta) = 1, \ \sigma(\eta, \zeta) = 1.$$

Then $(\mathcal{U}, \sigma)$ is a MS. Now define $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ by:

$$\Upsilon\eta = \eta, \Upsilon\xi = \xi \text{ and } \Upsilon\zeta = \eta.$$

Consider,

$$\begin{aligned} \sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\xi, \Upsilon\zeta) + \sigma(\Upsilon\eta, \Upsilon\zeta) &= \sigma(\eta, \xi) + \sigma(\xi, \zeta) + \sigma(\eta, \eta) \\ &= 1 + 1 + 0 \\ &\leq \alpha(\sigma(\eta, \xi) + \sigma(\xi, \zeta) + \sigma(\eta, \zeta)), \end{aligned}$$

for $\alpha \in (\frac{2}{3}, 1)$.

Now, we have to show that Υ have no periodic points of prime period 2.

$$\begin{aligned} \Upsilon\zeta &= \eta \\ \Upsilon(\Upsilon\zeta) &= \Upsilon(\eta) \\ \Upsilon^2(\zeta) &= \eta. \end{aligned}$$

Hence, Υ has no periodic points with prime period 2.

Therefore, all the assumptions of Theorem (3.4) are true.Thus, Υ has exactly two FPs namely η and ξ.

The following example demonstrates a perimeter-contracting map on triangles with no FP, if Υ has periodic point of prime period 2.

Example 3.2.6.

Let $\mathcal{U} = \{\eta, \xi, \zeta\}$, Define $\sigma : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{R}$ as follows:

$$\sigma(\eta, \xi) = 1,\ \sigma(\xi, \zeta) = 1,\ \sigma(\eta, \zeta) = 1.$$

Then one can prove that $(\mathcal{U}, \sigma)$ is a MS. Now define $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\Upsilon\eta = \xi, \Upsilon\xi = \eta \text{ and } \Upsilon\zeta = \eta.$$

Here, η and ξ are periodic points with prime period 2.
Since,

$$\begin{aligned} \Upsilon\eta =& \xi, \\ \Upsilon(\Upsilon\eta) =& \Upsilon(\xi), \\ \Upsilon^2(\eta) =& \eta. \end{aligned}$$

Also,

$$\begin{aligned} \Upsilon\xi =& \eta \\ \Upsilon(\Upsilon\xi) =& \Upsilon(\eta) \\ \Upsilon^2(\xi) =& \xi. \end{aligned}$$

As η and ξ are periodic points with prime period 2. One can see that Υ have no FP.

The following corollary provides a simple and direct proof of Banach FP theorem.

Corollary 3.2.1.

Suppose that $(\mathcal{U}, \sigma)$ be a complete MS where $\mathcal{U} \neq \emptyset$. Then the contraction mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ guarantees that Υ has a unique FP.

Proof. Suppose that $\mathcal{U}$ be a complete MS and $|\mathcal{U}| = 1$,
Let $\mathcal{U} = \{\eta\}$,
In this case, since $\mathcal{U}$ has only one element $\mathcal{U}$, the mapping Υ must map η to itself.
i-e $\Upsilon\eta = \eta$.

This is because there are no other elements in $\mathcal{U}$ for $\Upsilon\eta$ to map. So, we can see that η is indeed a FP of Υ, and it is unique. Therefore, the Banach FP theorem holds for a set $\mathcal{U}$ of order 1.

Now, if $|\mathcal{U}| = 2$,

Suppose that $\mathcal{U} = \{\eta, \xi\}$ and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a contraction mapping.

Assume, if possible, that Υ has two distinct FPs η and ξ. Then, $\Upsilon\eta = \eta$ and $\Upsilon\xi = \xi$.

By the definition of a contraction mapping, we have:

$$\sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha\sigma(\eta, \xi),$$
$$\Rightarrow \sigma(\eta, \xi) \leq \alpha\sigma(\eta, \xi).$$

Since $\alpha < 1$, we can divide both side by $\sigma(\eta, \xi)$, which is non zero since $\eta \neq \xi$.

$$1 \leq \alpha.$$

This contradicts the assumption that $\alpha < 1$.

Therefore, our assumption that Υ has two distinct FP is false. Hence, Υ can have at most one FP.

So, for $|\mathcal{U}| = 1,2$ the proof is complete.

Assuming $\mathcal{U}$ has at least three elements, i.e., $|\mathcal{U}| \geq 3$, if Υ has some $\eta \in \mathcal{U}$ with prime period 2, i.e., $\Upsilon(\Upsilon(\eta)) = \eta$, then

$$\sigma(\eta, \Upsilon\eta) = \sigma(\Upsilon\eta, \eta) = \sigma(\Upsilon\eta, \Upsilon(\Upsilon\eta)),$$

which is contradiction of (2.1). It follows that Υ has no periodic points with prime period 2. Considering pairwise distinct elements η, ξ, $\zeta \in \mathcal{U}$, and applying (2.1), we have:

$\sigma(\Upsilon(\eta), \Upsilon(\xi)) \leq \alpha\sigma(\eta, \xi)$, $\sigma(\Upsilon(\xi), \Upsilon(\zeta)) \leq \alpha\sigma(\xi, \zeta)$ and $\sigma(\Upsilon(\eta), \Upsilon(\zeta)) \leq \alpha\sigma(\eta, \zeta)$, which implies that

$\sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\xi, \Upsilon\zeta) + \sigma(\Upsilon\eta, \Upsilon\zeta) \leq \alpha(\sigma(\eta, \xi) + \sigma(\xi, \zeta) + \sigma(\eta, \zeta))$.

Hence, Υ is a MCPT on $\mathcal{U}$. By Theorem 3.4 a FP exists for the mapping Υ.

Uniqueness: Suppose that Υ has two FPs η and η^*. i-e $\Upsilon\eta = \eta$ and $\Upsilon\eta^* = \eta^*$.

Now, by the definition of metric and given assumption:

$$0 < \sigma(\eta, \eta^*) = \sigma(\Upsilon\eta, \Upsilon\eta^*),$$
$$\leq \alpha\sigma(\eta, \eta^*),$$

where $\alpha \in [0, 1)$,

which is only possible when

$\sigma(\eta, \eta^*) = 0 \Rightarrow \eta = \eta^*$.

Hence, Υ has a unique FP. □

Recall that for MS $\mathcal{U}$, a point $\eta \in \mathcal{U}$ is called a limit point of $\mathcal{U}$ if every neighborhood of η contains an infinite number of points from $\mathcal{U}$.

Proposition 3.2.7.

Consider a MS $(\mathcal{U}, \sigma)$ with at least three elements i.e, $|U| \geq 3$, and $\Upsilon : \mathcal{U} \to \mathcal{U}$ contracts triangle perimeters. Then, for all points $\xi \in \mathcal{U}$, Υ is a contraction mapping if η is a limit point of $\mathcal{U}$.

Proof. Consider an accumulation point $\eta \in \mathcal{U}$ and any point $\xi \in \mathcal{U}$. If $\xi = \eta$,then (2.1) is obviously satisfied. Now, consider the case where $\xi \neq \eta$. As η is a limit point, which implies the existence of a sequence $\{\zeta_n\}$ converging to η, satisfying $\zeta_n \neq \eta$, $\zeta_n \neq \xi$ with all distinct ζ_n. Consequently, applying (3.1) establishes the inequality:

$$\sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\xi, \Upsilon\zeta_n) + \sigma(\Upsilon\eta, \Upsilon\zeta_n) \leq \alpha(\sigma(\eta, \xi) + \sigma(\xi, \zeta_n) + \sigma(\eta, \zeta_n)),$$

which satisfies for all $n \in \mathbb{N}$. As $\sigma(\eta, \zeta_n) \longrightarrow 0$ and continuity of MS implies $\sigma(\xi, \zeta_n) \to \sigma(\eta, \xi)$. By the continuity of Υ, $\sigma(\Upsilon\eta, \Upsilon\zeta_n) \to 0$ and $\sigma(\Upsilon\xi, \Upsilon\zeta_n) \to \sigma(\Upsilon\eta, \Upsilon\xi)$. Hence, Taking limit $n \to \infty$, we get

$$\sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha(\sigma(\eta, \xi) + \sigma(\eta, \xi))$$
$$2\sigma(\Upsilon\eta, \Upsilon\xi) \leq 2\alpha\sigma(\eta, \xi)$$
$$\sigma(\Upsilon\eta, \Upsilon\xi) \leq \alpha\sigma(\eta, \xi).$$

Hence, if η is a limit point of $\mathcal{U}$, then $\forall\ \xi \in \mathcal{U}$, inequality (2.1) holds. □

Corollary 3.2.2.

Consider $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is a MCPT, and $(\mathcal{U}, \sigma)$ is a MS with at least three points i.e, $|\mathcal{U}| \geq 3$. Then Υ is a contracting mapping whenever every element of $\mathcal{U}$ is an accumulation point of $\mathcal{U}$.

In a MS $(\mathcal{U}, \sigma)$, ξ is an intermediate point for η and ζ whenever:

$$\sigma(\eta, \zeta) = \sigma(\eta, \xi) + \sigma(\xi, \zeta) \qquad \text{where} \quad \eta, \xi, \zeta \in \mathcal{U}. \tag{3.3}$$

Let us develop an example demonstrating the distinction between triangle-perimeter contraction and metric contraction.

Example 3.2.8.

Consider a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ that contracts triangle perimeters but exhibits non-contracting behavior in a MS $\mathcal{U}$.

Suppose $\mathcal{U}$ has countably infinite elements, $|\mathcal{U}| = \aleph_0$, specifically $\mathcal{U} = \{\eta^*, \eta_0, \eta_1, \ldots\}$. Let 'a' be a positive real number.

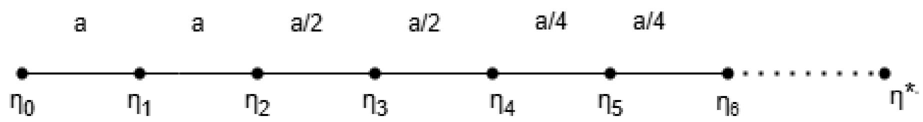

FIGURE 3.1: The points in the space $(\mathcal{U}, \sigma)$ with successive distances

Define σ on $\mathcal{U} \times \mathcal{U}$ as follows:

$$\sigma(\eta, \xi) = \begin{cases} \dfrac{\mathsf{a}}{2^{\lfloor i/2 \rfloor}}, & \text{if} \quad \eta = \eta_i, \xi = \eta_{i+1}, i = 1, 2, 3, \cdots., \\ \sigma(\eta_i, \eta_{i+1}) + \cdots + \sigma(\eta_{j-1}, \eta_j), & \text{if} \quad \eta = \eta_i, \xi = \eta_j, i + 1 < j, \\ 4\mathsf{a} - \sigma(\eta_0, \eta_i), & \text{if} \quad \eta = \eta_i, \xi = \eta^*, \\ 0, & \text{if} \quad \eta = \xi, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function defined as the greatest integer less than or equal to a given number.

Clearly, for every triplet of distinct points in $\mathcal{U}$, one point is situated between the remaining two, we can see in Fig.3.1. Furthermore, the space has a single accumulation point, η^*, and is therefore complete.

Consider the mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ given by $\Upsilon(\eta_i) = \eta_{i+1}$ for all $i \in \mathbb{N} \cup \{0\}$, and $\Upsilon(\eta^*) = \eta^*$. We can see by (2.1) that Υ is not contraction mapping, as $\sigma(x_{2n}, x_{2n+1}) = \sigma(\Upsilon\eta_{2n}, \Upsilon\eta_{2n+1})$, for all $n = 0,1,2,\cdots$.

Now we prove that in space $(\mathcal{U}, \sigma)$ the inequality (3.1) holds for each of the three pairwise distinct points. Consider the first triplets of points $\eta_i, \eta_j, \eta^* \in \mathcal{U}$ with $0 \leq i < j$. According to the definition of the metric σ we have

$$\sigma(\eta_i, \eta_j) + \sigma(\eta_j, \eta^*) = \sigma(\eta_i, \eta^*).$$

Adding $\sigma(\eta_i, \eta^*)$ both sides,

$$\begin{aligned}
\sigma(\eta_i, \eta_j) + \sigma(\eta_j, \eta^*) + \sigma(\eta_i, \eta^*) &= 2\sigma(\eta_i, \eta^*) \\
&= 2(4a - \sigma(\eta_0, \eta_i)) \\
&= 8a - 2\sigma(\eta_0, \eta_i).
\end{aligned}$$

Also,

$$\begin{aligned}
\sigma(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma(\Upsilon\eta_j, \Upsilon\eta^*) &= \sigma(\Upsilon\eta_i, \Upsilon\eta^*) \\
\sigma(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma(\Upsilon\eta_j, \Upsilon\eta^*) + \sigma(\Upsilon\eta_i, \Upsilon\eta^*) &= 2\sigma(\Upsilon\eta_i, \Upsilon\eta^*) \\
&= 2(\sigma(\eta_{i+1}, \eta^*)) \\
&= 2(4a - \sigma(\eta_0, \eta_{i+1})) \\
&= 8a - 2\sigma(\eta_0, \eta_{i+1}).
\end{aligned}$$

Using the formula for the sum of a geometric series with n terms, we get

$$\sigma(\eta_0, \eta_i) = \begin{cases} 4a\left(1 - \left(\frac{1}{2}\right)^n\right), & \text{if} \quad i = 2n, \\ & \qquad\qquad n = 1, 2, ... \\ 4a\left(1 - \left(\frac{1}{2}\right)^n\right) - \frac{a}{2^{n-1}}, & \text{if} \quad i = 2n - 1. \end{cases}$$

Now,

$$\sigma(\eta_0, \eta_{i+1}) = \sigma(\eta_0, \eta_i) + \sigma(\eta_i, \eta_{i+1}),$$

$$= \sigma(\eta_0, \eta_i) + a/(2^{\lfloor i/2 \rfloor}).$$

Consider,

$$\frac{\sigma(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma(\Upsilon\eta_j, \Upsilon\eta^*) + \sigma(\Upsilon\eta_i, \Upsilon\eta^*)}{\sigma(\eta_i, \eta_j) + \sigma(\eta_j, \eta^*) + \sigma(\eta_i, \eta^*)} = \frac{8a - 2\sigma(\eta_0, \eta_{i+1})}{8a - 2\sigma(\eta_0, \eta_i)}$$

$$= \frac{4a - \sigma(\eta_0, \eta_i) - \dfrac{a}{(2^{\lfloor i/2 \rfloor})}}{4a - \sigma(\eta_0, \eta_i)}.$$

$$= \begin{cases} \dfrac{4a - 4a\left(1 - \left(\frac{1}{2}\right)^n\right) - \dfrac{a}{(2^{\lfloor i/2 \rfloor})}}{4a - 4a\left(1 - \left(\frac{1}{2}\right)^n\right)}, & \text{if } i = 2n, \\ \\ \dfrac{4a - 4a\left(1 - \left(\frac{1}{2}\right)^n\right) + \dfrac{a}{2^{n-1}} - \dfrac{a}{(2^{\lfloor i/2 \rfloor})}}{4a - 4a\left(1 - \left(\frac{1}{2}\right)^n\right)) + \dfrac{a}{2^{n-1}}}, & \text{if } i = 2n-1, \end{cases}$$

$$= \begin{cases} \frac{3}{4}, & \text{if } i = 2n, \\ \frac{2}{3}, & \text{if } i = 2n-1. \end{cases}$$

Considering η_i, η_j, $\eta_k \in \mathcal{U}$ with $0 \leq i < j < k$, Fig.3.1 illustrates that;

$$\sigma(\eta_i, \eta_j) = \sigma(\eta_i, \eta_{i+1}) + \sigma(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma(\eta_{j-1}, \eta_j). \tag{3.4}$$

$$\sigma(\eta_j, \eta_k) = \sigma(\eta_j, \eta_{j+1}) + \sigma(\eta_{j+1}, \eta_{j+2}) + \cdots + \sigma(\eta_{k-1}, \eta_k). \tag{3.5}$$

$$\sigma(\eta_i, \eta_k) = \sigma(\eta_i, \eta_{i+1}) + \cdots + \sigma(\eta_{j-1}, \eta_j) + \cdots + \sigma(\eta_{k-1}, \eta_k). \tag{3.6}$$

Adding (3.4), (3.5) and (3.6). We obtain,

$$\sigma(\eta_i, \eta_j) + \sigma(\eta_j, \eta_k) + \sigma(\eta_i, \eta_k) = 2(\sigma(\eta_i, \eta_{i+1}) + \sigma(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma(\eta_{k-1}, \eta_k)). \tag{3.7}$$

Now, consider;

$$\sigma(\Upsilon\eta_i, \Upsilon\eta_j) = \sigma(\eta_{i+1}, \eta_{j+1}) = \sigma(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma(\eta_{j-1}, \eta_j). \tag{3.8}$$

$$\sigma(\Upsilon\eta_j, \Upsilon\eta_k) = \sigma(\eta_{j+1}, \eta_{k+1}) = \sigma(\eta_{j+1}, \eta_{j+2}) + \cdots + \sigma(\eta_k, \eta_{k+1}). \tag{3.9}$$

$$\sigma(\Upsilon\eta_i, \Upsilon\eta_k) = \sigma(\eta_{i+1}, \eta_{k+1}) = \sigma(\eta_{i+1}, \eta_{i+2}) + \cdots + \sigma(\eta_k, \eta_{k+1}). \tag{3.10}$$

Adding (3.8), (3.9) and (3.10). We obtain,

$$\sigma(\Upsilon\eta_i,\Upsilon\eta_j)+\sigma(\Upsilon\eta_j,\Upsilon\eta_k)+\sigma(\Upsilon\eta_i,\Upsilon\eta_k)=2(\sigma(\eta_{i+1},\eta_{i+2})+\cdots+\sigma(\eta_{k-1},\eta_k)+\sigma(\eta_k,\eta_{k+1})). \quad (3.11)$$

By subtracting (3.7) and (3.11)

$$\begin{aligned}\sigma(\eta_i,\eta_j)+\sigma(\eta_j,\eta_k)+\sigma(\eta_i,\eta_k)-(\sigma(\Upsilon\eta_i,\Upsilon\eta_j)+\sigma(\Upsilon\eta_j,\Upsilon\eta_k)+\sigma(\Upsilon\eta_i,\Upsilon\eta_k))\\ =2(\sigma(\eta_i,\eta_{i+1})+\sigma(\eta_k,\eta_{k+1}))\\ =2\left(\frac{a}{(2^{\lfloor i/2\rfloor})}-\frac{a}{(2^{\lfloor k/2\rfloor})}\right).\end{aligned}$$

Rearranging this equation,

$$\sigma(\Upsilon\eta_i,\Upsilon\eta_j)+\sigma(\Upsilon\eta_j,\Upsilon\eta_k)+\sigma(\Upsilon\eta_i,\Upsilon\eta_k)=\sigma(\eta_i,\eta_j)+\sigma(\eta_j,\eta_k)+\sigma(\eta_i,\eta_k)-2\left(\frac{a}{(2^{\lfloor i/2\rfloor})}-\frac{a}{(2^{\lfloor k/2\rfloor})}\right). \quad (3.12)$$

One can see that $i+1<k$, which implies:

$$\begin{aligned}2^{\lfloor i+1/2\rfloor}&<2^{\lfloor k/2\rfloor}\\ \Rightarrow 2.2^{\lfloor i/2\rfloor}&<2^{\lfloor k/2\rfloor}\\ \Rightarrow \frac{a}{2^{\lfloor k/2\rfloor}}&\leq\frac{a}{(2.2^{\lfloor i/2\rfloor})},\end{aligned}$$

using this in (3.12), we obtain

$$\sigma(\Upsilon\eta_i,\Upsilon\eta_j)+\sigma(\Upsilon\eta_j,\Upsilon\eta_k)+\sigma(\Upsilon\eta_i,\Upsilon\eta_k)\leq\sigma(\eta_i,\eta_j)+\sigma(\eta_j,\eta_k)+\sigma(\eta_i,\eta_k)-2\frac{a}{2^{\lfloor i/2\rfloor}}+\frac{a}{2^{\lfloor i/2\rfloor}},$$

$$\sigma(\Upsilon\eta_i,\Upsilon\eta_j)+\sigma(\Upsilon\eta_j,\Upsilon\eta_k)+\sigma(\Upsilon\eta_i,\Upsilon\eta_k)\leq\sigma(\eta_i,\eta_j)+\sigma(\eta_j,\eta_k)+\sigma(\eta_i,\eta_k)-\frac{a}{2^{\lfloor i/2\rfloor}}. \quad (3.13)$$

One can show that $\sigma(\eta_i,\eta^*)\leq 4\sigma(\eta_i,\eta_{i+1})$. We have, $\sigma(\eta_i,\eta_k)\leq\sigma(\eta_i,\eta^*)$ which implies that, $\sigma(\eta_i,\eta_k)\leq 4\sigma(\eta_i,\eta_{i+1})$. Using equality (3.3) and the preceding inequality, we obtain,

$$\sigma(\eta_i,\eta_j)+\sigma(\eta_j,\eta_k)+\sigma(\eta_i,\eta_k)=2\sigma(\eta_i,\eta_k)$$

$$\begin{aligned}&\leq 8\sigma(\eta_i,\eta_{i+1})\\ &=8\frac{a}{(2^{\lfloor i/2\rfloor})}.\end{aligned}$$

By putting this inequality in (3.13), yields:

$\sigma(\Upsilon\eta_i, \Upsilon\eta_j)+\sigma(\Upsilon\eta_j, \Upsilon\eta_k)+\sigma(\Upsilon\eta_i, \Upsilon\eta_k) \leq \sigma(\eta_i, \eta_j)+\sigma(\eta_j, \eta_k)+\sigma(\eta_i, \eta_k)-\frac{1}{8}(\sigma(\eta_i, \eta_j)+\sigma(\eta_j, \eta_k) + \sigma(\eta_i, \eta_k))$

$\sigma(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma(\Upsilon\eta_i, \Upsilon\eta_k) \leq \frac{7}{8}(\sigma(\eta_i, \eta_j) + \sigma(\eta_j, \eta_k) + \sigma(\eta_i, \eta_k))$.

With the coefficient $\alpha = \frac{7}{8} = \max\{\frac{2}{3}, \frac{3}{4}, \frac{7}{8}\}$, inequality (3.1) thus satisfies for any three pairwise distinct points from the space $\mathcal{U}$.

It should be noted that the sequence of iterates of any two points, η_i and η_j, in the preceding example overlap sets.

Let's create an example of a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ that contracts triangle perimeters and is not a contraction mapping. It has the feature that there are an infinite number of points such that the iteration sequences of these points are disjoint sets.

Example 3.2.9.

Consider the subset $\mathcal{U} \subseteq \mathbb{R}$ consisting of $\{\eta_0, \eta_1, ...\} \cup [0, 1]$, where $\eta_{2k} = \dfrac{-4}{2^k}$ and $\eta_{2k+1} = \dfrac{-3}{2^k}$ for $k \geq 0$, and equipped with the standard Euclidean metric σ, illustrated in Fig.3.2.

Consider the mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ defined by $\Upsilon\eta_i = \eta_{i+1}$ for all $i \in \{0\} \cup \mathbb{N}$ and $\Upsilon\eta = \dfrac{\eta}{2}$ for $\eta \in [0, 1]$. The mapping Υ satisfies the required condition for sequences of iterates of points in [0,1] of the form $\dfrac{\rho}{2^k}$, where ρ is a prime number greater than or equal to 3 and k is the smallest natural number ensuring $\dfrac{\rho}{2^k} \subseteq [0, 1]$.

Setting $\alpha = 1$ establishes an isometry between the previous example's MS and the subspace $(\{0, \eta_0, \eta_1, \cdots\}, \sigma)$ within $(\mathcal{U}, \sigma)$. The mapping Υ is defined

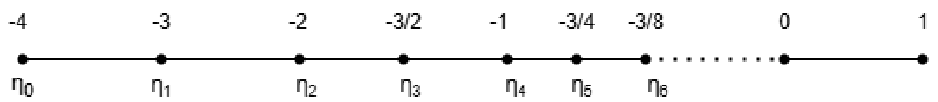

FIGURE 3.2: The MS ($\mathcal{U}$, σ)

in a similar manner for this subspace, and it follows that Υ is not contraction mapping.

We will demonstrate that, for each of the three pairwise distinct points from the space $(\mathcal{U}, \sigma)$, inequality (3.1) satisfies. The validity of this property for all distinct triplets in $(\{0, \eta_0, \eta_1, \cdots\}, \sigma)$ has been previously established. Since the metric σ is contractive on $([0,1],\sigma)$, and every contraction reduces triangle perimeters, we only need to prove inequality (3.1) for three pairwise distinct points $\eta,\xi,\zeta \in \mathcal{U}$ satisfying: $\eta<\xi<\zeta$ where, $\eta \in \{\eta_0,\eta_1,\cdots\}$ and $\zeta \in (0,1]$.

We begin by considering $\eta=\eta_{2k}=\frac{-4}{2^k}$. Subsequently,

$$\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta)=2\sigma(\eta,\zeta)=2\left(\frac{4}{2^k}+\zeta\right)=\frac{8}{2^k}+2\zeta. \tag{3.14}$$

From $\Upsilon\eta=\Upsilon\eta_{2k}$, it follows that $\Upsilon\eta_{2k}=\eta_{2k+1}=\frac{-3}{2^k}$. which implies that

$$\begin{aligned}\sigma(\Upsilon\eta,\Upsilon\xi)+\sigma(\Upsilon\xi,\Upsilon\zeta)+\sigma(\Upsilon\eta,\Upsilon\zeta)=2\sigma(\Upsilon\eta,\Upsilon\zeta)&=2\left(\frac{3}{2^k}+\frac{\zeta}{2}\right),\\&=6\left(\frac{1}{2^k}+\frac{\zeta}{6}\right),\\&=\frac{6}{8}\left(\frac{8}{2^k}+\frac{4\zeta}{3}\right),\\&\leq\frac{3}{4}\left(\frac{8}{2^k}+2\zeta\right),\end{aligned}$$

using equation (3.14)

$$\sigma(\Upsilon\eta,\Upsilon\xi)+\sigma(\Upsilon\xi,\Upsilon\zeta)+\sigma(\Upsilon\eta,\Upsilon\zeta)\leq\frac{3}{4}(\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta)).$$

We can see that, it satisfy the inequality (3.1).

Similarly, for $\eta=\eta_{2k+1}=\frac{-3}{2^k}$, we have

$$\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta)=2\sigma(\eta,\zeta)=2\left(\frac{3}{2^k}+\zeta\right)=\frac{6}{2^k}+2\zeta. \tag{3.15}$$

Applying Υ to η_{2k+1} yields $\Upsilon\eta_{2k+1}=\eta_{2(k+1)}=\frac{-4}{2^{k+1}}$. we get

$$\begin{aligned}\sigma(\Upsilon\eta,\Upsilon\xi)+\sigma(\Upsilon\xi,\Upsilon\zeta)+\sigma(\Upsilon\eta,\Upsilon\zeta)&=2\sigma(\Upsilon\eta,\Upsilon\zeta,\\&=2\left(\frac{4}{2^{k+1}}+\frac{\zeta}{2}\right),\\&=8\left(\frac{1}{2.2^k}+\frac{\zeta}{8}\right),\\&=4\left(\frac{1}{2^k}+\frac{\zeta}{4}\right),\\&=\frac{4}{6}\left(\frac{6}{2^k}+\frac{3\zeta}{2}\right),\end{aligned}$$

$$\leq \frac{2}{3}\left(\frac{6}{2^k} + 2\zeta\right).$$

Now, by using equation (3.15), we obtain;

$$\sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\xi, \Upsilon\zeta) + \sigma(\Upsilon\eta, \Upsilon\zeta) \leq \frac{2}{3}(\sigma(\eta, \xi) + \sigma(\xi, \zeta) + \sigma(\eta, \zeta)),$$

which implies that inequality (3.1) holds.

The previous example illustrates that inequality (3.1) is valid for any three pairwise distinct points in $\mathcal{U}$, with the coefficient $\alpha = \frac{7}{8} = \max\{\frac{2}{3}, \frac{3}{4}, \frac{7}{8}\}$.

Chapter 4

Fixed Point Theorem for Mapping Contracting Perimeter of Triangle Embedded with F-Contraction in b-Metric Spaces

4.1 Introduction

In this chapter the notion of mapping contracting perimeter of triangles (MCPT) embedded with F-contraction in the context of b-metric space (b-MS) is established. Moreover, by using this new contraction a FP result has been provided. In order to elaborate the idea an example is also provided.

4.2 MCPT Embedded with F-Contraction in b-MS

Definition 4.2.1.
Consider a b-MS $(\mathcal{U}, \sigma_{\mathsf{b}})$ with at least three elements i.e., $|\mathcal{U}| \geq 3$. A mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is said to be MCPT embedded with F-contraction on $\mathcal{U}$ if there exists

$\mathsf{s} \geq 1$, $\mathsf{F} \in \mathcal{F}$ and $\tau > 0$ such that the following inequality

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)) \leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right), \quad (4.1)$$

satisfies for all possible combinations of three pairwise distinct points η, ξ, ζ in $\mathcal{U}$.

Proposition 4.2.2.

Let $(\mathcal{U}, \sigma_{\mathsf{b}})$ be a complete b-MS and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a MCPT embedded with F-contraction. Then Υ is continuous.

Proof. Suppose that $(\mathcal{U}, \sigma_{\mathsf{b}})$ be a b-MS with $|\mathcal{U}| \geq 3$, $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a MCPT embedded with F-contraction on $\mathcal{U}$ and let η_0 be an isolated point in $\mathcal{U}$. Then, clearly, Υ is continuous at η_0. Let suppose that η_0 be a limit point of $\mathcal{U}$. Now we show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta) < \epsilon$ whenever $\sigma_{\mathsf{b}}(\eta_0, \eta) < \delta$.
Since η_0 is a limit point, for every $\delta > 0$ there exists $\xi \in \mathcal{U}$ such that $\sigma_{\mathsf{b}}(\eta_0, \xi) < \delta$. As, we have

$$\begin{aligned}\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta)) &\leq \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta)), \\ &\leq \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta) + \sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi)),\end{aligned}$$

Using (4.1),

$$\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta_0, \eta) + \sigma_{\mathsf{b}}(\eta_0, \xi) + \sigma_{\mathsf{b}}(\eta, \xi))\right).$$

Using the triangular inequality, we have

$$\begin{aligned}\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta)) &\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta_0, \eta) + \sigma_{\mathsf{b}}(\eta_0, \xi) + \mathsf{s}(\sigma_{\mathsf{b}}(\eta_0, \eta) + \sigma_{\mathsf{b}}(\eta_0, \xi)))\right), \\ &\leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(1 + \mathsf{s})(\sigma_{\mathsf{b}}(\eta_0, \eta) + \sigma_{\mathsf{b}}(\eta_0, \xi))\right), \\ &< \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(1 + \mathsf{s})(\delta + \delta)\right), \\ &= \mathsf{F}\left(2\frac{1}{\mathsf{s}^2}(1 + \mathsf{s})\delta\right). \quad (4.2)\end{aligned}$$

Setting $\delta = \dfrac{\epsilon \mathsf{s}^2}{2(1+\mathsf{s})}$, then equation (4.2) becomes

$$\mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta)) < \mathsf{F}(\epsilon).$$

Since, F in increasing, therefore,

$$\sigma_{\mathsf{b}}(\Upsilon\eta_0, \Upsilon\eta)) < \epsilon.$$

Hence, MCPT embedded with F-contraction are continuous. □

Definition 4.2.3.

Consider a mapping Υ on the b-MS $\mathcal{U}$. A point $\eta \in \mathcal{U}$ is said to be a periodic point of period n if $\Upsilon^{\mathsf{n}}(\eta) = \eta$, where n is the least positive integer for which $\Upsilon^{\mathsf{n}}(\eta) = \eta$, such a positive integer n is called the prime period of η.

Theorem 4.2.4.

Consider a complete b-MS $(\mathcal{U}, \sigma_{\mathsf{b}})$ with at least three elements, i.e., $|\mathcal{U}| \geq 3$ and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a MCPT embedded with F-contraction on $\mathcal{U}$. Then Υ has a fixed point (FP) if and only if Υ does not have periodic points with a prime period 2. Υ possesses at most two FPs.

Proof. Suppose that no point is periodic with prime period 2 under the mapping Υ. Our objective is to show that Υ has a FP. Let $\eta_0 \in \Upsilon$ and, $\Upsilon\eta_0 = \eta_1, \Upsilon\eta_1 = \eta_2, \cdots, \Upsilon\eta_{\mathsf{n}} = \eta_{\mathsf{n}+1}, \cdots$.

Assume that, for all $i = 0, 1, 2, \cdots$, there are no FP of the mapping Υ among the points η_{i}. Our goal is to demonstrate the distinctness of every point η_{i}. We have $\eta_{\mathsf{i}} \neq \eta_{\mathsf{i}+1} = \Upsilon\eta_{\mathsf{i}}$ because η_{i} is not a FP. We also know that $\eta_{\mathsf{i}+2} = \Upsilon(\Upsilon(\eta_{\mathsf{i}})) \neq \eta_{\mathsf{i}}$ since Υ lacks any periodic points of prime period 2. Moreover, $\eta_{\mathsf{i}+1} \neq \eta_{\mathsf{i}+2} = \Upsilon\eta_{\mathsf{i}+1}$ since $\eta_{\mathsf{i}+1}$ is not a FP. As a result, pairwise distinct points are η_{i}, $\eta_{\mathsf{i}+1}$, and $\eta_{\mathsf{i}+2}$. Furthermore, suppose that

$$\begin{aligned}
\gamma_0 &= \sigma_{\mathsf{b}}(\eta_0, \eta_1) + \sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_0), \\
\gamma_1 &= \sigma_{\mathsf{b}}(\eta_1, \eta_2) + \sigma_{\mathsf{b}}(\eta_2, \eta_3) + \sigma_{\mathsf{b}}(\eta_3, \eta_1), \\
&\vdots
\end{aligned}$$

$$\gamma_n = \sigma_b(\eta_n, \eta_{n+1}) + \sigma_b(\eta_{n+1}, \eta_{n+2}) + \sigma_b(\eta_{n+2}, \eta_n),$$
$$\vdots$$

Applying the contraction condition to the pairwise distinct points η_i, η_{i+1}, and η_{i+2}, we obtain

$$\begin{aligned} \mathsf{F}\left(\sigma_b(\eta_1,\eta_2)+\sigma_b(\eta_2,\eta_3)+\sigma_b(\eta_1,\eta_3)\right) &= \mathsf{F}\left(\sigma_b(\Upsilon\eta_0,\Upsilon\eta_1)+\sigma_b(\Upsilon\eta_1,\Upsilon\eta_2)+\sigma_b(\Upsilon\eta_0,\Upsilon\eta_2)\right), \\ &\leq \mathsf{F}\left(\frac{1}{s^2}((\sigma_b(\eta_0,\eta_1)+\sigma_b(\eta_1,\eta_2)+\sigma_b(\eta_0,\eta_2))\right)-\tau, \\ \mathsf{F}(\gamma_1) &\leq \mathsf{F}\left(\frac{1}{s^2}(\gamma_0)\right)-\tau, \end{aligned}$$

since, F is increasing, we can write above equation as;

$$\mathsf{F}(s^2\gamma_1) \leq \mathsf{F}(\gamma_0) - \tau.$$

Similarly, we can obtain,

$$\begin{aligned} \mathsf{F}(s^2\gamma_2) &\leq \mathsf{F}(\gamma_1) - \tau, \\ \mathsf{F}(s^2\gamma_3) &\leq \mathsf{F}(\gamma_2) - \tau, \\ &\vdots \\ \mathsf{F}(s^2\gamma_n) &\leq \mathsf{F}(\gamma_{n-1}) - \tau, \\ \mathsf{F}(s^2\gamma_{n+1}) &\leq \mathsf{F}(\gamma_n) - \tau. \end{aligned} \tag{4.3}$$

Since, $s \geq 1$

$$\Rightarrow \gamma_0 > \gamma_1 > \cdots > \gamma_n > \cdots. \tag{4.4}$$

Assume that $j \geq 3$ is the smallest natural number such that $\eta_j = \eta_i$ for some i satisfying $0 \leq i < j-2$. Then, we have $\eta_{j+1} = \eta_{i+1}$ and $\eta_{j+2} = \eta_{i+2}$. Consequently, $\gamma_i = \gamma_j$ which contradicts (4.4), This shows that all η_i's are distinct.
Further, We have to show that $\{\eta_n\}$ is a Cauchy sequence. It is clear that:

$$\mathsf{F}(s^2\gamma_{n+1}) \leq \mathsf{F}(\gamma_n) - \tau,$$

then, by F-4 (2.2),

$$\mathsf{F}(s^{n+2}\gamma_{n+1}) \leq \mathsf{F}(s^n\gamma_n) - \tau, \tag{4.5}$$

It follows by induction that;

$$\begin{aligned} \mathsf{F}(\mathsf{s}^{\mathsf{n}}\gamma_n) &\leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-2}\gamma_{n-1}) - \tau, \\ &\leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-4}\gamma_{n-2}) - 2\tau, \\ &\leq \mathsf{F}(\mathsf{s}^{\mathsf{n}-6}\gamma_{n-3}) - 3\tau, \end{aligned}$$

by continuing this process, one can obtain:

$$\mathsf{F}(\mathsf{s}^{\mathsf{n}}\gamma_n) \leq \mathsf{F}(\gamma_0) - \mathsf{n}\tau, \tag{4.6}$$

Taking limit as $\mathsf{n} \to \infty$ in (4.6) to obtain $\lim_{\mathsf{n}\to\infty} \mathsf{F}(\mathsf{s}^{\mathsf{n}}\gamma_{\mathsf{n}}) \to -\infty$ which together with F-2 (2.4.1) $\Rightarrow \lim_{\mathsf{n}\to\infty} \mathsf{s}^{\mathsf{n}}\gamma_{\mathsf{n}} = 0$.
According to F-3 (2.4.1), $\exists$ a $\mathsf{k} \in (0,1)$ such that $\lim_{\mathsf{n}\to\infty} (\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}\mathsf{F}(\mathsf{s}^{\mathsf{n}}\gamma_n) = 0$.
Multiply (4.6) by $(\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}$, yields;

$$0 \leq (\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}\mathsf{F}(\mathsf{s}^{\mathsf{n}}\gamma_{\mathsf{n}}) + (\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}\mathsf{n}\tau \leq (\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}\mathsf{F}(\gamma_0)$$

Taking limit as $\mathsf{n} \to \infty$, we get;

$$\lim_{\mathsf{n}\to\infty} (\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}\mathsf{n} = 0.$$

As a result, it can be concluded that $\exists\ \mathsf{n}_1 \in \mathbb{N}$ such that;

$$(\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}}\mathsf{n} \leq 1, \qquad \forall\ \mathsf{n} \geq \mathsf{n}_1.$$

Therefore,

$$(\mathsf{s}^{\mathsf{n}}\gamma_n)^{\mathsf{k}} \leq \frac{1}{\mathsf{n}}, \qquad \forall\ \mathsf{n} \geq \mathsf{n}_1.$$

$$\Rightarrow \mathsf{s}^{\mathsf{n}}\gamma_n \leq \frac{1}{\mathsf{n}^{\frac{1}{\mathsf{k}}}}, \qquad \forall\ \mathsf{n} \geq \mathsf{n}_1. \tag{4.7}$$

This implies that the series $\sum_{i=1}^{\infty} \mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}$ converges.
Now, by the triangular inequality, $\forall\ \mathsf{n}, p \in \mathbb{N}$,

$$\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{n+p}) \leq \mathsf{s}(\sigma_{\mathsf{b}}(\eta_{\mathsf{n}}, \eta_{n+1}) + \sigma_{\mathsf{b}}(\eta_{n+1}, \eta_{n+p})),$$

$$\sigma_b(\eta_n, \eta_{n+p}) \leq s\sigma_b(\eta_n, \eta_{n+1}) + s^2(\sigma_b(\eta_{n+1}, \eta_{n+2}) + \sigma_b(\eta_{n+2}, \eta_{n+p})).$$

Continuing in this way,

$$\sigma_b(\eta_n, \eta_{n+p}) \leq s\sigma_b(\eta_n, \eta_{n+1}) + s^2\sigma_b(\eta_{n+1}, \eta_{n+2}) + s^3\sigma_b(\eta_{n+2}, \eta_{n+3}) + \cdots + s^p\sigma_b(\eta_{n+p-1}, \eta_{n+p}). \tag{4.8}$$

Putting $n=0$, in (4.5);

$$F(s^2\gamma_1) \leq F(\gamma_0) - \tau, \tag{4.9}$$

As, $\gamma_1 = \sigma_b(\eta_1, \eta_2) + \sigma_b(\eta_2, \eta_3) + \sigma_b(\eta_3, \eta_1)$, (4.9) becomes,

$$F(s^2(\sigma_b(\eta_1, \eta_2) + \sigma_b(\eta_2, \eta_3) + \sigma_b(\eta_3, \eta_1))) \leq F(\gamma_0) - \tau,$$
$$\Rightarrow F(s^2(\sigma_b(\eta_1, \eta_2) + \sigma_b(\eta_2, \eta_3) + \sigma_b(\eta_3, \eta_1))) \leq F(\gamma_0),$$

since, F is increasing and $s \geq 1$,

$$\sigma_b(\eta_1, \eta_2) + \sigma_b(\eta_2, \eta_3) + \sigma_b(\eta_3, \eta_1)) \leq \gamma_0,$$
$$\Rightarrow \sigma_b(\eta_1, \eta_2) \leq \gamma_0,$$

continuing the same process, we obtained;

$$\sigma_b(\eta_2, \eta_3) \leq \gamma_1,$$
$$\sigma_b(\eta_3, \eta_4) \leq \gamma_2,$$
$$\vdots$$
$$\sigma_b(\eta_n, \eta_{n+1}) \leq \gamma_{n-1},$$
$$\sigma_b(\eta_{n+1}, \eta_{n+2}) \leq \gamma_n,$$
$$\vdots$$

substituting in (4.8),

$$\begin{aligned}\sigma_b(\eta_n, \eta_{n+p}) &\leq s\gamma_{n-1} + s^2\gamma_n + s^3\gamma_{n+1} + \cdots + s^p\gamma_{n+p-2},\\ &\leq s(\gamma_{n-1} + s\gamma_n + s^2\gamma_{n+1} + \cdots + s^{p-1}\gamma_{n+p-2}),\\ &\leq \frac{s}{s^{n-1}}(s^{n-1}\gamma_{n-1} + s^n\gamma_n + s^{n+1}\gamma_{n+1} + \cdots + s^{n+p-2}\gamma_{n+p-2}),\\ &\leq \frac{1}{s^{n-2}}\left(\sum_{i=n-1}^{n+p-2} s^i\gamma_i\right),\end{aligned}$$

$$\leq \frac{1}{\mathsf{s}^{n-2}}\left(\sum_{\mathsf{i}=n-1}^{\infty}\mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}\right).$$

Therefore, $\forall$ $\mathsf{n} \geq \mathsf{n}_1$ and $p \in \mathbb{N}$, (4.7) implies;

$$\sigma_{\mathsf{b}}(\eta_n, \eta_{n+p}) \leq \frac{1}{\mathsf{s}^{n-2}}\left(\sum_{\mathsf{i}=n-1}^{\infty}\mathsf{s}^{\mathsf{i}}\gamma_{\mathsf{i}}\right) \leq \frac{1}{\mathsf{s}^{n-2}}\left(\sum_{\mathsf{i}=n-1}^{\infty}\frac{1}{\mathsf{i}^{\frac{1}{k}}}\right).$$

Taking limit as $\mathsf{n} \longrightarrow \infty$,

$$\sigma_{\mathsf{b}}(\eta_n, \eta_{n+p}) \longrightarrow 0.$$

It follows that $\{\eta_n\}$ is a Cauchy sequence in $\mathcal{U}$. As $(\mathcal{U}, \sigma_{\mathsf{b}})$ is a complete MS, $\{\eta_n\}$ has a limit η^* in $\mathcal{U}$.

To show that $\Upsilon\eta^* = \eta^*$, we apply the triangular inequality and inequality (4.1)

$$\begin{aligned}
\sigma_{\mathsf{b}}(\eta^*, \Upsilon\eta^*) \leq& \mathsf{s}(\sigma_{\mathsf{b}}(\eta^*, \eta_n) + \sigma_{\mathsf{b}}(\eta_n, \Upsilon\eta^*)),\\
=& \mathsf{s}\sigma_{\mathsf{b}}(\eta^*, \eta_n) + \mathsf{s}\sigma_{\mathsf{b}}(\Upsilon\eta_{n-1}, \Upsilon\eta^*),\\
\leq& \mathsf{s}\sigma_{\mathsf{b}}(\eta^*, \eta_n) + \mathsf{s}(\sigma_{\mathsf{b}}(\Upsilon\eta_{n-1}, \Upsilon\eta^*) + \sigma_{\mathsf{b}}(\Upsilon\eta_{n-1}, \Upsilon\eta_n) + \sigma_{\mathsf{b}}(\Upsilon\eta_n, \Upsilon\eta^*)),\\
\leq& \mathsf{s}\sigma_{\mathsf{b}}(\eta^*, \eta_n) + \mathsf{s}(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta_{n-1}, \eta^*) + \sigma_{\mathsf{b}}(\eta_{n-1}, \eta_n) + \sigma_{\mathsf{b}}(\eta_n, \eta^*))),\\
\leq& \mathsf{s}\sigma_{\mathsf{b}}(\eta^*, \eta_n) + \mathsf{s}(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta_{n-1}, \eta^*) + \sigma_{\mathsf{b}}(\eta_{n-1}, \eta_n) + \sigma_{\mathsf{b}}(\eta_n, \eta^*)),)\\
\leq& \mathsf{s}\sigma_{\mathsf{b}}(\eta^*, \eta_n) + (\frac{1}{\mathsf{s}}(\sigma_{\mathsf{b}}(\eta_{n-1}, \eta^*) + \sigma_{\mathsf{b}}(\eta_{n-1}, \eta_n) + \sigma_{\mathsf{b}}(\eta_n, \eta^*))).
\end{aligned}$$

Taking the limit as $n \to \infty$, we note that each term in the preceding sum vanishes, yielding:
$\sigma_{\mathsf{b}}(\eta^*, \Upsilon\eta^*) = 0.$
Therefore, we conclude that $\Upsilon\eta^* = \eta^*$.

In order to prove there exists at most two FPs. Assume by contradiction that Υ has at least three pairwise distinct FPs, say η, ξ, and ζ. That is:
$\Upsilon\eta = \eta$, $\Upsilon\xi = \xi$ and $\Upsilon\zeta = \zeta$.
Then by contraction condition,

$$\begin{aligned}
\tau + \mathsf{F}\left(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta)\right) =& \tau + \mathsf{F}\left(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi) + \sigma_{\mathsf{b}}(\Upsilon\xi, \Upsilon\zeta) + \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\zeta)\right),\\
\leq& \mathsf{F}\left(\frac{1}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta))\right),
\end{aligned}$$

since, F is increasing,

$$\Rightarrow \sigma_b(\eta,\xi)+\sigma_b(\xi,\zeta)+\sigma_b(\eta,\zeta) \le \frac{1}{s^2}(\sigma_b(\eta,\xi)+\sigma_b(\xi,\zeta)+\sigma_b(\eta,\zeta)),$$

which is a contradiction, since $s^2 \ge 1$. Thus, we conclude that Υ possesses at most two FPs.

Conversely, Suppose that Υ possesses a FP η^*. We have to prove that there are no periodic points in Υ with a prime period 2. For this, suppose by contradiction that Υ has a periodic point η of prime period 2, that is $\Upsilon(\Upsilon\eta)=\eta$. Define $\xi=\Upsilon\eta$ and $\eta=\Upsilon\xi$. Then

$$\tau+F(\sigma_b(\Upsilon\eta,\Upsilon\xi)+\sigma_b(\Upsilon\xi,\Upsilon\eta^*)+\sigma_b(\Upsilon\eta,\Upsilon\eta^*)) = F(\sigma_b(\xi,\eta)+\sigma_b(\eta,\eta^*)+\sigma(\xi,\eta^*)),$$

which contradicts to (4.1). Thus, our supposition is wrong which implies that Υ does not have periodic points with a prime period 2. □

Remark:
In the assumption of Theorem (4.2.4), if we add an extra condition that the FP η^* is limit of some iterative sequence then Υ has a unique FP.
Now, we show that $\eta_n \neq \eta^*$ for all $n = 1,2,\cdots$. For this let η_0 be an initial point, and iterative sequence will be $\eta_1=\Upsilon\eta_0, \eta_2=\Upsilon\eta_1,\cdots$.
In that case, there is only one FP, η^*. Assume, in fact, that Υ has another FP x^{**}. For every $n = 1,2,\cdots$, it is evident that $x_n \neq \eta^{**}$. Thus, for every $n = 1,2,\cdots$, we have that the points η^*, η^{**}, and η_n are pairwise distinct.
Now, by contraction condition

$$\begin{aligned}\tau+F(\sigma(\eta^*,\eta^{**})+\sigma(\eta^*,\eta_{n+1})+\sigma(\eta^{**},\eta_{n+1})) &= \tau+F(\sigma(\Upsilon\eta^*,\Upsilon\eta^{**})+\sigma(\Upsilon\eta^*,\Upsilon\eta_n)+\sigma(\Upsilon\eta^{**},\Upsilon\eta_n))\\ &\le F\left(\frac{1}{s^2}(\sigma(\eta^*,\eta^{**})+\sigma(\eta^*,\eta_n)+\sigma(\eta^{**},\eta_n))\right)\end{aligned}$$

As $n \longrightarrow 0$ we get $\sigma(\eta^*,\ \eta_{n+1}) \longrightarrow 0$, $\sigma(\eta^*,\eta_n) \longrightarrow 0$, $\sigma(\eta^{**},\eta_{n+1}) \longrightarrow \sigma(x^{**},x^*)$ and $\sigma(\eta^{**},\eta_n) \longrightarrow \sigma(\eta^{**},\eta^*)$. Hence,

$$\tau+F(2\sigma(\eta^*,\eta^{**})) \le F(\frac{1}{s^2}2\sigma(\eta^*,\eta^{**})),$$

$$\Rightarrow F(2\sigma(\eta^*,\eta^{**})) \le F(\frac{2}{s^2}\sigma(\eta^*,\eta^{**})).$$

Since, F is increasing,

$$\Rightarrow 2\sigma(\eta^*,\eta^{**}) \leq \frac{2}{\mathsf{s}^2}\sigma(\eta^*,\eta^{**}).$$

which is contradiction as $\mathsf{s} \geq 1$. This shows that $\eta^* = \eta^{**}$. Therefore, Υ has a unique FP.

The following is an example of a MCPT embedded with F-contraction with exactly two FPs.

Example 4.2.5.

Let $\mathcal{U} = \{2,3,10\}$, Suppose $\sigma_{\mathsf{b}} : \mathcal{U}\times\mathcal{U} \longrightarrow \mathbb{R}$ be defined as follows:

$$\sigma_{\mathsf{b}}(\eta,\xi) = (\eta-\xi)^2, \qquad \forall\ \eta,\xi \in \mathcal{U}.$$

Then $(\mathcal{U},\sigma_{\mathsf{b}})$ is a b-MS with $\mathsf{s}=2$. Now define $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\Upsilon\eta = \eta, \Upsilon\xi = \xi \text{ and } \Upsilon\zeta = \eta.$$

Now,

$$\begin{aligned}
\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta)) &= \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\zeta)+\sigma(\eta,\eta)),\\
&= \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\eta,\xi)+\sigma_{\mathsf{b}}(\xi,\eta)+\sigma(\eta,\eta)),\\
&= \tau + \mathsf{F}(2\sigma_{\mathsf{b}}(\eta,\xi)),\\
&= \tau + \mathsf{F}(2(\eta-\xi)^2),\\
&= \tau + \mathsf{F}(2(2-3)^2),
\end{aligned}$$

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta)) = \tau + \mathsf{F}(2). \tag{4.10}$$

Now,

$$\begin{aligned}
\mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta))) =& \mathsf{F}(\frac{1}{\mathsf{s}^2}((\eta-\xi)^2+(\xi-\zeta)^2+(\eta-\zeta)^2)),\\
=& \mathsf{F}(\frac{1}{2^2}((2-3)^2+(3-10)^2+(2-10)^2),
\end{aligned}$$

by simplifying, one can get

$$\mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta))) = \mathsf{F}(28.5). \tag{4.11}$$

Hence, by (4.10) and (4.11), we conclude that:

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\xi)+\sigma_{\mathsf{b}}(\Upsilon\xi,\Upsilon\zeta)+\sigma_{\mathsf{b}}(\Upsilon\eta,\Upsilon\zeta)) \le \mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma(\eta,\xi)+\sigma(\xi,\zeta)+\sigma(\eta,\zeta)))$$

with $\mathsf{F}(\eta) = \ln(\eta)$ and $\tau = 1$.
Also, Υ have no periodic points of prime period 2,
since,

$$\begin{aligned} \Upsilon\zeta &= \eta, \\ \Upsilon(\Upsilon\zeta) &= \Upsilon(\eta), \\ \Upsilon^2(\mathsf{w}) &= \eta. \end{aligned}$$

Hence, Υ has no periodic points with prime period 2.
Therefore, all the assumptions of Theorem (4.2.4) are true.Thus, Υ has exactly two FPs namely η and ξ.

In next example we prove that if Υ has periodic points of prime period 2, then Υ has no FPs.

Example 4.2.6.

Let $\mathcal{U} = \{\eta,\xi,\zeta\}$, Define $\sigma_{\mathsf{b}} : \mathcal{U}\times\mathcal{U} \longrightarrow \mathbb{R}$ as follows:

$$\sigma_{\mathsf{b}}(\eta,\xi) = (\eta-\xi)^2, \qquad \forall\ \eta,\xi \in \mathcal{U}.$$

Then one can prove that $(\mathcal{U},\sigma)$ is a b-MS. Now define $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ by

$$\Upsilon\eta = \xi, \Upsilon\xi = \eta \text{ and } \Upsilon\zeta = \eta.$$

Then Υ has no FP.
Here, η and ξ are periodic points with prime period 2.

Since,

$$\Upsilon\eta = \xi,$$
$$\Upsilon(\Upsilon\eta) = \Upsilon(\xi),$$
$$\Upsilon^2(\eta) = \eta.$$

Also,

$$\Upsilon\xi = \eta,$$
$$\Upsilon(\Upsilon\xi) = \Upsilon(\eta),$$
$$\Upsilon^2(\xi) = \xi.$$

Definition 4.2.7.

Let $(\mathcal{U}, \sigma_{\mathsf{b}})$ be a b-MS. Then a F-mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is called a F-contraction mapping on $\mathcal{U}$ if there exists a positive real number $\mathsf{s} \geq 1$ and $\tau > 0$, such that

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi)) \leq \mathsf{F}\left(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)\right), \qquad \text{where } \mathsf{F} \in \mathcal{F} \text{ and } \forall\ \ \eta, \xi \in \mathcal{U}. \tag{4.12}$$

The following corollary provides a simple and direct proof of Banach FP theorem, in the framework of bMS.

Corollary 4.2.1.

Suppose that $(\mathcal{U}, \sigma_{\mathsf{b}})$ be a complete b-MS where $\mathcal{U} \neq \emptyset$. Then the F-contraction mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ guarantees that Υ has a unique FP.

Proof. Suppose that $\mathcal{U}$ be a complete b-MS and $|\mathcal{U}| = 1$. Let $\mathcal{U} = \{u\}$, in this case, since $\mathcal{U}$ has only one element η, the mapping Υ must map η to itself. i-e $\Upsilon\eta = \eta$. This is because there are no other elements in $\mathcal{U}$ for $\Upsilon\eta$ to map. So, we can see that η is indeed a FP of Υ, and it is unique. Therefore, the Banach FP theorem holds trivially for a set $\mathcal{U}$ of order 1.

Now, if $|\mathcal{U}| = 2$, suppose that $\mathcal{U} = \{u, v\}$ and $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ be a F-contraction mapping. Assume, if possible, that Υ has two distinct FPs η and ξ. Then, $\Upsilon\eta = \eta$ and $\Upsilon\xi = \xi$.

By the definition of a F-contraction mapping, we have:

$$\begin{aligned}\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\xi)) &\leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)),\\ \Rightarrow \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi)) &\leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)) - \tau,\\ \Rightarrow \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \xi)) &\leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)),\end{aligned}$$

since, F is increasing, therefore,

$$\Rightarrow \sigma_{\mathsf{b}}(\eta, \xi) \leq \frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi).$$

This is a contradiction, since $\mathsf{s} \geq 1$.

Therefore, our assumption that Υ has two distinct FP is false. Hence, Υ can have at most one FP.

So, for $|\mathcal{U}| = 1, 2$ the proof is complete.

Assuming $\mathcal{U}$ has at least three elements, i.e., $|\mathcal{U}| \geq 3$, if Υ has some $\eta \in \mathcal{U}$ with prime period 2, i.e., $\Upsilon(\Upsilon(\eta)) = \eta$, then

$$\sigma_{\mathsf{b}}(\eta, \Upsilon\eta) = \sigma_{\mathsf{b}}(\Upsilon\eta, \eta) = \sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon(\Upsilon\eta)).$$

which is contradiction of (4.12). It follows that Υ has no periodic points with prime period 2.

Considering pairwise distinct elements η, ξ, $\zeta \in \mathcal{U}$, and applying (4.12), we have:

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\xi))) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi))$$

Let $\mathsf{F}(\eta) = \ln(\eta)$,

$$\tau + \ln(\sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\xi))) \leq \ln(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi))$$

$$\Rightarrow e^{\tau}\sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\xi)) \leq \frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi)$$

$$\Rightarrow \sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\xi)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \xi). \tag{4.13}$$

Similarly, one can get

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon(\xi), \Upsilon(\zeta))) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\xi, \zeta))$$

$$\Rightarrow \sigma_{\mathsf{b}}(\Upsilon(\xi), \Upsilon(\zeta)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\xi, \zeta). \tag{4.14}$$

and

$$\tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\zeta))) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \zeta))$$

$$\Rightarrow \sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\zeta)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \zeta). \tag{4.15}$$

Adding (4.13), (4.14) and (4.15), gives

$$\sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\xi)) + \sigma_{\mathsf{b}}(\Upsilon(\xi), \Upsilon(\zeta)) + \sigma_{\mathsf{b}}(\Upsilon(\eta), \Upsilon(\zeta)) \leq \frac{e^{-\tau}}{\mathsf{s}^2}(\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta)),$$

with $\alpha = e^{-\tau}$.

Hence, Υ is a MCPT embedded with F-contraction on $\mathcal{U}$. By Theorem (4.2.4) a FP exists for the mapping Υ.

Uniqueness: Suppose that Υ has two FPs η and η^*. i-e $\Upsilon\eta = \eta$ and $\Upsilon\eta^* = \eta^*$. Now, by the definition of b-metric and given assumption:

$$\begin{aligned} 0 < \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \eta^*)) &= \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\eta^*)), \\ &< \tau + \mathsf{F}(\sigma_{\mathsf{b}}(\Upsilon\eta, \Upsilon\eta^*)), \\ \mathsf{F}(\sigma_{\mathsf{b}}(\eta, \eta^*)) &\leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \eta^*)), \end{aligned}$$

since F is increasing,

$$\sigma_{\mathsf{b}}(\eta, \eta^*) \leq (\frac{1}{\mathsf{s}^2}\sigma_{\mathsf{b}}(\eta, \eta^*)).$$

where $\mathsf{s} \geq 1$.

which is only possible when

$\sigma_{\mathsf{b}}(\eta, \eta^*) = 0 \Rightarrow \eta = \eta^*$.

Hence, Υ has a unique FP. □

Proposition 4.2.8.

Consider a b-MS $(\mathcal{U}, \sigma_{\mathsf{b}})$ with at least three elements i.e, $|\mathcal{U}| \geq 3$, and $\Upsilon : \mathcal{U} \rightarrow \mathcal{U}$

be a MCPT embedded with F-contraction. Then, for all points $\xi \in \mathcal{U}$, Υ is a F-contraction mapping if η is a limit point of $\mathcal{U}$.

Proof. Consider an accumulation point $\eta \in \mathcal{U}$ and any point $\xi \in \mathcal{U}$. If $\xi = \eta$, then (4.12) is obviously satisfied. Now, consider the case where $\xi \neq \eta$. As η is a limit point, which implies the existence of a sequence $\{\zeta_\mathsf{n}\}$ converging to η, satisfying $\zeta_\mathsf{n} \neq \mathsf{x}$, $\zeta_\mathsf{n} \neq \xi$ with all distinct ζ_n. Consequently, applying (4.1) establishes the inequality:

$$\tau + \mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi) + \sigma_\mathsf{b}(\Upsilon\xi, \Upsilon\zeta_\mathsf{n}) + \sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\zeta_\mathsf{n})) \leq \mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma_\mathsf{b}(\eta, \xi) + \sigma_\mathsf{b}(\xi, \zeta_\mathsf{n}) + \sigma_\mathsf{b}(\eta, \zeta_\mathsf{n}))), \quad (4.16)$$

which satisfies for all $\mathsf{n} \in \mathbb{N}$. As $\sigma_\mathsf{b}(\eta, \zeta_\mathsf{n}) \longrightarrow 0$, $\zeta_\mathsf{n} \longrightarrow \mathsf{x}$ and continuity of b-MS implies $\sigma_\mathsf{b}(\xi, \zeta_\mathsf{n}) \to \sigma_\mathsf{b}(\eta, \xi)$.

By continuity of Υ, $\sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\zeta_\mathsf{n}) \to \sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\eta) = 0$ and $\sigma_\mathsf{b}(\Upsilon\xi, \Upsilon\zeta_\mathsf{n}) \to \sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi)$. Taking limit $\mathsf{n} \to \infty$ in (4.16) gives,

$$\begin{aligned}
\tau + \mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\eta, \Upsilon\xi)) &\leq \mathsf{F}(\frac{1}{\mathsf{s}^2}(\sigma_\mathsf{b}(\eta, \xi) + \sigma_\mathsf{b}(\eta, \xi))), \\
\tau + \mathsf{F}(2\sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi)) &\leq \mathsf{F}(\frac{2}{\mathsf{s}^2}\sigma_\mathsf{b}(\eta, \xi)), \\
\tau + \mathsf{F}(\sigma_\mathsf{b}(\Upsilon\eta, \Upsilon\xi)) &\leq \mathsf{F}(\frac{1}{\mathsf{s}^2}\sigma_\mathsf{b}(\eta, \xi)).
\end{aligned}$$

Hence, if η is a limit point of $\mathcal{U}$, then Υ is a F-contraction. □

Corollary 4.2.2.

Consider $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ is a MCPT embedded with F-contraction , and $(\mathcal{U}, \sigma_\mathsf{b})$ is a b-MS with at least three points i.e, $|U| \geq 3$. Then Υ is a F-contracting mapping whenever every element of $\mathcal{U}$ is an accumulation point of $\mathcal{U}$.

In a MS $(\mathcal{U}, \sigma_\mathsf{b})$, ξ is an intermediate point for η and ζ whenever:

$$\sigma_\mathsf{b}(\eta, \zeta) = \sigma_\mathsf{b}(\eta, \xi) + \sigma_\mathsf{b}(\xi, \zeta) \qquad \text{where} \quad \eta, \xi, \zeta \in \mathcal{U}. \quad (4.17)$$

Let us develop an example demonstrating the distinction between MCPT embedded with F-contraction and F-contraction in the framework of b-MS.

Example 4.2.9.

Suppose $\mathcal{U}$ has countably infinite elements, $|\mathcal{U}| = \aleph_0$, specifically $\mathcal{U} = \{\eta^*, \eta_0, \eta_1, \ldots\}$. Consider a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ that is MCPT embedded with F-contraction but exhibits non-contracting behavior in a b-MS $\mathcal{U}$. Let 'a' be a positive real number.

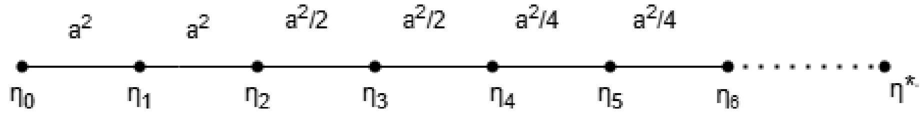

FIGURE 4.1: The points of the space $(\mathcal{U}, \sigma_b)$ with consecutive distances between them

Define σ_b on $\mathcal{U} \times \mathcal{U}$ as follows:

$$\sigma_b(\eta, \xi) = \begin{cases} a^2/2^{\lfloor i/2 \rfloor}, & \text{if} \quad \eta = \eta_i, \xi = \eta_{i+1}, i = 1, 2, 3, \cdots ., \\ \sigma_b(\eta_i, \eta_{i+1}) + \cdots + \sigma_b(\eta_{j-1}, \eta_j), & \text{if} \quad \eta = \eta_i, \xi = \eta_j, i+1 < j, \\ 4a^2 - \sigma_b(\eta_0, \eta_i), & \text{if} \quad \eta = \eta_i, \xi = \eta^*, \\ 0, & \text{if} \quad \eta = \xi, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function defined as the greatest integer less than or equal to a given number.

Clearly, for every triplet of distinct points in $\mathcal{U}$, one point is situated between the remaining two, we can see in Fig.4.1. Furthermore, the space has a single accumulation point, η^*, and is therefore complete.

Define the mapping $\Upsilon : \mathcal{U} \to \mathcal{U}$ given by $\Upsilon(\eta_i) = \eta_{i+1}$ for all $i \in \mathbb{N} \cup \{0\}$, and $\Upsilon(\eta^*) = \eta^*$.

We can see by that Υ is not F-contraction mapping, as $\sigma_b(\eta_{2n}, \eta_{2n+1}) = \sigma_b(\Upsilon\eta_{2n}, \Upsilon\eta_{2n+1})$, for all $n = 0,1,2,\cdots$.

Now we prove that Υ is MCPT embedded with F-contraction . Consider the first triplets of points $\eta_i, \eta_j, \eta^* \in \mathcal{U}$ with $0 \le i < j$. According to the definition of the σ_b given above,

$$\sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta^*) = \sigma_b(\eta_i, \eta^*).$$

Adding $\sigma_b(\eta_i, \eta^*)$ both sides,

$$\begin{aligned} \sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta^*) + \sigma_b(\eta_i, \eta^*) &= 2\sigma_b(\eta_i, \eta^*), \\ &= 2(4a^2 - \sigma_b(\eta_0, \eta_i)), \\ &= 8a^2 - 2\sigma_b(\eta_0, \eta_i). \end{aligned}$$

Also,

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta^*) = \sigma_b(\Upsilon\eta_i, \Upsilon\eta^*),$$

adding $\sigma_b(\Upsilon\eta_i, \Upsilon\eta^*)$ both sides,

$$\begin{aligned}
\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta^*) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta^*) &= 2\sigma_b(\Upsilon\eta_i, \Upsilon\eta^*),\\
&= 2(\sigma_b(\eta_{i+1}, \eta^*)),\\
&= 2(4a^2 - \sigma_b(\eta_0, \eta_{i+1})),\\
&= 8a^2 - 2\sigma_b(\eta_0, \eta_{i+1}).
\end{aligned}$$

Using the formula for the sum of a geometric series with n terms, we get

$$\sigma_b(\eta_0, \eta_i) = \begin{cases} 4a^2\left(1 - \left(\frac{1}{2}\right)^n\right), & \text{if} \quad i = 2n, \\ & \qquad\qquad\qquad n = 1, 2, ..., \\ 4a^2\left(1 - \left(\frac{1}{2}\right)^n\right) - \frac{a^2}{2^{n-1}}, & \text{if} \quad i = 2n - 1. \end{cases}$$

By the equation (4.17),

$$\begin{aligned}
\sigma_b(\eta_0, \eta_{i+1}) &= \sigma_b(\eta_0, \eta_i) + \sigma_b(\eta_i, \eta_{i+1}),\\
&= \sigma_b(\eta_0, \eta_i) + \frac{a^2}{(2^{\lfloor i/2 \rfloor})}.
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta^*) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta^*)}{\sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta^*) + \sigma_b(\eta_i, \eta^*)} &= \frac{8a^2 - 2\sigma_b(\eta_0, \eta_{i+1})}{8a^2 - 2\sigma_b(\eta_0, \eta_i)},\\
&= \frac{4a^2 - \sigma_b(\eta_0, \eta_i) - \dfrac{a^2}{(2^{\lfloor i/2 \rfloor})}}{4a - \sigma_b(\eta_0, \eta_i)},
\end{aligned}$$

$$= \begin{cases} \dfrac{4a^2 - 4a^2\left(1 - \left(\frac{1}{2}\right)^n\right) - \dfrac{a^2}{(2^{\lfloor i/2 \rfloor})}}{4a^2 - 4a^2(1 - (\frac{1}{2})^n)}, & \text{if } i = 2n, \\ \dfrac{4a^2 - 4a^2\left(1 - \left(\frac{1}{2}\right)^n\right) + \dfrac{a^2}{2^{n-1}} - \dfrac{a^2}{(2^{\lfloor i/2 \rfloor})}}{4a^2 - 4a^2(1 - (\frac{1}{2})^n) + \dfrac{a^2}{2^{n-1}}}, & \text{if } i = 2n - 1, \end{cases}$$

$$= \begin{cases} \frac{3}{4}, & \text{if} \quad \mathsf{i} = 2\mathsf{n}, \\ \\ \frac{2}{3}, & \text{if} \quad \mathsf{i} = 2\mathsf{n} - 1. \end{cases} \tag{4.18}$$

Considering η_i, η_j, $\eta_\mathsf{k} \in \mathcal{U}$ with $0 \leq \mathsf{i} < \mathsf{j} < \mathsf{k}$, Fig.4.1 illustrates that

$$\sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_\mathsf{j}) = \sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_{\mathsf{i}+1}) + \sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{i}+2}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{j}-1}, \eta_\mathsf{j}), \tag{4.19}$$

$$\sigma_\mathsf{b}(\eta_\mathsf{j}, \eta_\mathsf{k}) = \sigma_\mathsf{b}(\eta_\mathsf{j}, \eta_{\mathsf{j}+1}) + \sigma_\mathsf{b}(\eta_{\mathsf{j}+1}, \eta_{\mathsf{j}+2}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{k}-1}, \eta_\mathsf{k}), \tag{4.20}$$

and

$$\sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_\mathsf{k}) = \sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_{\mathsf{i}+1}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{j}-1}, \eta_\mathsf{j}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{k}-1}, \eta_\mathsf{k}). \tag{4.21}$$

Adding (4.19), (4.20) and (4.21) yields,

$$\sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_\mathsf{j}) + \sigma_\mathsf{b}(\eta_\mathsf{j}, \eta_\mathsf{k}) + \sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_\mathsf{k}) = 2(\sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_{\mathsf{i}+1}) + \sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{i}+2}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{k}-1}, \eta_\mathsf{k})). \tag{4.22}$$

Now, by the definition of σ_b,

$$\sigma_\mathsf{b}(\Upsilon\eta_\mathsf{i}, \Upsilon\eta_\mathsf{j}) = \sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{j}+1}) = \sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{i}+2}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{j}-1}, \eta_\mathsf{j}), \tag{4.23}$$

$$\sigma_\mathsf{b}(\Upsilon\eta_\mathsf{j}, \Upsilon\eta_\mathsf{k}) = \sigma_\mathsf{b}(\eta_{\mathsf{j}+1}, \eta_{\mathsf{k}+1}) = \sigma_\mathsf{b}(\eta_{\mathsf{j}+1}, \eta_{\mathsf{j}+2}) + \cdots + \sigma_\mathsf{b}(\eta_\mathsf{k}, \eta_{\mathsf{k}+1}), \tag{4.24}$$

and

$$\sigma_\mathsf{b}(\Upsilon\eta_\mathsf{i}, \Upsilon\eta_\mathsf{k}) = \sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{k}+1}) = \sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{i}+2}) + \cdots + \sigma_\mathsf{b}(\eta_\mathsf{k}, \eta_{\mathsf{k}+1}) \tag{4.25}$$

Adding (4.23), (4.24) and (4.25) yields,

$$\sigma_\mathsf{b}(\Upsilon\eta_\mathsf{i}, \Upsilon\eta_\mathsf{j}) + \sigma_\mathsf{b}(\Upsilon\eta_\mathsf{j}, \Upsilon\eta_\mathsf{k}) + \sigma_\mathsf{b}(\Upsilon\eta_\mathsf{i}, \Upsilon\eta_\mathsf{k}) = 2(\sigma_\mathsf{b}(\eta_{\mathsf{i}+1}, \eta_{\mathsf{i}+2}) + \cdots + \sigma_\mathsf{b}(\eta_{\mathsf{k}-1}, \eta_\mathsf{k}) + \sigma_\mathsf{b}(\eta_\mathsf{k}, \eta_{\mathsf{k}+1})) \tag{4.26}$$

By subtracting (4.22) and (4.26)

$$\sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_\mathsf{j}) + \sigma_\mathsf{b}(\eta_\mathsf{j}, \eta_\mathsf{k}) + \sigma_\mathsf{b}(\eta_\mathsf{i}, \eta_\mathsf{k}) - (\sigma_\mathsf{b}(\Upsilon\eta_\mathsf{i}, \Upsilon\eta_\mathsf{j}) + \sigma_\mathsf{b}(\Upsilon\eta_\mathsf{j}, \Upsilon\eta_\mathsf{k}) + \sigma_\mathsf{b}(\Upsilon\eta_\mathsf{i}, \Upsilon\eta_\mathsf{k})),$$

$$= 2(\sigma_b(\eta_i, \eta_{i+1}) + \sigma_b(\eta_k, \eta_{k+1})),$$
$$= 2\left(\frac{a^2}{(2^{\lfloor i/2 \rfloor})} - \frac{a^2}{(2^{\lfloor k/2 \rfloor})}\right).$$

Rearranging this equation,

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j)+\sigma_b(\Upsilon\eta_j, \Upsilon\eta_k)+\sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) = \sigma_b(\eta_i, \eta_j)+\sigma_b(\eta_j, \eta_k)+\sigma_b(\eta_i, \eta_k)-2\left(\frac{a^2}{(2^{\lfloor i/2 \rfloor})} - \frac{a^2}{(2^{\lfloor k/2 \rfloor})}\right). \tag{4.27}$$

We can see that $i + 1 < k$, which implies:

$$2^{\lfloor i/2+1 \rfloor} < 2^{\lfloor k/2 \rfloor},$$
$$\Rightarrow 2.2^{\lfloor i/2 \rfloor} < 2^{\lfloor k/2 \rfloor},$$
$$\Rightarrow \frac{a^2}{2^{\lfloor k/2 \rfloor}} \leq \frac{a^2}{(2.2^{\lfloor i/2 \rfloor})},$$

using this in (4.27) to obtain,

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) \leq \sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) - \frac{2a^2}{2^{\lfloor i/2 \rfloor}} + \frac{a^2}{2^{\lfloor i/2 \rfloor}}$$

$$\Rightarrow \sigma_b(\Upsilon\eta_i, \Upsilon\eta_j)+\sigma_b(\Upsilon\eta_j, \Upsilon\eta_k)+\sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) \leq \sigma_b(\eta_i, \eta_j)+\sigma_b(\eta_j, \eta_k)+\sigma_b(\eta_i, \eta_k)-\frac{a^2}{2^{\lfloor i/2 \rfloor}}. \tag{4.28}$$

One can show that $\sigma_b(\eta_i, \eta^*) \leq 4\sigma_b(\eta_i, \eta_{i+1})$.

We have $\sigma_b(\eta_i, \eta_k) \leq \sigma_b(\eta_i, \eta^*)$, consequently we obtain, $\sigma_b(\eta_i, \eta_k) \leq 4\sigma_b(\eta_i, \eta_{i+1})$.

Using equality (4.19) and the preceding inequality, we obtain,

$$\sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) = 2\sigma_b(\eta_i, \eta_k),$$
$$\leq 8\sigma_b(\eta_i, \eta_{i+1}),$$
$$= 8\frac{a^2}{(2^{\lfloor i/2 \rfloor})}.$$

By putting this inequality in (4.28), yields: $\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j)+\sigma_b(\Upsilon\eta_j, \Upsilon\eta_k)+\sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) \leq \sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k) - \frac{1}{8}(\sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k))$

$$\sigma_b(\Upsilon\eta_i, \Upsilon\eta_j) + \sigma_b(\Upsilon\eta_j, \Upsilon\eta_k) + \sigma_b(\Upsilon\eta_i, \Upsilon\eta_k) \leq \frac{7}{8}(\sigma_b(\eta_i, \eta_j) + \sigma_b(\eta_j, \eta_k) + \sigma_b(\eta_i, \eta_k)) \tag{4.29}$$

By equations (4.18) and (4.29), inequality (4.1) satisfies for any three pairwise distinct points from the space $\mathcal{U}$ with $F(\eta) = \ln(\eta)$ and $\frac{e^{-\tau}}{s^2} = \frac{7}{8} = \max\left\{\frac{2}{3}, \frac{3}{4}, \frac{7}{8}\right\}$.

It should be noted that the sequence of iterates of any two points, η_{i} and η_{j}, in the preceding example overlap sets.

Let's create an example of a mapping $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ that is MCPT embedded with F-contraction and is not a F-contraction mapping. It has the feature that there are an infinite number of points such that the iteration sequences of these points are disjoint sets.

Example 4.2.10.

Consider the subset $\mathcal{U} \subseteq \mathbb{R}$ consisting of $\{\eta_0, \eta_1, ...\} \cup [0, 1]$, where $\eta_{2\mathsf{k}} = \dfrac{-4}{2^k}$ and $\eta_{2\mathsf{k}+1} = \dfrac{-3}{2^k}$ for $k \geq 0$, illustrated in Fig.4.2.

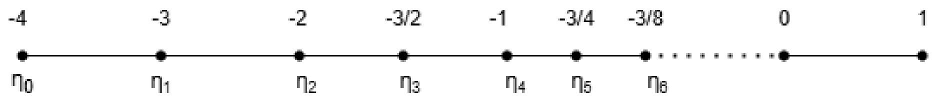

FIGURE 4.2: The b-MS ($\mathcal{U}$, σ_{b})

Let $\Upsilon : \mathcal{U} \longrightarrow \mathcal{U}$ defined by $\Upsilon\eta_{\mathsf{i}} = \eta_{\mathsf{i}+1}$ for all $\mathsf{i} \in \{0\} \cup \mathbb{N}$ and $\Upsilon\eta = \frac{\eta}{2}$ for $\eta \in [0, 1]$. The mapping Υ satisfies the required condition for sequences of iterates of points in [0,1] of the form $\dfrac{\mathsf{p}}{2^k}$, where p is a prime number greater than or equal to 3 and k is the smallest natural number ensuring $\dfrac{\mathsf{p}}{2^k} \subseteq [0, 1]$.

Setting $\mathsf{s} = 1$ establishes an isometry between the previous example's b-MS and the subspace $(\{0, \eta_0, \eta_1, \cdots\}, \sigma_{\mathsf{b}})$ within $(\mathcal{U}, \sigma_{\mathsf{b}})$.

The F-mapping Υ is defined in a similar manner for this subspace, and it follows that Υ is not F-contraction mapping.

We will demonstrate that, for each of the three pairwise distinct points from the space $(\mathcal{U}, \sigma_{\mathsf{b}})$, inequality (4.1) satisfies. The validity of this property for all distinct triplets in $(\{0, \eta_0, \eta_1, \cdots\}, \sigma_{\mathsf{b}})$ has been previously established. Since the b-metric σ_{b} is contractive on $([0, 1], \sigma_{\mathsf{b}})$, and every F-contraction reduces triangle perimeters, we only need to prove inequality (4.1) for three pairwise distinct points $\eta, \xi, \zeta \in \mathcal{U}$ satisfying: $\eta < \xi < \zeta$ where, $\eta \in \{\eta_0, \eta_1, \cdots\}$ and $\zeta \in (0, 1]$.

We begin by considering $\eta = \eta_{2\mathsf{k}} = \frac{-4}{2^k}$. Subsequently,

$$\sigma_{\mathsf{b}}(\eta, \xi) + \sigma_{\mathsf{b}}(\xi, \zeta) + \sigma_{\mathsf{b}}(\eta, \zeta) = 2\sigma_{\mathsf{b}}(\eta, \zeta) = 2\left(\frac{4}{2^k} + \zeta\right)^2. \tag{4.30}$$

From $\Upsilon\eta = \Upsilon\eta_{2k}$, it follows that $\Upsilon\eta_{2k} = \eta_{2k+1} = \frac{-3}{2^k}$. which implies that

$$\begin{aligned}
\sigma_b(\Upsilon\eta, \Upsilon\xi) + \sigma_b(\Upsilon\xi, \Upsilon\zeta) + \sigma_b(\Upsilon\eta, \Upsilon\zeta) = 2\sigma_b(\Upsilon\eta, \Upsilon\zeta) &= 2\left(\frac{3}{2^k} + \frac{\zeta}{2}\right)^2, \\
&= 3^2 \times 2\left(\frac{1}{2^k} + \frac{\zeta}{6}\right)^2, \\
&= \frac{9}{16} \times 2\left(\frac{4}{2^k} + \frac{4\zeta}{6}\right)^2, \\
&\le \frac{9}{16} \times 2\left(\frac{4}{2^k} + \zeta\right)^2,
\end{aligned}$$

by equation (4.30)

$$\sigma(\Upsilon\eta, \Upsilon\xi) + \sigma(\Upsilon\xi, \Upsilon\zeta) + \sigma(\Upsilon\eta, \Upsilon\zeta) \le \frac{9}{16}(\sigma(\eta, \xi) + \sigma(\xi, \zeta) + \sigma(\eta, \zeta))$$

We can see that, it satisfy the inequality (4.1).
Similarly, for $\eta = \eta_{2k+1} = \frac{-3}{2^k}$, we have

$$\sigma_b(\eta, \xi) + \sigma_b(\xi, \zeta) + \sigma_b(\eta, \zeta) = 2\sigma_b(\eta, \zeta) = 2\left(\frac{3}{2^k} + \zeta\right)^2. \tag{4.31}$$

Applying Υ to η_{2k+1} yields $\Upsilon\eta_{2k+1} = \eta_{2(k+1)} = \frac{-4}{2^{k+1}}$. we get

$$\begin{aligned}
\sigma_b(\Upsilon\eta, \Upsilon\xi) + \sigma_b(\Upsilon\xi, \Upsilon\zeta) + \sigma_b(\Upsilon\eta, \Upsilon\zeta) &= 2\sigma_b(\Upsilon\eta, \Upsilon\zeta), \\
&= 2\left(\frac{4}{2^{k+1}} + \frac{\zeta}{2}\right)^2, \\
&= 2\left(\frac{4}{2.2^k} + \frac{\zeta}{2}\right)^2, \\
&= 4 \times 2\left(\frac{1}{2^k} + \frac{\zeta}{4}\right)^2, \\
&= \frac{4}{9} \times 2\left(\frac{3}{2^k} + \frac{3\zeta}{4}\right)^2, \\
&\le \frac{4}{9} \times 2\left(\frac{3}{2^k} + 2\zeta\right)^2.
\end{aligned}$$

Now, by equation (4.31), we obtain;

$$\sigma_b(\Upsilon\eta, \Upsilon\xi) + \sigma_b(\Upsilon\xi, \Upsilon\zeta) + \sigma_b(\Upsilon\eta, \Upsilon\zeta) \le \frac{2}{3}(\sigma_b(\eta, \xi) + \sigma_b(\xi, \zeta) + \sigma_b(\eta, \zeta)),$$

which implies that inequality (4.1) holds with $\mathsf{F}(\eta) = \ln(\eta)$.

Chapter 5

Conclusion

A detailed review of Evgeniy Petrov [26] on "Fixed point (FP) theorem for mapping contracting perimeters of triangles" (MCPT) is given and elaborated.

A new class of mappings, referred to as MCPT, has been introduced in the platform of metric spaces (MS). The continuity of these mappings has been proven. The FP theorem has been established, and as a simple consequence, the classical Banach FP theorem has been obtained. Some non trivial examples are also provided for MCPT in MS.

Motivated by above work, the notion of "mapping contracting perimeters of triangles (MCPT) embedded with F-contraction in b-metric spaces (b-MS)" has been introduced.

The continuity of MCPT embedded with F-contractions has been demonstrated in the platform of b-MS. The FP theorem has been proved and classical Banach FP theorem is derived as a simple corollary. Examples of MCPT embedded with F-contraction which are not contraction mappings in the framework of b-MS has been established.

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