CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

The Combined Effect of Eringen Number, Micropolar Constant and Relaxation Factor on the Drag Force within a Cavity Flow

by

Ghulam Khadeeja

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This thesis is dedicated to my beloved mom, Zareena Kousar who always keeps me motivated and positive whenever I got fed up with this and wanted to quit, to my father, Muhammad Azam for letting me reach at this level and finally to Dr. Muhammad Sabeel Khan for his patience throughout my work.

CERTIFICATE OF APPROVAL

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by

Ghulam Khadeeja

(Registration No: MMT221013)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Ahmed Zeeshan	IIUI, Islamabad
(b)	Internal Examiner	Dr. Dur-e-Shehwar	CUST, Islamabad
(c)	Supervisor	Dr. Muhammad Sabeel Khan	CUST, Islamabad

Dr. Muhammad Sabeel Khan
Thesis Supervisor
February, 2025

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
February, 2025

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
February, 2025

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(Ghulam Khadeeja)

Registration No: MMT221013

Abstract

This thesis investigates the influence of relaxation time on the viscoelastic flow properties in a two-dimensional baffled cavity, an area not previously explored in literature. Using the upper convected Maxwell (UCM) model, both viscosity and relaxation time are incorporated into the governing equations, which are formulated as unsteady partial differential equations. A variational numerical method and the Galerkin finite element method (FEM) are employed to solve these equations, with FreeFem++ used for implementation and simulation. The study focuses on analyzing how relaxation time affects key flow parameters, such as drag and lift forces, and investigates flow characteristics through numerical simulations of velocity streamlines. The results, derived from non-dimensionalized equations, provide new insights into the impact of relaxation time on flow behavior and force distributions in confined geometries, contributing to the understanding of viscoelastic fluid dynamics.

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Abbreviations

FEM	Finite Element Method
GFEM	Galerkin Finite Element Method
UCM	Upper Convected Maxwell

Symbols

U	x-component of velocity
V	y-component of velocity
x, y	Cartesian coordinates
w_h	Approximate function
$\frac{\partial}{\partial x}$	Partial derivative w.r.t coordinate x
Pr	Prandtl number
Re	Reynolds number

Chapter 1

Literature Review

Numerous significant events are surrounded by complicated fluids in nature, such as viscoelastic or microfluidic mixtures; as a result, research into these fluids might be crucial to the advancement of numerous industrial operations. These industrial operations include the melting of polymers [1, 2], the production of pulp fibers [3], the processing of minerals [4], the processing of food [5], the production of cement pastes and cosmetics [6], and biophysical processes [7, 8].

It has also been observed that the vibration of nanostructures and the references they contain can induce viscoelastic flows in simple liquids. So far, the literature has explored viscoelastic flow behavior in various contexts. For instance, in his PhD dissertation [9], Boyang investigated the flow of a polymeric viscoelastic fluid across three different geometries and discovered that fluid relaxation influences the onset of turbulence in shear flows. The readers are directed to Yuan et al.'s study [10] for a review and state of the art on the impact of viscoelastic fluid flows and their applications. The reader is directed to Chen's work [7] for a review and up-to-date research on viscoelastic fluids in particle focusing and related particle manipulation applications. Zhu et al.'s work [11] examines the swimming of ciliated cells in a viscoelastic Giesekus fluid. When polymeric stresses are present, they see a decrease in the fluid's flow velocity. Their investigation's primary focus was on how the Weissenberg number affected the polymeric swimmers utilizing numerical simulation and the finite element method. Viscoelastic microfluidics are used to simulate blood flow in the human body and have applications in both health and sickness (see, for example, the review article by Sebastian and Dittrich [12]). Readers are directed to Haward et al.'s study [13] to comprehend the real-world use of viscoelastic fluids in porous media. Liu et al. [14] investigated the Bacillus population separation in microfluidics using viscoelastic fluids. The reader is directed to an extensive review article by Alves et al. [15] on the numerical approaches for viscoelastic fluid flows in order to understand how the flow of these complicated fluids is resolved utilizing a variety of numerical algorithms found in the literature.

The impact of fluid relaxation time in these models has hardly been examined, despite the fact that numerous significant applications are making the study of viscoelastic flows possible. There are a few studies on a small number in the body of current research examine fluid relaxation time, but they do so in many circumstances. For example, in the electroosmosis of a viscoelastic fluid, Mahapatra and Bandopadhyaya [16] examined the impact of fluid relaxation time and retardation time on a nonuniformly charged surface. Dey [17] found that the fluid relaxation time influences the hydromagnetic viscoelastic flow maximum near the geometry's center in an annular region. In one-dimensional situations, Eldesoky [18] examined the effects of relaxation time in a compressible Maxwellian peristaltic slip flow.

In recognition of its numerous commercial and technological uses, the study of conducting micropolar liquids has grown in importance in recent years. Colloidal suspensions, liquid crystals, unusual lubricants, and several other industrial liquids are examples of liquids used in the design of biomedical applications. François and Eugéne Cosserat, two French brothers, are credited for developing the theory of micropolar liquids in their monograph [19]. Since then, the theory has been widely accepted and has drawn attention because of its rich structure, particularly in light of [20] work. Numerous further research demonstrate its use in the rheological description of liquids having interior structure. The reader is directed to a recent study by [21] and the references therein for a thorough analysis of the uses of micropolar liquids. Another subclass of liquids known as micropolar liquids is made up of microelements that have internal structures. The nature of these liquids is anisotropic. Bar-like elements, liquid crystals, polymeric fluids, colloidal fluids, body fluids, paintings, and suspension liquids are examples of fluids that physically resemble micropolar liquids ([22], [23]). Numerous intriguing uses for the notion of micropolar fluids have been discovered. Sharma et al. [24] have utilized them to lubricate bearings in order to reduce skin friction and improve performance. In comparison to the conventional fluid theories, these theories have demonstrated a better load bearing capacity [25]. According to Karvelas et al. [26] and their sources, micropolar theories offer a more comprehensive explanation for complicated fluids such as blood in human carotid arteries. It is possible to model the flow of biofluids in human ureters, intestines, and arterioles using micropolar theory and conducting liquids.

Magnetization utilizing micropolar fluids has been the subject of numerous investigations. For example, Eringen [27] looked at the combined impact of electrodynamics and MHD of micropolar fluids. The nonlinear thermal radiation and viscous dissipation of MHD micropolar fluid through a porous channel were investigated by Saraswathy et al. [28]. They found that the spin gradient viscosity and vortex viscosity show reversal phenomena on the microrotation profile. A micropolar nanofluid model for MHD unsteady flow with entropy formation via a stretched surface on the boundary layer was introduced in [29]. Tulu [30] investigated a non-Fick mass flux and non-Fourier heat Cattaneo–Christov model of micropolar MHD nanofluid flow via a radially stretched disk. A nonlinear model of PDEs with boundary conditions for momentum, microrotation, temperature, and concentration that is subsequently numerically resolved using the spectral local linearization technique. The stagnation point flow of MHD nanofluid flow over a stretching sheet under the influence of magnetic induction was investigated in [31]. Khan et al. [32] looked at how MHD heat transfer affected upper-convected Maxwell micropolar flow with thermal radiation and joule heating. They calculated the angular-momentum balance equation while accounting for the microrotation of fluid particles and investigated the impact of microstructural factors on macroscopic velocity profiles and micro-rotations.

In a rotating reference frame, the study in [33] examined the effects of radiation absorption and hall current on MHD micropolar flow. It also examined the influence of the magnetic field normal to the porous surface that absorbs micropolar fluid with uniform suction velocity. The impact of heat generation and absorption in micropolar MHD flow past a porous material with porous suction and injection under the influence of thermal radiation and a magnetic field was examined by Goud [34]. The Runge–Kutta method of order 4 and the shooting approach are used to solve the resulting nonlinear ODEs. Khan et al. [35] looked into a finite element study of MHD micropolar flow over a rectangular channel using an iterative and numerical approach. They found that both magnetic induction and the micro-rotational velocity of particles decrease when the micropolar coupling value increases. The impact of MMR, a naturally occurring phenomena, in promoting micropolar liquids has not been taken into account in any of these earlier investigations.

The aim of this thesis is to present an investigation of fluid relaxation time on the evolution of viscoelastic flow dynamics where different parameters of study are taken into consideration. After modeling the governing flow equations, FreeFem++ [36] is used to create and implement the variational framework based on the typical Galerkin approach. Several significant aspects of the flow dynamics are seen and addressed by numerical simulations.

1.1 Thesis Contribution

This thesis analyzes Viscoelastic two-dimensional micropolar flow in a square baffled cavity, with a focus on the effects of relaxation time on drag forces. The governing equations are formulated using partial differential equations (PDEs) in vector-tensor notation, incorporating micropolar and viscoelastic properties. The system is non-dimensionalized to simplify the analysis, and a weak formulation is created for finite element solution. The FreeFEM++ software is used to simulate flow. A parametric study investigates how parameters like Reynolds number, Micropolar parameter, and Relaxation time affect drag forces, flow characteristics. The results highlight the complex interaction of these factors, emphasizing the role of relaxation time in drag forces and heat transfer. The study offers insights into micropolar fluid behavior in confined geometries, with potential applications in thermal management and microfluidics. The findings contribute to a better understanding of flow dynamics and heat transfer efficiency in such systems. The thesis also discusses the physical mechanisms influencing drag forces in viscoelastic-micropolar flows.

1.2 Thesis Layout

This thesis is further composed of the following chapters:

- **Chapter 2** explains the basic ideas of fluid dynamics, such as fluid types, flow classifications, and the difference between compressible and incompressible flows. Along with important equations like the Navier-Stokes, continuity, and energy equations, it addresses the fundamental principles of fluid motion, including mass, momentum, and energy conservation. A basis for examining viscoelastic and micropolar flow in subsequent chapters is laid by the discussion of dimensionalless quantities, such as Reynolds number, and the idea of non-dimensionalization. With an emphasis on fluid dynamic's applications, the chapter also presents the Finite Element Method (FEM), a computational method for solving partial differential equations. The FEM procedure, including discretization, weak formulation, and stiffness matrix generation, is demonstrated on a straightforward two-dimensional Poisson problem.

- **Chapter 3,** explores the impact of relaxation time on the characteristics of Viscoelastic Two-Dimensional Flow using FreeFEM++. The governing equations are non-dimensionalized, and boundary conditions are implemented, facilitating an analysis of how Reynolds number and relaxation time influence drag forces and flow behavior.

- **Chapter 4,** focuses on the analysis of viscoelastic two-dimensional micropolar flow, incorporating the effects of relaxation time on drag forces, using the Finite Element Method (FEM) in FreeFEM++. The governing equations are first non-dimensionalized, simplifying the problem and highlighting key parameters such as velocity, pressure, viscosity, relaxation time, and density. Boundary conditions are applied to the cavity walls, specifically influencing the velocity profiles. FreeFEM++ is then used to solve the dimensionless equations through finite element discretization, enabling the analysis of how parameters like the Reynolds number and relaxation time impact drag forces and flow patterns.

- **Chapter 5,** focuses on formulating the governing equations in their weak form for viscoelastic two-dimensional micropolar flow with relaxation time effects. The dimensional equations are first non-dimensionalized through scaling transformations, simplifying the problem by reducing physical parameters. The dimensionless equations are then integrated over the flow domain, transitioning to the weak form necessary for applying the Finite Element Method (FEM). The weak formulation is obtained by multiplying the equations by test functions and integrating by parts, making the system solvable numerically, particularly when dealing with boundary conditions and complex dynamics.

- In **Chapter 6,** results are presented through velocity profiles and streamlines, visually depicting the flow dynamics within the square baffled cavity, highlighting the effects of boundary conditions, thermal gradients, and micropolar interactions. Streamlines and graphs of drag forces, and velocity fluctuations offer insights into the influence of physical parameters like Reynolds number, and relaxation time on flow behavior.

- **Chapter 7** contains the conclusion of this work.

Chapter 2

Basic Terminologies

Basic ideas, definitions, and governing laws related to fluid dynamics have been discussed in this chapter. In particular, some important dimensionless quantities have been described, which appear to be useful in the next chapters.

2.1 Basic Definitions

2.1.1 Physical Properties of the Fluid

- **Mass Density or Density**

  Density or mass density of a fluid is defined as the ratio of the mass of a fluid to its volume. Thus mass per unit volume of a fluid is called density. It is denoted by ρ. The unit of mass density in SI unit is Kg per cubic metre, i.e., Kg/m^3.

  Mathematically, mass density is written as:

  $$\rho = \frac{\text{Mass of fluid}}{\text{Volume of fluid}}.$$

  The value of density of water is 1 gm/cm^3 or 1000 Kg/m^3.

- **Specific Volume**

  Specific Volume of a fluid occupied by a unit mass or volume per unit mass of a fluid is called specific volume. Mathematically, it is expressed as:

$$\text{Specific Volume} = \frac{\text{Volume of fluid}}{\text{Mass of fluid}} = \frac{1}{\frac{\text{Mass of fluid}}{\text{Volume of fluid}}}.$$

Thus, specific volume is the reciprocal of mass density.

- **Pressure**

  A fluid is enclosed in a vessel, it applies force to the container's top, bottom, and sides everywhere. Pressure is defined as the force per unit area.
  If,
  $P=$ The force, and
  $A=$ Area of force action, followed by pressure intensity,

  $$p = \frac{P}{A}$$

  A fluid's pressure acting on a surface will always be normal.

- **Viscosity**

  Viscosity is defined as the property of a fluid which offers resistance to the movement of one layer of fluid over another adjacent layer of the fluid. It is denoted by τ (called Tau).
  Mathematically,

  $$\tau \propto \frac{du}{dy}$$

  or

  $$\tau = \mu \frac{du}{dy},$$

  where, $\mu =$ constant of proportionality and $\frac{du}{dy}$= rate of shear stress or rate of shear deformation or velocity gradient.

  $$\mu = \frac{\tau}{\left[\frac{du}{dy}\right]}.$$

Thus, viscosity can alternatively be described by the shear stress required to produce a unit rate of shear strain.

- **Kinematic Viscosity**

  It is defined as the fluid's dynamic viscosity divided by its density. It is represented by the symbol ν, which can be read as **"nu"**.

  Mathematically,

$$\nu = \frac{\mu}{\rho}.$$

- **Newton's Law of Viscosity**

  According to this law, the rate of shear strain on a fluid element layer is precisely proportional to the shear stress (τ). The co-efficient of viscosity is the proportionality constant. Newtonian fluids are defined as those that obey this law.

- **Thermal Conductivity**

  The Fourier heat conduction law states that the heat flow is proportional to the temperature gradient. The coefficient of proportionality is a material parameter known as the thermal conductivity, which may be a function of several variables.

2.2 Types of Fluid Flow

2.2.1 Steady and Unsteady Flows

- **Steady Flow**

  Steady flow is the kind of flow where the fluid's properties, such as density, pressure, velocity, etc., at a given point in time do not change. Mathematically, we have:

$$\left(\frac{\partial u}{\partial t}\right)_{(x_0,y_0,z_0)} = 0; \quad \left(\frac{\partial v}{\partial t}\right)_{(x_0,y_0,z_0)} = 0; \quad \left(\frac{\partial w}{\partial t}\right)_{(x_0,y_0,z_0)} = 0;$$

$$\left(\frac{\partial p}{\partial t}\right)_{(x_0,y_0,z_0)} = 0; \quad \left(\frac{\partial \rho}{\partial t}\right)_{(x_0,y_0,z_0)} = 0; \quad \text{and so on}$$

  A fixed point in a fluid field where these variables are being measured with respect to time is denoted as (x_0, y_0, z_0).

Example: When the flow rate Qm^3/s is constant, the flow through a prismatic or non-prismatic conduit is steady. (A prismatic conduit's velocity equation, which is independent of time t and has a constant size shape, is $u = ax^2 + bx + c$.)

- **Unsteady Flow**

This kind of flow occurs when a point's velocity, pressure, or density varies over time. In terms of math, we have:

$$\left(\frac{\partial u}{\partial t}\right)_{x_0,y_0,z_0} \neq 0; \left(\frac{\partial v}{\partial t}\right)_{x_0,y_0,z_0} \neq 0; \left(\frac{\partial w}{\partial t}\right)_{x_0,y_0,z_0} \neq 0;$$

$$\left(\frac{\partial p}{\partial t}\right)_{x_0,y_0,z_0} \neq 0; \left(\frac{\partial \rho}{\partial t}\right)_{x_0,y_0,z_0} \neq 0 \text{ and so on}$$

Example: The flow through a pipe with a valve that is progressively opened or closed (the velocity equation is $u = ax^2 + bx(t)$).

2.2.2 Uniform and Non-Uniform Flows

- **Uniform Flow**

Uniform flow is the kind of flow where the velocity at any given moment does not vary with regard to space. Mathematically, we have:

$$\left(\frac{\partial V}{\partial s}\right)_{t=\text{constant}} = 0,$$

where ∂V = Change in velocity, and ∂s = Displacement in any direction.
Example: Flow through a straight prismatic conduit (i.e. flow through a straight pipe of constant diameter).

- **Non-Uniform Flow**

It is the kind of flow where the velocity varies with regard to space at any given time.

Mathematically,

$$\left(\frac{\partial V}{\partial s}\right)_{t=\text{constant}} \neq 0.$$

Examples:

(i) Pass through a conduit that isn't prismatic.

(ii) Circulate around a canal bend or pipe bend of uniform diameter.

2.2.3 One, Two and Three Dimensional Flows

- **One Dimensional Flow**

  This kind of flow occurs when a single spatial coordinate and time determine the flow characteristic, such as velocity. In terms of Mathematics:

$$u = f(x),$$

$$v = 0, \quad w \; = \; 0,$$

  where the velocity components in the x, y, and z directions are denoted by u, v, and w, respectively.

  Example:

  Flow in a pipe where analysis is done using average flow characteristics.

- **Two Dimensional Flow**

  Two-dimensional flow is a flow where the velocity depends on two rectangular spatial coordinates and time. In terms of Mathematics:

$$u = f_1(x, y),$$

$$u = f_2(x, y),$$

$$w = 0.$$

  Examples:

  (i) Flow of infinite magnitude between parallel plates.

  (ii) Flow in a broad river's main channel.

- **Three Dimensional Flow**

  This kind of flow has three mutually perpendicular directions and a velocity that depends on time. Mathematically:

$$u = f_1(x, y, z),$$

$$v = f_2(x, y, z),$$

$$w = f_3(x, y, z).$$

Examples:

(i) Flow in a channel or pipe that is either divergent or convergent.

(ii) Flow in an open, prismatic channel with water depth and width that are on the same order of magnitude.

2.2.4 Rotational and Irrotational Flows

- **Rotational Flow**

  If the fluid particles rotate around their mass centers while flowing in the direction of the flow, the flow is said to be rotational. Rotational flow occurs close to the solid boundaries.

  Example: Motion of liquid in a rotating tank.

- **Irrotational Flow**

  If the fluid particles do not rotate around their mass centers while traveling in the direction of the flow, the flow is said to be irrotational. In general, flow outside the boundary layer is regarded as irrotational.

  Example: Flow above a drain hole of a stationary tank or a wash basin.

2.2.5 Laminar and Turbulent Flows

- **Laminar Flow**

  When individual particles follow clearly defined routes and do not cross one another, the flow is said to be laminar.

  Examples:

  (i) Flow through a capillary tube.

(ii) Flow of blood in veins and arteries.

(iii) Ground water flow.

- **Turbulent Flow**

  A turbulent flow is characterized by the zigzag movement of fluid particles.

  Example: High velocity flow in a big conduit. More often than not, turbulent fluid flow problems occur in engineering practice.

2.2.6 Compressible and Incompressible Flows

- **Compressible Flow**

  This kind of flow occurs when the fluid's density ρ varies from one location to another; in other words, the flow's density is not constant. Mathematically:

$$\rho \neq \text{constant}.$$

  Example: Flow of gases through orifices, nozzles, gas turbines, etc.

- **Incompressible Flow**

  It is the kind of flow when the fluid's density remains constant. In general, liquids are thought to flow incompressibly. Mathematically:

$$\rho = \text{constant}.$$

  Example: Subsonic aerodynamics.

2.3 Fundamental Laws

2.3.1 Conservation of Mass

The principle of conservation of mass can be stated as the time rate of change of mass in a fixed volume is equal to the net rate of flow of mass across the surface. The Mathematical statement of the principle results in the following equation, known as the continuity (of mass) equation

$$\frac{\partial \rho}{\partial t} + \delta.\left(\rho V\right) = 0. \tag{2.1}$$

2.3.2 Conservation of Momentum

The principle of conservation of linear momentum (or Newton's Second Law of motion) states that the time rate of change of linear momentum of a given set of particles is equal to the vector sum of all the external forces acting on the particles of the set, provided Newton's Third Law of action and reaction governs the internal forces. Newton's second law can be written as

$$\frac{\partial}{\partial t}\left(\rho V\right) + \delta.\left(\rho V \otimes V\right) = \delta.\sigma + \rho f. \tag{2.2}$$

2.3.3 Conservation of Energy

The law of conservation of energy (or the First Law of Thermodynamics) states that the time rate of change of the total energy is equal to the sum of the rate of work done by applied forces and the change of heat content per unit time. In the general case, the First Law of Thermodynamics can be expressed in conservation form as

$$\frac{\partial \rho e^t}{\partial t} + \delta.\rho v e^t = -\delta.q + \delta.\left(\sigma.v\right) + Q + \rho f.v, \tag{2.3}$$

where $e^t = e + 1/2v^2$ is the total energy (J/m^3), e is the internal energy, q is he heat flux vector (W/m^2) and Q is the internal heat generation (W/m^3).

2.4 Heat and Mass Transfer Phenomenon and Related Properties

Heat transfer is the phenomenon of transferring energy and entropy from one place to another. The formal definition of heat transfer and its different types are given below.

2.4.1 Heat Transfer

Heat transfer is a branch of engineering that deals with the transfer of th ermal energy from one point to another within a medium from one mediumto another due to the occurrence of temperature difference. Heat transfer m ay take place in one or more of its three basic forms: conduction, convection, and radiation.

2.4.2 Mass Transfer

Mass transfer is the flow of molecules from one body to another when these bodies are in contact or within a system consisting of two components when the distribution of materials is not uniform. When a copper plate is placed on a steel plate, some molecules from either side will diffuse into the otherside. When salt is placed in a glass and water poured over it, after sufficient time the salt molecules will diffuse into the water body. A more common example is drying of clothes or the evaporation of water spilled on the floor when water molecules diffuse into the air surrounding it. Usually, mass transfer takes place from a location where the particular component is proportionately high to a location where the component is proportionately low. Mass transfer may also take place due to potentials other than concentration difference.

2.4.3 Conduction

Conduction is the transfer of heat from one part of a body at a higher temperature to another part of the same body at a lower temperature, or from one body at a higher temperature to another body in physical contact with it at a lower temperature. Energy is transferred from more energetic molecules to molecules with a lower energy level during the conduction process, which occurs at the molecular level. The average kinetic energy of molecules in the higher-temperature regions is higher than that of molecules in the lower-temperature regions, as can be readily seen in gases. Because they are constantly moving randomly, the more energetic molecules occasionally collide with molecules with lower energy levels, exchanging momentum and energy. In this way, energy is continuously transported from hotter parts to colder regions. Although the molecules in liquids are closer together than in gases, the process of molecular energy exchange is qualitatively comparable. Lattice waves produced by atomic motion carry heat in substances that are dielectrics, or non-conductors of electricity. This lattice vibration mechanism contributes very little to the energy transfer process in materials that are good electrical conductors; the main contribution comes from the mobility of free electrons, which move similarly to molecules in a gas.

2.4.4 Convection

The process of heat transfer between a surface and fluid flowing in contact with it is called convection.

Types of Convection:

(i) Natural Convection or Free Convection

(ii) Forced Convection

(iii) Mixed Convection

2.4.5 Natural Convection or Free Convection

If the flow caused by the buoyant forces generated by heating or cooling of the fluid the process is called as natural or free convection.

2.4.6 Forced Convection

If the flow is caused by an external device like a pump or blower, it is termed as forced convection.

2.4.7 Radiation

Radiation, or more correctly thermal radiation, is electromagnetic radiation emitted by a body by virtue of its temperature and at the expense of its internal energy. Thus thermal radiation is of the same nature as visible light, x-rays, and radio waves, the difference between them being in their wavelengths and the source of generation. The eye is sensitive to electromagnetic radiation in the region from 0.39 to $0.78\mu m$; this is identified as the visible region of the spectrum. Radio waves have a wavelength of 1×10^3 to $2\times10^{10}\mu m$, and x-rays have wavelengths of 1×10^{-5} to $2\times10^{-2}\mu m$, while the bulk of thermal radiation occurs in rays from approximately 0.1 to $100\mu m$. All heated solids and liquids, as well as some gases, emit thermal radiation. Conduction requires a material medium to transport energy, whereas radiation does not. In actuality, a vacuum is where radiation transfer happens most effectively. At the macroscopic level, thermal radiation is calculated using the Stefan-Boltzmann law, which links the energy flux released by a blackbody or ideal radiator to the fourth power of the absolute temperature.

2.5 Dimensionless Number

2.5.1 Reynolds Number

The Reynolds number (Re) is a dimensionless quantity used in fluid mechanics to predict the flow regime of a fluid, whether it will flow in a smooth (laminar) or chaotic (turbulent) manner. It is named after Osborne Reynolds, who first introduced it in the 1880s. The Reynolds number is critical in understanding and analyzing fluid flow in pipes, around objects, and in other engineering applications.

$$Re = \frac{\rho v L}{\mu} = \frac{vL}{\nu}$$

2.6 Micromagnetorotational (MMR)

The concept of Micromagnetorotational (MMR) fluids is a more specialized and advanced topic that arises in the context of magnetohydrodynamics (MHD), particularly in situations when magnetic fields and microstructured, rotating elements (microparticles or microelements) interact within a fluid. While this term is not universally standardized in the literature, MMR fluids typically refer to systems where microscopic particles within a fluid are influenced by both magnetic fields and rotational effects. This results in unique fluid behaviors that combine the principles of magnetism, rotation, and fluid dynamics.

2.6.1 Key Concepts of Micromagnetorotational (MMR) Fluids:

2.6.1.1 Magnetic Influence:

- In MMR systems, the magnetic field has a critical role in influencing the behavior the fluid.The magnetic field can interact with magnetic particles suspended within the fluid, suchiron particles, magnetic colloids, or other ferromagnetic substances.

- The fluid can exhibit magnetization or magnetic viscosity based on the type of magnetic particles involved.

- The magnetic forces acting on the particles can affect both their motion and rotation, leading to unique flow characteristics that would not be presentin purely non-magnetic systems.

2.6.1.2 Rotational Effects:

- The fluid is also influenced by rotational behavior at the microscale. This could involve the rotation of individual microparticles or microstructures within the fluid. In a manner similar to micropolar fluids, these particles might exhibit rotational motion that influences the macroscopic flow behavior.

- The rotation of these particles, when combined with external magnetic fields, can lead to torques or couple stresses within the fluid, which are important for understanding the flow of materials like magnetorheological fluids (MRFs).

2.6.1.3 Non-Newtonian Flow Behavior:

- The presence of rotating, magnetized microparticles in a fluid typically results in a non-Newtonian flow. The relationship between shear stress and shear rate is no longer linear, and can vary with changes in the magnetic field, the rotational velocity of the particles, and other factors.

- The material may exhibit yield stress, meaning it may not flow until a certain threshold of applied stress (e.g., when the magnetic field or rotational forces exceed a certain level).

2.6.1.4 Applications of MMR Fluids:

- Magnetorheological Fluids (MRFs): These are fluids that change their viscosity in response to a magnetic field. When subjected to a magnetic field, the microparticles align and cause the fluid to become more viscous, thereby offering applications in damping, clutch systems, shock absorbers, and actuators.
- Biomedical Applications: The MMR concept could apply to biological fluids like blood, especially in scenarios where magnetic particles (e.g., for drug delivery or imaging) interact with biological structures under rotational and magnetic influences.
- Fluid Flow Control: Systems requiring the control of flow properties through external fields, such as those used in smart materials, soft robotics, or microfluidic devices.
- Enhanced Heat Transfer: In some industrial applications, magnetic and rotational effects in the fluid can enhance heat transfer properties, which is useful in thermal management systems.

2.7 Relaxation Time Effect

The relaxation time effect refers to the characteristic time it takes for a system to return to equilibrium after a disturbance. In fluid dynamics, the relaxation time effect is particularly important in systems involving viscoelastic fluids, suspensions, or complex fluids, where the response to external forces (like shear or strain) is not instantaneous but instead evolves over time.
Key Concepts:

2.7.1.1 Relaxation Time:

- The relaxation time is a measure of the time it takes for the fluid's internal microstructure (e.g., polymer chains, particles, or molecules) to adjust or "relax" to a new equilibrium state after a change in external conditions.
- In the context of viscoelastic fluids or complex fluids, this time governs how quickly the fluid recovers from deformations (like shear stress or strain).

- For example, in polymer melts, the relaxation time corresponds to how long it takes for the polymer chains to disentangle or reorient after being stretched or deformed.

2.7.1.2 Viscoelastic Fluids:

- A viscoelastic fluid exhibits both elastic (solid-like) and viscous (fluid-like) behavior.
- The relaxation time characterizes how long it takes for the fluid's elastic properties to dissipate. When a stress is applied to a viscoelastic fluid, it deforms elastically (like a solid) initially, but over time, the deformation will relax as the fluid begins to flow and behave more like a viscous fluid.
- A key example is polymer solutions, where the molecular chains have to "relax" to their equilibrium configuration after being deformed.

2.7.1.3 Suspensions:

- In suspensions or colloidal fluids, the relaxation time refers to the time it takes for the particles or microstructures suspended in the fluid to return to equilibrium after experiencing an external force or disturbance.

- For example, when a force is applied to a suspension of particles, the particles might experience a delayed response due to inter-particle interactions or the time it takes for the particles to move or rotate in the fluid.

2.7.1.4 Time-dependent Rheological Behavior:

- The relaxation time can influence the rheological properties of the fluid, including its viscosity and shear modulus.

- In fluids with a significant relaxation time, the fluid will show time-dependent behavior (e.g., thixotropy or shear thinning), where its viscosity can change over time as the fluid relaxes or recovers.

2.8 Drag Force

The drag force is the resistive force experienced by an object moving through a fluid (liquid or gas). It acts in the direction opposite to the object's motion and is a result of the interaction between the object and the fluid. The drag force depends on several factors, including the velocity of the object, the fluid's properties, the shape and size of the object, and the surface roughness of the object. In engineering and physics, drag is an important consideration in fields such as aerospace, automotive design, marine engineering, and sports science.

Applications of Drag Force:

2.8.1.1 Automobile Design:

- Minimizing drag is crucial for improving fuel efficiency and top speed in cars and other vehicles. Streamlined shapes are used to reduce drag at high speeds.

2.8.1.2 Aerospace Engineering:

- The drag force is a major factor in the design of aircraft and spacecraft. Engineers optimize the shape of wings, fuselage, and tail sections to minimize drag and improve fuel efficiency and performance.

2.8.1.3 Sports Equipment:

- In sports like cycling, swimming, or car racing, drag is a critical factor. Athletes and designers use streamlined suits, aero helmets, and bike designs to reduce drag for better performance.

2.8.1.4 Marine Engineering:

- In ships and submarines, drag affects fuel efficiency and speed. Streamlined hull designs are used to minimize resistance in water.

2.8.1.5 Energy Efficiency:

- In wind turbines, drag is a factor in how much wind energy can be converted into usable power. The drag on turbine blades is minimized to improve efficiency.

2.9 Mass Flow Rate

Mass flow rate is the amount of mass of a substance passing through a given surface or volume per unit of time. It is a key concept in fluid dynamics and is widely used in engineering, physics, and various industrial applications to quantify how much mass is transported through a system, such as a pipe, duct, or channel. The mass flow rate is particularly important in understanding fluid systems in which mass conservation is critical, such as in ventilation, pipeline transportation, heating and cooling systems, and various chemical processes.

Applications of Mass Flow Rate:

2.9.1.1 Fluid Transport in Pipelines:

- In industrial applications, mass flow rate is essential for designing pipelines, pumps, and valves. Ensuring that the mass flow rate is properly controlled helps prevent over-pressurization and ensures consistent fluid delivery.

2.9.1.2 Ventilation and HVAC Systems:

- Mass flow rate is critical in ventilation systems to calculate air distribution and ensure proper air quality. HVAC systems use mass flow rates to determine the required airflow to maintain temperature and pressure.

2.9.1.3 Combustion Processes:

- In combustion engines, mass flow rate of air and fuel is important for controlling the stoichiometric mixture and ensuring efficient burning. In gas turbines, it is also crucial for maintaining optimal performance.

2.9.1.4 Chemical Processing:

- In chemical reactors, controlling the mass flow rate of reactants and products is necessary for optimizing reaction times and maintaining the correct stoichiometric ratios for desired chemical transformations.

2.9.1.5 Environmental and Geophysical Studies:

- Mass flow rate is used to study the movement of natural resources such as water in rivers, air in the atmosphere, or heat in geothermal systems.

Galerkin Finite Element Method for 2D Problems

The Galerkin Finite Element Method (FEM) is a numerical technique used to solve partial differential equations (PDEs), particularly in engineering and physics, by converting them into algebraic equations. The method is applied to discretized domains, often in the form of triangular or quadrilateral elements in 2D. In the Galerkin method, the test functions (weight functions) are chosen to be the same as the trial functions (shape functions) used to approximate the solution.

Steps to Solve a 2D Problem using the Galerkin Finite Element Method

We will apply the Galerkin FEM to a 2D boundary value problem. For simplicity, let's use the Poisson equation as the governing equation:

$$-\nabla^2 u(x,y) = f(x,y) \quad \text{in the domain } \Omega = [0, L_x] \times [0, L_y],$$

with boundary conditions:

$$u(x,y) = g(x,y) \quad \text{on } \partial\Omega,$$

where:
(i) $u(x,y)$ is the unknown solution (e.g., temperature),
(ii) $f(x,y)$ is a source term (e.g., heat generation),
(iii) ∇^2 is the Laplacian operator,
(iv) Ω is the 2D problem domain.

- Discretization of the Domain

The first step is to discretize the domain Ω into smaller subdomains called finite elements. These elements can be:
(i) Triangles (in 2D): Often used for irregular geometries.
(ii) Quadrilaterals: Used in structured grids.
We divide the domain Ω into finite elements (triangular or quadrilateral) and define the nodal points at the corners of the elements. The solution $u(x, y)$ will be approximated using shape functions defined at the nodes of the elements.

- Approximate Solution (Trial Functions)

In the Galerkin method, the trial (approximate) solution $u(x, y)$ is expressed as a linear combination of shape functions:

$$u(x, y) = \sum_{i=1}^{n} N_i(x, y) u_i,$$

where:
(i) $N_i(x, y)$ are the shape functions (typically polynomials defined over the elements),
(ii) u_i are the unknown nodal values (e.g., temperatures at the nodes),
(iii) n is the total number of nodes.
For example, if we are using linear shape functions for a triangular element, the approximation would be:

$$u(x, y) = N_1(x, y)u_1 + N_2(x, y)u_2 + N_3(x, y)u_3,$$

where $N_1(x, y)$, $N_2(x, y)$, $N_3(x, y)$ are the linear shape functions associated with the three nodes of the triangle.

- Formulate the Weak Form (Galerkin Formulation)

The governing equation is a second-order PDE. To solve it using the finite element method, we must first transform it into a weak form. Starting with the equation:

$$-\nabla^2 u(x, y) = f(x, y).$$

Multiply both sides by a test (weight) function $v(x, y)$ (in the Galerkin method, $v(x, y)$ is chosen to be the same as the trial function $N_i(x, y)$) and integrate over the domain Ω:

$$\int_\Omega v(x, y) \left[-\nabla^2 u(x, y) - f(x, y)\right] dA = 0.$$

Now, integrate by parts to eliminate the second derivative $\nabla^2 u(x, y)$. The integration by parts formula is:

$$\int_\Omega v(x, y)\nabla^2 u(x, y)dA = -\int_\Omega \nabla v(x, y) \cdot \nabla u(x, y)dA + \int_{\partial\Omega} v(x, y)\frac{\partial u}{\partial n}d\Gamma,$$

where:
(i) $\nabla v(x, y) \cdot \nabla u(x, y)$ is the inner product of the gradients of $v(x, y)$ and $u(x, y)$,
(ii) The boundary term $\int_{\partial\Omega} v(x, y)\frac{\partial u}{\partial n}d\Gamma$ accounts for the Neumann boundary conditions, where $\frac{\partial u}{\partial n}$ is the normal derivative of $u(x, y)$ on the boundary $\partial\Omega$.
For Dirichlet boundary conditions, this boundary term becomes zero.
Thus, the weak form of the governing equation is:

$$\int_\Omega \nabla v(x, y) \cdot \nabla u(x, y)dA = \int_\Omega v(x, y)f(x, y)dA.$$

- Discretize the weak form

Now, substitute the trial function $u(x, y) = \sum_{i=1}^{n} N_i(x, y)u_i$ and the test function $v(x, y) = N_j(x, y)$ into the weak form. For each element in the domain, we need to compute the integrals:
(i) Stiffness matrix: The left-hand side of the weak form involves the gradient terms and gives the stiffness matrix K_{ij}

$$K_{ij} = \int_\Omega \nabla N_i(x, y) \cdot \nabla N_j(x, y)dA.$$

(ii) Load vector: The right-hand side of the weak form involves the source term $f(x, y)$ and gives the load vector f_j

$$f_j = \int_\Omega N_j(x, y)f(x, y)dA.$$

- Assembly of the Global System

After calculating the element stiffness matrices and load vectors for all the finite elements, the global system of equations is assembled. The global stiffness matrix K is formed by summing the contributions from all elements:

$$K = \sum_{e=1}^{n_e} K^{(e)},$$

where $K^{(e)}$ is the stiffness matrix for element e, and n_e is the number of elements in the mesh. The global load vector f is similarly assembled by summing the contributions from each element's load vector:

$$f = \sum_{e=1}^{n_e} f^{(e)}.$$

- Solve the System of Equations

The system of linear equations is given by:

$$Ku = f,$$

where:
(i) K is the global stiffness matrix,
(ii) u is the vector of unknown nodal values (the approximate solution),
(iii) f is the global load vector.
This system is solved using standard linear algebra techniques (e.g., Gaussian elimination, LU decomposition, or iterative methods) to find the approximate values of u_i at the nodes.

- Apply Boundary Conditions

Boundary conditions are applied to the global system in one of two ways:
(i) Dirichlet Boundary Conditions (known values of $u(x, y)$): Modify the global system by setting the corresponding nodal values.
(ii) Neumann Boundary Conditions (known derivatives of $u(x, y)$): Adjust the global load vector.

Chapter 3

Investigation of Relaxation Time on Viscoelastic Two-Dimensional Flow

The viscoelastic fluids is the upper convected Maxwell (UCM) model that incorporates viscosity and relaxation time with the following constitutive equation:

$$\lambda\nabla_{\tau} + \tau = -2\eta\mathbf{d}(\mathbf{v}), \tag{3.1}$$

where λ and η are the fluid characteristic relaxation time and the fluid viscosity, respectively. The deformation strain tensor $\mathbf{d(v)}$ in (3.1), is defined as

$$\mathbf{d}(\mathbf{v}) = \frac{1}{2}\left(\nabla v + (\nabla v)^{T}\right). \tag{3.2}$$

∇_{τ} being the upper convected derivative is defined as

$$\nabla_{\tau} = \frac{\partial \tau}{\partial t} + v.\nabla\tau - (\nabla v).\tau - \tau.(\nabla v)^{T}. \tag{3.3}$$

Here, λ and η are assumed to be constant, whereas in general, they may depend on the local shear rate, pressure, and temperature, see for instance [15]. After following [37], the governing equation of motion for the unsteady incompressible flow is stated as

$$\rho\left(\frac{\partial v}{\partial t} + v.\nabla v\right) + \nabla p + \nabla.\tau = b = 0, \tag{3.4}$$

together with the mass conservation within the medium

$$\nabla . v = 0. \tag{3.5}$$

3.1 Problem Description

Consider a two-dimensional unsteady incompressible viscoelastic flow confined in a lid-driven baffled cavity as depicted in Figure 3.2. In the absence of body forces b, the momentum balance and continuity equations in (3.4) and (3.5) read in the component form as

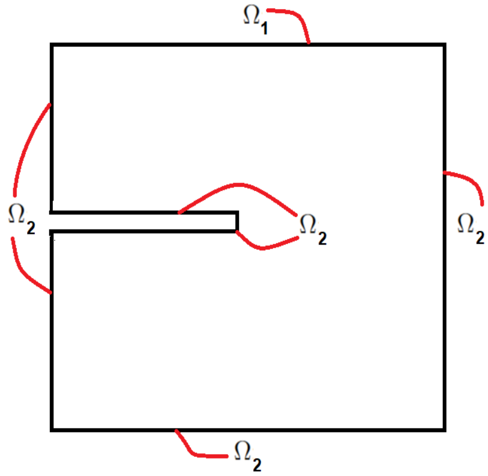


FIGURE 3.1: Geometry of the problem.

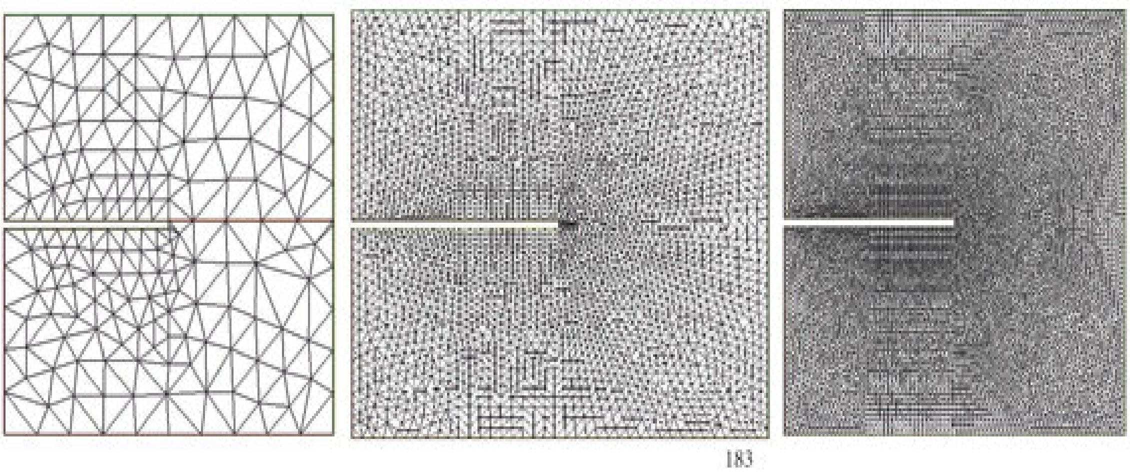


FIGURE 3.2: Computational meshes (from left to right): coarser, refined, and finer.

3.1.1 Dimensional Form of the Governing Equations

The dimensional form of the continuity, momentum and micromagnetorotational equations along with associated boundary conditions for the proposed problem are mentioned below.

Continuity Equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{3.6}$$

Momentum equation for u-velocity

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda\left(u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{3.7}$$

Momentum equation for v-velocity

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\lambda\left(u^2\frac{\partial^2 v}{\partial x^2}+v^2\frac{\partial^2 v}{\partial y^2}+2uv\frac{\partial^2 v}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{3.8}$$

Following boundary conditions are prescribed

$$u=u_0,\quad v=0 \qquad \text{on} \qquad \Omega_1 \quad \text{and} \quad u=v=0 \qquad \text{on} \qquad \Omega_2$$

3.1.2 Dimensionless Parameters

To non-dimensionalize the system (3.6)-(3.8), we use the following dimensionless parameters

$$U=\frac{u}{u_0},\quad V=\frac{v}{u_0},\quad T=\frac{u_0}{L}t,\quad X=\frac{x}{L},\quad Y=\frac{y}{L},\quad P=\frac{p}{\rho u_0^2}. \tag{3.9}$$

3.2 Conversion of Dimensional form into Dimensionless Form

To convert governing equations (3.6)-(3.8) into the dimensionless partial differential form, different derivatives are required which have been computed in the upcoming part of this sub-section.

- $X = \dfrac{x}{L} \quad \Rightarrow \quad \dfrac{\partial X}{\partial x} = \dfrac{1}{L}$

- $Y = \dfrac{y}{L} \quad \Rightarrow \quad \dfrac{\partial Y}{\partial y} = \dfrac{1}{L}$

- $U = \dfrac{u}{u_o} \quad \Rightarrow \quad u = u_o U$

- $V = \dfrac{v}{u_o} \quad \Rightarrow \quad v = u_o V$

- $T = \dfrac{u_o}{L}t \quad \Rightarrow \quad \dfrac{\partial T}{\partial t} = \dfrac{u_o}{L}$

- $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial T}\dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial T}(u)\dfrac{\partial}{\partial t}(T) = \dfrac{\partial}{\partial T}(u_o U)\dfrac{u_o}{L} = u_o\dfrac{\partial U}{\partial T}\dfrac{u_o}{L} = \dfrac{u_o^2}{L}\dfrac{\partial U}{\partial T}$

- $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{\partial}{\partial X}(u)\dfrac{1}{L} = \dfrac{\partial}{\partial X}(u_o U)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial U}{\partial X}$

- $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{\partial}{\partial Y}(u)\dfrac{1}{L} = \dfrac{\partial}{\partial Y}(u_o U)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}$

- $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{\partial}{\partial Y}(v)\dfrac{1}{L} = \dfrac{\partial}{\partial Y}(u_o V)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}$

- $u\dfrac{\partial u}{\partial x} = u_o U\dfrac{u_o}{L}\dfrac{\partial U}{\partial X} = \dfrac{u_o^2}{L}U\dfrac{\partial U}{\partial X}$

- $v\dfrac{\partial u}{\partial y} = u_o V\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y} = \dfrac{u_o^2}{L}V\dfrac{\partial U}{\partial Y}$

- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial X}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X^2}$

- $u^2\dfrac{\partial^2 u}{\partial x^2} = (u_o U)^2\dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X^2} = \dfrac{u_o^3}{L^2}U^2\dfrac{\partial^2 U}{\partial X^2}$

- $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{\partial Y}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial Y^2}$

- $v^2\dfrac{\partial^2 u}{\partial y^2} = (u_o V)^2\dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial Y^2} = \dfrac{u_o^3}{L^2}V^2\dfrac{\partial^2 U}{\partial Y^2}$

- $\dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X\partial Y}$

- $2uv\dfrac{\partial^2 u}{\partial x\partial y} = 2(u_o U)(u_o V)\dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X\partial Y} = 2\dfrac{u_o^3}{L^2}UV\dfrac{\partial^2 U}{\partial X\partial Y}$

- $\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial T}\dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial T}(v)\dfrac{\partial}{\partial t}(T) = \dfrac{\partial}{\partial T}(u_o V)\dfrac{u_o}{L} = u_o\dfrac{\partial V}{\partial T}\dfrac{u_o}{L} = \dfrac{u_o^2}{L}\dfrac{\partial V}{\partial T}$

- $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{\partial}{\partial X}(v)\dfrac{1}{L} = \dfrac{\partial}{\partial X}(u_o V)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial V}{\partial X} = u_o U \dfrac{u_o}{L}\dfrac{\partial V}{\partial X} = \dfrac{u_o^2}{L} U \dfrac{\partial V}{\partial X}$
- $v\dfrac{\partial v}{\partial y} = u_o V \dfrac{u_o}{L}\dfrac{\partial V}{\partial Y} = \dfrac{u_o^2}{L} V \dfrac{\partial V}{\partial Y}$
- $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial X}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X^2}$
- $u^2\dfrac{\partial^2 v}{\partial x^2} = (u_o U)^2 \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X^2} = \dfrac{u_o^3}{L^2} U^2 \dfrac{\partial^2 V}{\partial X^2}$
- $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{\partial Y}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial^2 v}{\partial y^2} = \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial Y^2}$
- $v^2\dfrac{\partial^2 v}{\partial y^2} = (u_o V)^2 \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial Y^2} = \dfrac{u_o^3}{L^2} V^2 \dfrac{\partial^2 V}{\partial Y^2}$
- $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X \partial Y}$
- $2uv\dfrac{\partial^2 v}{\partial x \partial y} = 2(u_o U)(u_o V)\dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X \partial Y} = 2\dfrac{u_o^3}{L^2} UV \dfrac{\partial^2 V}{\partial X \partial Y}$

Now, the dimensionless form of the continuity equation (3.6) can be obtained as follows,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{u_o}{L}\frac{\partial U}{\partial X} + \frac{u_o}{L}\frac{\partial V}{\partial Y} = 0,$$

$$\frac{u_o}{L}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) = 0,$$

As $\dfrac{u_o}{L} \neq 0$ but,

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \tag{3.10}$$

The dimensionless form of the u-momentum equation (3.7) can be obtained as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda\left(u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

$$\frac{u_o^2}{L}\frac{\partial U}{\partial T} + \frac{u_o^2}{L}U\frac{\partial U}{\partial X} + \frac{u_o^2}{L}V\frac{\partial U}{\partial Y} + \lambda\left(\frac{u_o^3}{L^2}U^2\frac{\partial^2 U}{\partial X^2} + \frac{u_o^3}{L^2}V^2\frac{\partial^2 V}{\partial Y^2} + 2\frac{u_o^3}{L^2}UV\frac{\partial^2 U}{\partial X \partial Y}\right)$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{u_o}{L^2}\frac{\partial^2 U}{\partial X^2} + \frac{u_o}{L^2}\frac{\partial^2 U}{\partial Y^2}\right)$$

$$\frac{u_o^2}{L}\left[\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+\lambda\frac{u_o}{L}\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)\right]$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{u_o}{L^2}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$
$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{L\mu}{\rho u_o^2}\frac{u_o}{L^2}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$
$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial}{\partial X}\frac{\partial X}{\partial x}(p)+\frac{\mu}{L\rho u_o}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$
$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-L\frac{\partial}{\partial X}\frac{1}{L}\left(\frac{p}{\rho u_o^2}\right)+\frac{1}{\frac{\rho L u_o}{\mu}}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)$$
$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial X}+\frac{1}{Re}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right). \tag{3.11}$$

Similarly, the dimensionless form of the v-momentum equation (3.8) can be obtained as follows:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\lambda\left(u^2\frac{\partial^2 v}{\partial x^2}+v^2\frac{\partial^2 v}{\partial y^2}+2uv\frac{\partial^2 v}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right),$$
$$\frac{u_o^2}{L}\frac{\partial V}{\partial T}+\frac{u_o^2}{L}U\frac{\partial V}{\partial X}+\frac{u_o^2}{L}V\frac{\partial V}{\partial Y}+\lambda\left(\frac{u_o^3}{L^2}U^2\frac{\partial^2 V}{\partial X^2}+\frac{u_o^3}{L^2}V^2\frac{\partial^2 V}{\partial Y^2}+2\frac{u_o^3}{L^2}UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{u_o}{L^2}\frac{\partial^2 V}{\partial X^2}+\frac{u_o}{L^2}\frac{\partial^2 V}{\partial Y^2}\right)$$
$$\frac{u_o^2}{L}\left[\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+\lambda\frac{u_o}{L}\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)\right]$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\frac{u_o}{L^2}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$
$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{L}{u_o^2}\frac{\mu}{\rho}\frac{u_o}{L^2}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial}{\partial Y}\frac{\partial Y}{\partial y}(p)+\frac{\mu}{\rho L u_o}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-L\frac{\partial}{\partial Y}\frac{1}{L}\left(\frac{p}{\rho u_o^2}\right)+\frac{1}{\frac{\rho L u_o}{\mu}}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial Y}+\frac{1}{Re}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right). \tag{3.12}$$

3.3 Dimensionless Governing Equations

The set of transformed dimensionless equations can be written as

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0. \tag{3.13}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial X}+\frac{1}{Re}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right). \tag{3.14}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial Y}+\frac{1}{Re}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right). \tag{3.15}$$

The associated dimensionless boundary conditions are as follows:

$$U=1.0,\quad V=0 \quad \text{on} \quad \Omega_1$$

$$U=V=0 \quad \text{on} \quad \Omega_2$$

In (3.13) and (3.15), the nondimensional parameters Re represent Reynolds number, R_f is the fluid relaxation time respectively, defined by

$$Re = \frac{\rho L u_o}{\mu}, \qquad \text{and} \qquad R_f = \frac{\lambda u_o}{L}. \tag{3.16}$$

Chapter 4

Micropolar Flow Analysis with Relaxation Time Effects

This chapter extends the analysis of viscoelastic two-dimensional micropolar flow by incorporating the effects of relaxation time on drag forces. The model investigates the behavior of viscoelastic fluids with micropolar properties under relaxation time influences. The procedure begins with deriving the dimensional governing partial differential equations (PDEs) for the flow dynamics. Next, similarity transformations are applied to convert these dimensional PDEs into non-dimensional form, simplifying the equations for easier analysis and solution. Boundary conditions are also defined to ensure accurate representation of the physical flow characteristics. The goal is to deepen the understanding of how relaxation time impacts drag forces in such flows, with the non-dimensional PDEs facilitating analysis and comparison of different fluid flow scenarios, benefiting practical engineering applications.

4.1 Dimensional Form of the Governing Equations

The dimensional form of the continuity, momentum and micromagnetorotational equations along with associated boundary conditions for the proposed problem are mentioned below.

Continuity Equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{4.1}$$

Momentum equation for u-velocity

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda\left(u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}(\mu+k)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{k}{\rho}\frac{\partial w}{\partial y}, \tag{4.2}$$

Momentum equation for v-velocity

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\lambda\left(u^2\frac{\partial^2 v}{\partial x^2}+v^2\frac{\partial^2 v}{\partial y^2}+2uv\frac{\partial^2 v}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{\rho}(\mu+k)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{k}{\rho}\frac{\partial w}{\partial x}, \tag{4.3}$$

Micromagnetorotational equation:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\frac{\gamma^*}{\rho j}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{2k}{\rho j}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}-2w\right). \tag{4.4}$$

Following boundary conditions are prescribed

$$u=u_0,\quad v=w=0 \qquad \text{on} \qquad \Omega_1 \quad \text{and} \quad u=v=w=0 \qquad \text{on} \qquad \Omega_2$$

4.1.1 Dimensionless Parameters

To non-dimensionalize the system (4.1)-(4.4), we use following dimensionless parameters

$$U=\frac{u}{u_0},\quad V=\frac{v}{u_0},\quad W=\frac{Lw}{u_0},\quad T=\frac{u_0}{L}t,\quad X=\frac{x}{L},\quad Y=\frac{y}{L},\quad P=\frac{p}{\rho u_0^2}. \tag{4.5}$$

4.2 Conversion of Dimensional form into Dimensionless Form

To convert governing equations (4.1)-(4.4) into the dimensionless partial differential form, different derivatives are required which have been computed in the upcoming part of this sub-section.

- $X = \dfrac{x}{L} \quad \Rightarrow \quad \dfrac{\partial X}{\partial x} = \dfrac{1}{L}$
- $Y = \dfrac{y}{L} \quad \Rightarrow \quad \dfrac{\partial Y}{\partial y} = \dfrac{1}{L}$
- $U = \dfrac{u}{u_o} \quad \Rightarrow \quad u = u_o U$
- $V = \dfrac{v}{u_o} \quad \Rightarrow \quad v = u_o V$
- $W = \dfrac{Lw}{u_o} \quad \Rightarrow \quad w = \dfrac{u_o W}{L}$
- $T = \dfrac{u_o}{L} t \quad \Rightarrow \quad \dfrac{\partial T}{\partial t} = \dfrac{u_o}{L}$
- $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial T}\dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial T}(u)\dfrac{\partial}{\partial t}(T) = \dfrac{\partial}{\partial T}(u_o U)\dfrac{u_o}{L} = u_o \dfrac{\partial U}{\partial T}\dfrac{u_o}{L} = \dfrac{u_o^2}{L}\dfrac{\partial U}{\partial T}$
- $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{\partial}{\partial X}(u)\dfrac{1}{L} = \dfrac{\partial}{\partial X}(u_o U)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial U}{\partial X}$
- $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{\partial}{\partial Y}(u)\dfrac{1}{L} = \dfrac{\partial}{\partial Y}(u_o U)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}$
- $\dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{\partial}{\partial Y}(v)\dfrac{1}{L} = \dfrac{\partial}{\partial Y}(u_o V)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}$
- $u\dfrac{\partial u}{\partial x} = u_o U \dfrac{u_o}{L}\dfrac{\partial U}{\partial X} = \dfrac{u_o^2}{L} U \dfrac{\partial U}{\partial X}$
- $v\dfrac{\partial u}{\partial y} = u_o V \dfrac{u_o}{L}\dfrac{\partial U}{\partial Y} = \dfrac{u_o^2}{L} V \dfrac{\partial U}{\partial Y}$
- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial X}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X^2}$
- $u^2\dfrac{\partial^2 u}{\partial x^2} = (u_o U)^2 \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X^2} = \dfrac{u_o^3}{L^2} U^2 \dfrac{\partial^2 U}{\partial X^2}$
- $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{\partial Y}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial Y^2}$
- $v^2\dfrac{\partial^2 u}{\partial y^2} = (u_o V)^2 \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial Y^2} = \dfrac{u_o^3}{L^2} V^2 \dfrac{\partial^2 U}{\partial Y^2}$
- $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial U}{\partial Y}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X \partial Y}$
- $2uv\dfrac{\partial^2 u}{\partial x \partial y} = 2(u_o U)(u_o V)\dfrac{u_o}{L^2}\dfrac{\partial^2 U}{\partial X \partial Y} = 2\dfrac{u_o^3}{L^2} UV \dfrac{\partial^2 U}{\partial X \partial Y}$

- $\dfrac{\partial v}{\partial t} = \dfrac{\partial v}{\partial T}\dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial T}(v)\dfrac{\partial}{\partial t}(T) = \dfrac{\partial}{\partial T}(u_oV)\dfrac{u_o}{L} = u_o\dfrac{\partial V}{\partial T}\dfrac{u_o}{L} = \dfrac{u_o^2}{L}\dfrac{\partial V}{\partial T}$
- $\dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{\partial}{\partial X}(v)\dfrac{1}{L} = \dfrac{\partial}{\partial X}(u_oV)\dfrac{1}{L} = \dfrac{u_o}{L}\dfrac{\partial V}{\partial X} = u_oU\dfrac{u_o}{L}\dfrac{\partial V}{\partial X} = \dfrac{u_o^2}{L}U\dfrac{\partial V}{\partial X}$
- $v\dfrac{\partial v}{\partial y} = u_oV\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y} = \dfrac{u_o^2}{L}V\dfrac{\partial V}{\partial Y}$
- $\dfrac{\partial^2 v}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial X}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X^2}$
- $u^2\dfrac{\partial^2 v}{\partial x^2} = (u_oU)^2\dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X^2} = \dfrac{u_o^3}{L^2}U^2\dfrac{\partial^2 V}{\partial X^2}$
- $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial v}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{\partial Y}{\partial y}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial^2 v}{\partial y^2} = \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial Y^2}$
- $v^2\dfrac{\partial^2 v}{\partial y^2} = (u_oV)^2\dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial Y^2} = \dfrac{u_o^3}{L^2}V^2\dfrac{\partial^2 V}{\partial Y^2}$
- $\dfrac{\partial^2 v}{\partial x\partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial v}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L}\dfrac{\partial V}{\partial Y}\right) = \dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X\partial Y}$
- $2uv\dfrac{\partial^2 v}{\partial x\partial y} = 2(u_oU)(u_oV)\dfrac{u_o}{L^2}\dfrac{\partial^2 V}{\partial X\partial Y} = 2\dfrac{u_o^3}{L^2}UV\dfrac{\partial^2 V}{\partial X\partial Y}$
- $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial T}\dfrac{\partial T}{\partial t} = \dfrac{\partial}{\partial T}(w)\dfrac{\partial}{\partial t}(T) = \dfrac{\partial}{\partial T}(\dfrac{u_oW}{L})\dfrac{u_o}{L} = \dfrac{u_o^2}{L^2}\dfrac{\partial W}{\partial T} = \dfrac{u_o^2}{L^2}\dfrac{\partial W}{\partial T}$
- $\dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{\partial}{\partial X}(w)\dfrac{1}{L} = \dfrac{\partial}{\partial X}(\dfrac{u_oW}{L})\dfrac{1}{L} = \dfrac{u_o}{L^2}\dfrac{\partial W}{\partial X}$
- $\dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial Y}\dfrac{\partial Y}{\partial y} = \dfrac{\partial}{\partial Y}(w)\dfrac{1}{L} = \dfrac{\partial}{\partial Y}(\dfrac{u_oW}{L})\dfrac{1}{L} = \dfrac{u_o}{L^2}\dfrac{\partial W}{\partial Y}$
- $u\dfrac{\partial w}{\partial x} = u_oU\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial X} = \dfrac{u_o^2}{L^2}U\dfrac{\partial W}{\partial X}$
- $v\dfrac{\partial w}{\partial y} = u_oV\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial Y} = \dfrac{u_o^2}{L^2}V\dfrac{\partial W}{\partial Y}$
- $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{\partial X}{\partial x}\left(\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial X}\right) = \dfrac{\partial}{\partial X}\dfrac{1}{L}\left(\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial X}\right) = \dfrac{u_o}{L^3}\dfrac{\partial^2 W}{\partial X^2}$
- $\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial w}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{\partial Y}{\partial y}\left(\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial Y}\right) = \dfrac{\partial}{\partial Y}\dfrac{1}{L}\left(\dfrac{u_o}{L^2}\dfrac{\partial W}{\partial Y}\right) = \dfrac{u_o}{L^3}\dfrac{\partial^2 W}{\partial Y^2}$

Now, the dimensionless form of the continuity equation (4.1) can be obtained as follows,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$\frac{u_o}{L}\frac{\partial U}{\partial X}+\frac{u_o}{L}\frac{\partial V}{\partial Y}=0,$$

$$\frac{u_o}{L}\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=0,$$

As $\frac{u_o}{L}\neq 0$ but,

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0. \tag{4.6}$$

The dimensionless form of the u-momentum equation (4.2) can be obtained as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\lambda\left(u^2\frac{\partial^2 u}{\partial x^2}+v^2\frac{\partial^2 u}{\partial y^2}+2uv\frac{\partial^2 u}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}(\mu+k)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{k}{\rho}\frac{\partial w}{\partial y},$$

$$\frac{u_o^2}{L}\frac{\partial U}{\partial T}+\frac{u_o^2}{L}U\frac{\partial U}{\partial X}+\frac{u_o^2}{L}V\frac{\partial U}{\partial Y}+\lambda\left(\frac{u_o^3}{L^2}U^2\frac{\partial^2 U}{\partial X^2}+\frac{u_o^3}{L^2}V^2\frac{\partial^2 V}{\partial Y^2}+2\frac{u_o^3}{L^2}UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho}(\mu+k)\left(\frac{u_o}{L^2}\frac{\partial^2 U}{\partial X^2}+\frac{u_o}{L^2}\frac{\partial^2 U}{\partial Y^2}\right)+\frac{k}{\rho}\frac{u_o}{L^2}\frac{\partial W}{\partial Y}$$

$$\frac{u_o^2}{L}\left[\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+\lambda\frac{u_o}{L}\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)\right]$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}(1+\frac{k}{\mu})\frac{u_o}{L^2}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{k}{\rho}\frac{u_o}{L^2}\frac{\partial W}{\partial Y}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{L\mu}{\rho u_o^2}(1+K)\frac{u_o}{L^2}\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{Lku_o}{\rho u_o^2 L^2}\frac{\partial W}{\partial Y}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial}{\partial X}\frac{\partial X}{\partial x}(p)+\frac{\mu}{L\rho u_o}(1+K)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{\frac{k}{\mu}}{\frac{\rho u_o L}{\mu}}\frac{\partial W}{\partial Y}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-L\frac{\partial}{\partial X}\frac{1}{L}\left(\frac{p}{\rho u_o^2}\right)+\frac{1}{\frac{\rho L u_o}{\mu}}(1+K)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{K}{Re}\frac{\partial W}{\partial Y}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial X}+\frac{1}{Re}(1+K)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{K}{Re}\frac{\partial W}{\partial Y}. \tag{4.7}$$

Similarly, the dimensionless form of the v-momentum equation (4.3) can be obtained as follows:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\lambda\left(u^2\frac{\partial^2 v}{\partial x^2}+v^2\frac{\partial^2 v}{\partial y^2}+2uv\frac{\partial^2 v}{\partial x\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{\rho}(\mu+k)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)-\frac{k}{\rho}\frac{\partial w}{\partial x},$$

$$\frac{u_o^2}{L}\frac{\partial V}{\partial T}+\frac{u_o^2}{L}U\frac{\partial V}{\partial X}+\frac{u_o^2}{L}V\frac{\partial V}{\partial Y}+\lambda\left(\frac{u_o^3}{L^2}U^2\frac{\partial^2 V}{\partial X^2}+\frac{u_o^3}{L^2}V^2\frac{\partial^2 V}{\partial Y^2}+2\frac{u_o^3}{L^2}UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{\rho}(\mu+k)\left(\frac{u_o}{L^2}\frac{\partial^2 V}{\partial X^2}+\frac{u_o}{L^2}\frac{\partial^2 V}{\partial Y^2}\right)-\frac{k}{\rho}\frac{u_o}{L^2}\frac{\partial W}{\partial X}$$

$$\frac{u_o^2}{L}\left[\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+\lambda\frac{u_o}{L}\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)\right]$$
$$=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}(1+\frac{k}{\mu})\frac{u_o}{L^2}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{k}{\rho}\frac{u_o}{L^2}\frac{\partial W}{\partial X}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{L}{u_o^2}\frac{\mu}{\rho}(1+K)\frac{u_o}{L^2}\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{L}{u_o^2}\frac{k}{\rho}\frac{u_o}{L^2}\frac{\partial W}{\partial X}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{L}{u_o^2}\frac{1}{\rho}\frac{\partial}{\partial Y}\frac{\partial Y}{\partial y}(p)+\frac{\mu}{\rho L u_o}(1+K)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{k}{\rho L u_o}\frac{\partial W}{\partial X}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-L\frac{\partial}{\partial Y}\frac{1}{L}\left(\frac{p}{\rho u_o^2}\right)+\frac{1}{\frac{\rho L u_o}{\mu}}(1+K)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{\frac{k}{\mu}}{\frac{\rho L u_o}{\mu}}\frac{\partial W}{\partial X}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial Y}+\frac{1}{Re}(1+K)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{K}{Re}\frac{\partial W}{\partial X}. \tag{4.8}$$

Now, the dimensionless form of the micromagnetorotational model (4.4) can be obtained as follows:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}=\frac{\gamma^*}{\rho j}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\frac{2k}{\rho j}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}-2w\right),$$

$$\frac{u_o^2}{L^2}\frac{\partial W}{\partial T}+\frac{u_o^2}{L^2}U\frac{\partial W}{\partial X}+\frac{u_o^2}{L^2}V\frac{\partial W}{\partial Y}$$
$$=\frac{\gamma^*}{\rho j}\left(\frac{u_o}{L^3}\frac{\partial^2 W}{\partial X^2}+\frac{u_o}{L^3}\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2k}{\rho j}\left(\frac{u_o}{L}\frac{\partial V}{\partial X}-\frac{u_o}{L}\frac{\partial U}{\partial Y}-2\frac{u_oW}{L}\right)$$

$$\frac{u_o^2}{L^2}\left[\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}\right]$$
$$=\frac{\gamma^*}{\rho j}\left(\frac{u_o}{L^3}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)\right)+\frac{2k}{\rho j}\left(\frac{u_o}{L}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}\right)-2\frac{u_oW}{L}\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}$$
$$=\frac{L^2}{u_o^2}\frac{\gamma^*}{\rho j}\frac{u_o}{L^3}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2k}{\rho j}\left(\frac{L^2}{u_o^2}\frac{u_o}{L}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}\right)-2\frac{u_oW}{L}\frac{L^2}{u_o^2}\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{\gamma^*}{\rho j}\frac{1}{u_oL}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2k}{\rho j}\left(\frac{L}{u_o}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}\right)-2\frac{WL}{u_o}\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{\gamma^*}{\rho j}\frac{1}{u_oL}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2k}{\rho j}\frac{L}{u_o}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{\gamma^*}{\frac{\rho u_oL}{\mu}j\mu}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2kL.L}{\rho u_oj.L}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{\gamma^*}{Re\mu j}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2kL^2}{\frac{\rho u_oL}{\mu}j\mu}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{K_m}{Re}\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2k}{\frac{Re}{L^2}j\mu}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right)$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{1}{Re}\left(1+\frac{K}{2}\right)\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\left(\frac{2K}{E_rRe}\right)\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right). \tag{4.9}$$

4.3 Dimensionless Governing Equations

The set of transformed dimensionless equations can be written as

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0. \tag{4.10}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+R_f\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial X}+\frac{1}{Re}(1+K)\left(\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right)+\frac{K}{Re}\frac{\partial W}{\partial Y}. \tag{4.11}$$

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+R_f\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)$$
$$=-\frac{\partial P}{\partial Y}+\frac{1}{Re}(1+K)\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-\frac{K}{Re}\frac{\partial W}{\partial X}. \tag{4.12}$$

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y}=\frac{1}{Re}\left(\left(1+\frac{K}{2}\right)\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+\frac{2K}{E_r}\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right)\right). \tag{4.13}$$

The associated dimensionless boundary conditions are as follows:

$$U=1.0,\quad V=W=0 \quad \text{on} \quad \Omega_1$$
$$U=V=W=0 \quad \text{on} \quad \Omega_2$$

In (4.10) and (4.13), the nondimensional parameters Re represent Reynolds number, R_f is the fluid relaxation time, K is Micropolar flow, γ^* is the spin gradient viscosity and K_m is micro-inertial parameter respectively, defined by

$$Re=\frac{\rho L u_o}{\mu},\quad R_f=\frac{\lambda u_o}{L},\quad K=\frac{k}{\mu},\quad K_m=\frac{\gamma^*}{\mu j},\quad E_r=\frac{j}{L^2},\quad \gamma^*=\mu\left(1+\frac{K}{2}\right)j \tag{4.14}$$

Chapter 5

Numerical Procedure of Galerkin based Finite Element Formulation

The Galerkin based Finite Element Method is used to solve the Nonlinear Partial Differential Equations with associated boundary conditions. First, the weak formulation of the governing equations is derived and then the solution is approximated by using the Galerkin approximation method.

5.1 Variational Formulation

The idea of variational formulation is to transform the governing equations into integral equations. The equations are integrated over the whole domain and are transformed from the strong form to the weak form by multiplying PDEs by test functions of the same space. Ultimately, we obtained an approximated solution by using the set of approximated trial functions that are valid only over a portion of the domain.

Strong Form of Governing Equations:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0. \tag{5.1}$$

$$\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+A_1\left(U^2\frac{\partial^2 U}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 U}{\partial X\partial Y}\right)$$

$$= -\frac{\partial P}{\partial X} + A_2\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + A_3\frac{\partial W}{\partial Y}. \quad (5.2)$$

$$\frac{\partial V}{\partial T} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} + A_4\left(U^2\frac{\partial^2 V}{\partial X^2} + V^2\frac{\partial^2 V}{\partial Y^2} + 2UV\frac{\partial^2 V}{\partial X \partial Y}\right)$$
$$= -\frac{\partial P}{\partial Y} + A_5\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) - A_6\frac{\partial W}{\partial X}. \quad (5.3)$$

$$\frac{\partial W}{\partial T} + U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} = A_7\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right) + A_8\left(\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} - 2W\right). \quad (5.4)$$

In the above set of equations, the values of $A_1,\ A_2, ...A_8$ are expressed as:

$$A_1 = A_4 = R_f, \quad A_2 = A_5 = \frac{1}{Re}, \quad A_3 = A_6 = \frac{K}{Re}, \quad A_7 = \frac{1}{Re}\left(1 + \frac{K}{2}\right), \quad A_8 = \frac{2K}{E_r Re}$$

Weak formulation/Variational form:

Weak formulation is a variational method that multiplies the dependent variables by an appropriate test function and then integrates the result over the entire computational domain to convert differential equations into integral form. In this case, the solution spaces U, V, W, and P are defined on the continuously varying infinite dimensional space Ω; in actuality, it is impossible to achieve the solution in such a large space. Finding some appropriate spaces to obtain functions with finite parameters or properties is the main goal. In order to find an approximate solution using the weak formulation, we must first define a few unique functions that we will refer to as test functions for the residuals. Assume that $\bar{W}$ and Q represent the infinite-dimensional test space where $\bar{W}= [H_1(\Omega),\ H_1(\Omega),\ H_1(\Omega)]$ and $Q = L_2(\Omega)$. The corresponding test functions, $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ and q should be such that $\tilde{U}$, $\tilde{V}$, $\tilde{W}\ \varepsilon\ \bar{W}$ and $q\ \varepsilon\ Q$. The test functions $\tilde{U}$, $\tilde{V}$, $\tilde{W}\ \varepsilon\ \bar{W}$ multiplies the momentum and micromagnetorotational equation components in the variational formulation. The weak formulation of the strong form of governing PDEs from eqs. (5.1) to (5.4) is written below:

The weak form for u-component of momentum equation (5.2) as follows:

$$\frac{\partial U}{\partial T} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + A_1\left(U^2\frac{\partial^2 U}{\partial X^2} + V^2\frac{\partial^2 V}{\partial Y^2} + 2UV\frac{\partial^2 U}{\partial X \partial Y}\right)$$
$$= -\frac{\partial P}{\partial X} + A_2\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) + A_3\frac{\partial W}{\partial Y}$$

Multiplying by test function $\tilde{U}$ first and then integrate over computational domain,

$$\frac{\partial U}{\partial T}\tilde{U} + \left[\left(U\frac{\partial U}{\partial X}\right) + \left(V\frac{\partial U}{\partial Y}\right)\right]\tilde{U} + A_1\Bigg\{U^2$$

$$\frac{\partial^2 U}{\partial X^2} + V^2\frac{\partial^2 V}{\partial Y^2} + 2UV\frac{\partial^2 U}{\partial X \partial Y}\Bigg\}\tilde{U} - q\frac{\partial U}{\partial X}\tilde{U} = -\frac{\partial P}{\partial X}\tilde{U} + A_2\left\{\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right\}\tilde{U} + A_3\frac{\partial W}{\partial Y}\tilde{U} \tag{5.5}$$

$$As \qquad \frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right) = U^2\frac{\partial U}{\partial X}\frac{\partial}{\partial X}(\tilde{U}) + U^2\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}\right)\tilde{U} + \frac{\partial}{\partial X}(U^2)\frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right) = U^2\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} + U^2\frac{\partial^2 U}{\partial X^2}\tilde{U} + 2U\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right) - U^2\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U} = U^2\frac{\partial^2 U}{\partial X^2}\tilde{U}$$

$$\frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right) = V^2\frac{\partial V}{\partial Y}\frac{\partial}{\partial Y}(\tilde{U}) + V^2\frac{\partial}{\partial Y}\left(\frac{\partial V}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}(V^2)\frac{\partial V}{\partial Y}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right) = V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} + V^2\frac{\partial^2 V}{\partial Y^2}\tilde{U} + 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right) - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} = V^2\frac{\partial^2 V}{\partial Y^2}\tilde{U}$$

$$2UV\frac{\partial^2 U}{\partial X \partial Y}\tilde{U} = \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U}$$

$$\begin{aligned}\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U} = {} & UV\frac{\partial U}{\partial Y}\frac{\partial}{\partial X}\tilde{U} + UV\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial Y}\right)\tilde{U} \\ & + U\frac{\partial}{\partial X}(V)\frac{\partial U}{\partial Y}\tilde{U} + \frac{\partial}{\partial X}(U)V\frac{\partial U}{\partial Y}\tilde{U} \\ & + UV\frac{\partial U}{\partial X}\frac{\partial}{\partial Y}\tilde{U} + UV\frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X}\right)\tilde{U} \\ & + U\frac{\partial}{\partial Y}(V)\frac{\partial U}{\partial X}\tilde{U} + \frac{\partial}{\partial Y}(U)V\frac{\partial U}{\partial X}\tilde{U}\end{aligned}$$

$$\begin{aligned}\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U} = {} & UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} + UV\frac{\partial^2 U}{\partial X \partial Y}\tilde{U} \\ & + U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} + \frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} \\ & + UV\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} + UV\frac{\partial^2 U}{\partial Y \partial X}\tilde{U} \\ & + U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} + \frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U}\end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U} - UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U}$$
$$- \frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} - \frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U}$$
$$= UV\frac{\partial^2 U}{\partial X\partial Y}\tilde{U} + UV\frac{\partial^2 U}{\partial Y\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U} - UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} -$$
$$\frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} - \frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U} = 2UV\frac{\partial^2 U}{\partial X\partial Y}\tilde{U}$$

and $$-\frac{\partial}{\partial X}(P\tilde{U}) + P\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial P}{\partial X}\tilde{U}$$

As $$\frac{\partial}{\partial X}(U\tilde{U}) = U\frac{\partial \tilde{U}}{\partial X} + \frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow \frac{\partial}{\partial X}(U\tilde{U}) - U\frac{\partial \tilde{U}}{\partial X} = \frac{\partial U}{\partial X}\tilde{U}$$

$$\Rightarrow -\left[\frac{\partial}{\partial X}(U\tilde{U}) - U\frac{\partial \tilde{U}}{\partial X}\right] = -\frac{\partial U}{\partial X}\tilde{U},$$

$$\Rightarrow -\frac{\partial}{\partial X}(U\tilde{U}) + U\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial U}{\partial X}\tilde{U},$$

$$\frac{\partial}{\partial Y}(W\tilde{U}) = W\frac{\partial \tilde{U}}{\partial Y} + \frac{\partial W}{\partial Y}\tilde{U},$$

$$\Rightarrow \frac{\partial}{\partial Y}(W\tilde{U}) - W\frac{\partial \tilde{U}}{\partial Y} = \frac{\partial W}{\partial Y}\tilde{U},$$

So, Equation (5.5) becomes:

$$\frac{\partial U}{\partial T}\tilde{U} + \left[\left(U\frac{\partial U}{\partial X}\right) + \left(V\frac{\partial U}{\partial Y}\right)\right]\tilde{U} + A_1\left\{\frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right) - U^2\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U}\right.$$
$$+ \frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right) - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} + \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U} + \frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U}$$
$$\left. - UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} - \frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} - \frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U}\right\}$$
$$- q\frac{\partial}{\partial X}(U\tilde{U}) + qU\frac{\partial \tilde{U}}{\partial X} = -\frac{\partial}{\partial X}(P\tilde{U}) + P\frac{\partial \tilde{U}}{\partial X} + A_2\left\{\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right\}\tilde{U} + A_3\frac{\partial}{\partial Y}(W\tilde{U})$$
$$- A_3W\frac{\partial \tilde{U}}{\partial Y}$$

$$
\begin{aligned}
&\int_{\Omega^n} \frac{\partial U}{\partial T}\tilde{U}\,d\Omega + \int_{\Omega^n}\left[\left(U\frac{\partial U}{\partial X}\right)+\left(V\frac{\partial U}{\partial Y}\right)\right]\tilde{U}\,d\Omega + A_1\int_{\Omega^n}\left\{\frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right) - U^2\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X}\right.\\
&- 2U\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U} + \frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right) - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} + \frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U}+\\
&\frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U} - UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} - \frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U}-\\
&\left.\frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U}\right\}d\Omega - q\int_{\Omega^n}\frac{\partial}{\partial X}(U\tilde{U})d\Omega + q\int_{\Omega^n}U\frac{\partial \tilde{U}}{\partial X}d\Omega = -\int_{\Omega^n}\frac{\partial}{\partial X}(P\tilde{U})d\Omega + P\int_{\Omega^n}\frac{\partial \tilde{U}}{\partial X}d\Omega\\
&\qquad + A_2\int_{\Omega^n}\left\{\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right\}\tilde{U}\,d\Omega + A_3\int_{\Omega^n}\frac{\partial}{\partial Y}(W\tilde{U})d\Omega - A_3\int_{\Omega^n}W\frac{\partial \tilde{U}}{\partial Y}d\Omega \qquad (5.6)
\end{aligned}
$$

As $\displaystyle\int_{\Omega^n}\frac{\partial U}{\partial T}\tilde{U}\,d\Omega + \int_{\Omega^n}\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]\tilde{U}\,d\Omega$

$$
\begin{aligned}
&= \int_{\Omega^n}\frac{(U^{n+1}-U^n)}{\delta t}\tilde{U}\,d\Omega + \int_{\Omega^n}\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]\tilde{U}\,d\Omega\\
&= \frac{1}{\delta t}\int_{\Omega^n}U^{n+1}\tilde{U}\,d\Omega - \frac{1}{\delta t}\int_{\Omega^n}U^n\tilde{U}\,d\Omega + \int_{\Omega^n}\left[U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]\tilde{U}\,d\Omega\\
&= \frac{1}{\delta t}\int_{\Omega^n}U^{n+1}\tilde{U}\,d\Omega - \frac{1}{\delta t}\left(U^n\ o\ X^n\right)\tilde{U}\,d\Omega
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int_{\Omega^n}U^{n+1}\tilde{U}\,d\Omega - \frac{1}{\delta t}\left(U^n\ o\ X^n\right)\tilde{U}\,d\Omega + A_1\int_{\Omega^n}\frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right)d\Omega + A_1\int_{\Omega^n}\frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right)\\
&d\Omega + A_1\int_{\Omega^n}\frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U}\,d\Omega + A_1\int_{\Omega^n}\frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U}\,d\Omega + A_1\int_{\Omega^n}\left\{-U^2\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} - 2U\right.\\
&\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U} - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} - \frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial X}\\
&\frac{\partial \tilde{U}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} - \left.\frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U}\right\}d\Omega - q\int_{\Omega^n}\frac{\partial}{\partial X}(U\tilde{U})d\Omega + q\int_{\Omega^n}U\frac{\partial \tilde{U}}{\partial X}d\Omega = -\int_{\Omega^n}\frac{\partial}{\partial X}(P\tilde{U})\\
&d\Omega + P\int_{\Omega^n}\frac{\partial \tilde{U}}{\partial X}d\Omega + A_2\int_{\Omega^n}\left\{\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right\}\tilde{U}\,d\Omega + A_3\int_{\Omega^n}\frac{\partial}{\partial Y}(W\tilde{U})d\Omega - A_3\int_{\Omega^n}W\frac{\partial \tilde{U}}{\partial Y}d\Omega
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\delta t}\int_{\Omega^n}U^{n+1}\tilde{U}\,d\Omega - \frac{1}{\delta t}\left(U^n\ o\ X^n\right)\tilde{U}\,d\Omega + \cancelto{0}{A_1\int_{\Omega^n}\frac{\partial}{\partial X}\left(U^2\frac{\partial U}{\partial X}\tilde{U}\right)d\Omega} +\\
&\cancelto{0}{A_1\int_{\Omega^n}\frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{U}\right)d\Omega} + \cancelto{0}{A_1\int_{\Omega^n}\frac{\partial}{\partial X}\left(UV\frac{\partial U}{\partial Y}\right)\tilde{U}\,d\Omega} + \cancelto{0}{A_1\int_{\Omega^n}\frac{\partial}{\partial Y}\left(UV\frac{\partial U}{\partial X}\right)\tilde{U}\,d\Omega} +
\end{aligned}
$$

$$A_1 \int\limits_{\Omega^n} \Bigg\{ - U^2 \frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} - 2U \frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U} - V^2 \frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} - 2V \frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} - UV \frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U \frac{\partial V}{\partial X}$$

$$\frac{\partial U}{\partial Y}\tilde{U} - \frac{\partial U}{\partial X} V \frac{\partial U}{\partial Y}\tilde{U} - UV \frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U \frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} - \frac{\partial U}{\partial Y} V \frac{\partial U}{\partial X}\tilde{U} \Bigg\} d\Omega - q \cancelto{0}{\int\limits_{\Omega^n} \frac{\partial}{\partial X}(U\tilde{U}) d\Omega}$$

$$+ q \int\limits_{\Omega^n} U \frac{\partial \tilde{U}}{\partial X} d\Omega = - \cancelto{0}{\int\limits_{\Omega^n} \frac{\partial}{\partial X}(P\tilde{U}) d\Omega} + P \int\limits_{\Omega^n} \frac{\partial \tilde{U}}{\partial X} d\Omega + A_2 \int\limits_{\Omega^n} \Bigg\{ \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \Bigg\} \tilde{U} d\Omega +$$

$$A_3 \cancelto{0}{\int\limits_{\Omega^n} \frac{\partial}{\partial Y}(W\tilde{U}) d\Omega} - A_3 \int\limits_{\Omega^n} W \frac{\partial \tilde{U}}{\partial Y} d\Omega$$

$$\frac{1}{\delta t} \int\limits_{\Omega^n} U^{n+1} \tilde{U} d\Omega - \frac{1}{\delta t} \Big(U^n \ o \ X^n \Big) \tilde{U} d\Omega + A_1 \int\limits_{\Omega^n} \Bigg\{ - U^2 \frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} - 2U \frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U} - V^2 \frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y}$$

$$- 2V \frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} - UV \frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U \frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} - \frac{\partial U}{\partial X} V \frac{\partial U}{\partial Y}\tilde{U} - UV \frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U \frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} -$$

$$\frac{\partial U}{\partial Y} V \frac{\partial U}{\partial X}\tilde{U} \Bigg\} d\Omega + q \int\limits_{\Omega^n} U \frac{\partial \tilde{U}}{\partial X} d\Omega = P \int\limits_{\Omega^n} \frac{\partial \tilde{U}}{\partial X} d\Omega + A_2 \int\limits_{\Omega^n} \Bigg\{ \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \Bigg\} \tilde{U} d\Omega$$

$$- A_3 \int\limits_{\Omega^n} W \frac{\partial \tilde{U}}{\partial Y} d\Omega \tag{5.7}$$

Using Green's theorem for Laplacian term as

$$\int\limits_{\Omega} \psi \Delta\phi \ d\Omega = - \int\limits_{\Omega} \nabla\phi \nabla\psi \ d\Omega + \int\limits_{\Omega} \psi(\nabla\phi n) \ d\Gamma$$

Here, $\psi = \tilde{U}, \ \Delta\phi = \frac{\partial^2 U}{\partial X^2}, \ \Delta\phi = \frac{\partial^2 U}{\partial Y^2}, \ \nabla\phi = \frac{\partial U}{\partial X}, \ \nabla\phi = \frac{\partial U}{\partial Y}, \ \nabla\psi = \frac{\partial \tilde{U}}{\partial X}, \ \nabla\psi = \frac{\partial \tilde{U}}{\partial Y}$

As $\quad \nabla\phi n = \frac{\partial \phi}{\partial n} = n_x \frac{\partial \phi}{\partial X} + n_y \frac{\partial \phi}{\partial Y} \qquad (\phi = U)$

So,

$$A_2 \int\limits_{\Omega^n} \Bigg\{ \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \Bigg\} \tilde{U} d\Omega =$$

$$A_2 \Bigg\{ - \int\limits_{\Omega^n} \frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} d\Omega + \oint\limits_{\Gamma} \tilde{U} \Big(n_x \frac{\partial U}{\partial X} \Big) d\Gamma - \int\limits_{\Omega} \frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial Y} d\Omega + \oint\limits_{\Gamma} \tilde{U} \Big(n_y \frac{\partial U}{\partial Y} \Big) d\Gamma \Bigg\}$$

Now (5.7) becomes as

$$\frac{1}{\delta t}\int\limits_{\Omega^n} U^{n+1}\tilde{U}d\Omega - \frac{1}{\delta t}\Big(U^n \ o \ X^n\Big)\tilde{U}d\Omega + A_1\int\limits_{\Omega^n}\Big\{ - U^2\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial U}{\partial X}\tilde{U} - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{U}}{\partial Y}$$
$$- 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial U}{\partial Y}\tilde{U} - \frac{\partial U}{\partial X}V\frac{\partial U}{\partial Y}\tilde{U} - UV\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial U}{\partial X}\tilde{U} -$$
$$\frac{\partial U}{\partial Y}V\frac{\partial U}{\partial X}\tilde{U}\Big\}d\Omega + q\int\limits_{\Omega^n} U\frac{\partial \tilde{U}}{\partial X}d\Omega = P\int\limits_{\Omega^n}\frac{\partial \tilde{U}}{\partial X}d\Omega + A_2\Big\{ - \int\limits_{\Omega^n}\frac{\partial U}{\partial X}\frac{\partial \tilde{U}}{\partial X}d\Omega + \oint\limits_{\Gamma}\tilde{U}\Big(n_x\frac{\partial U}{\partial X}\Big)d\Gamma$$
$$- \int\limits_{\Omega}\frac{\partial U}{\partial Y}\frac{\partial \tilde{U}}{\partial Y}d\Omega + \oint\limits_{\Gamma}\tilde{U}\Big(n_y\frac{\partial U}{\partial Y}\Big)d\Gamma\Big\} - A_3\int\limits_{\Omega^n} W\frac{\partial \tilde{U}}{\partial Y}d\Omega$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U^{n+1}\tilde{U}d\Omega - \frac{1}{\delta t}\Big(U^n \ o \ X^n\Big)\tilde{U}d\Omega + A_1\int\limits_{\Omega^{n+1}}\Big\{ - U^nU^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}} - 2U^{n+1}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\tilde{U} - V^nV^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{U}}{\partial Y^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\tilde{U}$$
$$- U^{n+1}V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\tilde{U} - \frac{\partial U^{n+1}}{\partial X^{n+1}}V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\tilde{U} - U^{n+1}V^{n+1}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{U}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\tilde{U} - \frac{\partial U^{n+1}}{\partial Y^{n+1}}V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\tilde{U}\Big\}d\Omega + q\int\limits_{\Omega^{n+1}} U^{n+1}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega$$
$$= P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega + A_2\Big\{ - \int\limits_{\Omega^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{U}\Big(n_x\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$
$$- \int\limits_{\Omega^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{U}}{\partial Y^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{U}\Big(n_y\frac{\partial U^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} - A_3\int\limits_{\Omega^{n+1}} W^{n+1}\frac{\partial \tilde{U}}{\partial Y^{n+1}}d\Omega$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U^{n+1}\tilde{U}d\Omega - \frac{1}{\delta t}\Big(U^n \ o \ X^n\Big)\tilde{U}d\Omega + A_1\int\limits_{\Omega^{n+1}}\Big\{\Big(- U^nU^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}} - U^{n+1}V^{n+1}$$
$$\frac{\partial U^{n+1}}{\partial Y^{n+1}}\Big)\frac{\partial \tilde{U}}{\partial X^{n+1}} + \Big(- V^nV^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)\frac{\partial \tilde{U}}{\partial Y^{n+1}} + \Big(- 2U^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)\tilde{U}\Big\}d\Omega + q\int\limits_{\Omega^{n+1}} U^{n+1}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega = P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega + A_2$$
$$\Big\{ - \int\limits_{\Omega^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{U}\Big(n_x\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$
$$- \int\limits_{\Omega^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{U}}{\partial Y^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{U}\Big(n_y\frac{\partial U^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} - A_3\int\limits_{\Omega^{n+1}} W^{n+1}\frac{\partial \tilde{U}}{\partial Y^{n+1}}d\Omega,$$

which is weak form of u-component of momentum equation.

Similarly, we obtain the weak form for v-component of momentum equation (5.3) as follows:

$$\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+A_4\left(U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right)=-\frac{\partial P}{\partial Y}+A_5\left(\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right)-A_6\frac{\partial W}{\partial X}$$

Multiplying by test function $\tilde{V}$ first and then integrate over computational domain,

$$\frac{\partial V}{\partial T}\tilde{V}+\left[\left(U\frac{\partial V}{\partial X}\right)+\left(V\frac{\partial V}{\partial Y}\right)\right]\tilde{V}+A_4\left\{U^2\frac{\partial^2 V}{\partial X^2}+V^2\frac{\partial^2 V}{\partial Y^2}+2UV\frac{\partial^2 V}{\partial X\partial Y}\right\}\tilde{V}-q\frac{\partial V}{\partial Y}\tilde{V}=-\frac{\partial P}{\partial Y}\tilde{V}+A_5\left\{\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right\}\tilde{V}-A_6\frac{\partial W}{\partial X}\tilde{V} \tag{5.8}$$

$$As \qquad \frac{\partial}{\partial X}\left(U^2\frac{\partial V}{\partial X}\tilde{V}\right)=U^2\frac{\partial V}{\partial X}\frac{\partial}{\partial X}(\tilde{V})+U^2\frac{\partial}{\partial X}\left(\frac{\partial V}{\partial X}\right)\tilde{V}+\frac{\partial}{\partial X}(U^2)\frac{\partial V}{\partial X}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(U^2\frac{\partial V}{\partial X}\tilde{V}\right)=U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X}+U^2\frac{\partial^2 V}{\partial X^2}\tilde{V}+2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(U^2\frac{\partial V}{\partial X}\tilde{V}\right)-U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X}-2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V}=U^2\frac{\partial^2 V}{\partial X^2}\tilde{V}$$

$$\frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{V}\right)=V^2\frac{\partial V}{\partial Y}\frac{\partial}{\partial Y}(\tilde{V})+V^2\frac{\partial}{\partial Y}\left(\frac{\partial V}{\partial Y}\right)\tilde{V}+\frac{\partial}{\partial Y}(V^2)\frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{V}\right)=V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y}+V^2\frac{\partial^2 V}{\partial Y^2}\tilde{V}+2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{V}\right)-V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y}-2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V}=V^2\frac{\partial^2 V}{\partial Y^2}\tilde{V}$$

$$2UV\frac{\partial^2 V}{\partial X\partial Y}\tilde{V}=\frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V}+\frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V}$$

$$\begin{aligned}\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V}+\frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V}&=UV\frac{\partial V}{\partial Y}\frac{\partial}{\partial X}\tilde{V}+UV\frac{\partial}{\partial X}\left(\frac{\partial V}{\partial Y}\right)\tilde{V}\\&+U\frac{\partial}{\partial X}(V)\frac{\partial V}{\partial Y}\tilde{V}+\frac{\partial}{\partial X}(U)V\frac{\partial V}{\partial Y}\tilde{V}\\&+UV\frac{\partial V}{\partial X}\frac{\partial}{\partial Y}\tilde{V}+UV\frac{\partial}{\partial Y}\left(\frac{\partial V}{\partial X}\right)\tilde{V}\\&+U\frac{\partial}{\partial Y}(V)\frac{\partial V}{\partial X}\tilde{V}+\frac{\partial}{\partial Y}(U)V\frac{\partial V}{\partial X}\tilde{V}\end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V} + \frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V} = UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} + UV\frac{\partial^2 V}{\partial X \partial Y}\tilde{V}$$
$$+ U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} + \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V}$$
$$+ UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} + UV\frac{\partial^2 V}{\partial Y \partial X}\tilde{V}$$
$$+ U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} + \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V} + \frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V}$$
$$- \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} - \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}$$
$$= UV\frac{\partial^2 V}{\partial X \partial Y}\tilde{V} + UV\frac{\partial^2 V}{\partial Y \partial X}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V} + \frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} -$$
$$\frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} - \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V} = 2UV\frac{\partial^2 V}{\partial X \partial Y}\tilde{V}$$

and $$-\frac{\partial}{\partial Y}(P\tilde{V}) + P\frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial P}{\partial Y}\tilde{V}$$

As $$\frac{\partial}{\partial Y}(V\tilde{V}) = V\frac{\partial \tilde{V}}{\partial Y} + \frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial Y}(V\tilde{V}) - V\frac{\partial \tilde{V}}{\partial Y} = \frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow -\left[\frac{\partial}{\partial Y}(V\tilde{V}) - V\frac{\partial \tilde{V}}{\partial Y}\right] = -\frac{\partial V}{\partial Y}\tilde{V}$$

$$\Rightarrow -\frac{\partial}{\partial Y}(V\tilde{V}) + V\frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial V}{\partial Y}\tilde{V}$$

$$\frac{\partial}{\partial X}(W\tilde{V}) = W\frac{\partial \tilde{V}}{\partial X} + \frac{\partial W}{\partial X}\tilde{V}$$

$$\Rightarrow \frac{\partial}{\partial X}(W\tilde{V}) - W\frac{\partial \tilde{V}}{\partial X} = \frac{\partial W}{\partial X}\tilde{V}$$

So, equation (5.8) becomes:

$$\frac{\partial V}{\partial T}\tilde{V} + \left[\left(U\frac{\partial V}{\partial X}\right) + \left(V\frac{\partial V}{\partial Y}\right)\right]\tilde{V} + A_4\left\{\frac{\partial}{\partial X}\left(U^2\frac{\partial V}{\partial X}\tilde{V}\right) - U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V} +\right.$$
$$\frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{V}\right) - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V} + \frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V} + \frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V} - UV$$

$$\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} - \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} - \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}\Big\} - q\frac{\partial}{\partial Y}($$

$$V\tilde{V}) + qV\frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial}{\partial Y}(P\tilde{V}) + P\frac{\partial \tilde{V}}{\partial Y} + A_5\Big\{\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\Big\}\tilde{V} - A_6\frac{\partial}{\partial X}(W\tilde{V}) + A_6 W\frac{\partial \tilde{V}}{\partial X}$$

$$\int_{\Omega^n}\frac{\partial V}{\partial T}\tilde{V}d\Omega + \int_{\Omega^n}\Big[\Big(U\frac{\partial V}{\partial X}\Big) + \Big(V\frac{\partial V}{\partial Y}\Big)\Big]\tilde{V}d\Omega + A_4\int_{\Omega^n}\Big\{\frac{\partial}{\partial X}\Big(U^2\frac{\partial V}{\partial X}\tilde{V}\Big) - U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X}$$

$$-2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V} + \frac{\partial}{\partial Y}\Big(V^2\frac{\partial V}{\partial Y}\tilde{V}\Big) - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V} + \frac{\partial}{\partial X}\Big(UV\frac{\partial V}{\partial Y}\Big)\tilde{V} +$$

$$\frac{\partial}{\partial Y}\Big(UV\frac{\partial V}{\partial X}\Big)\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} - \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V}$$

$$-\frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}\Big\}d\Omega - q\int_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega + q\int_{\Omega^n}V\frac{\partial \tilde{V}}{\partial Y}d\Omega = -\int_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})d\Omega$$

$$+P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega + A_5\int_{\Omega^n}\Big\{\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\Big\}\tilde{V}d\Omega - A_6\int_{\Omega^n}\frac{\partial}{\partial X}(W\tilde{V})d\Omega + A_6\int_{\Omega^n}W\frac{\partial \tilde{V}}{\partial X}d\Omega \quad (5.9)$$

As $\int_{\Omega^n}\frac{\partial V}{\partial T}\tilde{V}d\Omega + \int_{\Omega^n}\Big[U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\Big]\tilde{V}d\Omega$

$$= \int_{\Omega^n}\frac{(V^{n+1} - V^n)}{\delta t}\tilde{V}d\Omega + \int_{\Omega^n}\Big[U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\Big]\tilde{V}d\Omega$$

$$= \frac{1}{\delta t}\int_{\Omega^n}V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\int_{\Omega^n}V^n\tilde{V}d\Omega + \int_{\Omega^n}\Big[U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\Big]\tilde{V}d\Omega$$

$$= \frac{1}{\delta t}\int_{\Omega^n}V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\Big(V^n\ o\ Y^n\Big)\tilde{V}d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n}V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\Big(V^n\ o\ Y^n\Big)\tilde{V}d\Omega + A_4\int_{\Omega^n}\frac{\partial}{\partial X}\Big(U^2\frac{\partial V}{\partial X}\tilde{V}\Big)d\Omega + A_4\int_{\Omega^n}\frac{\partial}{\partial Y}\Big(V^2\frac{\partial V}{\partial Y}\tilde{V}\Big)$$

$$d\Omega + A_4\int_{\Omega^n}\frac{\partial}{\partial X}\Big(UV\frac{\partial V}{\partial Y}\Big)\tilde{V}d\Omega + A_4\int_{\Omega^n}\frac{\partial}{\partial Y}\Big(UV\frac{\partial V}{\partial X}\Big)\tilde{V}d\Omega + A_4\int_{\Omega^n}\Big\{-U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X} - 2U$$

$$\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V} - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} - \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}$$

$$\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} - \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}\Big\}d\Omega - q\int_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega + q\int_{\Omega^n}V\frac{\partial \tilde{V}}{\partial Y}d\Omega = -\int_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})$$

$$d\Omega + P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega + A_5\int_{\Omega^n}\Big\{\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\Big\}\tilde{V}d\Omega - A_6\int_{\Omega^n}\frac{\partial}{\partial X}(W\tilde{V})d\Omega + A_6\int_{\Omega^n}W\frac{\partial \tilde{V}}{\partial X}d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\left(V^n \ o \ Y^n\right)\tilde{V}d\Omega + A_4\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial X}\left(U^2\frac{\partial V}{\partial X}\tilde{V}\right)d\Omega} +$$

$$A_4\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial Y}\left(V^2\frac{\partial V}{\partial Y}\tilde{V}\right)d\Omega} + A_4\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial X}\left(UV\frac{\partial V}{\partial Y}\right)\tilde{V}d\Omega} + A_4\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial Y}\left(UV\frac{\partial V}{\partial X}\right)\tilde{V}d\Omega}$$

$$+ A_4\int_{\Omega^n}\Bigg\{ - U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V} - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y} - 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}$$

$$\frac{\partial V}{\partial Y}\tilde{V} - \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} - \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}\Bigg\}d\Omega - q\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial Y}(V\tilde{V})d\Omega}$$

$$+ q\int_{\Omega^n}V\frac{\partial \tilde{V}}{\partial Y}d\Omega = -\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial Y}(P\tilde{V})d\Omega} + P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega + A_5\int_{\Omega^n}\left\{\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right\}\tilde{V}d\Omega$$

$$- A_6\cancelto{0}{\int_{\Omega^n}\frac{\partial}{\partial X}(W\tilde{V})d\Omega} + A_6\int_{\Omega^n}W\frac{\partial \tilde{V}}{\partial X}d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\left(V^n \ o \ Y^n\right)\tilde{V}d\Omega + A_4\int_{\Omega^n}\Bigg\{ - U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V} - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y}$$
$$- 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} - \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V}$$
$$- \frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}\Bigg\}d\Omega + q\int_{\Omega^n}V\frac{\partial \tilde{V}}{\partial Y}d\Omega = P\int_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega + A_5\int_{\Omega^n}\left\{\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right\}\tilde{V}d\Omega$$
$$+ A_6\int_{\Omega^n}W\frac{\partial \tilde{V}}{\partial X}d\Omega \tag{5.10}$$

Using Green's theorem for Laplacian term as

$$\int_{\Omega}\psi\Delta\phi \ d\Omega = -\int_{\Omega}\nabla\phi\nabla\psi \ d\Omega + \int_{\Omega}\psi(\nabla\phi n) \ d\Gamma$$

Here, $\psi = \tilde{V}, \ \Delta\phi = \dfrac{\partial^2 V}{\partial X^2}, \ \Delta\phi = \dfrac{\partial^2 V}{\partial Y^2}, \ \nabla\phi = \dfrac{\partial V}{\partial X}, \ \nabla\phi = \dfrac{\partial V}{\partial Y}, \ \nabla\psi = \dfrac{\partial \tilde{V}}{\partial X}, \ \nabla\psi = \dfrac{\partial \tilde{V}}{\partial Y}$

As $\quad \nabla\phi n = \dfrac{\partial\phi}{\partial n} = n_x\dfrac{\partial\phi}{\partial X} + n_y\dfrac{\partial\phi}{\partial Y} \qquad (\phi = V)$

$$A_5\int_{\Omega^n}\left\{\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right\}\tilde{V}d\Omega =$$

$$A_5\Bigg\{ -\int_{\Omega^n}\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X}d\Omega + \oint_{\Gamma}\tilde{V}\left(n_x\frac{\partial V}{\partial X}\right)d\Gamma - \int_{\Omega}\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y}d\Omega + \oint_{\Gamma}\tilde{V}\left(n_y\frac{\partial V}{\partial Y}\right)d\Gamma\Bigg\}$$

Now (5.10) becomes as

$$\frac{1}{\delta t}\int\limits_{\Omega^n} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\Big(V^n\ o\ Y^n\Big)\tilde{V}d\Omega + A_4\int\limits_{\Omega^n}\Big\{ - U^2\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X} - 2U\frac{\partial U}{\partial X}\frac{\partial V}{\partial X}\tilde{V} - V^2\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y}$$
$$- 2V\frac{\partial V}{\partial Y}\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial X} - U\frac{\partial V}{\partial X}\frac{\partial V}{\partial Y}\tilde{V} - \frac{\partial U}{\partial X}V\frac{\partial V}{\partial Y}\tilde{V} - UV\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial Y} - U\frac{\partial V}{\partial Y}\frac{\partial V}{\partial X}\tilde{V} -$$
$$\frac{\partial U}{\partial Y}V\frac{\partial V}{\partial X}\tilde{V}\Big\}d\Omega + q\int\limits_{\Omega^n} V\frac{\partial \tilde{V}}{\partial Y}d\Omega = P\int\limits_{\Omega^n}\frac{\partial \tilde{V}}{\partial Y}d\Omega + A_5\Big\{ - \int\limits_{\Omega^n}\frac{\partial V}{\partial X}\frac{\partial \tilde{V}}{\partial X}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Big(n_x\frac{\partial V}{\partial X}\Big)d\Gamma$$
$$- \int\limits_{\Omega}\frac{\partial V}{\partial Y}\frac{\partial \tilde{V}}{\partial Y}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Big(n_y\frac{\partial V}{\partial Y}\Big)d\Gamma\Big\} + A_6\int\limits_{\Omega^n} W\frac{\partial \tilde{V}}{\partial X}d\Omega$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\Big(V^n\ o\ Y^n\Big)\tilde{V}d\Omega + A_4\int\limits_{\Omega^{n+1}}\Big\{ - U^nU^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{V}}{\partial X^{n+1}} - 2U^{n+1}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\tilde{V} - V^nV^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\tilde{V}$$
$$- U^{n+1}V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{V}}{\partial X^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\tilde{V} - \frac{\partial U^{n+1}}{\partial X^{n+1}}V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\tilde{V} - U^{n+1}V^{n+1}$$
$$\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\tilde{V} - \frac{\partial U^{n+1}}{\partial Y^{n+1}}V^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\tilde{V}\Big\}d\Omega + q\int\limits_{\Omega^{n+1}} V^{n+1}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega$$
$$= P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega + A_5\Big\{ - \int\limits_{\Omega^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{V}}{\partial X^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Big(n_x\frac{\partial V^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$
$$- \int\limits_{\Omega^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Big(n_y\frac{\partial V^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} + A_6\int\limits_{\Omega^{n+1}} W^{n+1}\frac{\partial \tilde{V}}{\partial X^{n+1}}d\Omega$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\Big(V^n\ o\ Y^n\Big)\tilde{V}d\Omega + A_4\int\limits_{\Omega^{n+1}}\Big\{\Big(- U^nU^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}} - U^{n+1}V^{n+1}$$
$$\frac{\partial V^{n+1}}{\partial Y^{n+1}}\Big)\frac{\partial \tilde{V}}{\partial X^{n+1}} + \Big(- V^nV^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}V^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\Big)\frac{\partial \tilde{V}}{\partial Y^{n+1}} + \Big(- 2U^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}$$
$$\frac{\partial V^{n+1}}{\partial X^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}$$
$$\frac{\partial V^{n+1}}{\partial X^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\Big)\tilde{V}\Big\}d\Omega + q\int\limits_{\Omega^{n+1}} V^{n+1}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega = P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega + A_5$$
$$\Big\{ - \int\limits_{\Omega^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{V}}{\partial X^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Big(n_x\frac{\partial V^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$

$$-\int_{\Omega^{n+1}} \frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}} d\Omega + \oint_{\Gamma} \tilde{V}\left(n_y \frac{\partial V^{n+1}}{\partial Y^{n+1}}\right) d\Gamma \Bigg\} + A_6 \int_{\Omega^{n+1}} W^{n+1} \frac{\partial \tilde{V}}{\partial X^{n+1}} d\Omega,$$

which is weak form of v-component of momentum equation.

The micromagnetorotational equation in the same way becomes:

$$\frac{\partial W}{\partial T}+U\frac{\partial W}{\partial X}+V\frac{\partial W}{\partial Y} = A_7\left(\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial Y^2}\right)+A_8\left(\frac{\partial V}{\partial X}-\frac{\partial U}{\partial Y}-2W\right)$$

Multiplying by test function $\tilde{W}$ first and then integrate over computational domain,

$$\frac{\partial W}{\partial T}\tilde{W} + \left[\left(U\frac{\partial W}{\partial X}\right) + \left(V\frac{\partial W}{\partial Y}\right)\right]\tilde{W}$$
$$= A_7\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right)\tilde{W} + A_8\left(\frac{\partial V}{\partial X}\tilde{W} - \frac{\partial U}{\partial Y}\tilde{W} - 2W\tilde{W}\right) \tag{5.11}$$

$$\text{As } \frac{\partial}{\partial X}(V\tilde{W}) = V\frac{\partial \tilde{W}}{\partial X} + \frac{\partial V}{\partial X}\tilde{W}$$
$$\Rightarrow \frac{\partial}{\partial X}(V\tilde{W}) - V\frac{\partial \tilde{W}}{\partial X} = \frac{\partial V}{\partial X}\tilde{W}$$
$$\frac{\partial}{\partial Y}(U\tilde{W}) = U\frac{\partial \tilde{W}}{\partial Y} + \frac{\partial U}{\partial Y}\tilde{W}$$
$$\Rightarrow \frac{\partial}{\partial Y}(U\tilde{W}) - U\frac{\partial \tilde{W}}{\partial Y} = \frac{\partial U}{\partial Y}\tilde{W}$$
$$\Rightarrow -\left[\frac{\partial}{\partial Y}(U\tilde{W}) - U\frac{\partial \tilde{W}}{\partial Y}\right] = -\frac{\partial U}{\partial Y}\tilde{W}$$
$$\Rightarrow -\frac{\partial}{\partial Y}(U\tilde{W}) + U\frac{\partial \tilde{W}}{\partial Y} = -\frac{\partial U}{\partial Y}\tilde{W}$$

$$\int_{\Omega^n} \frac{\partial W}{\partial T}\tilde{W} d\Omega + \int_{\Omega^n} \left[\left(U\frac{\partial W}{\partial X}\right) + \left(V\frac{\partial W}{\partial Y}\right)\right]\tilde{W} d\Omega = A_7 \int_{\Omega^n}$$
$$\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right)\tilde{W} d\Omega + A_8 \int_{\Omega^n} \left(\frac{\partial}{\partial X}(V\tilde{W}) - V\frac{\partial \tilde{W}}{\partial X} - \frac{\partial}{\partial Y}(U\tilde{W}) + U\frac{\partial \tilde{W}}{\partial Y} - 2W\tilde{W}\right) d\Omega$$

$$\text{As } \int_{\Omega^n} \frac{\partial W}{\partial T}\tilde{W} d\Omega + \int_{\Omega^n} \left[U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y}\right]\tilde{W} d\Omega$$
$$= \int_{\Omega^n} \frac{(W^{n+1} - W^n)}{\delta t}\tilde{W} d\Omega + \int_{\Omega^n} \left[U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y}\right]\tilde{W} d\Omega$$

$$= \frac{1}{\delta t}\int_{\Omega^n} W^{n+1}\tilde{W} d\Omega - \frac{1}{\delta t}\int_{\Omega^n} W^{n}\tilde{W} d\Omega + \int_{\Omega^n}\left[U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y}\right]\tilde{W} d\Omega$$

$$= \frac{1}{\delta t}\int_{\Omega^n} W^{n+1}\tilde{W} d\Omega - \frac{1}{\delta t}\Big(W^n \ o \ X^n\Big)\tilde{W} d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n} W^{n+1}\tilde{W} d\Omega - \frac{1}{\delta t}\Big(W^n \ o \ X^n\Big)\tilde{W} d\Omega = A_7\int_{\Omega^n}$$
$$\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right)\tilde{W} d\Omega + A_8\int_{\Omega^n}\left(\frac{\partial}{\partial X}(V\tilde{W}) - V\frac{\partial \tilde{W}}{\partial X} - \frac{\partial}{\partial Y}(U\tilde{W}) + U\frac{\partial \tilde{W}}{\partial Y} - 2W\tilde{W}\right)d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n} W^{n+1}\tilde{W} d\Omega - \frac{1}{\delta t}\Big(W^n \ o \ X^n\Big)\tilde{W} d\Omega = A_7\int_{\Omega^n}\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right)\tilde{W} d\Omega$$
$$+ A_8\int_{\Omega^n}\left(\cancelto{0}{\frac{\partial}{\partial X}(V\tilde{W})} - V\frac{\partial \tilde{W}}{\partial X} - \cancelto{0}{\frac{\partial}{\partial Y}(U\tilde{W})} + U\frac{\partial \tilde{W}}{\partial Y} - 2W\tilde{W}\right)d\Omega$$

$$\frac{1}{\delta t}\int_{\Omega^n} W^{n+1}\tilde{W} d\Omega - \frac{1}{\delta t}\Big(W^n \ o \ X^n\Big)\tilde{W} d\Omega$$
$$= A_7\int_{\Omega^n}\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right)\tilde{W} d\Omega + A_8\int_{\Omega^n}\left(-V\frac{\partial \tilde{W}}{\partial X} + U\frac{\partial \tilde{W}}{\partial Y} - 2W\tilde{W}\right)d\Omega \tag{5.12}$$

Using Green's theorem for Laplacian term as

$$\int_{\Omega}\psi\Delta\phi \ d\Omega = -\int_{\Omega}\nabla\phi\nabla\psi \ d\Omega + \int_{\Omega}\psi(\nabla\phi n) \ d\Gamma$$

Here,

$\Psi = \tilde{W}, \ \Delta\phi = \frac{\partial^2 W}{\partial X^2}, \ \Delta\phi = \frac{\partial^2 W}{\partial Y^2}, \ \nabla\phi = \frac{\partial W}{\partial X}, \ \nabla\phi = \frac{\partial W}{\partial Y}, \ \nabla\psi = \frac{\partial \tilde{W}}{\partial X}, \ \nabla\psi = \frac{\partial \tilde{W}}{\partial Y}$

As $\quad \nabla\phi n = \frac{\partial\phi}{\partial n} = n_x\frac{\partial\phi}{\partial X} + n_y\frac{\partial\phi}{\partial Y} \qquad (\phi = W)$

So,

$$\int_{\Omega^n}\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right)\tilde{W} d\Omega = \Bigg(-\int_{\Omega^n}\frac{\partial W}{\partial X}\frac{\partial \tilde{W}}{\partial X} d\Omega + \oint_{\Gamma}\tilde{W}\left(n_x\frac{\partial W}{\partial X}\right)d\Gamma$$
$$-\int_{\Omega}\frac{\partial W}{\partial Y}\frac{\partial \tilde{W}}{\partial Y} d\Omega + \oint_{\Gamma}\tilde{W}\left(n_y\frac{\partial W}{\partial Y}\right)d\Gamma\Bigg)$$

Now (5.12) becomes as

$$\frac{1}{\delta t}\int_{\Omega^n} W^{n+1}\tilde{W}d\Omega - \frac{1}{\delta t}\Big(W^n \ o \ X^n\Big).\tilde{W}d\Omega = A_7\Big\{ -\int_{\Omega^n}\frac{\partial W}{\partial X}\frac{\partial \tilde{W}}{\partial X}d\Omega + \oint_{\Gamma}\tilde{W}\Big(n_x\frac{\partial W}{\partial X}\Big)d\Gamma$$
$$-\int_{\Omega^n}\frac{\partial W}{\partial Y}\frac{\partial \tilde{W}}{\partial Y}d\Omega + \oint_{\Gamma}\tilde{W}\Big(n_y\frac{\partial W}{\partial Y}\Big)d\Gamma\Big\} + A_8\int_{\Omega^n}\Big(-V\frac{\partial \tilde{W}}{\partial X} + U\frac{\partial \tilde{W}}{\partial Y} - 2W\tilde{W}\Big)d\Omega$$

Taking Domain as current time step value, we have

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} W^{n+1}\tilde{W}d\Omega - \frac{1}{\delta t}\Big(W^n \ o \ X^n\Big)\tilde{W}d\Omega = A_7\Big\{ -\int_{\Omega^{n+1}}\frac{\partial W^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{W}}{\partial X^{n+1}}d\Omega$$
$$+\oint_{\Gamma}\tilde{W}\Big(n_x\frac{\partial W^{n+1}}{\partial X^{n+1}}\Big)d\Gamma - \int_{\Omega^{n+1}}\frac{\partial W^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{W}}{\partial Y^{n+1}}d\Omega + \oint_{\Gamma}\tilde{W}\Big(n_y\frac{\partial W^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\}$$
$$+ A_8\int_{\Omega^{n+1}}\Big(-V^{n+1}\frac{\partial \tilde{W}}{\partial X^{n+1}} + U^{n+1}\frac{\partial \tilde{W}}{\partial Y^{n+1}} - 2W^{n+1}\tilde{W}\Big)d\Omega$$

Find $(U, \ V, \ W) \ \varepsilon \ \bar{W}$ and $P \ \varepsilon \ Q$ such that

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} U^{n+1}\tilde{U}d\Omega - \frac{1}{\delta t}\Big(U^n \ o \ X^n\Big)\tilde{U}d\Omega + A_1\int_{\Omega^{n+1}}\Big\{\Big(-U^nU^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}} - U^{n+1}V^{n+1}$$
$$\frac{\partial U^{n+1}}{\partial Y^{n+1}}\Big)\frac{\partial \tilde{U}}{\partial X^{n+1}} + \Big(-V^nV^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)\frac{\partial \tilde{U}}{\partial Y^{n+1}} + \Big(-2U^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}$$
$$\frac{\partial U^{n+1}}{\partial X^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)\tilde{U}\Big\}d\Omega + q\int_{\Omega^{n+1}}U^{n+1}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega - P^{n+1}\int_{\Omega^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega - A_2$$
$$\Big\{-\int_{\Omega^{n+1}}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{U}}{\partial X^{n+1}}d\Omega + \oint_{\Gamma}\tilde{U}\Big(n_x\frac{\partial U^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$
$$-\int_{\Omega^{n+1}}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{U}}{\partial Y^{n+1}}d\Omega + \oint_{\Gamma}\tilde{U}\Big(n_y\frac{\partial U^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} + A_3\int_{\Omega^{n+1}}W^{n+1}\frac{\partial \tilde{U}}{\partial Y^{n+1}}d\Omega = 0$$

(5.13)

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} V^{n+1}\tilde{V}d\Omega - \frac{1}{\delta t}\Big(V^n \ o \ Y^n\Big)\tilde{V}d\Omega + A_4\int_{\Omega^{n+1}}\Big\{\Big(-U^nU^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}} - U^{n+1}V^{n+1}$$

$$\frac{\partial V^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial \tilde{V}}{\partial X^{n+1}} + \Bigg(- V^{n}V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}V^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial \tilde{V}}{\partial Y^{n+1}} + \Bigg(- 2U^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}$$
$$\frac{\partial V^{n+1}}{\partial X^{n+1}} - 2V^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial X^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}} - U^{n+1}\frac{\partial V^{n+1}}{\partial Y^{n+1}}$$
$$\frac{\partial V^{n+1}}{\partial X^{n+1}} - V^{n+1}\frac{\partial U^{n+1}}{\partial Y^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\Bigg)\tilde{V}\Bigg\}d\Omega + q\int\limits_{\Omega^{n+1}} V^{n+1}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega - P^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega - A_5$$
$$\Bigg\{ - \int\limits_{\Omega^{n+1}}\frac{\partial V^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{V}}{\partial X^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Bigg(n_x\frac{\partial V^{n+1}}{\partial X^{n+1}}\Bigg)d\Gamma$$
$$- \int\limits_{\Omega^{n+1}}\frac{\partial V^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{V}}{\partial Y^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{V}\Bigg(n_y\frac{\partial V^{n+1}}{\partial Y^{n+1}}\Bigg)d\Gamma\Bigg\} - A_6\int\limits_{\Omega^{n+1}} W^{n+1}\frac{\partial \tilde{V}}{\partial X^{n+1}}d\Omega = 0 \tag{5.14}$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} W^{n+1}\tilde{W}d\Omega - \frac{1}{\delta t}\Bigg(W^{n}\ o\ X^{n}\Bigg)\tilde{W}d\Omega - A_7\Bigg\{ - \int\limits_{\Omega^{n+1}}\frac{\partial W^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{W}}{\partial X^{n+1}}d\Omega$$
$$+ \oint\limits_{\Gamma}\tilde{W}\Bigg(n_x\frac{\partial W^{n+1}}{\partial X^{n+1}}\Bigg)d\Gamma - \int\limits_{\Omega^{n+1}}\frac{\partial W^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{W}}{\partial Y^{n+1}}d\Omega + \oint\limits_{\Gamma}\tilde{W}\Bigg(n_y\frac{\partial W^{n+1}}{\partial Y^{n+1}}\Bigg)d\Gamma\Bigg\}$$
$$- A_8\int\limits_{\Omega^{n+1}}\Bigg(- V^{n+1}\frac{\partial \tilde{W}}{\partial X^{n+1}} + U^{n+1}\frac{\partial \tilde{W}}{\partial Y^{n+1}} - 2W^{n+1}\tilde{W}\Bigg)d\Omega = 0 \tag{5.15}$$

for all $\tilde{U}$, $\tilde{V}$, $\tilde{W}\ \varepsilon\ \bar{W}$ and $P\ \varepsilon\ Q$.

For the Galerkin discretization, the infinite dimensional test and trial spaces are approximated by finite dimensional spaces. In particular, following are the trial and test spaces

Trial Spaces:

$$U \approx U_h, \quad V \approx V_h, \quad W \approx W_h \quad \text{and} \quad P \approx P_h.$$

Test Spaces:

$$\bar{W} \approx \bar{W}_h \quad \text{and} \quad \mathrm{Q} \approx \mathrm{Q}_h.$$

Find $(U_h,\ V_h,\ W_h)\ \varepsilon\ \bar{W}_h$ and $\mathrm{P}_h\ \varepsilon\ \mathrm{Q}_h$ such that

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} U_h^{n+1}\tilde{U}_h d\Omega - \frac{1}{\delta t}\Bigg(U_h^{n}\ o\ X^{n}\Bigg)\tilde{U}_h d\Omega + A_1\int\limits_{\Omega^{n+1}}\Bigg\{\Bigg(- U_h^{n}U_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}} - U_h^{n+1}V_h^{n+1}$$
$$\frac{\partial U_h^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial \tilde{U}_h}{\partial X^{n+1}} + \Bigg(- V_h^{n}V_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} - U_h^{n+1}V_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial \tilde{U}_h}{\partial Y^{n+1}} + \Bigg(- 2U_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}$$

$$\frac{\partial U_h^{n+1}}{\partial X^{n+1}} - 2V_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} - U_h^{n+1}\frac{\partial V_h^{n+1}}{\partial X^{n+1}}\frac{\partial U_h^{n+1}}{\partial Y^{n+1}} - V_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\frac{\partial U_h^{n+1}}{\partial Y^{n+1}} - U_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}$$
$$\frac{\partial U_h^{n+1}}{\partial X^{n+1}} - V_h^{n+1}\frac{\partial U_h^{n+1}}{\partial Y^{n+1}}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\Big)\tilde{U}_h\Big\}d\Omega + q_h\int_{\Omega^{n+1}} U_h^{n+1}\frac{\partial \tilde{U}_h}{\partial X^{n+1}}d\Omega - P_h^{n+1}\int_{\Omega^{n+1}}\frac{\partial \tilde{U}_h}{\partial X^{n+1}}d\Omega$$
$$- A_2\Big\{ - \int_{\Omega^{n+1}}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{U}_h}{\partial X^{n+1}}d\Omega + \oint_{\Gamma}\tilde{U}_h\Big(n_x\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$
$$- \int_{\Omega^{n+1}}\frac{\partial U_h^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{U}_h}{\partial Y^{n+1}}d\Omega + \oint_{\Gamma}\tilde{U}_h\Big(n_y\frac{\partial U_h^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} + A_3\int_{\Omega^{n+1}} W_h^{n+1}\frac{\partial \tilde{U}_h}{\partial Y^{n+1}}d\Omega = 0 \tag{5.16}$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} V_h^{n+1}\tilde{V}_h d\Omega - \frac{1}{\delta t}\Big(V_h^n\ o\ Y^n\Big)\tilde{V}_h d\Omega + A_4\int_{\Omega^{n+1}}\Big\{\Big(- U_h^n U_h^{n+1}\frac{\partial V_h^{n+1}}{\partial X^{n+1}} - U_h^{n+1}V_h^{n+1}$$
$$\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}\Big)\frac{\partial \tilde{V}_h}{\partial X^{n+1}} + \Big(- V_h^n V_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} - U_h^{n+1}V_h^{n+1}\frac{\partial V_h^{n+1}}{\partial X^{n+1}}\Big)\frac{\partial \tilde{V}_h}{\partial Y^{n+1}} + \Big(- 2U_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}$$
$$\frac{\partial V_h^{n+1}}{\partial X^{n+1}} - 2V_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} - U_h^{n+1}\frac{\partial V_h^{n+1}}{\partial X^{n+1}}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} - V_h^{n+1}\frac{\partial U_h^{n+1}}{\partial X^{n+1}}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}} - U_h^{n+1}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}$$
$$\frac{\partial V_h^{n+1}}{\partial X^{n+1}} - V_h^{n+1}\frac{\partial U_h^{n+1}}{\partial Y^{n+1}}\frac{\partial V_h^{n+1}}{\partial X^{n+1}}\Big)\tilde{V}_h\Big\}d\Omega + q_h\int_{\Omega^{n+1}} V_h^{n+1}\frac{\partial \tilde{V}_h}{\partial Y^{n+1}}d\Omega - P_h^{n+1}\int_{\Omega^{n+1}}\frac{\partial \tilde{V}_h}{\partial Y^{n+1}}d\Omega -$$
$$A_5\Big\{ - \int_{\Omega^{n+1}}\frac{\partial V_h^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{V}_h}{\partial X^{n+1}}d\Omega + \oint_{\Gamma}\tilde{V}_h\Big(n_x\frac{\partial V_h^{n+1}}{\partial X^{n+1}}\Big)d\Gamma$$
$$- \int_{\Omega^{n+1}}\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{V}_h}{\partial Y^{n+1}}d\Omega + \oint_{\Gamma}\tilde{V}_h\Big(n_y\frac{\partial V_h^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} - A_6\int_{\Omega^{n+1}} W_h^{n+1}\frac{\partial \tilde{V}_h}{\partial X^{n+1}}d\Omega = 0 \tag{5.17}$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} W_h^{n+1}\tilde{W}_h d\Omega - \frac{1}{\delta t}\Big(W_h^n\ o\ X^n\Big)\tilde{W}_h d\Omega - A_7\Big\{ - \int_{\Omega^{n+1}}\frac{\partial W_h^{n+1}}{\partial X^{n+1}}\frac{\partial \tilde{W}_h}{\partial X^{n+1}}d\Omega$$
$$+ \oint_{\Gamma}\tilde{W}_h\Big(n_x\frac{\partial W_h^{n+1}}{\partial X^{n+1}}\Big)d\Gamma - \int_{\Omega^{n+1}}\frac{\partial W_h^{n+1}}{\partial Y^{n+1}}\frac{\partial \tilde{W}_h}{\partial Y^{n+1}}d\Omega + \oint_{\Gamma}\tilde{W}_h\Big(n_y\frac{\partial W_h^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\}$$
$$- A_8\int_{\Omega^{n+1}}\Big(- V_h^{n+1}\frac{\partial \tilde{W}_h}{\partial X^{n+1}} + U_h^{n+1}\frac{\partial \tilde{W}_h}{\partial Y^{n+1}} - 2W_h^{n+1}\tilde{W}_h\Big)d\Omega = 0 \tag{5.18}$$

for all $\tilde{U}_h$, $\tilde{V}_h$, $\tilde{W}_h\ \varepsilon\ \bar{W}_h$ and $P_h\ \varepsilon\ Q_h$.

FEM approximation is achieved by using the approximate trial solution functions and trial test functions. These functions are the linear combination of nodal unknowns and shape functions which are linearly independent. Given below are the trial solution functions:

$$U_h = \sum_{j=1}^{m} U_j\xi_j, \qquad V_h = \sum_{j=1}^{m} V_j\xi_j, \qquad W_h = \sum_{j=1}^{m} W_j\xi_j, \qquad P_h = \sum_{j=1}^{l} P_j\eta_j.$$

Similarly following trial approximated functions are defined for test spaces:

$$\tilde{U}_h, \tilde{V}_h, \tilde{W}_h = \sum_{i=1}^{m} (\tilde{U}_i, \tilde{V}_i, \tilde{W}_i)\xi_i, \qquad q_h = \sum_{i=1}^{l} q_i\xi_i.$$

In all above relations ξ_j and η_j are the shape functions. By using these approximations in Eqs. (5.16) to (5.18), weak formulation can be expressed as

(5.16)⇒

$$\frac{1}{\delta t}\int_{\Omega^{n+1}} \left(\sum_{j=1}^{m} U_j\xi_j\right)^{n+1} \sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega - \frac{1}{\delta t}\left(\left(\sum_{j=1}^{m} U_j\xi_j\right)^{n} o\ X^n\right) \sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega + A_1 \int_{\Omega^{n+1}} \Bigg\{\Bigg(-$$
$$\left(\sum_{j=1}^{m} U_j\xi_j\right)^{n}\left(\sum_{j=1}^{m} U_j\xi_j\right)^{n+1} \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j\Bigg)\frac{\partial}{\partial X}\sum_{i=1}^{m} \tilde{U}_i\xi_i +$$
$$\Bigg(-\left(\sum_{j=1}^{m} V_j\xi_j\right)^{n}\left(\sum_{j=1}^{m} V_j\xi_j\right)^{n+1} \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j\Bigg)\frac{\partial}{\partial Y}\sum_{i=1}^{m} \tilde{U}_i\xi_i$$
$$+\Bigg(-2\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j - 2\left(\sum_{j=1}^{m} V_j\xi_j\right)\frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j - \sum_{j=1}^{m}$$
$$U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j$$
$$\frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j\Bigg)\sum_{i=1}^{m} \tilde{U}_i\xi_i\Bigg\}d\Omega + \sum_{i=1}^{l} q_i\xi_i \int_{\Omega^{n+1}} \sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}$$
$$\sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega - \sum_{j=1}^{l} P_j\eta_j \int_{\Omega^{n+1}} \frac{\partial}{\partial X}\sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega - A_2\Bigg\{-\int_{\Omega^{n+1}} \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega + \oint_{\Gamma}$$
$$\sum_{i=1}^{m} \tilde{U}_i\xi_i\left(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j\right)d\Gamma - \int_{\Omega^{n+1}} \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial Y}\sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega + \oint_{\Gamma} \sum_{i=1}^{m} \tilde{U}_i\xi_i\Bigg(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}$$
$$U_j\xi_j\Bigg)d\Gamma\Bigg\} + A_3\int_{\Omega^{n+1}} \sum_{j=1}^{m} W_j\xi_j \frac{\partial}{\partial Y}\sum_{i=1}^{m} \tilde{U}_i\xi_i d\Omega = 0$$

By Galerkins,

$$\tilde{U}_h = \sum_{i=1}^{m} \xi_i$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n+1} \sum_{i=1}^{m} \xi_i d\Omega - \frac{1}{\delta t}\bigg(\Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n}\ o\ X^n\bigg) \sum_{i=1}^{m} \xi_i d\Omega + A_1 \int\limits_{\Omega^{n+1}} \bigg\{\bigg(-$$

$$\Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n+1} \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j\bigg) \frac{\partial}{\partial X}\sum_{i=1}^{m} \xi_i +$$

$$\bigg(-\Big(\sum_{j=1}^{m} V_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m} V_j\xi_j\Big)^{n+1} \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j\bigg) \frac{\partial}{\partial Y}\sum_{i=1}^{m} \xi_i +$$

$$\bigg(-2\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j - 2\Big(\sum_{j=1}^{m} V_j\xi_j\Big) \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j - \sum_{j=1}^{m} U_j\xi_j$$

$$\frac{\partial}{\partial X}\sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j - \sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m}$$

$$U_j\xi_j - \sum_{j=1}^{m} V_j\xi_j \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j\bigg) \sum_{i=1}^{m} \xi_i\bigg\} d\Omega + \sum_{i=1}^{l} q_i\xi_i \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{i=1}^{m} \xi_i d\Omega -$$

$$\sum_{j=1}^{l} P_j\eta_j \int\limits_{\Omega^{n+1}} \frac{\partial}{\partial X}\sum_{i=1}^{m} \xi_i d\Omega - A_2\bigg\{-\int\limits_{\Omega^{n+1}} \frac{\partial}{\partial X}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial X}\sum_{i=1}^{m} \xi_i d\Omega + \oint\limits_{\Gamma} \sum_{i=1}^{m} \xi_i\bigg(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}$$

$$U_j\xi_j\bigg) d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j \frac{\partial}{\partial Y}\sum_{i=1}^{m} \xi_i d\Omega + \oint\limits_{\Gamma} \sum_{i=1}^{m} \xi_i\bigg(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m} U_j\xi_j\bigg) d\Gamma\bigg\}$$

$$+ A_3 \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} W_j\xi_j \frac{\partial}{\partial Y}\sum_{i=1}^{m} \xi_i d\Omega = 0$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n+1} \sum_{i=1}^{m} \xi_i d\Omega - \frac{1}{\delta t}\bigg(\Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n}\ o\ X^n\bigg) \sum_{i=1}^{m} \xi_i d\Omega + A_1 \int\limits_{\Omega^{n+1}} \bigg\{\bigg(-\Big(\sum_{j=1}^{m}$$

$$U_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m} U_j\xi_j\Big)^{n+1} \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X} U_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} V_j\xi_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} U_j\bigg) \sum_{i=1}^{m} \frac{\partial\xi_i}{\partial X} + \bigg(-\Big(\sum_{j=1}^{m} V_j\xi_j$$

$$\Big)^{n}\Big(\sum_{j=1}^{m} V_j\xi_j\Big)^{n+1} \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} V_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} V_j\xi_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X} U_j\bigg) \sum_{i=1}^{m} \frac{\partial\xi_i}{\partial Y} + \bigg(-2\sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m}$$

$$\frac{\partial\xi_j}{\partial X} U_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X} U_j - 2\Big(\sum_{j=1}^{m} V_j\xi_j\Big) \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} V_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} V_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X} V_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} U_j -$$

$$\sum_{j=1}^{m} V_j\xi_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X} U_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} U_j - \sum_{j=1}^{m} U_j\xi_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} V_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X} U_j - \sum_{j=1}^{m} V_j\xi_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial Y} U_j \sum_{j=1}^{m} \frac{\partial\xi_j}{\partial X}$$

$$U_j\bigg) \sum_{i=1}^{m} \xi_i\bigg\} d\Omega + \sum_{i=1}^{l} q_i\xi_i \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} U_j\xi_j \sum_{i=1}^{m} \frac{\partial\xi_i}{\partial X} d\Omega - \sum_{j=1}^{l} P_j\eta_j \int\limits_{\Omega^{n+1}} \sum_{i=1}^{m} \frac{\partial\xi_i}{\partial X} d\Omega - A_2\bigg\{-\int\limits_{\Omega^{n+1}}$$

$$\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial X}U_j\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial X}d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\xi_i\Big(n_x\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial X}U_j\Big)d\Gamma-\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial Y}U_j\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial Y}d\Omega+\oint\limits_{\Gamma}\sum_{i=1}^{m}\xi_i\Big(n_y\sum_{j=1}^{m}\frac{\partial\xi_j}{\partial Y}U_j\Big)d\Gamma\Big\}+A_3\int\limits_{\Omega^{n+1}}\sum_{j=1}^{m}W_j\xi_j\sum_{i=1}^{m}\frac{\partial\xi_i}{\partial Y}d\Omega=0$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\Big(U_j\xi_j\Big)^{n+1}\xi_i d\Omega-\frac{1}{\delta t}\Big(\Big(U_j\xi_j\Big)^{n}\ o\ X^n\Big)\xi_i d\Omega+A_1\int\limits_{\Omega^{n+1}}\Big\{\Big(-\Big(U_j\xi_j\Big)^{n}\Big(U_j\xi_j\Big)^{n+1}\frac{\partial\xi_j}{\partial X}-U_j\xi_jV_j\xi_j\frac{\partial\xi_j}{\partial Y}\Big)\frac{\partial\xi_i}{\partial X}+\Big(-\Big(V_j\xi_j\Big)^{n}\Big(V_j\xi_j\Big)^{n+1}\frac{\partial\xi_j}{\partial Y}-U_j\xi_jV_j\xi_j\frac{\partial\xi_j}{\partial X}\Big)\frac{\partial\xi_i}{\partial Y}+\Big(-2U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial X}-2\Big(V_j\xi_j\Big)\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial Y}-U_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial Y}-V_j\xi_j\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_j}{\partial Y}-U_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial X}-V_j\xi_j\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_j}{\partial X}\Big)\xi_i\Big\}d\Omega+\xi_i\int\limits_{\Omega^{n+1}}\xi_j\frac{\partial\xi_i}{\partial X}d\Omega-\eta_j\int\limits_{\Omega^{n+1}}\frac{\partial\xi_i}{\partial X}d\Omega-A_2\Big\{-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j}{\partial X}\frac{\partial\xi_i}{\partial X}d\Omega-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j}{\partial Y}\frac{\partial\xi_i}{\partial Y}d\Omega\Big\}$$
$$=\oint\limits_{\Gamma}\xi_i\Big(n_x\frac{\partial\xi_j}{\partial X}\Big)d\Gamma+\oint\limits_{\Gamma}\xi_i\Big(n_y\frac{\partial\xi_j}{\partial Y}\Big)d\Gamma+A_3\int\limits_{\Omega^{n+1}}W_j\xi_j\frac{\partial\xi_i}{\partial Y}d\Omega$$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}U_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega-\frac{1}{\delta t}\Big(U_j^n\xi_j^n\ o\ X^n\Big)\xi_i^{n+1}d\Omega+A_1\int\limits_{\Omega^{n+1}}\Big\{\Big(-U_j^n\xi_j^nU_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Big)\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}+\Big(-V_j^n\xi_j^nV_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Big)\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}+\Big(-2U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}-2V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}-U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}-V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}-U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}-V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Big)\xi_i^{n+1}\Big\}d\Omega+\xi_i^{n+1}\int\limits_{\Omega^{n+1}}\xi_j^{n+1}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega-\eta_j^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega-A_2\Big\{-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega-\int\limits_{\Omega^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega\Big\}$$
$$=\oint\limits_{\Gamma}\xi_i^{n+1}\Big(n_x\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Big)d\Gamma+\oint\limits_{\Gamma}\xi_i^{n+1}\Big(n_y\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma+A_3\int\limits_{\Omega^{n+1}}W_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega \tag{5.19}$$

(5.17)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n+1}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega-\frac{1}{\delta t}\Big(\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n}\ o\ Y^n\Big)\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega+A_4\int\limits_{\Omega^{n+1}}\Big\{\Big(-\Big(\sum_{j=1}^{m}U_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m}U_j\xi_j\Big)^{n+1}\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\Big)\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{V}_i\xi_i+$$

$$\Big(-\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n+1}\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\Big)\frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{V}_i\xi_i$$
$$+\Big(-2\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j-2\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}$$
$$U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j$$
$$\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\Big)\sum_{i=1}^{m}\tilde{V}_i\xi_i\Big\}d\Omega+\sum_{i=1}^{l}q_i\xi_i\int_{\Omega^{n+1}}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}$$
$$\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega-\sum_{j=1}^{l}P_j\eta_j\int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega-A_5\Big\{-\int_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega+\oint_{\Gamma}$$
$$\sum_{i=1}^{m}\tilde{V}_i\xi_i\Big(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\Big)d\Gamma-\int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega+\oint_{\Gamma}\sum_{i=1}^{m}\tilde{V}_i\xi_i\Big(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}$$
$$V_j\xi_j\Big)d\Gamma\Big\}-A_6\int_{\Omega^{n+1}}\sum_{j=1}^{m}W_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\tilde{V}_i\xi_i d\Omega=0$$

By Galerkins,

$$\tilde{V}_h=\sum_{i=1}^{m}\xi_i$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n+1}\sum_{i=1}^{m}\xi_i d\Omega-\frac{1}{\delta t}\Big(\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n}o\ Y^n\Big)\sum_{i=1}^{m}\xi_i d\Omega+A_4\int_{\Omega^{n+1}}\Big\{\Big(-$$
$$\Big(\sum_{j=1}^{m}U_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m}U_j\xi_j\Big)^{n+1}\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\Big)\frac{\partial}{\partial X}\sum_{i=1}^{m}\xi_i+$$
$$\Big(-\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n}\Big(\sum_{j=1}^{m}V_j\xi_j\Big)^{n+1}\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\Big)\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i+$$
$$\Big(-2\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j-2\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j$$
$$\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j-\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}$$
$$V_j\xi_j-\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{j=1}^{m}U_j\xi_j\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\Big)\sum_{i=1}^{m}\xi_i\Big\}d\Omega+\sum_{i=1}^{l}q_i\xi_i\int_{\Omega^{n+1}}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i d\Omega-$$
$$\sum_{j=1}^{l}P_j\eta_j\int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i d\Omega-A_5\Big\{-\int_{\Omega^{n+1}}\frac{\partial}{\partial X}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^{m}\xi_i d\Omega+\oint_{\Gamma}\sum_{i=1}^{m}\xi_i\Big(n_x\frac{\partial}{\partial X}\sum_{j=1}^{m}$$
$$V_j\xi_j\Big)d\Gamma-\int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^{m}\xi_i d\Omega+\oint_{\Gamma}\sum_{i=1}^{m}\xi_i\Big(n_y\frac{\partial}{\partial Y}\sum_{j=1}^{m}V_j\xi_j\Big)d\Gamma\Big\}$$

$$- A_6 \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} W_j \xi_j \frac{\partial}{\partial X} \sum_{i=1}^{m} \xi_i d\Omega = 0$$

$$\frac{1}{\delta t} \int\limits_{\Omega^{n+1}} \left(\sum_{j=1}^{m} V_j \xi_j \right)^{n+1} \sum_{i=1}^{m} \xi_i d\Omega - \frac{1}{\delta t} \Bigg(\left(\sum_{j=1}^{m} V_j \xi_j \right)^{n} o\ Y^n \Bigg) \sum_{i=1}^{m} \xi_i d\Omega + A_4 \int\limits_{\Omega^{n+1}} \Bigg\{ \Bigg(- \Bigg(\sum_{j=1}^{m} U_j \xi_j \Bigg)^n \left(\sum_{j=1}^{m} U_j \xi_j \right)^{n+1} \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j - \sum_{j=1}^{m} U_j \xi_j \sum_{j=1}^{m} V_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j \Bigg) \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial X} + \Bigg(- \Bigg(\sum_{j=1}^{m} V_j \xi_j \Bigg)^n \left(\sum_{j=1}^{m} V_j \xi_j \right)^{n+1} \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j - \sum_{j=1}^{m} U_j \xi_j \sum_{j=1}^{m} V_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j \Bigg) \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial Y} + \Bigg(- 2 \sum_{j=1}^{m} U_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} U_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j - 2 \sum_{j=1}^{m} V_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j - \sum_{j=1}^{m} U_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j - \sum_{j=1}^{m} V_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} U_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j - \sum_{j=1}^{m} U_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j - \sum_{j=1}^{m} V_j \xi_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} U_j \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j \Bigg) \sum_{i=1}^{m} \xi_i \Bigg\} d\Omega + \sum_{i=1}^{l} q_i \xi_i \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} V_j \xi_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial Y} d\Omega - \sum_{j=1}^{l} P_j \eta_j \int\limits_{\Omega^{n+1}} \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial Y} d\Omega - A_5 \Bigg\{ - \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial X} d\Omega + \oint\limits_{\Gamma} \sum_{i=1}^{m} \xi_i \left(n_x \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} V_j \right) d\Gamma - \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial Y} d\Omega + \oint\limits_{\Gamma} \sum_{i=1}^{m} \xi_i \left(n_y \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} V_j \right) d\Gamma \Bigg\} - A_6 \int\limits_{\Omega^{n+1}} \sum_{j=1}^{m} W_j \xi_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial X} d\Omega = 0$$

$$\frac{1}{\delta t} \int\limits_{\Omega^{n+1}} \left(V_j \xi_j \right)^{n+1} \xi_i d\Omega - \frac{1}{\delta t} \Bigg(\left(V_j \xi_j \right)^{n} o\ Y^n \Bigg) \xi_i d\Omega + A_4 \int\limits_{\Omega^{n+1}} \Bigg\{ \Bigg(- \left(U_j \xi_j \right)^n \left(U_j \xi_j \right)^{n+1} \frac{\partial \xi_j}{\partial X} - U_j \xi_j V_j \xi_j \frac{\partial \xi_j}{\partial Y} \Bigg) \frac{\partial \xi_i}{\partial X} + \Bigg(- \left(V_j \xi_j \right)^n \left(V_j \xi_j \right)^{n+1} \frac{\partial \xi_j}{\partial Y} - U_j \xi_j V_j \xi_j \frac{\partial \xi_j}{\partial X} \Bigg) \frac{\partial \xi_i}{\partial Y} + \Bigg(- 2 U_j \xi_j \frac{\partial \xi_j}{\partial X} \frac{\partial \xi_j}{\partial X} - 2 V_j \xi_j \frac{\partial \xi_j}{\partial Y} \frac{\partial \xi_j}{\partial Y} - U_j \xi_j \frac{\partial \xi_j}{\partial X} \frac{\partial \xi_j}{\partial Y} - V_j \xi_j \frac{\partial \xi_j}{\partial X} \frac{\partial \xi_j}{\partial Y} - U_j \xi_j \frac{\partial \xi_j}{\partial Y} \frac{\partial \xi_j}{\partial X} - V_j \xi_j \frac{\partial \xi_j}{\partial Y} \frac{\partial \xi_j}{\partial X} \Bigg) \xi_i \Bigg\} d\Omega + \xi_i \int\limits_{\Omega^{n+1}} \xi_j \frac{\partial \xi_i}{\partial Y} d\Omega - \eta_j \int\limits_{\Omega^{n+1}} \frac{\partial \xi_i}{\partial Y} d\Omega - A_5 \Bigg\{ - \int\limits_{\Omega^{n+1}} \frac{\partial \xi_j}{\partial X} \frac{\partial \xi_i}{\partial X} d\Omega - \int\limits_{\Omega^{n+1}} \frac{\partial \xi_j}{\partial Y} \frac{\partial \xi_i}{\partial Y} d\Omega \Bigg\} = \oint\limits_{\Gamma} \xi_i \left(n_x \frac{\partial \xi_j}{\partial X} \right) d\Gamma + \oint\limits_{\Gamma} \xi_i \left(n_y \frac{\partial \xi_j}{\partial Y} \right) d\Gamma - A_6 \int\limits_{\Omega^{n+1}} W_j \xi_j \frac{\partial \xi_i}{\partial X} d\Omega$$

$$\frac{1}{\delta t} \int\limits_{\Omega^{n+1}} V_j^{n+1} \xi_j^{n+1} \xi_i^{n+1} d\Omega - \frac{1}{\delta t} \Big(V_j^n \xi_j^n\ o\ Y^n \Big) \xi_i^{n+1} d\Omega + A_4 \int\limits_{\Omega^{n+1}} \Bigg\{ \Bigg(- U_j^n \xi_j^n U_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} - U_j^{n+1} \xi_j^{n+1} V_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \Bigg) \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} + \Bigg(- V_j^n \xi_j^n V_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1} \xi_j^{n+1} V_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \Bigg) \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} + \Bigg(- 2 U_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} - 2 V_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1} \xi_j^{n+1}$$

$$\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} - V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} - V_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\Big)\xi_i^{n+1}\Big\}d\Omega + \xi_i^{n+1}\int_{\Omega^{n+1}} \xi_j^{n+1}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}d\Omega - \eta_j^{n+1}\int_{\Omega^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}d\Omega - A_5\Big\{-\int_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}d\Omega - \int_{\Omega^{n+1}}\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}}d\Omega\Big\}$$
$$= \oint_\Gamma \xi_i^{n+1}\Big(n_x\frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\Big)d\Gamma + \oint_\Gamma \xi_i^{n+1}\Big(n_y\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma - A_6\int_{\Omega^{n+1}} W_j^{n+1}\xi_j^{n+1}\frac{\partial \xi_i^{n+1}}{\partial X^{n+1}}d\Omega \tag{5.20}$$

(5.18)⇒

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\Big(\sum_{j=1}^m W_j\xi_j\Big)^{n+1}\sum_{i=1}^m \tilde{W}_i\xi_i d\Omega - \frac{1}{\delta t}\Big(\Big(\sum_{j=1}^m W_j\xi_j\Big)^n o\ X^n\Big)\sum_{i=1}^m \tilde{W}_i\xi_i d\Omega - A_7\Big\{-\int_{\Omega^{n+1}} \frac{\partial}{\partial X}\sum_{j=1}^m W_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^m \tilde{W}_i\xi_i d\Omega + \oint_\Gamma \sum_{i=1}^m \tilde{W}_i\xi_i\Big(n_x\frac{\partial}{\partial X}\sum_{j=1}^m W_j\xi_j\Big)d\Gamma - \int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^m W_j\xi_j\frac{\partial}{\partial Y} \sum_{i=1}^m \tilde{W}_i\xi_i d\Omega + \oint_\Gamma \sum_{i=1}^m \tilde{W}_i\xi_i\Big(n_y\frac{\partial}{\partial Y}\sum_{j=1}^m W_j\xi_j\Big)d\Gamma\Big\} - A_8\int_{\Omega^{n+1}}\Big(-\sum_{j=1}^m V_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^m \tilde{W}_i\xi_i + \sum_{j=1}^m U_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^m \tilde{W}_i\xi_i - 2\sum_{j=1}^m W_j\xi_j\sum_{i=1}^m \tilde{W}_i\xi_i\Big)d\Omega = 0$$

By Galerkins,

$$\tilde{W}_h = \sum_{i=1}^m \xi_i$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\Big(\sum_{j=1}^m W_j\xi_j\Big)^{n+1}\sum_{i=1}^m \xi_i d\Omega - \frac{1}{\delta t}\Big(\Big(\sum_{j=1}^m W_j\xi_j\Big)^n o\ X^n\Big)\sum_{i=1}^m \xi_i d\Omega - A_7\Big\{-\int_{\Omega^{n+1}}\frac{\partial}{\partial X} \sum_{j=1}^m W_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^m \xi_i d\Omega + \oint_\Gamma \sum_{i=1}^m \xi_i\Big(n_x\frac{\partial}{\partial X}\sum_{j=1}^m W_j\xi_j\Big)d\Gamma - \int_{\Omega^{n+1}}\frac{\partial}{\partial Y}\sum_{j=1}^m W_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^m \xi_i d\Omega + \oint_\Gamma \sum_{i=1}^m \xi_i\Big(n_y\frac{\partial}{\partial Y}\sum_{j=1}^m W_j\xi_j\Big)d\Gamma\Big\} - A_8\int_{\Omega^{n+1}}\Big(-\sum_{j=1}^m V_j\xi_j\frac{\partial}{\partial X}\sum_{i=1}^m \xi_i + \sum_{j=1}^m U_j\xi_j\frac{\partial}{\partial Y}\sum_{i=1}^m \xi_i - 2\sum_{j=1}^m W_j\xi_j\sum_{i=1}^m \xi_i\Big)d\Omega = 0$$

$$\frac{1}{\delta t}\int_{\Omega^{n+1}}\Big(\sum_{j=1}^m W_j\xi_j\Big)^{n+1}\sum_{i=1}^m \xi_i d\Omega - \frac{1}{\delta t}\Big(\Big(\sum_{j=1}^m W_j\xi_j\Big)^n o\ X^n\Big)\sum_{i=1}^m \xi_i d\Omega - A_7\Big\{-\int_{\Omega^{n+1}}\sum_{j=1}^m$$

$$\frac{\partial \xi_j}{\partial X} W_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial X} d\Omega + \oint_\Gamma \sum_{i=1}^{m} \xi_i \left(n_x \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial X} W_j \right) d\Gamma - \int_{\Omega^{n+1}} \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} W_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial Y} d\Omega + \oint_\Gamma \sum_{i=1}^{m} \xi_i$$
$$\left(n_y \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial Y} W_j \right) d\Gamma \Bigg\} - A_8 \int_{\Omega^{n+1}} \Bigg(- \sum_{j=1}^{m} V_j \xi_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial X} +$$
$$\sum_{j=1}^{m} U_j \xi_j \sum_{i=1}^{m} \frac{\partial \xi_i}{\partial Y} - 2 \sum_{j=1}^{m} W_j \xi_j \sum_{i=1}^{m} \xi_i \Bigg) d\Omega = 0$$

$$\frac{1}{\delta t} \int_{\Omega^{n+1}} \left(W_j \xi_j \right)^{n+1} \xi_i d\Omega - \frac{1}{\delta t} \left(\left(W_j \xi_j \right)^n o\ X^n \right) \xi_i d\Omega - A_7 \Bigg\{ - \int_{\Omega^{n+1}} \frac{\partial \xi_j}{\partial X} \frac{\partial \xi_i}{\partial X} d\Omega - \int_{\Omega^{n+1}} \frac{\partial \xi_j}{\partial Y}$$
$$\frac{\partial \xi_i}{\partial Y} d\Omega \Bigg\} - A_8 \int_{\Omega^{n+1}} \left(- \xi_j \frac{\partial \xi_i}{\partial X} + \xi_j \frac{\partial \xi_i}{\partial Y} - 2 \xi_j \xi_i \right) d\Omega = \oint_\Gamma \xi_i \left(n_x \frac{\partial \xi_j}{\partial X} \right) d\Gamma + \oint_\Gamma \xi_i \left(n_y \frac{\partial \xi_j}{\partial Y} \right) d\Gamma$$

$$\frac{1}{\delta t} \int_{\Omega^{n+1}} W_j^{n+1} \xi_j^{n+1} \xi_i^{n+1} d\Omega - \frac{1}{\delta t} \left(W_j^n \xi_j^n\ o\ X^n \right) \xi_i^{n+1} d\Omega - A_7 \Bigg\{ - \int_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} d\Omega - \int_{\Omega^{n+1}}$$
$$\frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} d\Omega \Bigg\} - A_8 \int_{\Omega^{n+1}} \left(- \xi_j^{n+1} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} + \xi_j^{n+1} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} - 2 \xi_j^{n+1} \xi_i^{n+1} \right) d\Omega$$
$$= \oint_\Gamma \xi_i^{n+1} \left(n_x \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \right) d\Gamma + \oint_\Gamma \xi_i^{n+1} \left(n_y \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \right) d\Gamma \tag{5.21}$$

From Eqs. (5.19) to (5.21), we get the discretized system of nonlinear algebraic equations as

$$[K^*(U, V)]\{X^*\} = \{Q^*\}.$$

The matrix notation of $K^*(U, V)$, X^* and Q^* can be written as

$$\underbrace{\begin{bmatrix} A_{11} & A_{12} & B_1 & A_{14} \\ A_{21} & A_{22} & B_2 & A_{24} \\ B_1^t & B_2^t & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}}_{K^*} \underbrace{\begin{bmatrix} U_h \\ V_h \\ P_h \\ W_h \end{bmatrix}}_{X^*} = \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix}}_{Q^*}. \tag{5.22}$$

Here K^*, X^* and Q^* are called block stiffness matrix, block solution vector and block boundary vector respectively. The local elemental entries of block stiffness matrix are given as

$$A_{11} = \frac{1}{\delta t}\int_{\Omega^{n+1}} U_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\Big(U_j^n\xi_j^n \ o \ X^n\Big)\xi_i^{n+1}d\Omega + A_1\int_{\Omega^{n+1}}\Bigg\{\Bigg(-U_j^n\xi_j^nU_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}} - U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}} + \Bigg(-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}} + \Bigg(-2U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}} - U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\xi_i^{n+1}\Bigg\}d\Omega + \xi_i^{n+1}\int_{\Omega^{n+1}}\xi_j^{n+1}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega - A_2\Bigg\{-\int_{\Omega^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega - \int_{\Omega^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega\Bigg\},$$

$$A_{12} = A_1\int_{\Omega^{n+1}}\Bigg\{\Bigg(-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}} + \Bigg(-V_j^n\xi_j^nV_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}} + \Bigg(-2V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\xi_i^{n+1}\Bigg\}d\Omega,$$

$$A_{14} = A_3\int_{\Omega^{n+1}} W_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega,$$

$$A_{21} = A_4\int_{\Omega^{n+1}}\Bigg\{\Bigg(-U_j^n\xi_j^nU_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}} - U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}} + \Bigg(-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}} + \Bigg(-2U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}} - U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\xi_i^{n+1}\Bigg\}d\Omega,$$

$$A_{22} = \frac{1}{\delta t}\int_{\Omega^{n+1}} V_j^{n+1}\xi_j^{n+1}\xi_i^{n+1}d\Omega - \frac{1}{\delta t}\Big(V_j^n\xi_j^n \ o \ Y^n\Big)\xi_i^{n+1}d\Omega + A_4\int_{\Omega^{n+1}}\Bigg\{\Bigg(-U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}} + \Bigg(-V_j^n\xi_j^nV_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - U_j^{n+1}\xi_j^{n+1}V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}} + \Bigg(-2V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}} - V_j^{n+1}\xi_j^{n+1}\frac{\partial\xi_j^{n+1}}{\partial Y^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\Bigg)\xi_i^{n+1}\Bigg\}d\Omega + \xi_i^{n+1}\int_{\Omega^{n+1}}\xi_j^{n+1}\frac{\partial\xi_i^{n+1}}{\partial Y^{n+1}}d\Omega - A_5\Bigg\{-\int_{\Omega^{n+1}}\frac{\partial\xi_j^{n+1}}{\partial X^{n+1}}\frac{\partial\xi_i^{n+1}}{\partial X^{n+1}}d\Omega$$

$$-\int\limits_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} d\Omega\Big\},$$

$$A_{24} = A_6 \int\limits_{\Omega^{n+1}} W_j^{n+1} \xi_j^{n+1} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} d\Omega,$$

$$A_{44} = \frac{1}{\delta t} \int\limits_{\Omega^{n+1}} W_j^{n+1} \xi_j^{n+1} \xi_i^{n+1} d\Omega - \frac{1}{\delta t}\Big(W_j^n \xi_j^n \ o \ X^n\Big) \xi_i^{n+1} d\Omega - A_7 \Big\{ - \int\limits_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} d\Omega$$

$$- \int\limits_{\Omega^{n+1}} \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} d\Omega \Big\} - A_8 \int\limits_{\Omega^{n+1}} \Big(- 2\xi_j^{n+1} \xi_i^{n+1} \Big) d\Omega$$

$$A_{33} = 0,$$

$$A_{34} = 0,$$

$$A_{41} = \xi_j^{n+1} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}},$$

$$A_{42} = -\xi_j^{n+1} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}},$$

$$A_{43} = 0.$$

The entries $A_{13},\ A_{23}$ and $A_{31},\ A_{32}$ are the pressure matrices with their respective transposes can be written as

$$B_1^{ij} = -\eta_j^{n+1} \int\limits_{\Omega^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial X^{n+1}} d\Omega,$$

$$B_2^{ij} = -\eta_j^{n+1} \int\limits_{\Omega^{n+1}} \frac{\partial \xi_i^{n+1}}{\partial Y^{n+1}} d\Omega,$$

$$(B_1^{ij})^t = -\xi_i^{n+1} \int\limits_{\Omega^{n+1}} \frac{\partial \eta_j^{n+1}}{\partial X^{n+1}} d\Omega,$$

$$(B_2^{ij})^t = -\xi_i^{n+1} \int\limits_{\Omega^{n+1}} \frac{\partial \eta_j^{n+1}}{\partial Y^{n+1}} d\Omega,$$

$$Q_1 = \oint\limits_{\Gamma} \xi_i^{n+1} \Big(n_x \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \Big) d\Gamma + \oint\limits_{\Gamma} \xi_i^{n+1} \Big(n_y \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \Big) d\Gamma,$$

$$Q_2 = \oint\limits_{\Gamma} \xi_i^{n+1} \Big(n_x \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}} \Big) d\Gamma + \oint\limits_{\Gamma} \xi_i^{n+1} \Big(n_y \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}} \Big) d\Gamma,$$

$Q_3 = 0,$

$$Q_4 = \oint_\Gamma \xi_i^{n+1}\left(n_x \frac{\partial \xi_j^{n+1}}{\partial X^{n+1}}\right) d\Gamma + \oint_\Gamma \xi_i^{n+1}\left(n_y \frac{\partial \xi_j^{n+1}}{\partial Y^{n+1}}\right) d\Gamma.$$

The discrete system of non-linear algebraic equations in matrix form can be written as:

$$\begin{bmatrix} A_{11} & A_{12} & B_1 & A_{14} \\ A_{21} & A_{22} & B_2 & A_{24} \\ B_1^T & B_2^T & 0 & 0 \\ A_{41} & A_{42} & 0 & A_{44} \end{bmatrix} \begin{bmatrix} U_h \\ V_h \\ P_h \\ W_h \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \\ Q_4 \end{bmatrix} \tag{5.23}$$

An examination of the weak form of (5.16) to (5.18) and the finite element matrices in (5.22) shows that ξ_i should be atleast linear functions of x and y. Selected two-dimensional finite elements are used to discretize the domain. The three-noded triangular element is among the most basic two-dimensional elements. Another name for this is a linear triangular element. The element is displayed in Figure 5.1. The element's variable interpolation is linear in x and y, with three nodes at the triangle's vertices.

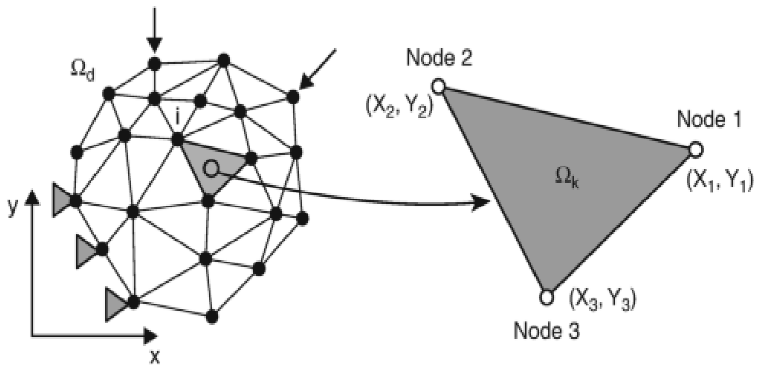


FIGURE 5.1: Systematic Computational Domain

$$u = a_1 + a_2 x + a_3 y \tag{5.24}$$

or

$$u = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \tag{5.25}$$

where a_i is the constant to be determined. The interpolation function, Eq. (5.24) should represent the nodal variables at the three nodal points. Therefore, substituting the x and y values at each nodal point gives

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \tag{5.26}$$

Here, x_i and y_i are the coordinate values at the i^{th} node and u_i is the nodal variable as seen in Figure 5.1. Inverting the matrix and rewriting Eq. (5.26) give

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{|A|} \begin{bmatrix} X_2Y_3 - X_3Y_2 & X_3Y_1 - X_1Y_3 & X_1Y_2 - X_2Y_1 \\ Y_2 - Y_3 & Y_3 - Y_1 & Y_1 - Y_2 \\ X_3 - X_2 & X_1 - X_3 & X_2 - X_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{5.27}$$

For the finite element computation, the element nodal sequence must be in the same direction for every element in the domain. Substituting the Eq. (5.27) into Eq. (5.25), we obtain

$$u = [1 \quad x \quad y] \frac{1}{|A|} \begin{bmatrix} X_2Y_3 - X_3Y_2 & X_3Y_1 - X_1Y_3 & X_1Y_2 - X_2Y_1 \\ Y_2 - Y_3 & Y_3 - Y_1 & Y_1 - Y_2 \\ X_3 - X_2 & X_1 - X_3 & X_2 - X_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \frac{1}{|A|} \begin{bmatrix} \xi_1, & \xi_2, & \xi_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$= \sum_{i=1}^{3} \xi_i u_i$$

where,

$$\xi_1 = \frac{1}{|A|}[(X_2Y_3 - X_3Y_2) + x(Y_2 - Y_3) + y(X_3 - X_2)],$$
$$\xi_2 = \frac{1}{|A|}[(X_3Y_1 - X_1Y_3) + x(Y_3 - Y_1) + y(X_1 - X_3)],$$
$$\xi_3 = \frac{1}{|A|}[(X_1Y_2 - X_2Y_1) + x(Y_1 - Y_2) + y(X_2 - X_1)].$$

For a linear triangular element, (5.13) to (5.15), becomes

$$U = \sum_{i=1}^{3} U_i\xi_i,\ \ V = \sum_{i=1}^{3} V_i\xi_i,\ \ W = \sum_{i=1}^{3} W_i\xi_i,\ \ P = \sum_{i=1}^{3} P_i\eta_i,\ \ \tilde{U},\tilde{V},\tilde{W} = \sum_{j=1}^{3}(\tilde{U}_j,\tilde{V}_j,\tilde{W}_j)\eta_j.$$

(5.13)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}} \left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\left(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\right)d\Omega - \frac{1}{\delta t}\left(\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n} o\ X^n\right)\left(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\right)d\Omega + A_1$$

$$\int\limits_{\Omega^{n+1}} \Bigg\{\Bigg(-\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n}\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}} - \left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}$$

$$\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial Y^{n+1}}\Bigg)\frac{\partial\left(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\right)}{\partial X^{n+1}} + \Bigg(-\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}}{\partial Y^{n+1}} - \Bigg($$

$$\sum_{i=1}^{3} U_i\xi_i\Bigg)^{n+1}\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}}\Bigg)\frac{\partial\left(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\right)}{\partial Y^{n+1}} + \Bigg(-2\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}$$

$$\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}} - 2\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}}{\partial Y^{n+1}}\frac{\partial\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}}{\partial Y^{n+1}} - \Bigg(\sum_{i=1}^{3}$$

$$U_i\xi_i\Bigg)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}}{\partial X^{n+1}}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial Y^{n+1}} - \left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial Y^{n+1}}$$

$$-\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}}{\partial Y^{n+1}}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}} - \left(\sum_{i=1}^{3} V_i\xi_i\right)^{n+1}\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial Y^{n+1}}$$

$$\frac{\partial\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}}{\partial X^{n+1}}\Bigg)\left(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\right)\Bigg\}d\Omega + q\int\limits_{\Omega^{n+1}}\left(\sum_{i=1}^{3} U_i\xi_i\right)^{n+1}\frac{\partial\sum_{j=1}^{3}\tilde{U}_j\eta_j}{\partial X^{n+1}}d\Omega - \Bigg(\sum_{i=1}^{3} P_i$$

$$\eta_i\Big)^{n+1}\int\limits_{\Omega^{n+1}}\frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\Big)}{\partial X^{n+1}}d\Omega-A_2\Big\{-\int\limits_{\Omega^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\Big)}{\partial X^{n+1}}d\Omega+\oint\limits_{\Gamma}\Big(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\Big)\Big(n_x\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\Big)d\Gamma-\int\limits_{\Omega^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\Big)}{\partial Y^{n+1}}d\Omega+\oint\limits_{\Gamma}\Big(\sum_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\Big)\Big(n_y\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\}+A_3\int\limits_{\Omega^{n+1}}\Big(\sum_{i=1}^{3}W_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{U}_j\eta_j^{n+1}\Big)}{\partial Y^{n+1}}d\Omega=0, \tag{5.28}$$

(5.14)⇒

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\Big(\sum_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)d\Omega-\frac{1}{\delta t}\Big(\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n}\ o\ Y^n\Big)\Big(\sum_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)d\Omega+A_4\int\limits_{\Omega^{n+1}}\Big\{\Big(-\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n}\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}-\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\Big)\frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)}{\partial X^{n+1}}+\Big(-\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n}\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}-\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\Big)\frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)}{\partial Y^{n+1}}+\Big(-2\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}-2\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}-\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}-\Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}-\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\Big(\sum\limits_{i=1}^{3}U_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\Big)\Big(\sum_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)\Big\}d\Omega+q\int\limits_{\Omega^{n+1}}\Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j}{\partial Y^{n+1}}d\Omega-\Big(\sum_{i=1}^{3}P_i$$

$$\eta_i\Big)^{n+1} \int\limits_{\Omega^{n+1}} \frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)}{\partial Y^{n+1}} d\Omega - A_5\Big\{ - \int\limits_{\Omega^{n+1}} \frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}} \frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)}{\partial X^{n+1}} d\Omega + \oint\limits_{\Gamma}\Big(
\sum_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)\Big(n_x\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\Big)d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}} \frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)}{\partial Y^{n+1}} d\Omega + \oint\limits_{\Gamma}\Big(\sum_{j=1}^{3}
\tilde{V}_j\eta_j^{n+1}\Big)\Big(n_y\frac{\partial\Big(\sum\limits_{i=1}^{3}V_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\Big)d\Gamma\Big\} - A_6 \int\limits_{\Omega^{n+1}} \Big(\sum_{i=1}^{3}W_i\xi_i\Big)^{n+1} \frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{V}_j\eta_j^{n+1}\Big)}{\partial X^{n+1}} d\Omega = 0, \tag{5.29}$$

(5.15)$\Rightarrow$

$$\frac{1}{\delta t}\int\limits_{\Omega^{n+1}}\Big(\sum_{i=1}^{3}W_i\xi_i\Big)^{n+1}\Big(\sum_{j=1}^{3}\tilde{W}_j\eta_j^{n+1}\Big)d\Omega - \frac{1}{\delta t}\Big(\Big(\sum_{i=1}^{3}W_i\xi_i\Big)^{n} o\ X^n\Big)\Big(\sum_{j=1}^{3}\tilde{W}_j\eta_j^{n+1}\Big)d\Omega -
A_7\Big\{ - \int\limits_{\Omega^{n+1}} \frac{\partial\Big(\sum\limits_{i=1}^{3}W_i\xi_i\Big)^{n+1}}{\partial X^{n+1}} \frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{W}_j\eta_j^{n+1}\Big)}{\partial X^{n+1}} d\Omega + \oint\limits_{\Gamma}\Big(\sum_{j=1}^{3}\tilde{W}_j\eta_j^{n+1}\Big)\Big(n_x\frac{\partial\Big(\sum\limits_{i=1}^{3}W_i\xi_i\Big)^{n+1}}{\partial X^{n+1}}\Big)
d\Gamma - \int\limits_{\Omega^{n+1}} \frac{\partial\Big(\sum\limits_{i=1}^{3}W_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}} \frac{\partial\Big(\sum\limits_{j=1}^{3}\tilde{W}_j\eta_j^{n+1}\Big)}{\partial Y^{n+1}} d\Omega + \oint\limits_{\Gamma}\Big(\sum_{j=1}^{3}\tilde{W}_j\eta_j^{n+1}\Big)\Big(n_y\frac{\partial\Big(\sum\limits_{i=1}^{3}W_i\xi_i\Big)^{n+1}}{\partial Y^{n+1}}\Big)
d\Gamma\Big\} - A_8\int\limits_{\Omega^{n+1}}\Big(- \Big(\sum_{i=1}^{3}V_i\xi_i\Big)^{n+1}\frac{\partial\sum\limits_{j=1}^{3}\tilde{W}_j\eta_j}{\partial X^{n+1}} + \Big(\sum_{i=1}^{3}U_i\xi_i\Big)^{n+1}\frac{\partial\sum\limits_{j=1}^{3}\tilde{W}_j\eta_j}{\partial Y^{n+1}}
- 2\Big(\sum_{i=1}^{3}W_i\xi_i\Big)^{n+1}\sum_{j=1}^{3}\tilde{W}_j\eta_j\Big)d\Omega = 0. \tag{5.30}$$

Equations (5.28) to (5.30) are solved through FEM code Free Fem++. These equations are along with Boundary Conditions are implemented in Free Fem++, the code is used to compute the solution for variant parameters on the computational domain.

Chapter 6

Results and Discussion

6.1 Validation of Numerical Results

In order to validate the numerical results we calculate minimum and maximum values of the stream function and the minimum and maximum values of the velocity field in order to compare our results with the work of Khan et al. [38]. In Table 6.1, we have calculated the u-velocity component $u(x, y)$ for different values of the fluid relaxation factor R_f while keeping the Reynolds number fixed at $Re = 200$. The table presents velocity values at various spatial positions along the y-axis, allowing us to observe how the velocity distribution changes with different relaxation factors. The values of R_f considered are 0.0, 5.0e-03, and 1.0e-02, which influence the numerical stability and convergence of the finite element method (FEM) simulation. By comparing the velocity values across these different R_f settings, we can analyze the impact of relaxation on the solution accuracy and stability.

TABLE 6.1: U velocity table:

y	$R_f = 0.0$	$R_f = 5.0\text{e-}03$	$R_f = 1.0\text{e-}02$
-0.4375	u(0,-0.4375)	u(0,-0.4375)	u(0,-0.4375)
-0.375	u(0,-0.375)	u(0,-0.375)	u(0,-0.375)
0.4375	u(0,0.4375)	u(0,0.4375)	u(0,0.4375)

From the comparison, we observe that the velocity values vary slightly between different relaxation factors, with more noticeable differences at certain locations, such as

$u(0, -0.4375)$ and $u(0, -0.375)$. At these positions, the effect of the relaxation factor is more significant, indicating that relaxation influences the velocity field more prominently in specific regions of the computational domain. On the other hand, at higher y coordinates, such as $u(0, 0.4375)$, the values appear to stabilize, suggesting a steady-state behavior in the solution. These results are particularly relevant in a finite element analysis (FEA) framework, where relaxation techniques are often employed to enhance numerical stability and ensure convergence. The findings in this table could be compared with other numerical methods, such as finite difference or finite volume methods, to evaluate the accuracy and efficiency of FEM. Additionally, the observed trends can help in optimizing computational parameters to achieve a balance between stability and accuracy. Overall, this comparison highlights the importance of the fluid relaxation factor in FEM-based fluid dynamics simulations and its role in refining numerical predictions.

TABLE 6.2: Stream Functions and Velocities

Stream Function	**Velocity Components**
Ψ_{min} increases slightly with increasing relaxation time, from -2.30958×10^{-4} at $R_f = 0.0$ to -2.48093×10^{-4} at $R_f = 1.0 \times 10^{-2}$.	u_{min} becomes less negative as R_f increases, from -0.452793 at $R_f = 0.0$ to -0.379307 at $R_f = 1.0 \times 10^{-2}$.
Ψ_{max} also shows a slight increase, from 9.1456×10^{-2} to 9.28754×10^{-2}.	v_{min} becomes more negative as R_f increases, from -0.742481 to -0.845826. v_{max} also increased slightly, from 0.621798 to 0.673491.

In Table 6.2, the trends in the flow function and the velocity components can be attributed to the effect of the relaxation time on the kinetic energy of the system. As the relaxation factor increases, the kinetic energy of the fluid particles also increases, allowing for greater translational motion. Relaxation time reduces frictional resistance, enabling particles to move more freely and thus increasing the overall magnitude of velocities in both the x and y directions. As a result, the flow becomes smoother and the motion of the particles intensifies with higher relaxation time. Overall, the relaxation time has a significant impact on the fluid's velocity field, stream function, and the kinetic energy of the system's particles, with higher values of R_f leading to more stable and less fluctuating flow characteristics.

Fluid Relaxation Factor Rf				
x-axis Position	0.1	0.2	0.3	0.4
-10	-0.5	-0.4	-0.3	-0.2
-5	-0.3	-0.2	-0.1	0.0
0	0.0	0.1	0.2	0.3
5	0.2	0.3	0.4	0.5
10	0.4	0.5	0.6	0.7

TABLE 6.3: v-velocity component of fluid flow at various positions along the x-axis, for different values of the fluid relaxation factor Rf, while keeping the Reynolds number Re constant at 200.

Table 6.3 presents the v-velocity component of fluid flow at various positions along the x-axis, for different values of the fluid relaxation factor Rf, while keeping the Reynolds number Re constant at 200. V-velocity component represents the fluid velocity in the vertical direction (perpendicular to the x-axis), a critical parameter in fluid dynamics, especially for analyzing vertical flow behavior. Each data point in the table corresponds to the vertical velocity at specific x-coordinates, ranging from negative to positive values, reflecting how the fluid moves in the vertical direction at those positions.

As the relaxation factor changes, the v-velocity fluctuates, highlighting how different relaxation rates influence the vertical movement of the fluid. The values v-velocity indicate that the fluid moves downward (negative values) at some positions and upward (positive values) at others, with the movement transitioning from negative to positive as we move along the x-axis. The data shows slight variations in v-velocity as the relaxation factor changes, suggesting Rf impacts the flow dynamics in these regions, although the effect is more pronounced at some points than others. Table demonstrates the influence of the fluid relaxation factor on the vertical velocity of the flow. By keeping the Reynolds number fixed, the results offer insights into how Rf affects the fluid's vertical movement and flow behavior.

Rf	0.0	5.0e-03	1.0e-02	0.0	5.0e-03	1.0e-02
Parameter	Khan et al. [38]			Present		
Ψ_{min}	-2.30958e-04	-2.39755e-04	-2.48093e-04	-0.000228599	-0.000237125	-0.000245135
Ψ_{max}	9.1456e-02	9.21415e-02	9.28754e-02	0.091412	0.0920927	0.092821
u_{min}	-4.52796e-01	-4.26949e-01	-3.79307e-01	-0.452793	-0.426947	-0.379307
v_{min}		-7.42482e-01	-8.45826e-01	-0.695244	-0.742481	-0.845826
v_{max}		6.21795e-01	6.7349e-01	0.593753	0.621798	0.673491

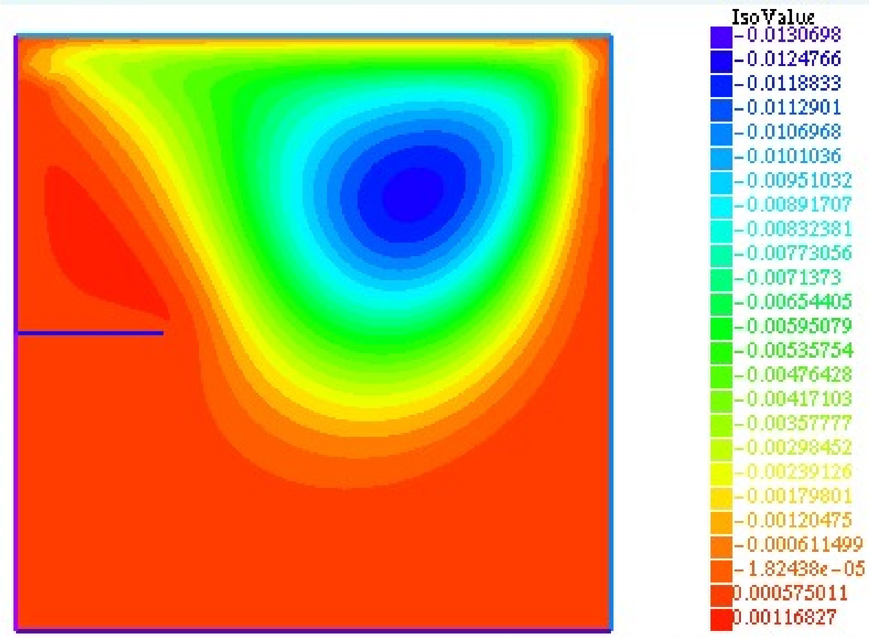
TABLE 6.4: Variable of interest for varying values of the fluid relaxation factor R_f with fixed Reynolds number $Re = 200$.

Rf	0.0	5.0e-03	1.0e-02	0.0	5.0e-03	1.0e-02
	Khan et al. [38]			Present		
$u(0,-0.4375)$	-0.0137483	-0.01343	-0.01307	-0.0301441	-0.0297663	-0.0526012
$u(0,-0.375)$	-0.030435	-0.0301	-0.0525019	-0.0813211	-0.0819026	-0.08209
$u(0,-0.25)$	-0.0814902	-0.11768	-0.11925	-0.115901	-0.117358	-0.118891
$u(0,-0.125)$	-0.152827	-0.15526	-0.15788	-0.152479	-0.154873	-0.157444
$u(0,-0.0625)$	-0.183365	-0.18637	-0.18965	-0.18304	-0.186006	-0.189248
$u(0,0)$	-0.197098	-0.19985	-0.20293	-0.196879	-0.199608	-0.202667
$u(0,0.0625)$	-0.183887	-0.18551	-0.18744	-0.183818	-0.185434	-0.187365
$u(0,0.125)$	-0.140708	-0.14078	-0.1411	-0.140779	-0.140866	-0.141201
$u(0,0.1875)$	-0.0739784	-0.07263	-0.07142	-0.0741538	-0.0728288	-0.071643
$u(0,0.25)$	0.00552624	0.007896	0.010288	0.00527294	0.00761584	0.00997587
$u(0,0.3125)$	0.0919126	0.094687	0.097687	0.0915941	0.0943351	0.0972965
$u(0,0.375)$	0.208309	0.21068	0.213527	0.207985	0.210322	0.21313
$u(0,0.4375)$	0.471098	0.472924	0.475312	0.470901	0.472707	0.475071

TABLE 6.5: u-velocity component for varying values of the fluid relaxation factor R_f with fixed Reynolds number $Re = 200$.

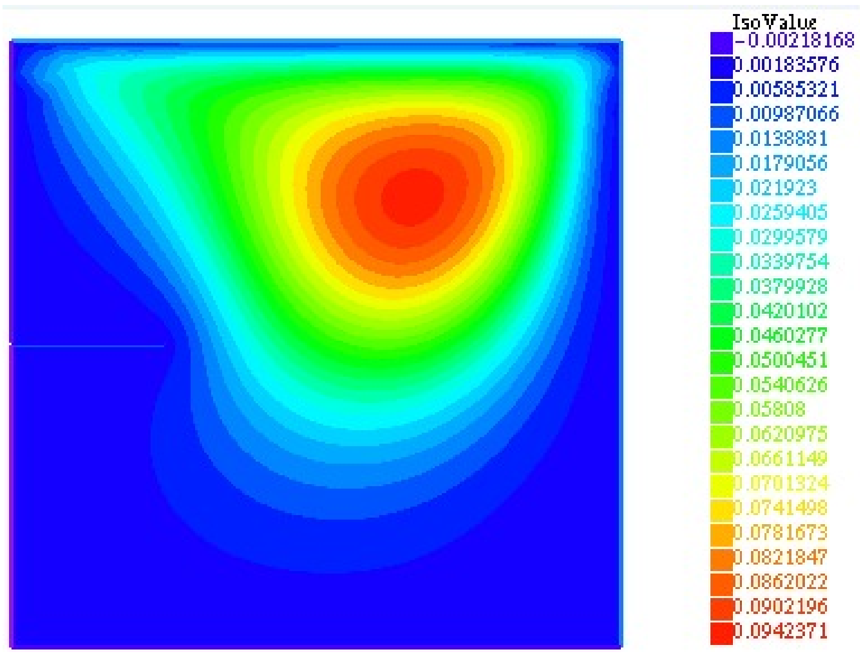
Rf	0.0	5.0e-03	1.0e-02	0.0	5.0e-03	1.0e-02
	-	Khan et al. [38]	-	-	Present	-
$v(-0.4375,0.0625)$	-0.0028463	-0.00295	-0.00304	-0.00282089	-0.00291704	-0.00300874
$v(-0.375,0.0625)$	0.00278488	0.002681	0.002603	0.00280836	0.0027071	0.00263342
$v(-0.3125,0.0625)$	0.0223357	0.022341	0.022417	0.0223205	0.0223253	0.0224004
$v(-0.25,0.0625)$	0.11937	0.120209	0.121301	0.119151	0.119968	0.121033
$v(-0.1875,0.0625)$	0.240211	0.243157	0.246468	0.239771	0.242665	0.245913
$v(-0.125,0.0625)$	0.245026	0.248092	0.251353	0.244635	0.247655	0.250862
$v(-0.0625,0.0625$	0.211699	0.214401	0.217148	0.211439	0.21411	0.216822
$v(0.0,0.0625)$	0.159801	0.161952	0.164013	0.1597	0.16184	0.163889
$v(0.0625,0.0625)$	0.0885403	0.08944	0.090932	0.0885837	0.0898926	0.0909865
$v(0.125,0.0625)$	-0.0018371	-0.00158	-0.00163	-0.00169661	-0.00142305	-0.00145684
$v(0.1875,0.0625)$	-0.107099	-0.10807	-0.10939	-0.106906	-0.107851	-0.109151
$v(0.25,0.0625)$	-0.214806	-0.21728	-0.22007	-0.214573	-0.217018	-0.219782
$v(0.3125,0.0625)$	-0.29301	-0.29694	-0.30103	-0.292732	-0.296631	-0.300682
$v(0.375,0.0625)$	-0.288203	-0.29211	-0.29585	-0.287915	-0.291785	-0.295484
$v(0.4375,0.0625)$	-0.168197	-0.16994	-0.17139	-0.168013	-0.169736	-0.17116

TABLE 6.6: v-velocity component for varying values of the fluid relaxation factor R_f with fixed Reynolds number $Re = 200$.



(a)

$Er = 0.5$



(b)

$Er = 5.5$

FIGURE 6.1: Plot of Stream function with varying Er

The observed increasing trend in Figure 6.1, in streamlines as Er values are varied can be explained by the changing balance between viscous and inertial forces in the fluid flow.

If Er is related to viscosity, an increase could lead to a more viscous fluid, which stabilizes the flow and causes streamlines to become more orderly and confined. Conversely, if Er is linked to inertial forces (such as Reynolds number), increasing Er could make the flow more chaotic or turbulent, though the streamlines might still appear more confined within the cavity if instability doesn't occur. The randomness in streamlines may indicate a transition between laminar and transitional flow regimes. Overall, altering Er values affects the fluid's behavior, leading to streamlines that are either more ordered or constrained, depending on the specific role of Er in the system.

In Figure 6.2, we observe that the moving velocities and the streamlines of microrotation velocity are showing a trend. As the micropolar constant is changed, we notice that the velocities tend to increase. Specifically, when we increase the values of the micropolar constant, the velocities increase, and this leads to an increase in the trend of microrotations. The reason behind this behavior is related to the nature of microrotation in micropolar fluids. Micropolar fluids, which are characterized by additional degrees of freedom (microrotation), respond differently than classical fluids. As the micropolar constant increases, it reflects the increase in the microstructural rotation of fluid particles, which contributes to enhanced flow velocities. The microrotation introduces rotational motion to the fluid elements, and this rotational motion increases as the micropolar constant is increased. This is why we observe a rise in velocities and microrotations with the increase in the micropolar constant. In summary, the data shows that with an increase in the micropolar constant, there is a corresponding increase in the velocities of the fluid, which further leads to an increase in the microrotational behavior. This phenomenon occurs due to the intrinsic properties of micropolar fluids, where the interactions between the fluid's microstructure and the flow dynamics result in this observed trend. In Figure 6.3 the increase in drag force Fd with rising Er values is explained by the fluid's increasing resistance to motion. If Er is related to viscosity, higher values indicate a more viscous fluid, which increases internal friction and leads to a higher drag force on objects moving through it. On the other hand, if Er corresponds to Reynolds number, increasing values could transition the flow from laminar to more turbulent or inertial regimes, where both viscous and inertial forces contribute to drag. In both cases, whether due to increased viscosity or more chaotic flow, the resistance the fluid provides to motion rises, resulting in a higher drag force. Thus, the observed trend of increasing drag force with increasing Er values is linked to the growing friction or turbulence in the fluid. In Figure 6.4, the relaxation time effect, represent models used for calculation. If we refer to the previous Khan model, velocities would be generated accordingly. However, when we look at the present model. In the current model, Figures show an increase, meaning the maximum values increase as the relaxation time parameter increases. As compared to the previous Khan model, with the increase in the relaxation time effect, the stream function values will also rise. This happens due to the nature of the relaxation time, which represents the time required for a fluid element to return to equilibrium after a disturbance.

As the relaxation time increases, the fluid takes longer to adjust to the applied forces, leading to enhanced flow behavior and higher velocities. In the case of the present model, when the relaxation time increases, the system exhibits a higher level of inertia, meaning that the fluid experiences a delayed response to changes in velocity. This delay allows the system to adjust more gradually, leading to an increase in the stream function values. The increased relaxation time effectively allows more time for the flow patterns to develop and stabilize, which contributes to the observed increase in velocities and stream function values. In summary, as the relaxation time increases in the present model, there is a corresponding increase in the stream function and velocities. This is because the relaxation time introduces a delayed response in the fluid's adjustment to forces, leading to a more pronounced flow behavior. In comparison to the previous Khan model, the present model shows a more significant increase in the stream function values due to the nature of the relaxation time effect. The increase in drag force Fd with rising time relaxation Rf can be seen in Figure 6.5 by the fluid's delayed response to changes in the object's movement. As Rf increases, the fluid takes longer to adjust to changes in velocity, causing it to exert greater resistive forces over time. This "lag" in the fluid's reaction leads to higher drag because the fluid cannot efficiently follow the object's motion, resulting in increased friction and stress. Additionally, in cases involving viscoelastic fluids, higher Rf can enhance stress buildup, further increasing drag. Essentially, the slower adjustment of the fluid due to higher time relaxation leads to greater frictional resistance, causing the drag force to rise. In Figure 6.6, the decrease in drag force Fd with increasing micropolar fluid parameters can be attributed to the unique properties of micropolar fluids, which have microstructures like suspended particles or vortices that exhibit independent rotation. Unlike traditional Newtonian fluids, the micro-rotation of particles in micropolar fluids reduces the fluid's resistance to motion, leading to a decrease in drag force. This is because the internal rotation decouples some of the resistance typically generated by viscosity in classical fluids, allowing the fluid to move more easily around the object.

Additionally, micropolar fluids tend to have lower effective viscosity, particularly at higher micropolar effects, which leads to more efficient energy dissipation and less drag. The micro-rotation also alters the flow structure, often resulting in a more streamlined, less turbulent flow around the object, further reducing drag. Overall, as the micropolar effect increases, the drag force decreases due to reduced resistance from the fluid's micro-rotational behavior and lower effective viscosity.

The observed increase in drag force Fd with rising Er values can be seen in Figure 6.7 by the changing characteristics of the fluid flow. If Er relates to viscosity, higher values indicate a more viscous fluid, which increases friction and results in a greater drag force on an object. Alternatively, if Er is linked to Reynolds number, increasing values may push the flow from a laminar state to a more turbulent or inertial regime, where both viscous and inertial forces contribute to drag. In both cases, the fluid's resistance to motion rises, either due to enhanced viscosity or more chaotic flow dynamics, leading to an increase in drag force. Thus, the trend of increasing drag force with increasing Er values is driven by higher friction in more viscous fluids or greater turbulence in more inertial flows.

In Figure 6.8, the observed increase in drag force Fd with rising Er values in a lid-driven cavity is observed by changes in the relative importance of inertial and viscous forces in the fluid. If Er corresponds to Reynolds number Re, an increase in Er implies higher flow velocity or larger characteristic length, leading to more inertial-dominated flow, which can transition from laminar to turbulent, increasing drag force due to greater turbulence and energy dissipation. Alternatively, if Er refers to the Ekman number, which relates viscous forces to Coriolis forces, increasing Er could indicate stronger viscous forces, making the fluid more resistant to the movement of the lid and increasing drag force due to higher friction. In both cases, whether Er affects inertial or viscous forces, the drag force increases as Er rises because the flow either becomes more turbulent or more resistant, both contributing to higher drag.

In Figure 6.9, the increase in drag force Fd with rising time relaxation Rf in a lid-driven cavity is due to the fluid's slower response to changes in the lid's motion. As Rf increases, the fluid takes longer to adjust to the shear forces generated by the lid's movement, leading to a delayed stabilization of the flow. This slower adjustment results in higher friction between the lid and the fluid, increasing drag force. The fluid behaves more sluggishly, generating greater resistance to acceleration, similar to a more viscous fluid. The time delay in the fluid's response causes a buildup of resistance over time, amplifying the drag force. In summary, the increase in Rf leads to greater drag due to the fluid's slower adjustment, which increases frictional resistance against the lid's motion.

In Figure 6.10, it is observed that increasing trend in streamlines as Er values are varied can be explained by the changing balance between viscous and inertial forces in the fluid flow. If Er is related to viscosity, an increase could lead to a more viscous fluid, which stabilizes the flow and causes streamlines to become more orderly and confined. This effect is often seen in systems where higher viscosity suppresses chaotic behavior, leading to smoother, more predictable fluid motion. Conversely, if Er is linked to inertial forces (such as Reynolds number), increasing Er could make the flow more chaotic or turbulent, though the streamlines might still appear more confined within the cavity if instability doesn't occur. In this case, the increase in Er could result in a more dynamic flow, with local fluctuations in velocity and pressure, but without fully transitioning into turbulence. The randomness in streamlines may indicate a transition between laminar and transitional flow regimes, where the flow undergoes small instabilities that disrupt the smooth motion of the fluid but do not yet result in full turbulence. This transition zone is highly sensitive to changes in parameters like Er, and slight increases can significantly impact the flow's stability.

Furthermore, as Er increases, the flow may encounter different forms of energy dissipation, leading to altered vortex patterns, more complex streamlines, and varying pressure distributions throughout the cavity. Overall, altering Er values affects the fluid's behavior, leading to streamlines that are either more ordered or constrained, depending on the specific role of Er in the system. If Er is primarily affecting viscosity, we expect smoother, more stable flow and tightly confined streamlines. If Er is primarily affecting inertial forces, we might observe greater variability in the streamlines as the fluid attempts to adjust to the increasing momentum, with the potential for chaotic or transitional flow characteristics. Thus, understanding how Er influences both viscosity and inertial forces is crucial to predicting the flow pattern and behavior within the system.

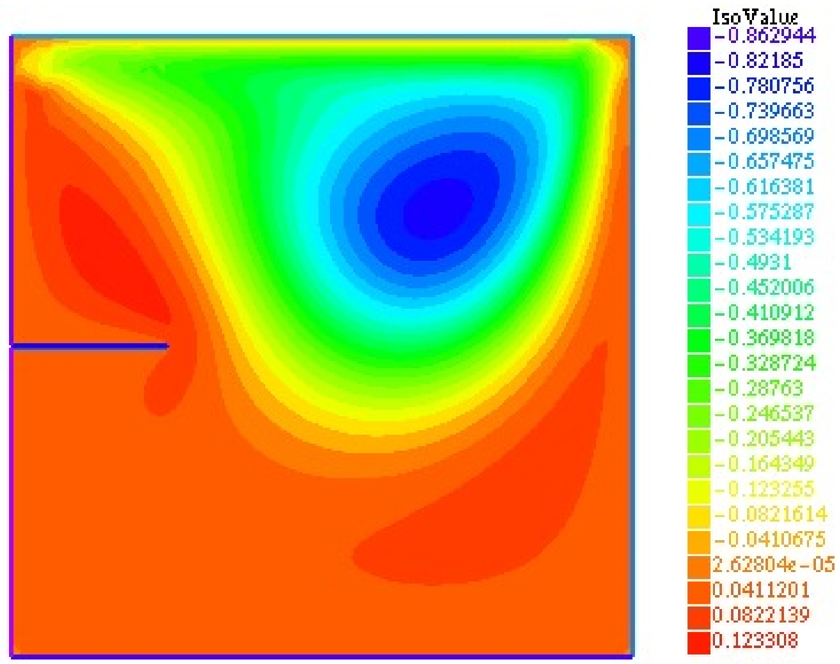
In Figure 6.11, the absence of micropolar effects, if the magnitude of microrotations is small, the streamline values will be lower. However, when the micropolar effect is added, the stream function values will increase. As the micropolar effect values are further increased, the stream functions will continue to rise. This happens because the introduction of micropolar effects adds additional degrees of freedom to the fluid's behavior. Micropolar fluids have not only translational motion but also rotational motion, meaning that individual fluid elements can rotate in addition to moving through the flow field. When these microrotations are introduced into the system, they enhance the flow dynamics, causing the streamlines to become more organized and the fluid to exhibit a more complex flow pattern. The increase in micropolar effects means that the fluid's microstructure is able to rotate more, leading to an increase in the momentum transfer within the fluid. This rotational motion contributes to the overall flow, enhancing the movement and behavior of the fluid. As a result, the flow becomes more energetic, leading to an increase in the stream function values. Furthermore, as the micropolar effect increases, the fluid's resistance to deformation decreases, which allows the flow to become more sensitive to small changes in external forces. This results in an increase in the overall flow intensity, which manifests as higher stream function values. Essentially, the microrotational component of the micropolar fluid leads to a more dynamic and complex flow pattern, causing the stream function to increase as the micropolar effect is amplified.

In Figure 6.12, the increase in kinetic energy (KE) with rising Er values in a lid-driven cavity can be explained by the changing balance between inertial and viscous forces as Er increases. If Er is associated with a parameter like Reynolds number Re, higher values indicate a shift toward inertial-dominated flow, where the fluid velocity increases. This higher velocity, due to more dynamic and turbulent flow, directly increases kinetic energy, as KE is proportional to the square of the velocity. As Er increases, the flow may transition from laminar to turbulent, with the formation of eddies and vortices, which enhance energy transfer and further increase kinetic energy. Overall, as Er increases, the fluid becomes faster and more complex in its motion, leading to an increase in kinetic energy within the cavity.

In Figure 6.13, the effect of time relaxation on microrotations and velocities is seen, it is observed that the rotation of particles increases. Time relaxation, in the context of a fluid, refers to the time it takes for the fluid to return to equilibrium after a disturbance or change in its state. During the process of motion, time relaxation has a significant impact on the behavior of microrotations. If time relaxation decreases, microrotations also tend to decrease, while the presence of time relaxation causes an increase in microrotations. The reason for this behavior is that time relaxation influences how the fluid responds to changes in its motion. In the absence of time relaxation, the fluid's response to disturbances can be more abrupt and less smooth. As a result, microrotations, or the rotational motion of fluid particles, may not be fully developed or may be suppressed.

On the other hand, when time relaxation is present and increased, the fluid takes longer to reach equilibrium, which allows for a more gradual adjustment. This extended period enables the fluid to experience more complete microstructural rotation, leading to an increase in microrotations. The longer the relaxation time, the more pronounced the rotational effects become, as the fluid has more time to respond and adjust to external forces and internal flow conditions. Thus, the presence of time relaxation fosters an environment where microrotations are enhanced.

In summary, the absence of time relaxation causes microrotations to decrease or remain subdued, as the fluid adjusts more quickly and abruptly. However, when time relaxation is introduced and increased, the fluid has more time to develop its microstructural rotational motion, leading to an increase in microrotations. Therefore, time relaxation directly influences the magnitude of microrotations, with a longer relaxation time resulting in a more significant increase in these rotations.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-0.452006
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-0.123255
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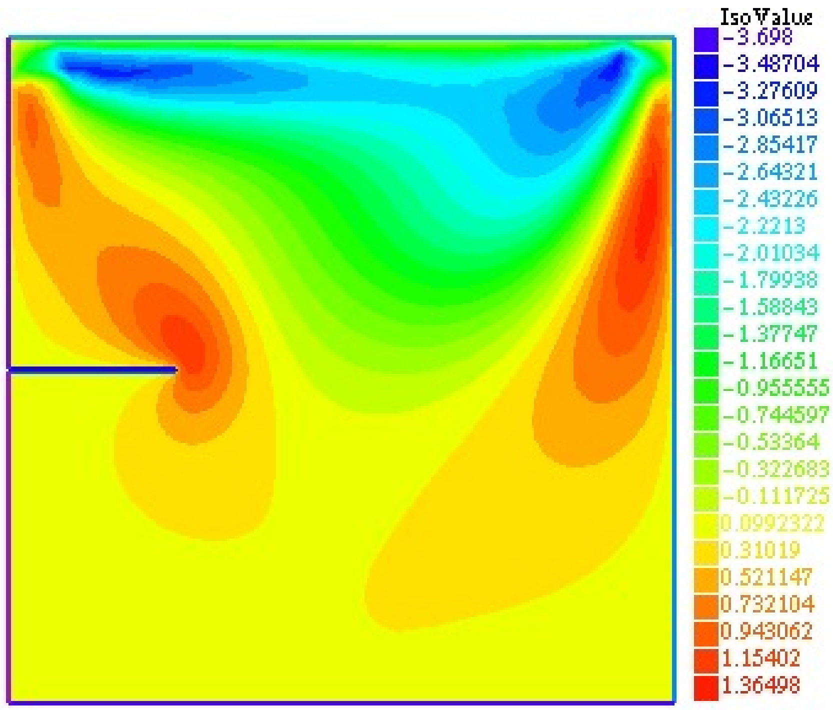
(a)

$K = 0.05$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-1.16651
-0.955555
-0.744597
-0.53364
-0.322683
-0.111725
0.0992322
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0.521147
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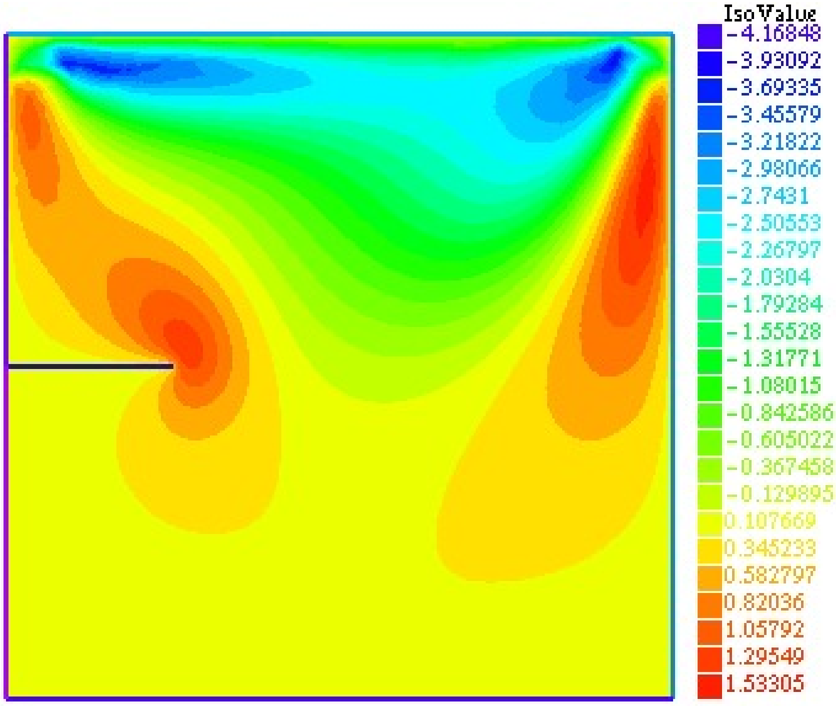
0.943062
1.15402
1.36498

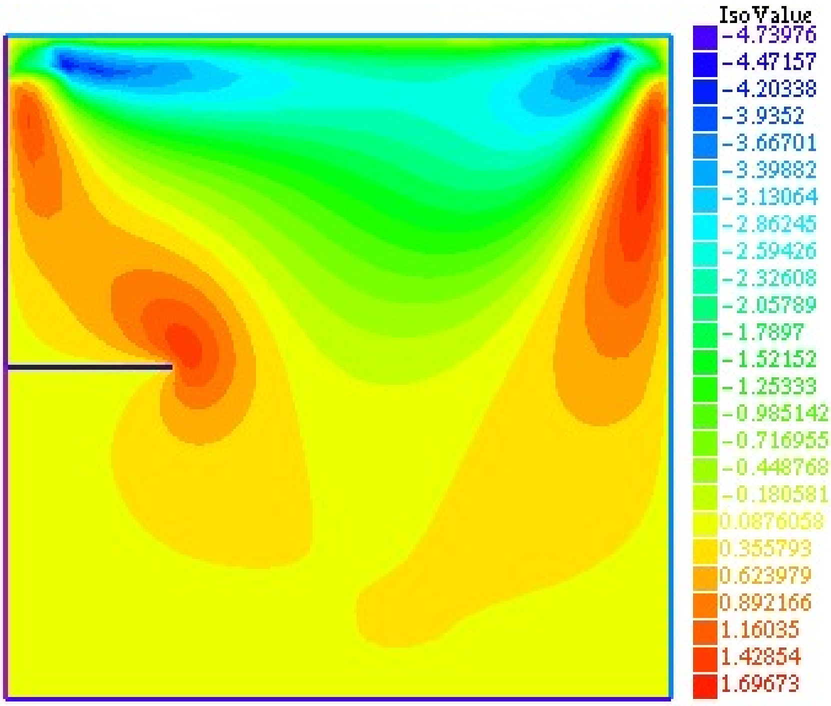
(b)

$K = 2.0$



(c)

$K = 3.0$



(d)

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FIGURE 6.2: Microrotation velocities with varying values of micropolar constant

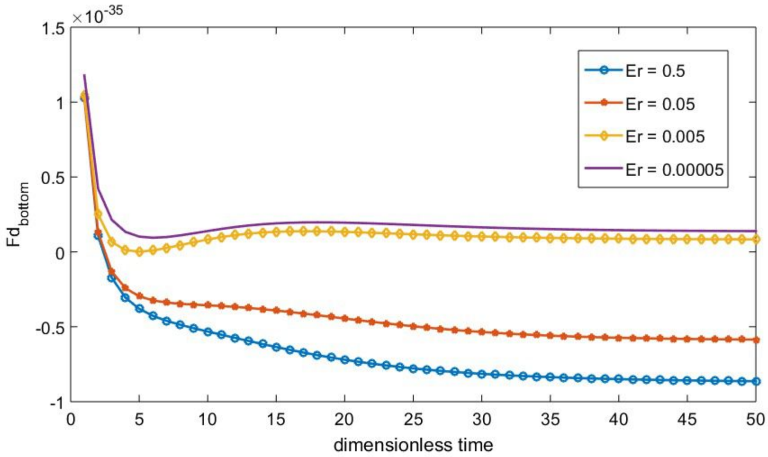
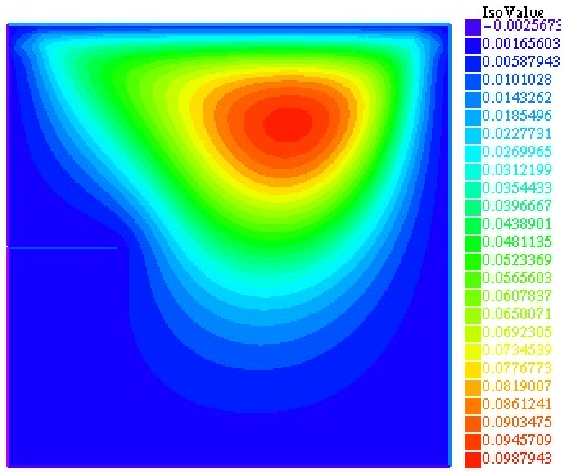
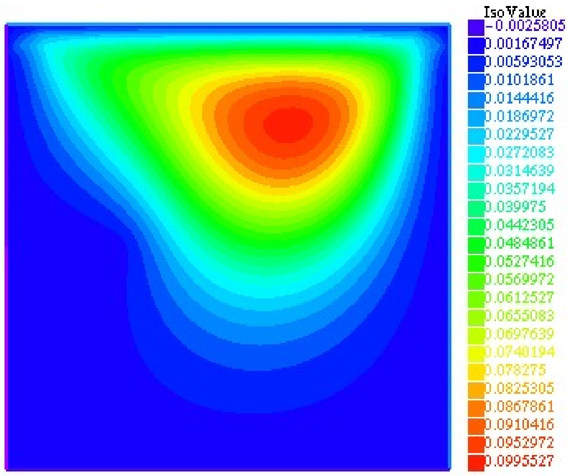


FIGURE 6.3: Fd Bottom with varying Er



(a)

$Rf = 0.0$



(b)

$Rf = 0.01$

FIGURE 6.4: Plot of Velocity Stream Values varying Rf

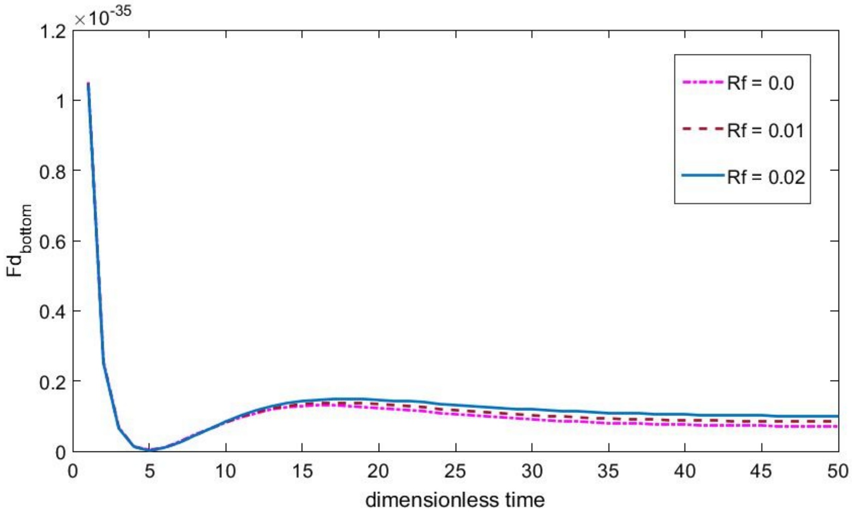


FIGURE 6.5: Fd Bottom with varying Rf.

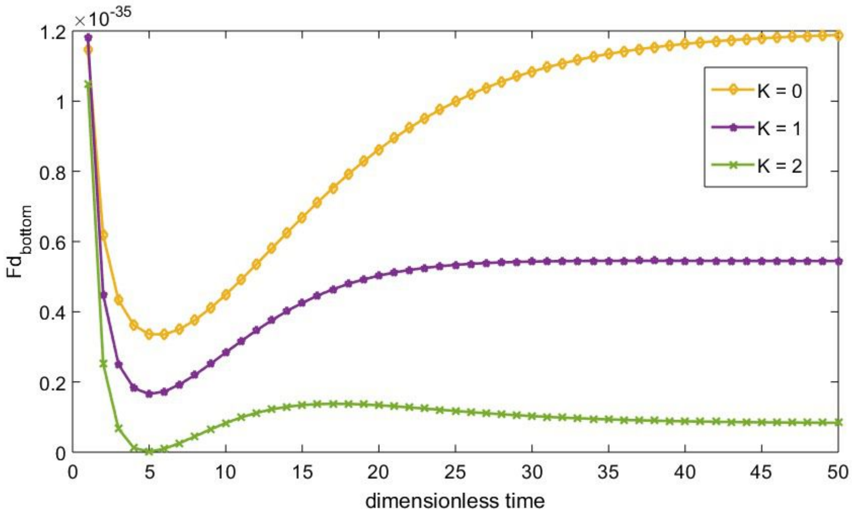


FIGURE 6.6: Fd Bottom with varying K

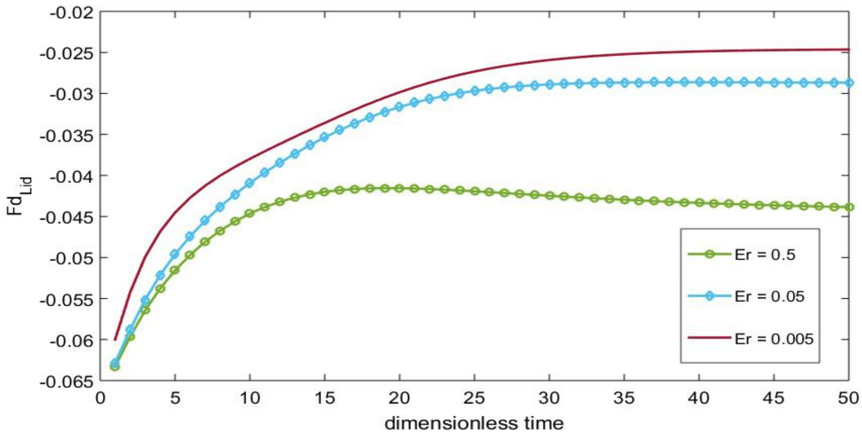


FIGURE 6.7: Fd lid with varying Er

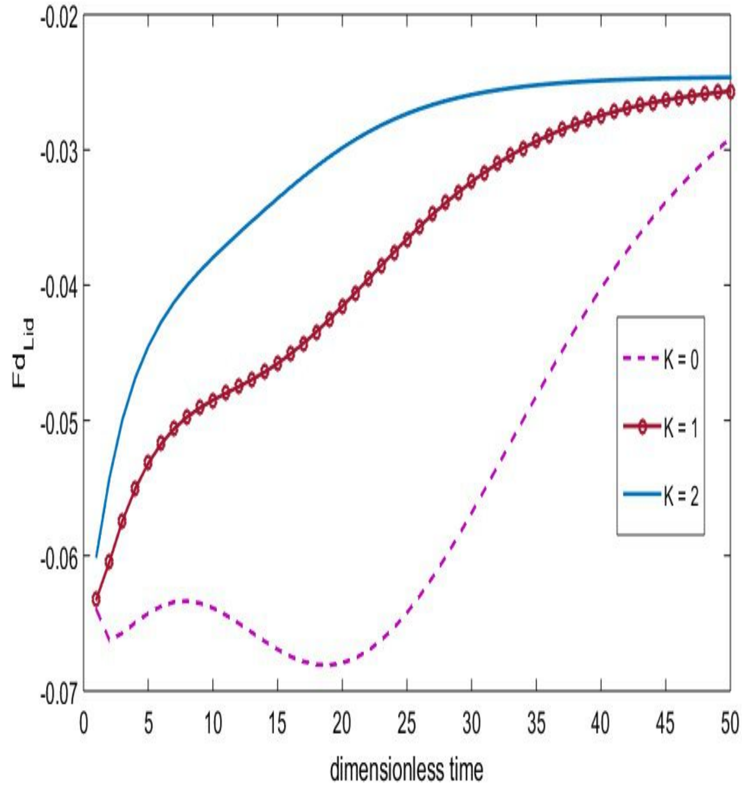


FIGURE 6.8: Fd Lid with varying values of K

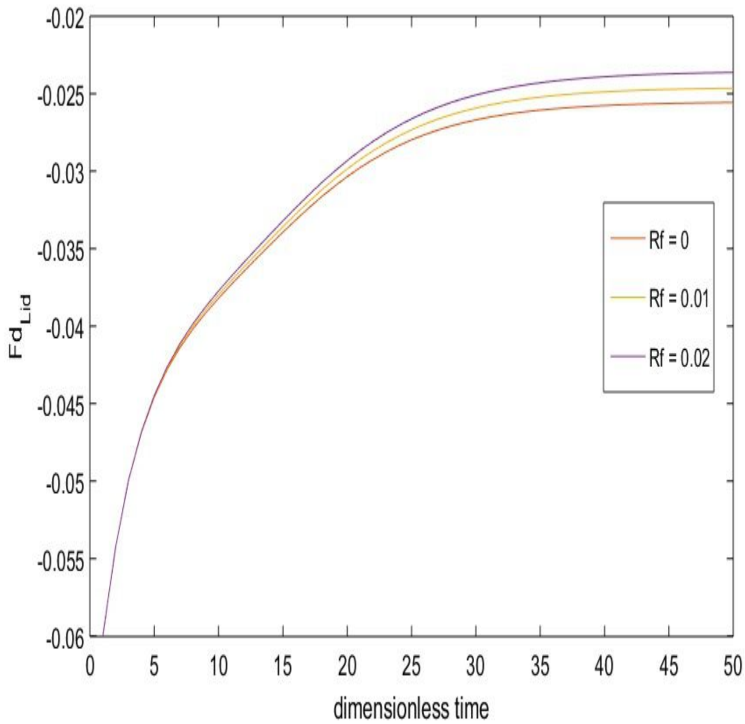
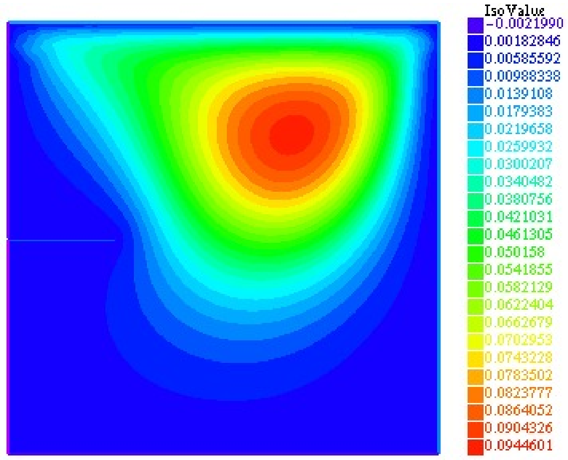
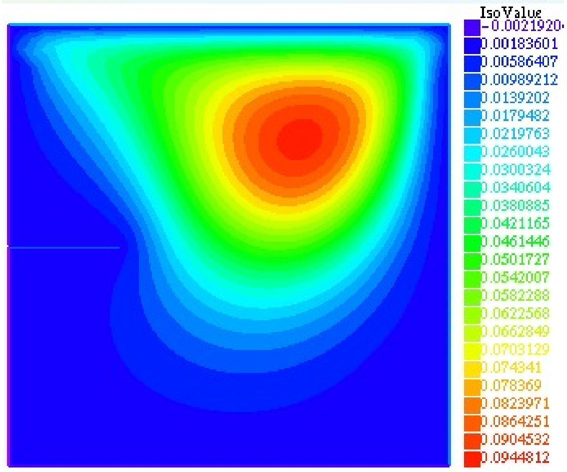


FIGURE 6.9: Fd Lid with varying values of Rf



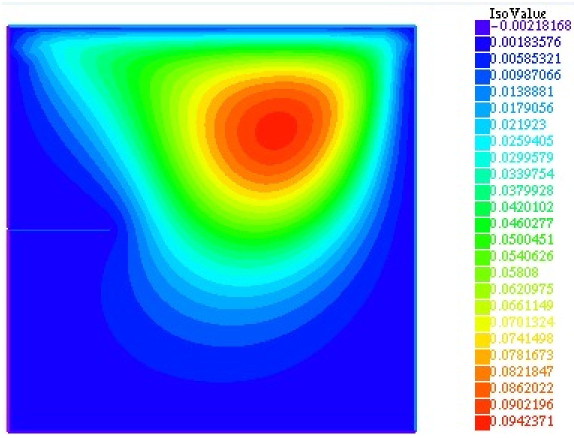
(a)

$Er = 0.00005$



(b)

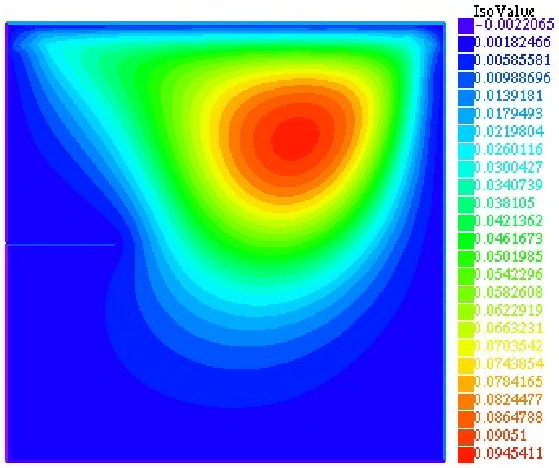
$Er = 0.005$



(c)

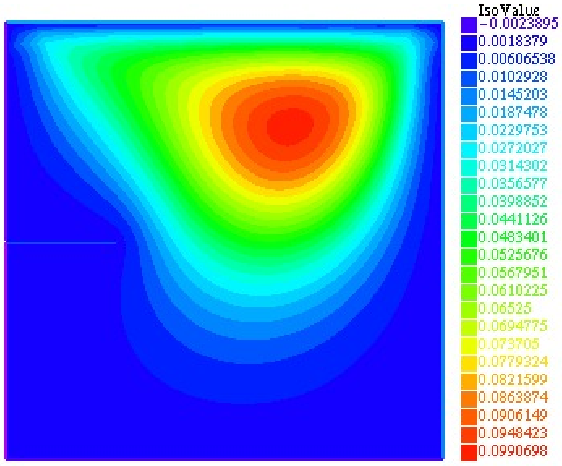
$Er = 5.5$

FIGURE 6.10: Plot of Stream function with varying Er



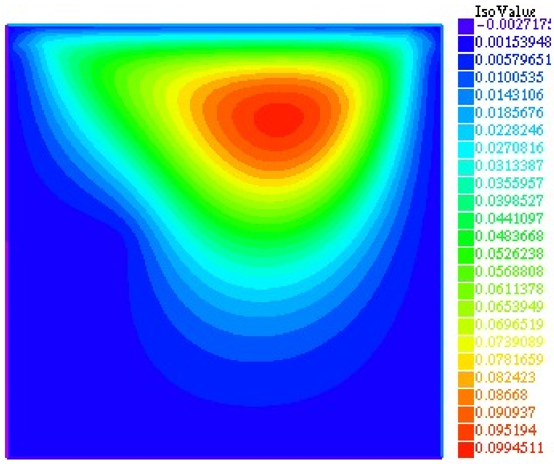
(a)

$K = 0.0$



(b)

$K = 1.0$



(c)

$K = 3.0$

FIGURE 6.11: Plot of Stream function with varying K

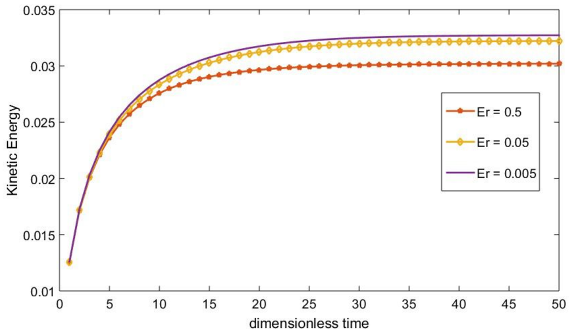
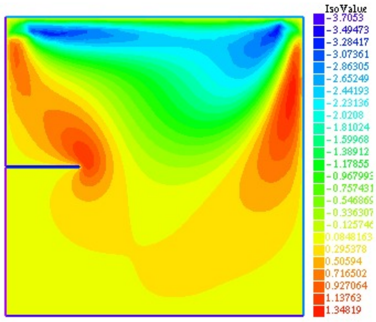
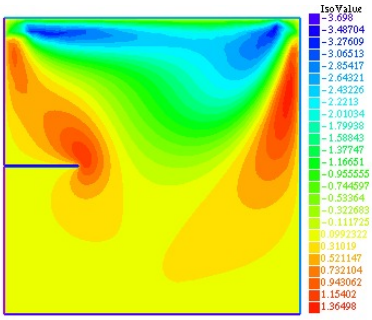


FIGURE 6.12: Kinetic Energy with varying Er



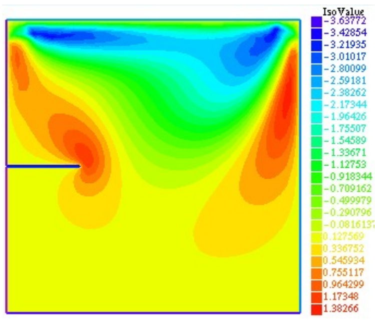
(a)

$Rf = 0.0$



(b)

$Rf = 0.01$



(c)

$Rf = 0.02$

FIGURE 6.13: Magnitude of micro-rotations are increasing in the presence of relaxation time

Chapter 7

Conclusion

This study focuses on understanding the rheology of complex fluids, which is crucial for modeling various industrial applications. To achieve this, a nonlinear partial differential equation (PDE) model incorporating fluid relaxation time is presented and analyzed numerically. The primary goal is to investigate the influence of fluid relaxation time on the dynamics of viscoelastic flows. A two-dimensional lid-driven baffled cavity serves as a model test problem for this analysis. The study employs the characteristic Galerkin finite element method for numerical computations. A custom code developed using the FreeFem++ platform is implemented to perform the analysis. Some of the main findings of this study are as follows:

- In the laminar flow regime, increasing the relaxation time leads to an increase in the mass flow rate. However, this trend reverses in the transitional flow regime, where an increase in relaxation time results in a decrease in the mass flow rate.
- Increasing the fluid relaxation time leads to an increase in the kinetic energy of the system of particles in flow. This finding aligns with observations from previous studies, where an increase in swimmer speed was observed with increasing fluid relaxation time. Relaxation time is a crucial parameter in viscoelastic fluids. It represents the time required for the fluid to return to its equilibrium state after a deformation. Increasing relaxation time generally leads to an increase in the fluid's elasticity.

- The analysis of drag forces is critical in various engineering applications, such as aerodynamics and hydrodynamics. Understanding how relaxation time affects drag forces in viscoelastic fluids can have significant implications for the design and optimization of systems involving these fluids.
- The drag force at the bottom surface decreases as the relaxation time increases in both laminar and transitional flow regimes.
- The drag force exhibits non-monotone behavior at low Reynolds numbers. However, it decreases with increasing relaxation time as the flow approaches the transitional regime.
- The minimum stream value decreases with increasing relaxation time.
- For $Re = 200$, ψ_{max} increases monotonically with increasing relaxation time.
- For $Re = 1000$, the relationship between ψ_{max} and relaxation time becomes non-monotone. Furthermore, at $Re = 1000$, the maximum value of ψ_{max} reaches $R_f = 2.0x10^3$ for the chosen relaxation times.

7.1 Future Research Directions

Some of the future research directions that can be studied in the context of present study are as follows:

- Investigating the sensitivity of the flow characteristics to variations in other parameters, such as Reynolds number, baffle geometry, and fluid properties.
- Exploring strategies for controlling and optimizing flow characteristics in viscoelastic fluids using techniques such as active flow control or shape optimization.

By continuing to investigate the effects of relaxation time on viscoelastic flows, researchers can gain a deeper understanding of these complex fluids and develop more accurate and efficient models for their behavior in various applications.

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