CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

An Unsteady Squeezing Casson Fluid Flow Under the Effects of Darcy Number

by

Ayesha Begum

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*First of all, I dedicate this research project to **Allah** Almighty, The most merciful and beneficent, creator and Sustainer of the earth*

And

*Dedicated to **Prophet Muhammad (peace be upon him)** whom, the world where we live and breathe owes its existence to his blessings*

And

*Dedicated to my beloved **Parents**, **Husband**, and **Siblings**, who pray for me and always pave the way to success for me*

And

*Dedicated to my **Teachers**, who are a persistent source of inspiration and encouragement for me*

CERTIFICATE OF APPROVAL

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All the praise and appreciation are for almighty **ALLAH** who is the most beneficent and the most merciful created the universe and blessed the mankind with Intelligence and wisdom to explore His secrets. Countless respect and endurance for **Prophet Muhammad (Peace Be Upon Him)**, the fortune of knowledge, who took the humanity out of ignorance and shows the right path.

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Abstract

This thesis is focused on the analysis of an unsteady squeezing Casson fluid flow between two parallel plates under the effect of Darcy Number. Further the effect of magnetic parameter, the magnetic inclination angle, the squeeze number, the lower plate stretching parameter, the lower suction/injection parameter and Eckert number on the velocity and temperature are also investigated. The non-linear partial differential equations are transformed into ordinary differential equations by using similarity transformation. Shooting method is adopted for solving the set of ordinary differential equations, with the help of computational software MATLAB. It is observed that the velocity profile is an increasing function of the Darcy number. An enhancement in the temperature profile is observed due to a rise in the inclination of the magnetic field

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Abbreviations

IVP	Initial value problem
MHD	Magnetohydrodynamics
ODEs	Ordinary differential equation
PDEs	Partial differential equations
RK	Runge-Kutta

Symbols

u	Velocity in x-direction
v	Velocity in y-direction
C_f	Skin Friction Coefficient
Nu	Nusselt Number
Pr	Prandtl Number
Ec	Eckert Number
S	Squeezing Number
γ	Chemical Reaction Parameter
T	Temperature
P	Pressure
ρ	Fluid Density
ν	Kinematic Viscosity
μ	Fluid Viscosity
κ	Thermal Conductivity
C_p	Specific Heat
$h(t)$	Distance Between two Plates
T_H	Temperature of the Upper Plate
T_o	Temperature of the Lower Plate
θ	Dimensionless Temperature
σ	Electrical Conductivity
M	Magnetic Parameter
R	Lower-Plate Stretching Parameter
S_b	Lower-Plate suction/injection Parameter

B	Magnetic Field
Da	Darcy Number
β	Casson Fluid Parameter

Chapter 1

Introduction

1.1 Background

Fluid is helping as a cause of life for human beings and human beings have always interest for locating nature and fluid is an important factor of the world, so it attracts human. Archimedes was the first who explored statics on fluid, lightness and compose his well known rule known as the Archimedes principle, which came out in his work. On floating bodies normally considered to be the first major work on fluid mechanics [1]. In the fifteenth century rapid development in fluid mechanics started. Da Vinci Leonardo acknowledged this field by observing and recording the situation that we realize today as a fundamental law of physics; namely the law of mass conservation. In this view, Da Vinci was the first person who took the task of making outline of different fields of flow.

Flow property of fluid between two parallel plates have attracted many research interests. This is due to their several applications in engineering such as food refinement, administration model, contraction, cooling water and so on. Duwairi et al. [2] reported the impact of squeezing parameters on the rate of heat conduction of the squeezed thick fluid in the center of two similar plates. They observed that the rate of hotness conduction increases and the local coefficient of friction decelerates by increasing the squeezing parameter, whereas the heat transport charge decreases and the skin resistance volumes enhanced by an increment in the extrusion parameter. Heat conduction and unsteady motion of nano-fluid by a smoothly moving plate was examine by Ahmadi et al. [3]. It is discussed that the unsteady variable perform the main role on the velocity profile which means that velocity profile is compliment by increasing unsteady parameter.

In heat shifting process, nano-sized particles are added in base fluid to get better thermic properties. A nano-fluid consists nano meter-sized particles called nano-particles. Water, gasoline, ethylene glycol etc. are common base fluids. Nano-fluid enhance the heat transmitting rate of the base fluid. Nada [4] numerically investigate the heat transfer using different types of nano-fluid. It was noticed that the nano-particles having high thermal conductivity outside the recirculation zones have more enhancement in Nusselt number. Khan et al. [5] developed estimated solutions under viscous dissipation control and velocity slip for the gripping flow of nano-fluid.

Squeezing flow explains the motion of a droplet of material. Squeezing flow has many applications in science i.e, rheological testing, composite material joining etc. Bhatta et al. [6] observed the unsteady squeezing nano-fluid flow based on water between two disks held parallel to each other. The unstable MHD gripping Eyring-Powell fluid flow across an infinite channel was examined in Adesanya et al. [7]. They concluded that the profile of the concentration decreases with respect to the parameter of chemical reaction. It concluded that transfering heat rate increases by expanding the thermal radiation and channel walls than the transfering heat rate decreases with in the heat absorption. Farooq et al. [8] presented the effect of melting transfering heat over a Darcy pore media in the pressing nano-fluid flux. Hayat et al. [9] reviewed the similar solution of an incompressible squeezing micro-polar fluid flow between two disks held parallel to each other along with magnetic effect.

Magnetohydrodynamics deals with the behaviour of electrically conducted fluids and magnetic properties, such as salt water, liquid metals, electrolytes and plasmas. Gholinia et al. [10] analysed the different internal effects like slip and magnetic field on Eyring-Powell fluid along with the reactions due to rotating disk and conclude that temperature profile is decreased by increasing the Prandtal number. Hayat et al. [11] observed the magnetic effect on an unsteady 2D second-degree fluid flow between two parallel disks. Jha and Aina [12] computed an approximate solution for MHD flow of an incompressible which fluid is viscous and to perform electrically in a vertical micro-porous channel developed by electrically non-conducting vertical plates held parallel to each other in the presence of induced magnetic effect. They noted that the fluid velocity and slip velocity is enhanced with the effect of suction/injection parameter. It was also observed that the volume flow rate is reduced by an increase in magnetic parameter and Hartmann number. Khan et al. [13] reviewed the transfering heat in the nano-fluid flow between two plates held parallel to each other along with the magnetic effect. They noted that the shape factor doesn't affect the velocity of the fluid. Siddiqui et al. [14] computed the solution by using Homotopy perturbation method of an unsteady 2D squeezing MHD fluid flow between two plates held parallel to each other.

Casson Fluid is a non-Newtonian fluid, first discovered by Casson [15, 16] in 1959. It is a shear-thinning liquid of infinite viscosity at zero shear rate, a yield stress under which the infinite shear rate does not move and zero viscosity. It has ability to capture complex rheological properties of a fluid, unlike other simplified models such as the power law [17] and second, third of fourth-grade models [18]. In fact, the Casson flow model more precisely explains the flow properties of blood at low shear levels and it passes through small blood cells [19]. So, human blood can also be treated as a Casson fluid in the presence of several substances such as fibrinogen, globulin in aqueous base plasma, protein, and human blood cells. On the other hand, the flow behaviours of the Casson fluid in the presence of magnetic field and heat transfer is also an important research area. Therefore, Khalid et al. [20] focused on the unsteady flow of a Casson fluid past an oscillating vertical plate with constant wall temperature under the non-slip conditions. Application of Casson fluid flow between two rotating cylinders is performed in [21]. The effect of magnetohydrodynamic (MHD) Casson fluid flow in a lateral direction past linear stretching sheet was explained by Nadeem et al. [22].

A literature survey has also reported that an evolved variant is recognized as Brinkman Forchheimer, an expanded Darcy model. The combined convection flow in a transparent vertical channel with heat sources at the walls was concluded numerically Chen and Hadim [23]. With the impact of porosity, as Darcy number decreases, Nusselt number is increases in the vertical flow. Similar model for porous enclosed geometry with nano-particles in the liquid observed by Muthtamilselvan and Sureshkumar [24] recently considered in 2016. Conclusions are made to find the heat transfer rate effect. Nu decreases for a small Darcy amount whereas porosity is fixed because smaller value of Darcy number decreases the flow conductance with the permeability in the fluid. Naqarajan and Akbar [25] investigated the computational simulation of mixed convection with nano-particles in a square filled enclosure bisecting the travelling plate held in the centre. Higher transfering heat rate concludes for low Darcy number due to greater porosity impact on energy and momentum equation. Likewise Kumar and Gupta investigated transfering heat and flow in non-Darcy porous media in [26, 27]. In addition, Chen et al. [28] also discussed the fluid movement that nano-particles performed in narrow pipeline.

1.2 Thesis Contributions

In this thesis, a detailed review of [29] is conducted and the results have been imitated by considering the additional impact of Darcy number and Casson fluid parameter. In this work, through an appropriate transformation, the governing PDEs are converted into the dimensionless ODEs. The numerical results are calculated by using the shooting technique. The impact of various physical parameters on the flow and heat conduction are also explained using the graphs.

1.3 Thesis Outlines

A brief overview of the content of the thesis is provided as:

In **Chapter 2**, we describe a few fundamental definitions and terminologies. Morever few basic laws and dimensionless physical parameters are also included.

In **Chapter 3**, includes a thorough analysis of [29] which considers the impact of magnetic field effects with suction/injection between parallel plates on the unsteady squeezing flow.

In **Chapter 4**, we extend the model given in [29] by considering the additional impact of Casson fluid parameter and Darcy number in momentum equation. The dimensionless ODEs are solved mathematically by method of shooting. Different physical parameters are illustrated using graphs.

In **Chapter 5**, we recapitulate the thesis and give the conclusion from the whole work and a proposal for the future work.

All the references used in this research work are listed in Bibliography.

Chapter 2

Fundamental Definitions and Governing Mathematical Statements

The purpose of this chapter is to state some simple concepts, regulatory rules and dimensional quantities that are useful for further discussion.

2.1 Important Definitions

Definition 2.1.1. [30]
"A fluid is a material which has the ability to flow. Further, fluids are categorised into liquids and gases. Liquids take the shape of the container while gases do not."

Definition 2.1.2. [31]
"Fluid mechanics is defined as the science that deals with the behaviour of fluids at rest or in motion and the interaction of fluids with solid or other fluids at the boundaries."

Definition 2.1.3. [30]
"Fluid static is the part of fluid mechanics, that deals with the fluid and its characteristics at the constant position."

Definition 2.1.4. [31]

"It is the study of the motion of liquids, gases and plasmas from one place to another. Fluid dynamics has a wide range of applications like calculating force and moments on aircraft, mass flow rate of petroleum passing through pipelines, prediction of weather, etc."

Definition 2.1.5. [30]

"The nano-particles used in nano-fluids are typically made of metals, oxides, copper, carbides or carbon nano-tubes."

Definition 2.1.6. [30]

"Another class of fluid that contains nanometre-sized particles known as nano-particles, typically made up of oxides, metals, carbon nano-tubes or carbides or carbon nano-tubes. These are the fluids in which nano-particles are suspended in the base fluid."

Definition 2.1.7. [32]

"Casson fluid can be defined as a shear thinning liquid which is assumed to have an infinite viscosity at zero rate of shear, a yield stress below which no flow occurs, and a zero viscosity at an infinite rate of shear."

Definition 2.1.8. [30]

"The porosity is the relationship of the volume of void space to the bulk volume of a permeable medium. A permeable medium is often identified by its porosity."

2.2 Physical Properties of Fluid

Definition 2.2.1. [30]

"This is the internal property of a fluid by virtue of which it offers resistance to the flow. Mathematically it is defined as the ratio of the shear stress to the rate of shear strain. i.e,

$$\mu = \frac{\text{shear stress}}{\text{shear strain}}, \tag{2.1}$$

In the above definition, μ is the coefficient of viscosity or absolute viscosity or dynamics viscosity or simply viscosity having dimension $[\frac{M}{LT}]$. Its unit is $Pa.s = \frac{kg}{(s.m)}$"

Definition 2.2.2. [30]

"The kinematic viscosity represents the ratio of dynamic viscosity μ to the density of the fluid ρ, Mathematically can be written as

$$\nu = \frac{\mu}{\rho}, \tag{2.2}$$

where ν is coefficient of viscosity. The dimension of kinematic viscosity is $[L^2T^{-1}]$ and its unit in SI system is m^2/s."

Definition 2.2.3. [30]

"Stress is a force acted upon a material per unit of its area and is denoted by τ. Mathematically, it can be written as:

$$\tau = \frac{F}{A},$$

where F denote the force and A denote the area."

Definition 2.2.4. [30]

"It is a type of stress in which the force vector acts parallel to the material surface or the cross section of a material."

Definition 2.2.5. [30]

"It is a type of stress in which the force vector acts perpendicular to the material surface or the cross section of a material."

2.3 Classification of Fluids

This section contains definition of different types of fluids.

Definition 2.3.1. [33]

"An ideal fluid is defined as the fluid which is incompressible and has no viscosity.

It is also called inviscid fluid.

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{2.3}$$

where τ_{yx} is shear stress."

Definition 2.3.2. [33]

"A fluid which is compressible in nature and contains some viscosity ($\mu > 0$) is said to be viscous fluid. As this fluid moves, certain amount of resistance is always offered by the fluid."

Definition 2.3.3. [33]

"The fluid for which the shear stress varies directly and linearly with the deformation rate is known as Newtonian fluid. Shear stress of Newtonian fluid is mathematically defined as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{2.4}$$

where τ_{yx} is shear stress and u denotes the x-component of velocity and μ denotes the dynamic viscosity." The common examples of Newtonian fluids are mercury, water, oxygen, gas and milk.

Definition 2.3.4. [33]

"The fluids for which the shear stress does not vary linearly with the deformation rate are known as Non-Newtonian Fluids. Mathematically, it can be expressed as

$$\tau_{xy} \propto \left(\frac{du}{dy}\right)^m, m \neq 1.$$

$$\Longrightarrow \tau_{xy} = \nu \left(\frac{du}{dy}\right), \tag{2.5}$$

where μ denotes the apparent viscosity and m is the index of the flow performance." The common examples are toothpaste, ketchup and blood.

2.4 Types of Flow

This section is dedicated to different types of flow.

Definition 2.4.1. [30]

"It is the deformation of the material under the influence of different forces. If the deformation increase is continuous without any limit then the process is known as flow."

Several types of flow are as follow:

Definition 2.4.2. [33]

"In fluid dynamics, laminar flow occurs when a flow is in parallel/closed channel or flat plates with no interruption between the plates. Typically, each particle has a definite path and the particles of the path in the fluid do not cross each other. Rising of cigarette smoke is an example of laminar flow."

Definition 2.4.3. [33]

"When the fluid undergoes irregular fluctuation or flowing faster, this type of flow (liquid or gas) is called turbulent flow. Turbulent flow moves randomly in any direction and has no definite path and cannot be handled easily."

Definition 2.4.4. [33]

"The flow in which fluid properties do not change with respect to time is called steady flow. Mathematically it can be written as

$$\frac{d\eta^*}{dt} = 0, \tag{2.6}$$

where η^* is a fluid property."

Definition 2.4.5. [33]

"The flow that continuously changes with respect to time, is expressed as unsteady flow. Mathematically this behaviour can be expressed as

$$\frac{d\eta^*}{dt} \neq 0, \tag{2.7}$$

where η^* is a fluid property."

Definition 2.4.6. [33]

"The flow in which the material density varies during fluid flow is said to be compressible flow. Compressible fluid flow is used in high-speed jet engines, aircraft, rocket motors also in high-speed usage in a planetary atmosphere, gas pipelines and in commercial fields. Mathematically, it is expressed as

$$\rho(x, y, z, t) \neq c, \tag{2.8}$$

where 'c' is a constant."

Definition 2.4.7. [33]

"A type of fluid flow in which material density during the flow remains constant is said to be incompressible flow. Mathematically, it can be expressed as

$$\rho(x, y, z, t) = c, \tag{2.9}$$

where 'c' is a constant."

Definition 2.4.8. [33]

"A flow, where the velocity of each fluid particle remains unchanged at any instant of time is called uniform flow. Mathematically, it can be written as

$$\frac{\partial V}{\partial s} = 0, \tag{2.10}$$

where V is the velocity and s is the displacement."

Definition 2.4.9. [33]

"A flow in which the velocity of fluid particles varies from point to point at a given instant of time is known as non-uniform flow. Mathematically, it is expressed as

$$\frac{\partial V}{\partial s} \neq 0, \tag{2.11}$$

where V is the velocity and s is the displacement in any direction."

Definition 2.4.10. [33]

"The flow which is not bounded by the solid surface, is known as external flow. The flow of water in the ocean or in the river is an example of the external flow."

Definition 2.4.11. [33]
"Fluid flow which is bounded by the solid surface. The examples of the internal flow are the flow through pipes or glass."

2.5 Heat Transfer Mechanism and related properties

This section provides different modes of heat transfer.

Definition 2.5.1. [30]
"It is the energy transfer due to the temperature difference. At the point when there is a temperature contrast in a medium or between media, heat transfer must take place. Heat transfer is normally in an object from high temperature to a lower temperature."

Definition 2.5.2. [30]
"Conduction is the process in which heat is transferred through the material between the objects that are in physical contact. For example: picking up a hot cup of tea."

Definition 2.5.3. [30]
"Convection is a mechanism in which heat is transferred through fluids (gases or liquids) from a hot place to a cool place. For example:

- Macaroni rising and falling in a pot of boiling water,
- Streaming cup of hot tea. The steam is showing heat transferred into the air."

Definition 2.5.4. [30]
"Forced convection is a process in which fluid motion is produced by an external source. It is a special type of heat transfer in which fluid moves in order to increase the heat transfer. In other words, a method of heat transfer in which heat transfer is caused by dependent source like a fan and pump etc, is called forced convection.

For example: gas convection heaters have a gas burner to generate the heat and fan to force the heated air to circulate around the room."

Definition 2.5.5. [30]

"Natural convection is a heat transport process, in which the heat transfer is not caused by an external source, like pump, fan and suction. It happens due to the temperature differences which affect the density of the fluid. It is also called free convection. For example: Daily weather."

Definition 2.5.6. [30]

"A method in which both forced and natural convection processes simultaneously and significantly involve in the heat transfer is called mixed convection."

Definition 2.5.7. [30]

"A process in which heat is transferred directly by electromagnetic waves is known as radiation and it occurs when two bodies of different temperature are aligned."

Definition 2.5.8. [31]

"A heat which is produced due to flow of current through conductor is called joule heating. Joule heating is also known as Ohmic heating and resistive heating."

Definition 2.5.9. [30]

"Thermal diffusivity is material's property which identifies the unsteady heat conduction. Mathematically, it can be written as,

$$\alpha = \frac{k}{\rho C_p}, \tag{2.12}$$

where k, ρ and C_p represents the thermal conductivity of material, the density and the specific heat capacity.

The SI units and dimension of thermal diffusivity are $m^2 s^{-1}$ and $[LT^{-1}]$ respectively."

Definition 2.5.10. [30]

"Thermal conductivity (k) is the property of a material related to its ability to transfer heat. Mathematically,

$$\frac{dQ}{dT} = -kA\frac{dT}{dx}, \tag{2.13}$$

where A, k, $\frac{dQ}{dT}$, $\frac{dT}{dx}$ are the area, the thermal conductivity, the rate of heat transfer and the temperature gradient respectively. With the increase of temperature, thermal conductivity of the most of the liquids decreases except water.

The SI unit of thermal conductivity is $\frac{Kg.m}{s^3.K}$ and its dimension is $[MLT^{-3}\theta^{-1}]$."

Definition 2.5.11. [30]

"The study of the dynamics of electrically conducting fluids for example plasmas or electrolytes, acted on by magnetic field is known as magnetohydrodynamics. It is denoted by (MHD)."

2.6 Dimensionless Numbers

Definition 2.6.1. [30]

"It is the ratio of the convective to the conductive heat transfer to the boundary. Mathematically,

$$Nu = \frac{hL}{k}, \tag{2.14}$$

where h stands for convective heat transfer, L for the characteristics length and k stands for the thermal conductivity."

Definition 2.6.2. [30]

"The ratio between the momentum diffusivity ν and thermal diffusivity α. Mathematically, it can be defined as

$$Pr = \frac{\nu}{\alpha}$$

$$\Longrightarrow Pr = \frac{\mu}{\rho}.\frac{\rho C_p}{k}$$

$$\Longrightarrow Pr = \frac{\mu C_p}{k}, \tag{2.15}$$

where μ represents the dynamic viscosity, C_p denotes the specific heat and k stands for thermal conductivity. The relative thickness of thermal and momentum boundary layer is controlled by Prandtl number."

Definition 2.6.3. [30]

"It is a dimensionless number which is used to clarify the different flow behaviours like turbulent or laminar flow. It helps to measure the ratio between inertial force and the viscous force. Mathematically,

$$Re = \frac{LU}{\nu}, \tag{2.16}$$

where U denotes the free stream velocity, L the characteristics length. At low Reynolds number, laminar flow arises where the viscous forces are dominant. At high Reynolds number, turbulent flow arises where the inertial forces are dominant."

Definition 2.6.4. [30]

"The Richardson number was first introduced by Lewis Fry Richardson as a dimensionless parameter that expresses the relationship of buoyancy term with flow shear term. It is denoted by Ri and mathematically it can be written as

$$Ri = \frac{Gr}{Re^2}, \tag{2.17}$$

where Gr represents the Grashof number and Re is the Reynolds number. Richardson number is used for weather forecast and in the investigation of density, lakes oceans and reservoirs."

Definition 2.6.5. [30]

"It is the dimensionless number used in continuum mechanics. It describes the relation between flows and the boundary layer enthalpy difference and it is used for characterized heat dissipation. Mathematically,

$$Ec = \frac{u^2}{C_p \bigtriangledown T}, \tag{2.18}$$

where u^2 is the characteristic flow velocity, C_p the specific heat and $\bigtriangledown T$ is the difference between wall temperature."

Definition 2.6.6. [30]

"The Darcy number Da represents the effect of the permeability of medium according to its cross sectional area.

$$Da = \frac{\kappa}{H^2}, \tag{2.19}$$

where κ shows the permeability of porous medium and H is the length of prescribed geometry. It was first introduced by Henry Darcy. It is transformed by the non-dimensionalizing differential form of Darcy's law."

2.7 Basic Equations

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Definition 2.7.1. [30]

"Mass conservation law states that fluid mass can neither be created nor destroyed. It is expressed mathematically for compressible fluids.

$$\frac{\partial \rho}{\partial t} + \bigtriangledown . \rho V = 0,$$

Where t is time, the fluid density is ρ, and the fluid velocity is v. When the fluid density is unchanged the shape is provided by consistency for incompressible liquids."

$$\bigtriangledown . V = 0$$

Definition 2.7.2. [30]

"The sum of a body's mass and velocity is called the linear momentum or simply the body's momentum, and the momentum of a solid body with mass m travelling at a velocity with V is mV Newton's second law notes that the motion of a body is equal to the total force operating on it and is inversely proportional to its speed, and that the rate of change in a body's momentum is identical to the rate of change. Therefore, if the total force acting on it is negative, the momentum of a device stays unchanged, and therefore the energy of it is conserved. This is regarded as the theory of the conservation of momentum."

Definition 2.7.3. [30]
"One of the most fundamental laws in nature is the first law of thermodynamics, also known as the conservation of energy principle. It states that energy can be neither created nor destroyed during a process, it can only change forms. In two dimensional system the energy equation for base fluid can be expressed as:"

$$v.\nabla T = \alpha \nabla^2 T \tag{2.20}$$

2.8 Solution Methodology

"Shooting method is used to solve the higher order non-linear ordinary differential equations. To implement this technique, we first convert the higher order ODEs to the system of first order ODEs. After that we assume the missing initial conditions and the differential equations are then integrated numerically using the Runge-Kutta method as an initial value problem. The accuracy of the assumed missing initial condition is then checked by comparing the calculated values of the dependent variables at the terminal point with their given value there. If the boundary conditions are not fulfilled up to the required accuracy, with the new set of initial conditions, then they are modified by Newtons method. The process is repeated again until the required accuracy is achieved. To explain the shooting method, we consider the following general second order boundary value problem:[34]

$$y''(x) = f(x, y, y'(x)) \tag{2.21}$$

along with the boundary conditions

$$y(0) = 0, \qquad y(L) = B. \tag{2.22}$$

To have a system of first order ODEs, used the notations:

$$y = y_1, \qquad y' = y_2. \tag{2.23}$$

By using the notations (2.23) in (2.21) and (2.22) can be written as

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= 0,\\ y_2' &= f(x, y_1, y_2), & y_1(L) &= B. \end{aligned}\right\} \tag{2.24}$$

Choose the missing initial condition $y_2(0) = h$ we have the following IVP

$$\left.\begin{aligned} y_1' &= y_2, & y_1(0) &= 0,\\ y_2' &= f(x, y_1, y_2), & y_2(0) &= h. \end{aligned}\right\} \tag{2.25}$$

Now, the initial value problem satisfy the boundary condition $y_2(L) = B$

$$y_1(L, h) - B = \phi(h) = 0. \tag{2.26}$$

To find an approximate root of (2.26) by the Newton's method, is written as

$$h_{n+1} = h_n - \frac{\phi h_n}{\phi' h_n}, \tag{2.27}$$

or

$$h_{n+1} = h_n - \frac{y_1(L, h_n) - B}{\frac{\partial}{\partial h}[y_1(L, h_n) - B}. \tag{2.28}$$

To implement the Newton's method, consider the following notations

$$\frac{\partial y_1}{\partial h} = y_3, \qquad \frac{\partial y_2}{\partial h} = y_4. \tag{2.29}$$

Differentiating Eq. (2.25) with respect to h we get the following four first order ODEs along wi

$$\left.\begin{aligned} y_3' &= y_4, & y_3(0) &= 0,\\ y_4' &= y_3\frac{\partial f}{\partial y_1} + y_4\frac{\partial f}{\partial y_2}, & y_4(0) &= 1. \end{aligned}\right\} \tag{2.30}$$

Now, solving the IVP (2.30), we get y_3 at L. This value is actually the derivative of y_1 with respect to h compute at L. Using the value of $y_3(L, h)$ in Eq. (2.28), the modified value of h can be achieved. This new value of h is used to solve the (2.30) and the process is repeated until the require accuracy."

Chapter 3

Magnetic Field Effects with Suction/Injection between Parallel Plates on the Unsteady Squeezing Flow

3.1 Introduction

In this chapter the detailed analysis of S. Xiaohong and Y. Yunxing is discussed [29]. The description of the empirical research of inclined magnetic field effects with suction/injection between parallel on the unsteady squeezing flow is reviewed in this study. By using suitable similarity transformations, the controlling partial differential equations are converted into ordinary differential equations. The mathematical solution for the differential equations are obtained by utilizing the shooting technique. Graphs are represented to show the physical significance of distinct dimensionless quantities. By varying the values of the different parameters, we observed the trend of the velocity and temperature distributions.

3.2 Mathematical Modeling

Considering the unsteady squeezing flow of an incompressible fluid that conducts electrically and is squeezed between two infinite parallel plates. The lower plate channel is along the x-axis, so it is normal to the y-axis. Here $B = (B_m \cos\gamma, B_m \sin\gamma, 0)$, is the time-variable magnetic field in which B_m denotes $B_o(1-\alpha t)^{\frac{-1}{2}}$. $H(t) = l(1-\alpha t)^{\frac{1}{2}}$ is the difference between the plates that varies with the time t, where l is the difference between plates [35]. Geometry of flow model is in FIGURE 3.1.

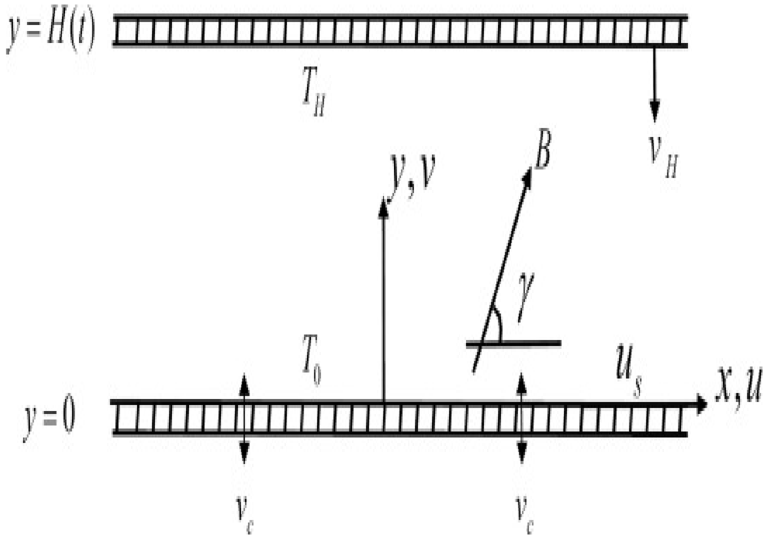

FIGURE 3.1: Geometry of the problem

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The flow is described by considering the continuity, momentum equation and energy equation are as follows:
Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{3.1}$$

Momentum equation for u-velocity:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) + (v\cos\gamma - u\sin\gamma) \sin\gamma\frac{\sigma B_m^2}{\rho}. \tag{3.2}$$

Momentum equation for v-velocity:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + u\sin\gamma - v\cos\gamma)$$
$$cos\gamma\;\frac{\sigma B_m^2}{\rho}\;. \tag{3.3}$$

Energy equation:

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\frac{k}{\rho c_p} + \Big[2\Big(\frac{\partial u}{\partial x}\Big)^2 + 2\Big(\frac{\partial v}{\partial y}\Big)^2 +$$
$$\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)^2\Big]\,\frac{\mu}{\rho c_p} + \frac{\sigma B_m^2}{\rho c_p}\,(u\sin\gamma - v\cos\gamma)^2. \tag{3.4}$$

Here u is fluid motion in the direction of x and v is fluid motion in the direction of y. The temperature is T, the total dynamic viscosity is ν, the density is ρ, the real heat capacity of the fluid is C_p, and the thermal conductivity of the fluid is κ respectively.

Boundary conditions of lower and upper plates are:

$$\left.\begin{aligned}
& u = 0, \\
& v = v_H = \frac{dH}{dt} = -\frac{\alpha l}{2\sqrt{1-\alpha t}}, \\
& T = T_H = T_o + \left(\frac{T_o}{1-\alpha t}\right) \;\; at\;\, y = H(t), \\
& u = u_s = \frac{bx}{1-\alpha t}\;, \\
& v = v_c = -\,\frac{v_o}{\sqrt{1-\alpha t}}\;, \\
& T = T_o \;\;\; at \;\;\; y = 0.
\end{aligned}\right\} \tag{3.5}$$

Here, u_s denotes lower-plate stretching velocity, v_c represents lower-plate mass flux velocity, v_H denotes upper-plate velocity, T_o is lower-plate surface temperature and T_H denotes upper-plate surface temperature.

Furthermore, similarity transformations are used to convert partial differential equations into set of ordinary differential equations for concluding the desired results.

By using the following dimensionless parameters. (3.1) - (3.4) are converted into the dimensionless form [36].

$$\left.\begin{aligned} v &= v_H f(\eta), \\ \eta &= \frac{y}{H(t)}, \\ u &= v_H f'(\eta)\frac{-x}{H(t)}, \\ \theta(\eta) &= \frac{T - T_o}{T_H - T_o}. \end{aligned}\right\} \tag{3.6}$$

3.3 Dimensionless Structure of the Governing Equations

Consider the continuity equation. By substituting (3.6) into (3.1):

$$\frac{\partial\Big(\frac{-x}{H(t)}\, v_H f'(\eta)\Big)}{\partial x} + \frac{\partial\Big(v_H f(\eta)\Big)}{\partial y} = 0$$

Substituting $v_H = \frac{dH}{dt} = -\frac{\alpha l}{2\sqrt{1-\alpha t}}$ and $\eta = \frac{y}{H(t)}$ in the above relation. We obtain,

$$\Rightarrow \frac{\alpha l}{2\sqrt{1-\alpha t}H(t)} f'(\eta) - \frac{\alpha l}{2\sqrt{1-\alpha t}H(t)} f'(\eta) = 0$$

Hence the continuity function is satisfied identically.

Now we include the following dimensionless parameters for the conversion of (3.2) into the dimensionless form.

- $$\begin{aligned} u &= \frac{-x}{H(t)} v_H f'(\eta) \\ &= \frac{-x\eta}{y}\Big(\frac{-\alpha l}{2\sqrt{1-\alpha t}}\Big) f'(\eta) \\ &= \frac{\alpha x}{2(1-\alpha t)}\, f'(\eta) \\ &\Big(\because \eta = \frac{y}{H(t)} \quad \text{and} \quad H(t) = l\sqrt{1-\alpha t}\Big) \end{aligned}$$

- $$\frac{\partial u}{\partial x} = \frac{\alpha}{2(1-\alpha t)}\, f'(\eta)$$

- $\dfrac{\partial^2 u}{\partial x^2} = 0$
- $\dfrac{\partial u}{\partial y} = \dfrac{\alpha x}{2l(1-\alpha t)^{\frac{3}{2}}} \, f''(\eta)$
- $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\alpha x}{2l^2(1-\alpha t)^2} \, f'''(\eta)$
- $\dfrac{\partial u}{\partial t} = \dfrac{\alpha^2 x}{2l(1-\alpha t)^2} \, f'(\eta) + \dfrac{\alpha^2 xy}{4l(1-\alpha t)^{\frac{5}{2}}} \, f''(\eta)$
- $\dfrac{\mu}{\rho} \, \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\alpha x}{2(1-\alpha t)^2} \left(\dfrac{\nu}{l^2} \right) f''' \quad \left(\because \nu = \dfrac{\mu}{\rho} \right)$
- $B_m^2 = B_o^2(1{-}\alpha t)$
- $$\begin{aligned} \frac{\sigma B_m^2}{\rho} \sin\gamma(v\cos\gamma - u\sin\gamma) &= \frac{\sigma B_m^2}{\rho} \sin\gamma \Big[\frac{-\alpha l}{2\sqrt{1-\alpha t}} \, f\cos\gamma \\ &\quad - \frac{\alpha x}{2(1-\alpha t)} \, f'\sin\gamma \Big] \\ &= \frac{\sigma B_o^2}{\rho(1-\alpha t)} \sin\gamma \Big[\frac{-\alpha l}{2\sqrt{1-\alpha t}} \, f\cos\gamma \\ &\quad - \frac{\alpha x}{2(1-\alpha t)} \, f'\sin\gamma \Big] \end{aligned}$$
- $\delta = \dfrac{H}{x} = \dfrac{l(1-\alpha t)^2}{x}$

The dimensionless form of (3.2) is:

$$\begin{aligned} &\Longrightarrow \frac{\alpha^2 x}{2l(1-\alpha t)^2} f'(\eta) + \frac{\alpha^2 xy}{4l(1-\alpha t)^{\frac{5}{2}}} f''(\eta) + f'^2(\eta) \frac{-\alpha^2 x}{4(1-\alpha t)^2} + f(\eta)f''(\eta) \\ &\qquad \frac{\alpha^2 x}{2(1-\alpha t)^2} = -\frac{1}{\rho} \, \frac{\partial \rho}{\partial x} + \Big[\frac{\nu}{l^2} + \frac{\sigma B_o^2}{\rho} \sin\gamma(\delta f\cos\gamma + f'\sin\gamma) \Big] \, \frac{\alpha x}{2(1-\alpha t)^2} \\ &\Longrightarrow \Big[f' + \frac{y}{2l(1-\alpha t)^{\frac{1}{2}}} f'' + \frac{1}{2} f'^2 - \frac{1}{2} f f'' \Big] \frac{\alpha^2 x}{2(1-\alpha t)^2} = -\frac{1}{\rho} \, \frac{\partial \rho}{\partial x} \\ &\qquad + \frac{\alpha x}{2(1-\alpha t)^2} \, \Big[\frac{\nu}{l^2} + \frac{\sigma B_o^2}{\rho} \sin\gamma(\delta f\cos\gamma + f'\sin\gamma) \Big] \end{aligned} \tag{3.7}$$

Differentiate (3.7) w.r.t y, we get:

$$\Longrightarrow \frac{\alpha^2 x}{4(1-\alpha t)^2} \, \frac{1}{l(1-\alpha t)^{\frac{1}{2}}} \, [3f'' + \eta f''' + f'f'' - ff'''] = -\frac{1}{\rho} \, \frac{\partial^2 \rho}{\partial x \partial y}$$

$$+\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\ \frac{\nu}{l^3}\left[f'''' - \frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma + f''\sin\gamma)\right]$$

$$\Longrightarrow \frac{\alpha^2 x}{2(1-\alpha t)^2}\ \frac{1}{2l(1-\alpha t)^{\frac{1}{2}}}\ [3f'' + \eta f''' + f'f'' - ff'''] = -\frac{1}{\rho}\ \frac{\partial^2\rho}{\partial x\partial y}$$

$$+\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\ \frac{\nu}{l^3}\left[f'''' - \frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma + f''\sin\gamma)\right]$$

$$\Longrightarrow \frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}\ [3f'' + \eta f''' + f'f'' - ff'''] + \frac{1}{\rho}\ \frac{\partial^2\rho}{\partial x\partial y} -$$

$$\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\ \frac{\nu}{l^3}\left[\frac{\nu}{l^2}\ f'''' + \frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma + f''\sin\gamma)\right] = 0 \qquad (3.8)$$

Now we include the following derivatives for the conversion of momentum equation (3.3) into the dimensionless form.

$$\frac{\partial v}{\partial x} = 0 \quad \left(\because v = v_H = -\left(\frac{\alpha l}{2\sqrt{1-\alpha t}}\right) f(\eta)\right)$$

$$\frac{\partial^2 v}{\partial x^2} = 0$$

$$\frac{\partial v}{\partial t} = \frac{-\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}}\ f(\eta) - \frac{\alpha l}{2\sqrt{1-\alpha t}}\ \frac{\partial\eta}{\partial t} f'(\eta)$$

$$\frac{\partial v}{\partial y} = \frac{-\alpha}{2(1-\alpha t)}\ f'(\eta)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{-\alpha l}{2\sqrt{1-\alpha t}}\ \frac{\partial^2\eta}{\partial y^2}\ f'(\eta) - \frac{\alpha l}{2\sqrt{1-\alpha t}}\left(\frac{\partial\eta}{\partial y}\right)^2 f''(\eta)$$

The dimensionless form of (3.3) can be written as

$$\Rightarrow \quad \frac{-\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}}\ f - \frac{\alpha l}{2(1-\alpha t)^{\frac{1}{2}}}\ \frac{\partial\eta}{\partial t}\ f' + \frac{\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}}\ ff') = -\frac{1}{\rho}\ \frac{\partial p}{\partial y} +$$
$$\nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\sigma B_o^2}{2\rho(1-\alpha t)}\cos\gamma\left(\frac{\alpha x}{1-\alpha t}\ f'(\eta)\sin\gamma + \frac{\alpha l}{\sqrt{1-\alpha t}}\right.$$
$$\left.\cos\gamma f\right) \qquad (3.9)$$

Differentiating (3.9) w.r.t x, we get:

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{-\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}}\ f - \frac{\alpha l}{2(1-\alpha t)^{\frac{1}{2}}}\ \frac{\partial\eta}{\partial t}\ f' + \frac{\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}}\ ff'\right) =$$

$$
\begin{aligned}
&-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\Big[\frac{1}{\rho}+\nu\Big(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\Big)+\frac{\sigma B_o^2}{2\rho(1-\alpha t)}\cos\gamma\Big(\frac{\alpha x}{1-\alpha t}f'(\eta)\sin\gamma\\
&+\frac{\alpha l}{\sqrt{1-\alpha t}}\cos\gamma f\Big)\Big]\\
\Rightarrow & 0=-\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f'\\
\Rightarrow & -\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}+\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f'=0\\
\Rightarrow & \frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}=\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f' \qquad (3.10)
\end{aligned}
$$

Substituting (3.10) in (3.8) we get,

$$
\begin{aligned}
&\frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}[3f''+\eta f'''+f'f''-ff''']+\frac{1}{\rho}\frac{\partial^2\rho}{\partial x\partial y}-\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\frac{\nu}{l^3}\Big[f''''\\
&+\frac{-\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)\Big]=\frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}[3f''+\eta f'''+f'f''\\
&-ff''']+\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f'-\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\frac{\nu}{l^3}\Big[f''''-\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma\\
&(\delta f'\cos\gamma+f''\sin\gamma)\Big] \qquad (3.11)\\
\Longrightarrow & \frac{\alpha}{2}(3f''+\eta f'''+f'f''-ff''')-\frac{\nu}{l^2}\Big[f''''-\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)\Big]\\
&+\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f'\Big(\frac{2(1-\alpha t)^{\frac{5}{2}}}{\alpha x}\Big)=0\\
\Longrightarrow & \frac{\alpha}{2}(3f''+\eta f'''+f'f''-ff''')-\frac{\nu}{l^2}\Big[f''''-\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)\Big]\\
&+\frac{\sigma B_o^2 l\sqrt{1-\alpha t}}{\rho x}\alpha cos\gamma\sin\gamma f'=0\\
\Longrightarrow & \frac{\alpha}{2}(3f''+\eta f'''+f'f''-ff''')-\frac{\nu}{l^2}\Big[f''''-\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)\Big]\\
&+\frac{\sigma B_o^2\delta}{\rho}\cos\gamma\sin\gamma f'=0\\
\Longrightarrow & \frac{\alpha l^2}{2\nu}(3f''+\eta f'''-ff'''+f'f'')-\frac{\nu l^2}{l^2\nu}\Big[f''''-\frac{\sigma l^2 B_o^2 l^2}{\rho\nu}(\delta\cos\gamma f'+\sin\gamma f'')\\
&\sin\gamma\Big]+\frac{\sigma B_o^2 l^2\delta}{\rho\nu}\cos\gamma\sin\gamma f'=0\\
\Longrightarrow & \frac{\alpha l^2}{2\nu}(3f''+\eta f'''-ff'''+f'f'')-\frac{\sigma B_o^2 l^2}{\rho\nu}(\delta f'\cos\gamma+\sin\gamma f'')\sin\gamma-f''''\\
&+\frac{\sigma B_o^2 l^2\delta}{\rho\nu}\cos\gamma sin\gamma f'=0
\end{aligned}
$$

where $M^2 = \frac{\sigma B_o^2 l^2}{\rho\nu}$, $S = \frac{\alpha l^2}{2\nu}$ and $\delta = \frac{H}{x} = \frac{l(1-\alpha t)^2}{x}$.

$$\Longrightarrow S(3f'' + \eta f''' - f''f + f'''f') - f'''' + M^2\delta \sin\gamma f' \cos\gamma + M^2 \sin\gamma f'' + M^2 \sin\gamma \cos\gamma f' = 0$$

$$\Longrightarrow f'''' - S(3f'' - f'''f + \eta f''' + f''f') - \sin\gamma(2\delta \cos\gamma f' + \sin\gamma f'')M^2 = 0 \tag{3.12}$$

For the conversion of temperature equation (3.4) into an ordinary differential equation, following derivatives are evaluated:

$$\frac{\partial v}{\partial y} = \frac{-\alpha}{2(1-\alpha t)} f'(\eta)$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\alpha}{2(1-\alpha t)} f'(\eta)$$

$$\frac{\partial u}{\partial y} = \frac{\alpha x}{2l(1-\alpha t)^{\frac{3}{2}}} f''(\eta)$$

$$T = T_H = T_o + \left(\frac{T_o}{1-\alpha t}\right) \quad at \quad y = H(t)$$

$$\theta(\eta) = \frac{T - T_o}{T_H - T_o}$$

$$T = T_o\left(1 + \frac{1}{1-\alpha t}\,\theta(\eta)\right)$$

$$\frac{\partial T}{\partial t} = \frac{\alpha T_o}{(1-\alpha t)^2}\,\theta + \frac{\alpha T_o y}{2l(1-\alpha t)^{\frac{5}{2}}}\,\theta'$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial^2 T}{\partial x^2} = 0$$

$$\frac{\partial T}{\partial y} = \frac{T_o}{l(1-\alpha t)^{\frac{3}{2}}}\,\theta'$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_o}{l^2(1-\alpha t)^2}\,\theta''$$

$$\frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} = \frac{k}{\rho c_p}\left(0+\frac{T_o}{l^2(1-\alpha t)^2}\,\theta''\right) = \frac{1}{c_p(1-\alpha t)^2}\,\frac{kT_o\theta''}{\rho l^2}$$

$$\begin{aligned}\frac{\mu}{\rho c_p}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right] &= \frac{\nu}{c_p}\left[2\,\frac{\alpha^2}{4(1-\alpha t)^2}\,f'^2(\eta)\right.\\ \left.+2\,\frac{\alpha^2}{4(1-\alpha t)^2}\,f'^2(\eta)+\left(\frac{\alpha x}{2l(1-\alpha t)^{\frac{3}{2}}}\,f''(\eta)+0\right)^2\right] &= \frac{\nu}{c_p}\left[\frac{\alpha^2}{(1-\alpha t)^2}\left(f'^2\right.\right.\\ \left.\left.+\frac{x^2}{4l^2(1-\alpha t)}\,f''^2\right)\right] &= \frac{1}{c_p(1-\alpha t)}\left[\frac{\nu\alpha^2x^2}{4l^2(1-\alpha t)}\left(4\delta^2 f'^2+f''^2\right)\right]\end{aligned}$$

$$\begin{aligned}\frac{\sigma B_m^2}{\rho c_p}\,(u\sin\gamma - v\cos\gamma)^2 &= \frac{\sigma B_o^2}{c_p\rho(1-\alpha t)}\left[\frac{\alpha x}{2(1-\alpha t)}\,f'(\eta)sin\gamma\right.\\ \left.+\frac{\alpha l}{2\sqrt{1-\alpha t}}\cos\gamma f\right]^2 &= \frac{1}{c_p(1-\alpha t)}\,\frac{\sigma B_o^2}{\rho}\,\frac{\alpha^2x^2}{4(1-\alpha t)^2}\left[f'^2\sin^2\gamma\right.\\ &\left.+\frac{l^2(1-\alpha t)^2}{x^2}f^2\cos^2\gamma+\frac{2l\sqrt{1-\alpha t}}{x}\,ff'\cos\gamma\sin\gamma\right]\end{aligned}$$

Now consider the L.H.S of equation (3.4),

$$\begin{aligned}\Longrightarrow \frac{\partial T}{\partial t}+u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y} &= \frac{\alpha T_o}{(1-\alpha t)^2}\,\theta+\frac{\alpha T_o y}{2l(1-\alpha t)^{\frac{5}{2}}}\,\theta'+u(0)\\ &= -\left(\frac{\alpha l}{2\sqrt{1-\alpha t}}\right)f(\eta)\left[\frac{T_o}{l(1-\alpha t)^{\frac{3}{2}}}\,\theta'\right]\\ \Longrightarrow \frac{\partial T}{\partial t}+u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y} &= \frac{\alpha T_o}{2(1-\alpha t)^2}\,(2\theta+\eta\theta')+\frac{\alpha T_o}{2(1-\alpha t)^2}\,(-f\theta')\\ \Longrightarrow \frac{\partial T}{\partial t}+u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y} &= \frac{\alpha T_o}{2(1-\alpha t)^2}\,(2\theta+\eta\theta'-f\theta')\end{aligned} \tag{3.13}$$

R.H.S of equation (3.4) is;

$$\begin{aligned}&\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu}{\rho c_p}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]+\frac{\sigma B_m^2}{\rho c_p}\\ &(u\sin\gamma-v\cos\gamma)^2 = \frac{1}{c_p(1-\alpha t)^2}\,\frac{kT_o\theta''}{\rho l^2}+\frac{1}{c_p(1-\alpha t)}\left[\frac{\nu\alpha^2x^2}{4l^2(1-\alpha t)}\right.\\ &\left.(4\delta^2 f'^2+f''^2)\right]+\frac{1}{c_p(1-\alpha t)}\,\frac{\sigma B_o^2}{\rho}\,\frac{\alpha^2x^2}{4(1-\alpha t)^2}\left[f'^2\sin^2\gamma+\frac{l^2(1-\alpha t)^2}{x^2}\right.\\ &\left.f^2\cos^2\gamma+\frac{2l\sqrt{1-\alpha t}}{x}\,ff'\cos\gamma\sin\gamma\right]\end{aligned}$$

where $\quad H = l\sqrt{1-\alpha t}, \qquad H^2 = l^2(1-\alpha t) \quad$ and $\quad \delta = \dfrac{H}{x}$.

Hence

$$\Longrightarrow \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{\mu}{\rho c_p}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]$$
$$+\frac{\sigma B_m^2}{\rho c_p}(u\sin\gamma - v\cos\gamma)^2 = \frac{1}{c_p(1-\alpha t)^2}\left[\frac{kT_o}{\rho l^2\theta''}+\frac{\nu\alpha^2x^2}{4l^2(1-\alpha t)}(4\delta^2 f'^2\right.$$
$$\left.+f''^2)+\frac{\sigma B_o^2\alpha^2x^2}{4\rho(1-\alpha t)}(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f'\cos\gamma\sin\gamma)\right] \tag{3.14}$$

Combining (3.13)and (3.14);

$$\Longrightarrow \frac{\alpha T_o}{2(1-\alpha t)^2}[2\theta+\eta\theta'-f\theta'] = \frac{1}{c_p(1-\alpha t)^2}\left[\frac{kT_o\theta'}{\rho l^2}+\frac{\nu\alpha^2x^2}{4l^2(1-\alpha t)}\right.$$
$$\left.(4\delta^2 f'^2+f''^2)+\frac{\sigma B_o^2\alpha^2x^2}{4\rho(1-\alpha t)}(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f'\cos\gamma\sin\gamma)\right]$$
$$\Longrightarrow \frac{\alpha T_o}{2}[2\theta+\eta\theta'-f\theta'] = \frac{1}{c_p}\left[\frac{kT_o}{\rho l^2\theta''}+\frac{\nu\alpha^2x^2}{4l^2(1-\alpha t)}(4\delta^2 f'^2+f''^2)\right.$$
$$\left.+\frac{\sigma B_o^2\alpha^2x^2}{4\rho(1-\alpha t)}(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f'\cos\gamma\sin\gamma)\right]$$
$$\Longrightarrow \frac{\alpha T_o}{2}[2\theta+\eta\theta'-f\theta'] = \frac{\nu}{c_p l^2}\left[\frac{kT_o\theta''}{\rho\nu}+\frac{\alpha^2x^2}{4(1-\alpha t)}(4\delta^2 f'^2+f''^2)\right.$$
$$\left.+\frac{\sigma B_o^2 l^2\alpha^2x^2}{4\rho\nu(1-\alpha t)}(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f' cos\gamma\sin\gamma)\right]$$
$$\Longrightarrow \frac{\alpha T_o}{2}[2\theta+\eta\theta'-f\theta'] = \frac{\nu}{c_p l^2}\left[\frac{kT_o\theta''}{\rho\nu}+\frac{\alpha^2x^2}{4(1-\alpha t)}(4\delta^2 f'^2+f''^2)\right.$$
$$\left.+M^2\frac{\alpha^2x^2}{4(1-\alpha t)}(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f'\cos\gamma\sin\gamma)\right]$$
$$\Longrightarrow \frac{\alpha T_o}{2}[2\theta+\eta\theta'-f\theta'] = \frac{\nu}{c_p l^2}\left[\frac{kT_o\theta''}{\rho\nu}+\frac{\alpha^2x^2}{4(1-\alpha t)}(4\delta^2 f'^2+f''^2)\right.$$
$$\left.+\frac{\sigma B_o^2 l^2\alpha^2x^2}{4\rho\nu(1-\alpha t)}(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f'\cos\gamma\sin\gamma)\right]$$
$$\Longrightarrow \frac{\alpha T_o}{2}[2\theta+\eta\theta'-f\theta'] = \frac{\nu}{c_p l^2}\frac{\alpha^2x^2}{4(1-\alpha t)}\left[\frac{4kT_o(1-\alpha t)\theta''}{\rho\nu\alpha^2x^2}+(4\delta^2 f'^2+f''^2)\right.$$
$$\left.+M^2(f'^2\sin^2\gamma+\delta^2 f^2\cos^2\gamma+2\delta f f'\cos\gamma\sin\gamma)\right]$$
$$\Longrightarrow \frac{c_p l^2}{\nu}\frac{4(1-\alpha t)}{\alpha^2x^2}\frac{\alpha T_o}{2}[f\theta'-\eta\theta'-2\theta]+\left[\frac{4kT_o(1-\alpha t)\theta''}{\rho\nu\alpha^2x^2}+(4\delta^2 f'^2+f''^2)\right.$$

$$\Longrightarrow \frac{c_p l^2}{\nu}\ \frac{4(1-\alpha t)}{\alpha^2 x^2}\ \frac{\alpha T_o}{2}\ [f\theta' - \eta\theta' - 2\theta] + \Big[\ \frac{4kT_o(1-\alpha t)\theta''}{\rho\nu\alpha^2 x^2} + (4\delta^2 f'^2 + f''^2)$$
$$+ M^2(f'^2 \sin^2\gamma + \delta^2 f^2 \cos^2\gamma + 2\delta f f' \cos\gamma \sin\gamma)\Big] = 0$$
$$\Longrightarrow \frac{2c_p T_o H^2}{\nu x^2 \alpha}\ [f\theta' - \eta\theta' - 2\theta] + \frac{4kT_o(1-\alpha t)}{\rho\nu\alpha^2 x^2}\ \Big[\theta'' + \frac{\rho\nu\alpha^2 x^2}{4kT_o(1-\alpha t)}\ (4\delta^2 f'^2$$
$$+ f''^2) + M^2(f'^2 \sin^2\gamma + \delta^2 f^2 \cos^2\gamma + 2\delta f f' \cos\gamma \sin\gamma)\Big] = 0$$
$$\Longrightarrow \frac{2c_p T_o \rho\nu x^2 l^2(1-\alpha t)}{4kT_o \nu x^2 \alpha(1-\alpha t)}\ [f\theta' - \eta\theta' - 2\theta] + \frac{4kT_o(1-\alpha t)}{\rho\nu\alpha^2 x^2}\ \Big[\theta'' + \frac{\rho\nu\alpha^2 x^2}{4kT_o(1-\alpha t)}$$
$$(4\delta^2 f'^2 + f''^2) + M^2(f'^2 \sin^2\gamma + \delta^2 f^2 \cos^2\gamma + 2\delta f f' \cos\gamma \sin\gamma)\Big] = 0$$
$$\Longrightarrow \frac{\mu c_p \rho l^2}{2\rho\nu k}\ [f\theta' - \eta\theta' - 2\theta] + \theta'' + \Big[\ \frac{\rho\nu\alpha^2 x^2}{4kT_o(1-\alpha t)}\ (4\delta^2 f'^2 + f''^2) + M^2(f'^2$$
$$\sin^2\gamma + \delta^2 f^2 \cos^2\gamma + 2\delta f f' \cos\gamma \sin\gamma)\Big] = 0$$
$$\Longrightarrow P_r S[f\theta' - \eta\theta' - 2\theta] + \frac{\rho\nu\alpha^2 x^2}{4kT_o(1-\alpha t)}\ (4\delta^2 f'^2 + f''^2) + \theta'' + \big(f'^2 \sin^2\gamma$$
$$+ \delta^2 f^2 \cos^2\gamma + 2\delta f f' \cos\gamma \sin\gamma\big)M^2 = 0 \tag{3.15}$$

where;

$$Pr = \frac{\mu c_p}{k}$$

$$S = \frac{\alpha l^2}{2\nu}$$

$$Ec = \frac{u_o^2}{c_p R^2 (T_H - T_o)}$$

$$R = \frac{u_s \delta}{v_H}$$

$$u_s = \frac{bx}{1-\alpha t}$$

$$v_H = \frac{-\alpha l}{2\sqrt{1-\alpha t}}$$

$$T_H - T_o = \frac{T_o}{1-\alpha t}$$

$$R = \frac{u_s \delta}{v_H}$$

$$T_o = (T_H - T_o)(1 - \alpha t)$$

$$S_b = \frac{2v_o}{\alpha l}$$

$$\begin{aligned}
\frac{\mu\alpha^2x^2c_p}{4kc_pT_o(1-\alpha t)} &= \frac{\mu c_p}{k}\frac{\alpha^2x^2}{4c_pT_o(1-\alpha t)} \\
&= Pr\ \frac{\alpha^2x^2u_s^2\delta^2}{4c_pT_o(1-\alpha t)R^2v_H^2} \\
&= \frac{b^2x^2Pr}{\alpha c_pT_o(1-\alpha t)R^2} \\
&= \frac{b^2x^2Pr}{\alpha c_p(T_H-T_o)(1-\alpha t)^2R^2} \\
&= \frac{Pr}{c_p(T_H-T_o)R^2}\ \frac{b^2x^2}{(1-\alpha t)^2} \\
&= \frac{Pru_o^2}{c_p(T_H-T_o)R^2} \\
&= PrEc
\end{aligned}$$

Collectively the following equation is obtained:

$$PrS[f\theta'-\eta\theta'-2\theta]+\frac{\rho\mu\alpha^2x^2c_p}{4k\rho c_pT_o(1-\alpha t)}(4\delta^2f'^2+f''^2)+\theta''+M^2(f'^2\sin^2\gamma$$
$$+\delta^2f^2\cos^2\gamma+2\delta ff'\cos\gamma\sin\gamma)=0$$

$\Longrightarrow$

$$PrS(f\theta'-\eta\theta'-2\theta)+PrEc(4\delta^2f'^2+f''^2)+\theta''+M^2(f'^2\sin^2\gamma+\delta^2f^2\cos^2\gamma$$
$$+2\delta ff'\cos\gamma\sin\gamma)=0 \tag{3.16}$$

Finally, two ordinary differential equations are obtained with the following system:

$$f''''-S\big(3f''+f'''\eta+f'f''-ff'''\big)-\sin\gamma\big(2\delta\cos\gamma f'+\sin\gamma f''\big)M^2=0 \tag{3.17}$$

$$\Longrightarrow PrS\big(f\theta'-\eta\theta'-2\theta\big)+PrEc\big(4\delta^2f'^2+f''^2\big)+\theta''+M^2\big(f'^2\sin^2\gamma$$
$$+\delta^2f^2\cos^2\gamma+2\delta ff'\cos\gamma\sin\gamma\big)=0, \tag{3.18}$$

Subject to the Boundary conditions:

$$f(0)=S_b,\quad f'(0)=R,\quad \theta(0)=0,\ f(1)=1,\quad f'(1)=0,\quad \theta(1)=1. \tag{3.19}$$

where the squeezing number is S, the prandtl number is Pr, the magnetic parameter is M, the Eckert number is Ec and the lower-plate stretching parameter is R. S_b reflects a function of the lower-plate suction/injuction with $S_b < 0$ for damage and $S_b > 0$ for suction.

The following formulation is available for various parameters used in the above equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\alpha x}{2l(1-\alpha t)^{\frac{3}{2}}} f''(\eta), & \frac{\partial T}{\partial y} &= \frac{T_o}{l(1-\alpha t)^{\frac{3}{2}}} \theta', \\ v_H &= \frac{-\alpha l}{2\sqrt{1-\alpha t}}, & T_H - T_O &= \frac{T_o}{(1-\alpha t)}, \\ Re_x &= \frac{u_s x}{\nu}, & u_s &= \frac{bx}{(1-\alpha t)}. \end{aligned}\right\} \tag{3.20}$$

Before going towards the mathematical solution the skin friction coefficient C_f or the shear stress and the Nusselt number Nu or heat transfer coefficient on the lower plate surface are represented as:

$$C_f = \frac{\mu\left(\frac{\partial u}{\partial y}\right)_{y=H(t)}}{v_H^2 \rho}$$

$$Nu = \frac{l\left(\frac{\partial T}{\partial y}\right)_{y=H(t)}}{T_H - T_o}$$

As for equation (3.6), we have:

$$C_f^* = \frac{\alpha l^3 (1-\alpha t)^{\frac{3}{2}}}{bx^3} Re_x C_f = f''(1),$$

$$Nu^* = \left(\frac{\nu}{b}\right)^{\frac{1}{2}} x^{-1} (Re_x)^{\frac{-1}{2}} Nu = \theta'(1).$$

where $Re_x = \frac{u_s x}{\nu}$ represents the local Reynolds number.

$$\Longrightarrow C_f = \frac{\mu(\frac{\partial u}{\partial y})_{y=H(t)}}{\rho v_H^2}, \quad N_u = \frac{1}{T_H - T_O}\left(\frac{\partial T}{\partial y}\right)_{y=H(t)} \tag{3.21}$$

The skin friction coefficient, is given as follows:

$$\Rightarrow \quad C_f = -\frac{\mu x v_H}{\rho v_H^2 l^2(1-\alpha t)} f''(\eta) = -\frac{\mu x}{\rho v_H l^2(1-\alpha t)} f''(\eta)$$

$$\Rightarrow \quad C_f = -\frac{\mu x}{\rho \left(\frac{-\alpha l}{2\sqrt{1-\alpha t}}\right) l^2(1-\alpha t)} f''(\eta)$$

$$\Rightarrow \quad C_f = \frac{2\mu x}{\rho \alpha l^3 \sqrt{1-\alpha t}} f''(\eta)$$

$$\Rightarrow \quad f''(\eta) = \frac{\rho \alpha l^3 \sqrt{1-\alpha t}}{\mu x} C_f, \quad \text{where} \quad y = H(t) \quad \text{and} \quad \eta = 1.$$

$$\Rightarrow \quad f''(1) = \frac{\rho \alpha l^3 \sqrt{1-\alpha t}}{\nu \rho x} C_f = \frac{\alpha l^3 \sqrt{1-\alpha t}}{\frac{u_s x}{Re_x} x} C_f$$

$$\Rightarrow \quad f''(1) = \frac{\alpha l^3 \sqrt{1-\alpha t}}{u_s x^2} Re_x C_f$$

$$\Rightarrow \quad f''(1) = \frac{\alpha l^3 \sqrt{1-\alpha t}}{\frac{bx}{(1-\alpha t)} x^2} Re_x C_f$$

$$\Rightarrow \quad f''(1) = \frac{\alpha l^3 (1-\alpha t)^{\frac{3}{2}}}{b x^3} Re_x C_f$$

The local Nusselt number is defined as follows:

- $$Nu = \frac{1}{T_H - T_O} \left(\frac{\partial T}{\partial y}\right)_{y=H(t)}$$

$$\Rightarrow \quad Nu = \frac{1}{T_H - T_O} \frac{T_o}{l(1-\alpha t)^{\frac{3}{2}}} \theta'(\eta)$$

$$\Rightarrow \quad Nu = \frac{1}{T_H - T_O} (T_H - T_O) \frac{1}{l(1-\alpha t)^{\frac{1}{2}}} \theta'(\eta)$$

$$\Rightarrow \quad Nu = \frac{1}{l(1-\alpha t)^{\frac{1}{2}}} \theta'(\eta),$$

where $\quad y = H(t) \quad$ and $\quad \eta = 1.$

$$\Rightarrow \quad \theta'(1) = l(1-\alpha t)^{\frac{1}{2}} Nu$$

$$\Rightarrow \quad \theta'(1) = \frac{l(bx)^{\frac{1}{2}}}{(u_s)^{\frac{1}{2}}} Nu (1-\alpha t)^{\frac{1}{2}}$$

$$\Rightarrow \quad \theta'(1) = \frac{(bx)^{\frac{1}{2}}}{(u_s)^{\frac{1}{2}}} Nu$$

$$\Rightarrow \quad \theta'(1) = \frac{l(bx)^{\frac{1}{2}}}{(\frac{Re_x \nu}{x})^{\frac{1}{2}}} Nu$$

3.4 Numerical Treatment

This section is dedicated to the implementation of the shooting method to solve the transformed ODEs (3.17) and (3.18) subject to the Boundary Conditions (3.19). One can easily observe that (3.18) is independent of variable θ, hence (3.17) is first solved by using the shooting technique. For this purpose, following notations are used:

$$f = y, \quad f' = y', \quad f'' = y'', \quad f''' = y''', \quad f'''' = y''''.$$

Further denote

$$y = y_1, \quad y' = y_1' = y_2, \quad y'' = y_2' = y_3, \quad y''' = y_3' = y_4, \quad y'''' = y_4'.$$

Equations are,

$$\left.\begin{aligned}
&y_1' = y_2; && y_1(0) = S_b = \frac{2v_o}{\alpha l}\ ,\\
&y_2' = y_3; && y_2(0) = R = \frac{u_s\delta}{v_H}\ ,\\
&y_3' = y_4; && y_3(0) = \alpha_1,\\
&y_4' = S(3y_3 + \eta y_4 + y_2y_3 - y_1y_4) + M^2 sin\gamma(2\delta cos\gamma y_2 + sin\gamma y_2); &&\\
& && y_4(0) = \alpha_2.
\end{aligned}\right\} \tag{3.22}$$

In the above system of equations the missing conditions α_1 and α_2 are to be chosen such that:

$$y_4(\eta, \alpha_1, \alpha_2)_{\eta=1} - 1 = 0,$$
$$y_5(\eta, \alpha_1, \alpha_2)_{\eta=1} - 1 = 0.$$

Now

$$y_3(0) = y''(0) = \alpha_1, \quad y_4(0) = y'''(0) = \alpha_2.$$

The Newton method is used to solve algebraic equations system and has the following iterative scheme:

$$\begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = \begin{pmatrix} u^{n} \\ v^{n} \end{pmatrix} - \begin{pmatrix} \frac{\partial y_1}{\partial \alpha_1} & \frac{\partial y_1}{\partial \alpha_2} \\ \frac{\partial y_2}{\partial \alpha_1} & \frac{\partial y_2}{\partial \alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} y_1(1) - 1 \\ y_2(1) - 0 \end{pmatrix} \tag{3.23}$$

Further use the notations:

$$\frac{\partial y_1}{\partial \alpha_1} = y_5, \quad \frac{\partial y_2}{\partial \alpha_1} = y_6, \quad \frac{\partial y_3}{\partial \alpha_1} = y_7, \quad \frac{\partial y_4}{\partial \alpha_1} = y_8,$$
$$\frac{\partial y_1}{\partial \alpha_2} = y_9, \quad \frac{\partial y_2}{\partial \alpha_2} = y_{10}, \quad \frac{\partial y_3}{\partial \alpha_2} = y_{11}, \quad \frac{\partial y_4}{\partial \alpha_2} = y_{12}.$$

As the result of these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = \begin{pmatrix} u^{n} \\ v^{n} \end{pmatrix} - \begin{pmatrix} y_5 & y_9 \\ y_6 & y_{10} \end{pmatrix}^{-1} \begin{pmatrix} y_1(1) - 1 \\ y_2(1) - 0 \end{pmatrix} \tag{3.24}$$

Now differentiate the above system of four first order ODEs (3.22) with respect to each of the variables α_1 and α_2 to have another system of eight ODEs . Writing all these twelve ODEs together, the following IVP has:

$$\begin{aligned}
y_1' &= y_2; & y_1(0) &= S_b = \frac{2v_o}{\alpha l}, \\
y_2' &= y_3; & y_2(0) &= R = \frac{u_s \delta}{v_H}, \\
y_3' &= y_4; & y_3(0) &= \alpha_1, \\
y_4' &= S(3y_3 + \eta y_4 + y_2 y_3 - y_1 y_4) + M^2 sin\gamma & & \\
&\quad (2\delta cos\gamma y_2 + sin\gamma y_2); & y_4(0) &= \alpha_2, \\
y_5' &= y_6; & y_5(0) &= 0, \\
y_6' &= y_7; & y_6(0) &= 0, \\
y_7' &= y_8; & y_7(0) &= 0, \\
y_8' &= S(3y_7 + \eta y_8 + y_5 y_3 - y_5 y_4 + y_2 y_7 - y_1 y_8) & & \\
&\quad + M^2 \sin\gamma (2\delta \cos\gamma y_6 + \sin\gamma y_7); & y_8(0) &= 0,
\end{aligned}$$

$$
\begin{aligned}
y_9' &= y_{10}; & y_9(0) &= 0,\\
y_{10}' &= y_{11}; & y_{10}(0) &= 0,\\
y_{11}' &= y_{12}; & y_{11}(0) &= 0,\\
y_{12}' &= S(3y_{11} + \eta y_{12} + y_{10}y_3 - y_9y_4 + y_2y_{11} - y_1y_{12}) & &\\
&\quad + M^2 \sin\gamma(2\delta\cos\gamma y_{10} + \sin\gamma y_{11}); & y_{12}(0) &= 0.
\end{aligned}
$$

The fourth order Runge-Kutta procedure is used to solve the above system of twelve equations with α_1 and α_2 initial guess. Such estimates are modified by the scheme of the Newton. The iterative method is performed before the conditions here are met:

$$max\big[\mid {\alpha_1}^{n+1} - {\alpha_1}^{n} \mid, \quad \mid {\alpha_2}^{n+1} - {\alpha_2}^{n} \mid\big] < \epsilon$$

for an arbitrarily small positive value of ϵ . Throughout this chapter ϵ has been taken as $(10)^{-6}$

Since (3.17) and (3.18) are coupled equations. So (3.18) will be solved seperately by incorporating the solution of (3.17). For this purpose let us denote:

$$y_1 = \theta, \qquad y_1' = y_{13} = \theta', \qquad y_2' = y_{14} = \theta''.$$

to get the following first order ODEs.

$$
\left.
\begin{aligned}
y_{13}' &= y_2; & y_{13}(0) &= 0,\\
y_{14}' &= -\big[PrS(fy_2 - \eta y_2 - 2y_1) + PrEc\big(f''^2 + 4\delta^2 f'^2 + & &\\
&\quad M^2(f'^2\sin^2\gamma + f^2\delta^2\cos^2\gamma + 2\delta f f'\cos\gamma\sin\gamma)\big)\big]; & y_{14}(0) &= m.
\end{aligned}
\right\}
\tag{3.25}
$$

The above IVP is solved numerically by Runge-Kutta strategy of the fourth order. In the above initial value problem, the missing condition m is to be chosen such that:

$$y_{13}(\eta, m)_{\eta=1} - 1 = 0, \tag{3.26}$$

To solve the above algebraic equation (3.26) the Newton's method is used which has the following iterative scheme:

$$m^{n+1} = m^n - \left(\frac{\partial y_{13}}{\partial m}\right)^{-1} \big(y_{13}(\eta, m^n)_{\eta=1} - 1\big).$$

Further considering the following derivatives:

$$\frac{\partial y_{13}}{\partial m} = y_{15}, \qquad \frac{\partial y_{14}}{\partial m} = y_{16}.$$

to formulate the following Newton's iterative scheme:

$$m^{n+1} = m^n - \big[y_{15}(\eta, m^n)_{\eta=1}\big]^{-1} y_{13}(\eta, m^n)_{\eta=1} - 1. \tag{3.27}$$

Here n is the number of iterations (n = 0,1,2,3...).
Now differentiate the above system of two first order ODEs (3.26) with respect to m to have another system of four ODEs . Writing all these four ODEs together, the following IVP has:

$$\begin{aligned}
&y'_{13} = y_2; && y_{13}(0) = 0,\\
&y'_{14} = -PrS(D_1 y_2 - \eta y_2 - 2y_1) - PrEc\big[D_3^2 + 4\delta^2 D_2^2 \\
&\quad + M^2(D^2 - 2\sin^2\gamma + D_1^2\delta^2\cos^2\gamma + 2\delta D_1 D_2 \cos\gamma \sin\gamma)\big]; && y_{14}(0) = m,\\
&y'_{15} = y_4; && y_{15}(0) = 0,\\
&y'_{16} = -PrS(D_1 y_4 - \eta y_4 - 2y_3); && y_{16}(0) = 1.
\end{aligned}$$

The RK-4 method has been used to solve the IVP consisting of the above four ODEs for some suitable choices of m. The missing condition m is updated by using Newtons scheme (3.27). If the following criterion is fulfilled the iterative process is stopped:

$$|\ m^{n+1} - r^m\ | < \epsilon$$

for an arbitrarily small positive value of ϵ. Throughout this chapter ϵ was taken as $(10)^{-6}$.

3.5 Results with discussion

In this section, the numerical results are displayed graphically to perceive the physical properties of flow more transparently. The variation in the velocity and temperature profiles are represented in graphs below, by varying the parameter of the lower-plate stretching, the angle of inclination, the magnetic parameter, the squeeze number, the Eckert number and the parameter of the lower-plate suction/injuction.

Figure 3.2 and Figure 3.3 displays the influence of the squeeze number S for the temperature $\theta(\eta)$ and velocity profile $f'(\eta)$. FIGURE 3.2 shows the influence of S on the profile of velocity. Remember that the velocity of the fluid decreases by growing the squeezing parameter values. Figure 3.3 indicates that decreased in S causes a decrease in fluid temperature across parallel plates.

Figure 3.4 and Figure 3.5 demonstrate the velocity and temperature profile of the fluid with different magnetic parameter values. From Figure 3.4, it is noticed that a particular time, a rising magnetic parameter causes the fluid's velocity to increase in regions similar to the upper or lower plates, while the fluid's velocity in the central region indicates an obvious decrease. The fluid in the central area has a higher velocity relative to the viscous fluid plates and it is found from Figure 3.5 that temperature rises with magnetic parameter change. Moreover, the temperature of the fluid decreases from the lower to the upper plate area while the parameter of the magnetic field is small. The fluid temperature offers maximum values for larger magnetic parameter values not on the upper plate area but in the middle area between the walls. However, stronger magnetic fields naturally influence the temperature distribution.

Figure 3.6 and Figure 3.7 represent the velocity and temperature behaviours by rising the inclination angle of the magnetic field applied, respectively. The angles of magnetic inclination vary between 0 and $\pi/2$. Related velocity and temperature patterns were obtained from the two estimates as applied to the corresponding velocity and temperature profiles with specific magnetic parameter values. The influencing of the inclination angle on both fluid temperature and velocity profile is close to the magnetic parameters.

Figure 3.8 and Figure 3.9 illustrate the impact on dimensionless temperature of the lower-surface and velocity of stretching parameter, respectively. Figure 3.8 indicates that the fluid velocity near the lower plate rises in order to raise the magnitude of the lower-plate stretching function while the velocity of the fluid near the upper-plate falls with the fluid. Since the parameter for lower-plate stretching is increasing gradually, the fluid with the velocity of maximum value does not appear in the central area between the plates, but on the lower-plate side. Figure 3.9 shows that the increasing lower-plate stretching velocity is initially decreases fluid temperature below the lower-plate.

Figure 3.10 and Figure 3.11 depicts the lower-plate suction/injection affect on temperature profile and velocity profile, respectively. Figure 3.10 indicates that for the lower-plate injection/suction function, the velocity profile is rising. The fluid with peak velocity will not occur in the central area when extending the lower-plate for better suction across the lower-plate and the velocity of the fluid decreases from the lower-plate area to the upper-plate. Temperature profiles decrease as the injection/suction parameter increases. In particular, it is found out that when the injection/suction parameter S_b falls in the central area, the fluid has high temperature could not occur at the upper-plate area between the two plates.

In Figure 3.12 by the increment of Eckert number their is rise in temperature profile. It is obvious the temperature rises to raise Eckert number values. Joule heating and viscous dissipation is due to Ec, which significantly increases the fluid temperature between two surfaces. Figure 3.12 also indicates that the maximum fluid temperature for the larger Eckert number occurs in the central area between the two plates and for the smaller Eckert number it exists on the upper plate side.

In Figure 3.13 and Figure 3.14 the results of the magnetic field's inclination angle, the squeeze number on the skin friction coefficient and the Nusselt number are displayed. It is noticed that the Nusselt number is decreasing function of inclination angle of magnetic field and on the contrary to this total value of skin friction coefficient raises by the effect of γ.

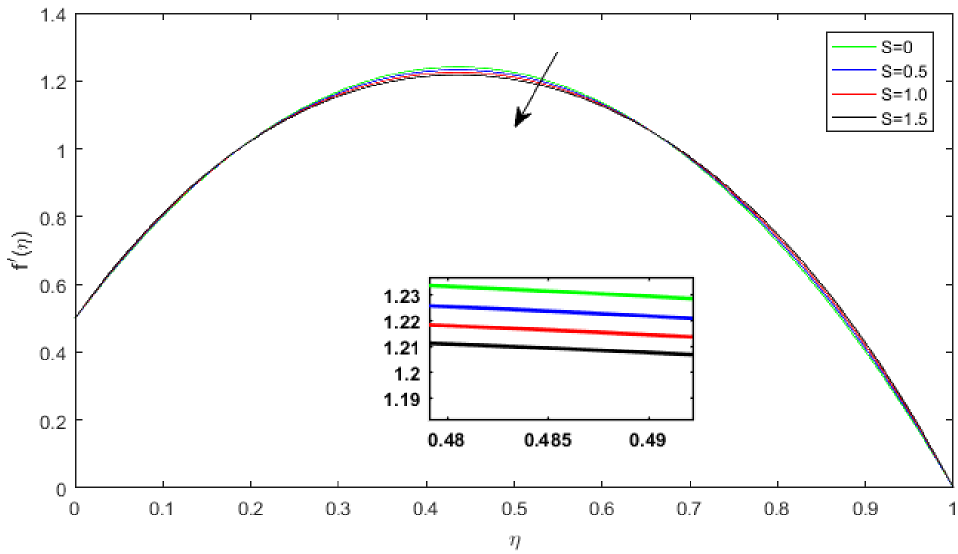

FIGURE 3.2: Influence of S on $f'(\eta)$. when
$R = M = 0.5,\ Pr = 1.0,\ \gamma = \frac{\pi}{6}$ and $Ec = \delta = S_b = 0.1$

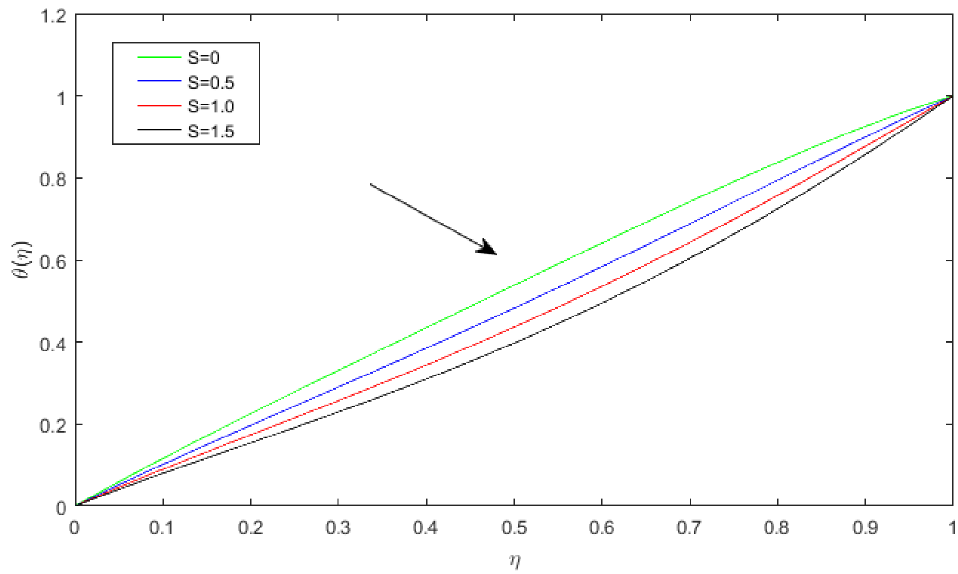

FIGURE 3.3: Impact of S on $\theta(\eta)$. when
$R = M = 0.5,\ Pr = 1.0,\ \gamma = \frac{\pi}{6}$ and $Ec = \delta = S_b = 0.1$

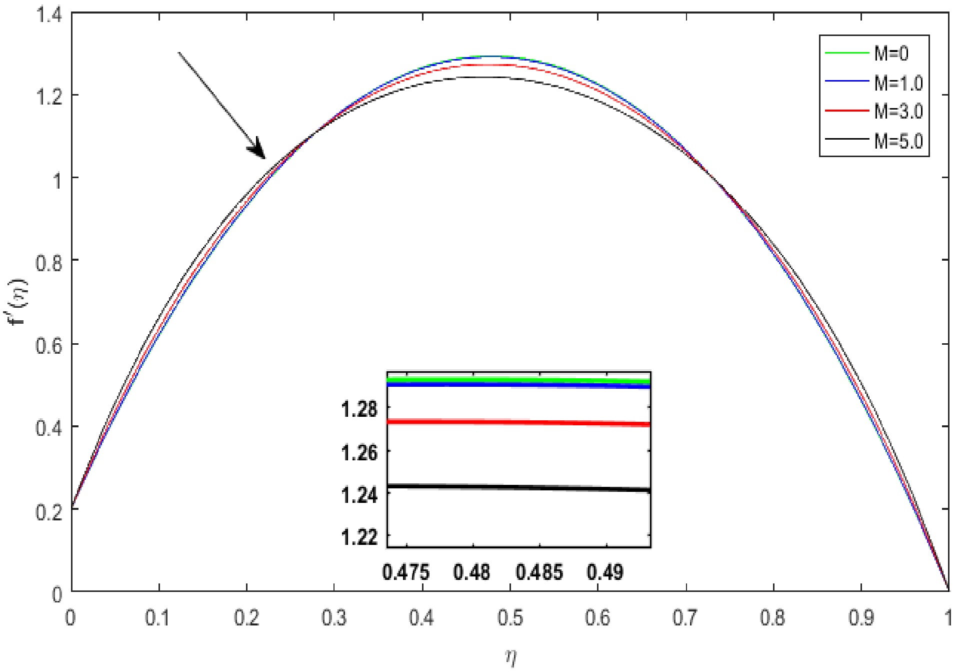

FIGURE 3.4: Impact of M on $f'(\eta)$. when
$R = 0.2$, $S = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.3$ and $\delta = S_b = 0.1$

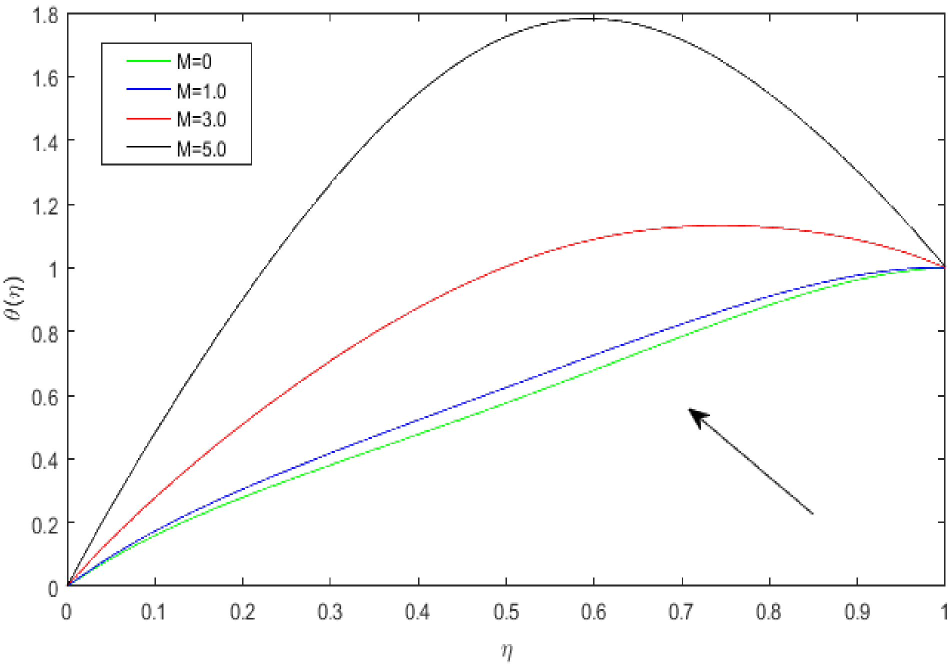

FIGURE 3.5: Impact of M on $\theta(\eta)$. when
$R = 0.2$, $S = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.3$ and $\delta = S_b = 0.1$

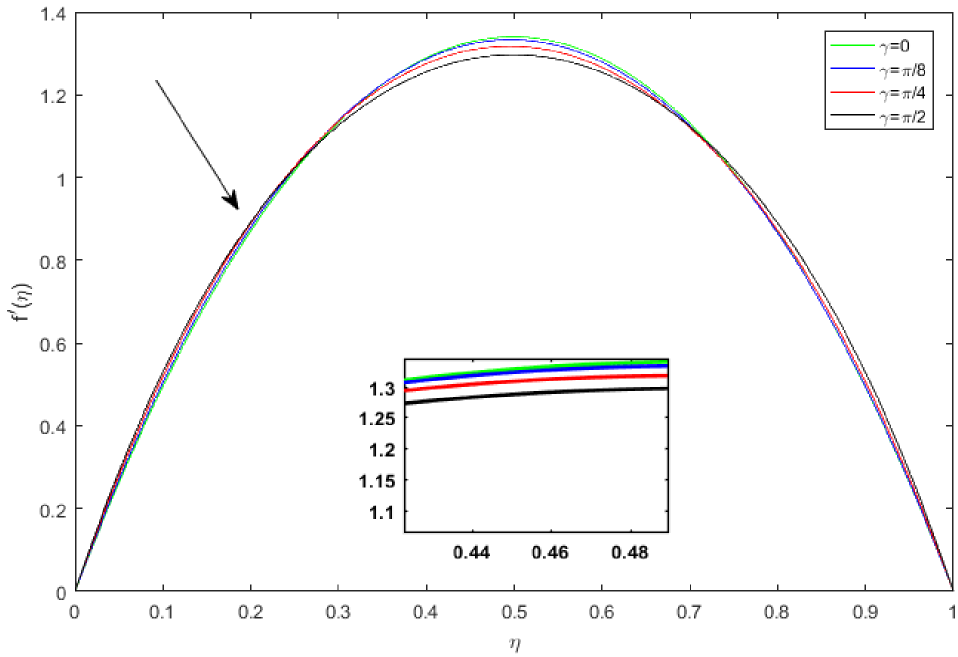

FIGURE 3.6: Impact of γ on $f'(\eta)$. when
$R = 0$, $S = 0.5$, $Pr = 1.0$, $M = 3.0$, $Ec = 0.3$ and $\delta = S_b = 0.1$

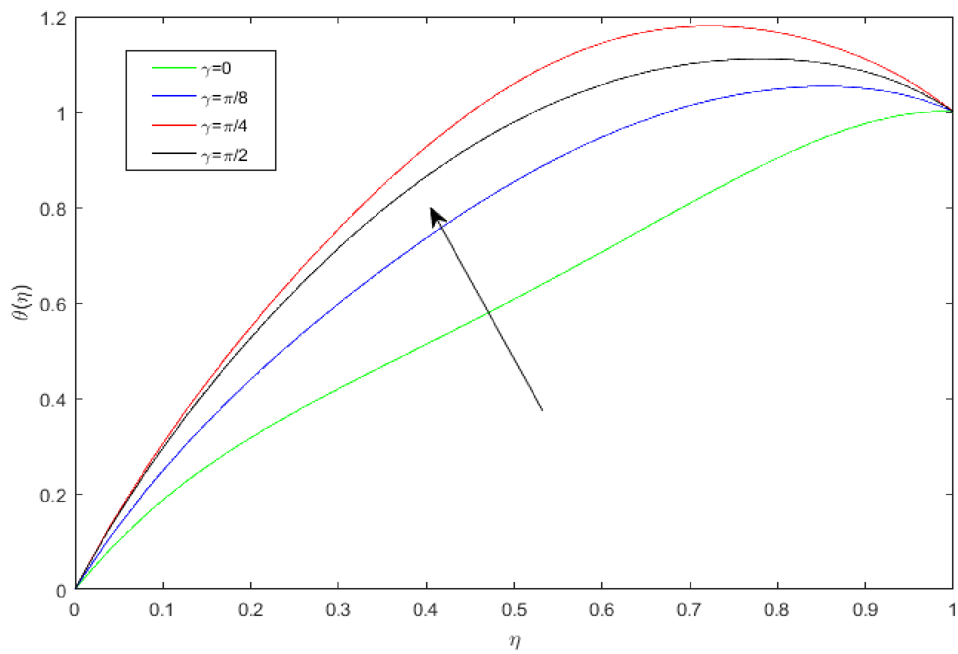

FIGURE 3.7: Impact of γ on $\theta(\eta)$. when
$R = 0$, $S = 0.5$, $Pr = 1.0$, $M = 3.0$, $Ec = 0.3$ and $\delta = S_b = 0.1$

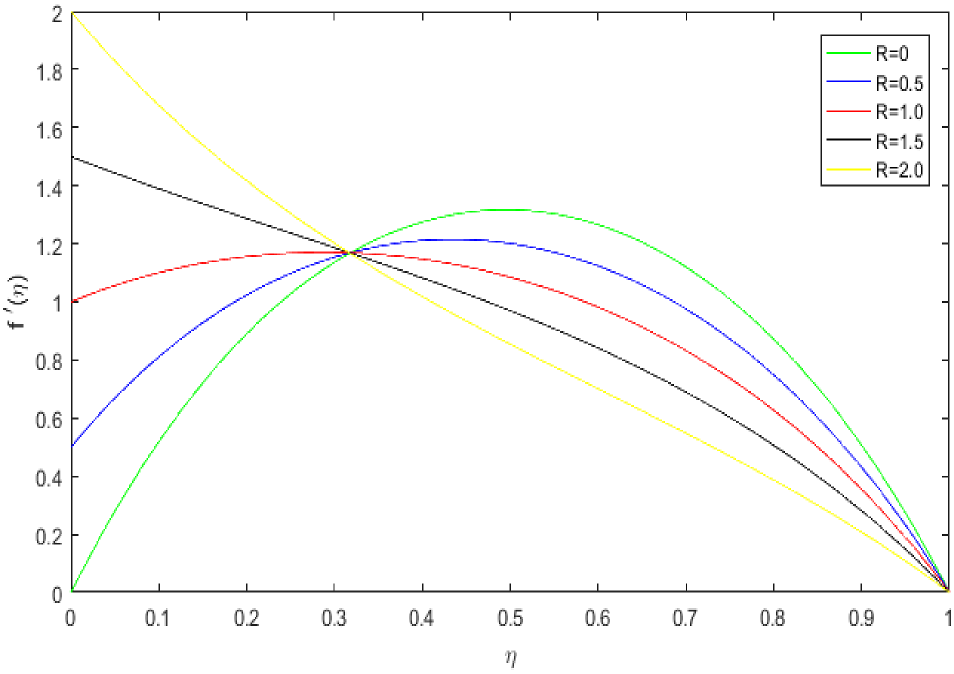

FIGURE 3.8: Impact of R on $f'(\eta)$. when
$S = 0.5$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.6$ and $\delta = S_b = 0.1$

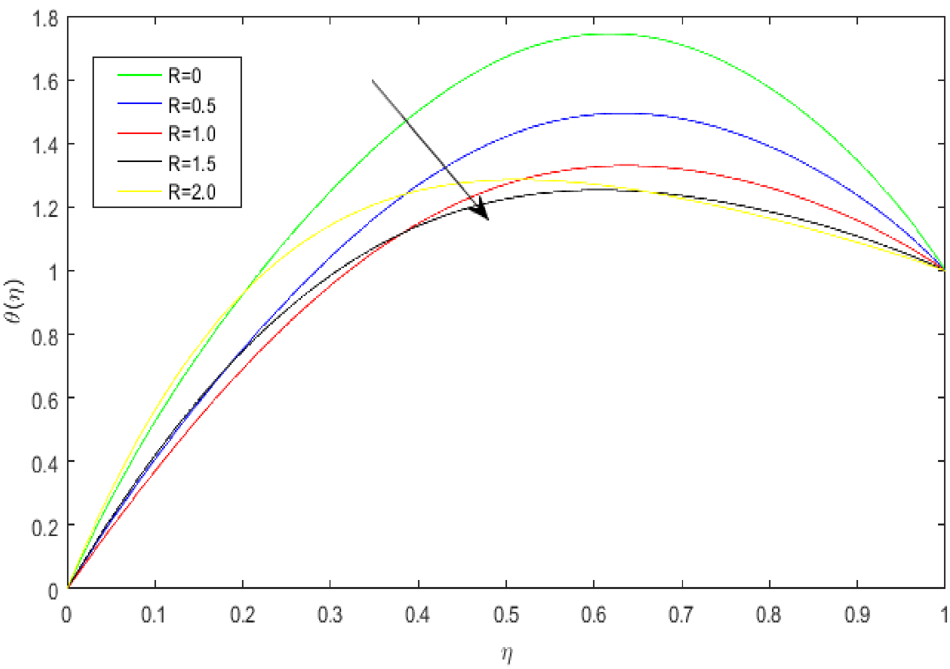

FIGURE 3.9: Influence of R on $\theta(\eta)$. when
$M = 3.0$, $S = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.3$ and $\delta = S_b = 0.1$

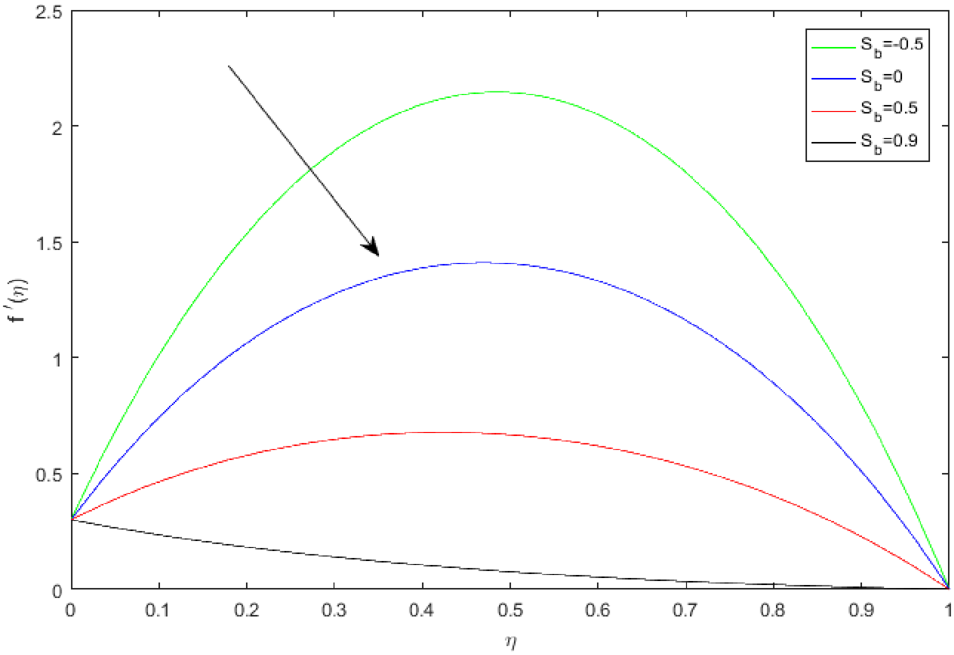

FIGURE 3.10: Influence of S_b on $f'(\eta)$. when
$R = Ec = 0.3$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $\delta = 0.1$ and $S = 0.5$

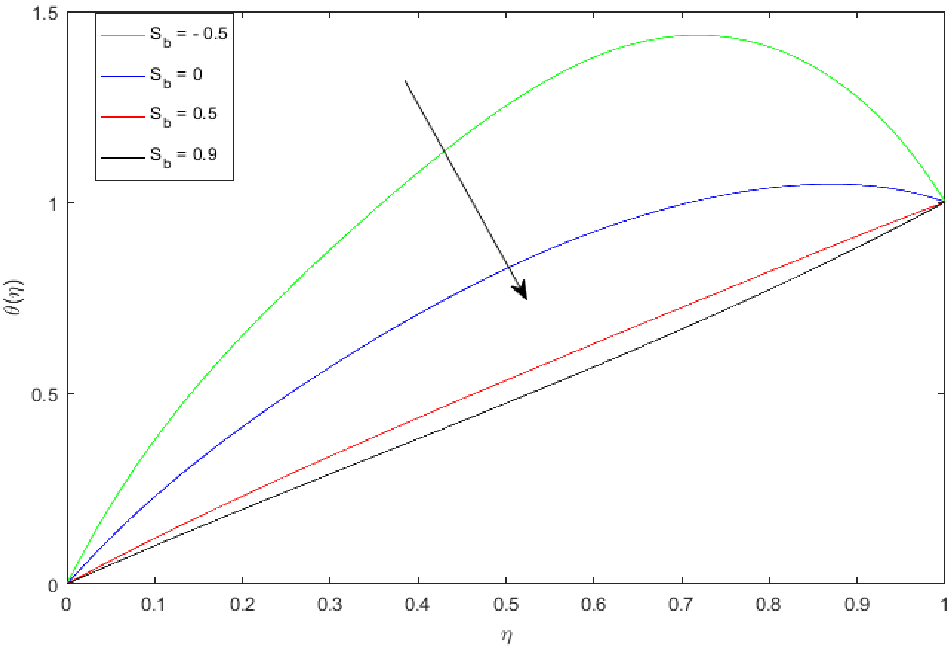

FIGURE 3.11: Impact of S_b on $\theta(\eta)$. when
$R = Ec = 0.3$, $M = 2.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $\delta = 0.1$ and $S = 0.5$

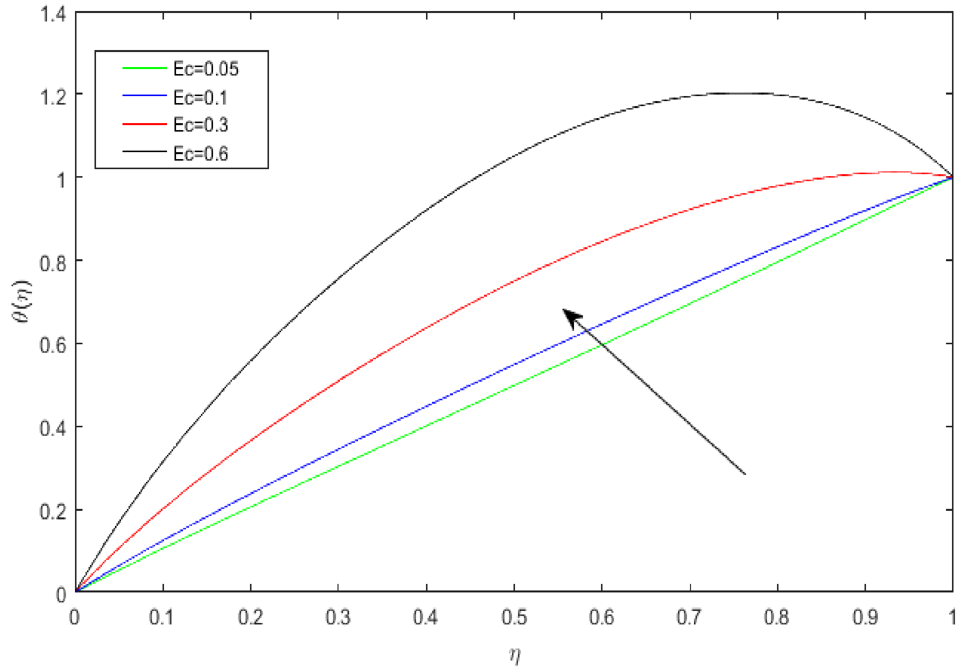

FIGURE 3.12: Influence of Ec on $\theta(\eta)$. when
$R = 0.3$, $M = 2.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $S_b = \delta = 0.1$ and $S = 0.5$

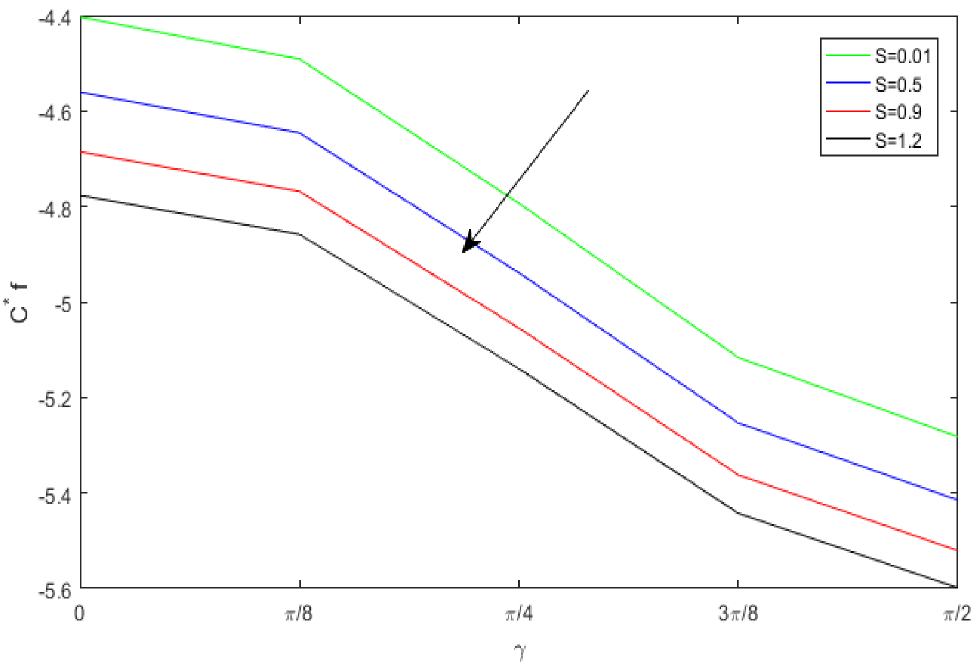

FIGURE 3.13: Effect of S on C^*f. when
$R = 0.5$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{2}$ and $S_b = \delta = Ec = 0.1$

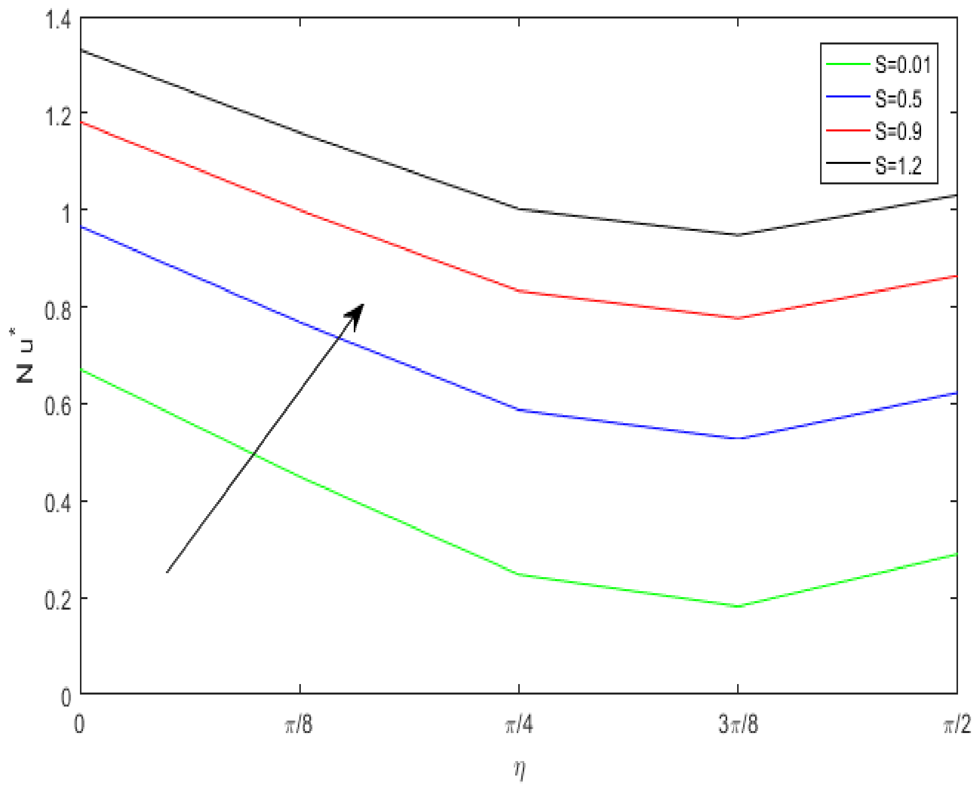

FIGURE 3.14: Effect of S on N^*. when
$R = 0.5,\ M = 3.0,\ Pr = 1.0,\ \gamma = \frac{\pi}{2}$ and $S_b = \delta = Ec = 0.1$

Chapter 4

An Unsteady Squeezing Casson Fluid Flow Under the Effect of Darcy Number

4.1 Introduction

The major goal of this section is to extend the model of [37] by considering the additional effect of the parameter Darcy number Da and Casson fluid parameter β on the unstable squeezing flow of an incompressible electrically conductive fluid contained between two infinite parallel walls. The non-linear partial differential equations of heat and momentum are changed into a set of ordinary differential equations through effectively transforming similarities. Using the shooting process, numerical solutions are obtained. The impact of different physical parameter values is discussed and the observations are in outstanding structure. The numerical computed effects of different parameters on the dimensionless velocity and temperature are calculated and presented in the form of graphs.

4.2 Mathematical Modeling

A Darcy number Da and Casson fluid β parameter, along with the inclined magnetic field effect has been considered on the unsteady squeezing flow of an electrically conductive incompressible fluid confined between two infinite parallel plates. The lower plate of the channel is along the x-axis and the y-axis is normal to it. $B = (B_m \cos\gamma, B_m \sin\gamma, 0)$, is the time-variable magnetic field in which B_m denotes $B_o(1-\alpha t)^{\frac{-1}{2}}$, is applied at an inclination angle γ with respect to the x-axis. The distance between two plates is $H(t) = l(1-\alpha t)^{\frac{1}{2}}$ changes with the time t, where l at the time $t = 0$ is the initial distance between the plates [35].
Flow model geometry is shown in FIGURE 4.1

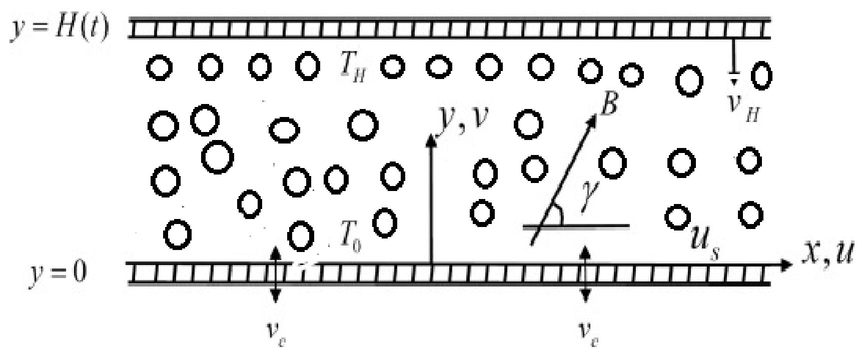

FIGURE 4.1: Geometry of the problem

.

Equation of continuity, equation of momentum and the equation of energy are given as:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.1}$$

Momentum equation for u-velocity:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\Big(1+\frac{1}{\beta}\Big)\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big)+(\cos\gamma v-\sin\gamma u)$$
$$\sin\gamma\frac{\sigma B_m^2}{\rho}-\frac{\mu u}{\rho K_p} \tag{4.2}$$

Momentum equation for v-velocity:

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}=-\,\frac{1}{\rho}\,\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\,\Big(1+\frac{1}{\beta}\Big)\Big(\,\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\,\Big)+\sin\gamma u-\cos\gamma v)$$
$$cos\gamma\,\frac{\sigma B_m^2}{\rho}-\frac{\nu u}{\rho K_p} \tag{4.3}$$

Energy equation:

$$\frac{\partial T}{\partial t}+u\,\frac{\partial T}{\partial x}+v\,\frac{\partial T}{\partial y}=\Big(\,\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\,\Big)\,\frac{k}{\rho c_p}+\Big[2\Big(\,\frac{\partial u}{\partial x}\,\Big)^2+2\Big(\,\frac{\partial v}{\partial y}\,\Big)^2+$$
$$\Big(\,\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\,\Big)^2\Big]\,\frac{\mu}{\rho c_p}+\frac{\sigma B_m^2}{\rho c_p}\,(u\sin\gamma-v\cos\gamma)^2. \tag{4.4}$$

Here u is fluid motion in the direction of x and v is fluid motion in the direction of y. The temperature is T, the total dynamic viscosity is ν, the density is ρ, the real heat capacity of the fluid is C_p, and the thermal conductivity of the fluid is κ respectively. Lower and upper plate boundary conditions are based on:

$$\left.\begin{aligned}
&u=0,\\
&v=v_H=\frac{dH}{dt}=-\frac{\alpha l}{2\sqrt{1-\alpha t}},\\
&T=T_H=T_o+\left(\frac{T_o}{1-\alpha t}\right)\ at\ y=H(t),\\
&u=u_s=\frac{bx}{1-\alpha t}\,,\\
&v=v_c=-\,\frac{v_o}{\sqrt{1-\alpha t}}\,,\\
&T=T_o\ \ at\ \ y=0.
\end{aligned}\right\} \tag{4.5}$$

Here, u_s denotes lower-plate stretching velocity, v_c represents lower-plate mass flux velocity, v_H denotes upper-plate velocity, T_o is lower-plate surface temperature and T_H denotes upper-plate surface temperature.

4.3 Dimensionless Structure of the Governing Equations

By using following dimensionless parameters (4.1) - (4.4) are transformed into the dimensionless form.

$$\left.\begin{aligned} v &= v_H f(\eta), \\ \eta &= \frac{y}{H(t)}, \\ u &= v_H f'(\eta)\frac{-x}{H(t)}, \\ \theta(\eta) &= \frac{T-T_o}{T_H-T_o}. \end{aligned}\right\} \tag{4.6}$$

The detailed procedure for the verification of the continuity equation (4.1) has been discussed in **Chapter 3**. The conversion of (4.2) - (4.4) into dimensionless form is described into the upcoming discussion.

Following derivatives are calculated for the conversion of (4.2) into the dimensionless form.

- $$\begin{aligned} u &= \frac{-x}{H(t)} v_H f'(\eta) \\ &= \frac{-x\eta}{y}\Big(\frac{-\alpha l}{2\sqrt{1-\alpha t}}\Big) f'(\eta) \\ &= \frac{\alpha x}{2(1-\alpha t)} f'(\eta) \\ &\left(\because \eta = \frac{y}{H(t)} \quad \textit{and} \quad H(t) = l\sqrt{1-\alpha t}\right) \end{aligned}$$

- $$\frac{\partial u}{\partial x} = \frac{\alpha}{2(1-\alpha t)} f'(\eta)$$

- $$\frac{\partial^2 u}{\partial x^2} = 0$$

- $\dfrac{\partial u}{\partial y} = \dfrac{\alpha x}{2l(1-\alpha t)^{\frac{3}{2}}} f''(\eta)$
- $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\alpha x}{2l^2(1-\alpha t)^2} f'''(\eta)$
- $\dfrac{\partial u}{\partial t} = \dfrac{\alpha^2 x}{2l(1-\alpha t)^2} f'(\eta) + \dfrac{\alpha^2 xy}{4l(1-\alpha t)^{\frac{5}{2}}} f''(\eta)$
- $\dfrac{\mu}{\rho} \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\alpha x}{2(1-\alpha t)^2} \left(\dfrac{\nu}{l^2} \right) f''' \quad \left(\because \nu = \dfrac{\mu}{\rho} \right)$
- $B_m^2 = B_o^2(1-\alpha t)$
- $\dfrac{\sigma B_m^2}{\rho} (v \cos\gamma - u \sin\gamma) \sin\gamma$

$$- \frac{\alpha x}{2(1-\alpha t)} f' \sin\gamma \Big]$$

$$= \frac{\sigma B_o^2}{\rho(1-\alpha t)} \sin\gamma \Big[\frac{-\alpha l}{2\sqrt{1-\alpha t}} f \cos\gamma - \frac{\alpha x}{2(1-\alpha t)} f' \sin\gamma \Big]$$

- $\delta = \dfrac{H}{x} = \dfrac{l(1-\alpha t)^2}{x}$
- $Da = \dfrac{K_p}{H^2}$
- $K_p = DaH^2$
- $\dfrac{\mu u}{pK_p} = \dfrac{\mu x \alpha}{2Dal^2\rho(1-\alpha t)^2} f'(\eta)$
- $\dfrac{\mu v}{pK_p} = \dfrac{-\alpha \nu}{2Dal\rho(1-\alpha t)^{\frac{3}{2}}} f$

Using the above expression L.H.S of (4.2) yields:

$$\Longrightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\alpha^2 x}{2l(1-\alpha t)^2} f'(\eta) + \frac{\alpha^2 xy}{4l(1-\alpha t)^{\frac{5}{2}}} f''(\eta) + \frac{-\alpha^2 x}{4(1-\alpha t)^2} f'^2(\eta) + \frac{\alpha^2 x}{2(1-\alpha t)^2} f(\eta) f''(\eta)$$

$$\Longrightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\alpha^2 x}{2(1-\alpha t)^2} \Big[f' + \frac{y}{2l(1-\alpha t)^{\frac{1}{2}}} f'' + \frac{1}{2} f'^2 - \frac{1}{2} f f'' \Big] \tag{4.7}$$

Now consider R.H.S of (4.2) and using the above relations, we get

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \Big) + \frac{\sigma B_m^2}{\rho} \sin\gamma (v \cos\gamma - u \sin\gamma) = -\frac{1}{\rho}\frac{\partial \rho}{\partial x}$$

$$+\frac{\alpha x}{2(1-\alpha t)^2}\Big[\frac{\nu}{l^2}+\frac{\sigma B_o^2}{\rho}(\delta f\cos\gamma+f'\sin\gamma)\sin\gamma-\frac{\mu}{Dal^2\rho}f'(\eta)\Big] \tag{4.8}$$

Differentiate w.r.t 'y' of equation (4.7).

$$\Longrightarrow \frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v=\frac{\alpha^2 x}{2(1-\alpha t)^2}\,\frac{1}{2l(1-\alpha t)^{\frac{1}{2}}}\,[3f''+\eta f'''+f'f''-ff''']$$

$$\Longrightarrow \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}\,[3f''+\eta f'''+f'f''-ff'''] \tag{4.9}$$

Differentiate w.r.t 'y' of equation (4.8).

$$\Longrightarrow \frac{\partial}{\partial y}\Big[-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big)+\frac{\sigma B_m^2}{\rho}\sin\gamma(v\cos\gamma-u\sin\gamma)\Big]=$$
$$-\frac{1}{\rho}\frac{\partial^2\rho}{\partial x\partial y}+\frac{\alpha x}{2(1-\alpha t)^2}\,\frac{1}{l(1-\alpha t)^{\frac{1}{2}}}\,\Big[\frac{\nu}{l^2}f''''-\frac{\sigma B_o^2}{\rho}\sin\gamma(\delta f'\cos\gamma$$
$$+f''\sin\gamma)-\frac{\mu}{Dal^2p}f'(\eta)\Big]$$

$$\Longrightarrow=-\frac{1}{\rho}\frac{\partial^2\rho}{\partial x\partial y}+\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\,\frac{\nu}{l^3}\,\Big[f''''-\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)$$
$$-\frac{1}{Da}f''(\eta)\Big] \tag{4.10}$$

Hence combining equations (4.9) and (4.10), we get:

$$\Longrightarrow \frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}[3f''+\eta f'''+f'f''-ff''']+\frac{1}{\rho}\frac{\partial^2\rho}{\partial x\partial y}-\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\frac{\nu}{l^3}\Big[\frac{\nu}{l^2}f''''$$
$$+\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)-\frac{1}{Da}f''(\eta)\Big]=0 \tag{4.11}$$

Now we include the following dimensionless parameters for the conversion of momentum equation(4.3) into the dimensionless form.

- $\frac{\partial v}{\partial x}=0 \quad \Big(\because v=v_H=-\Big(\frac{\alpha l}{2\sqrt{1-\alpha t}}\Big)f(\eta)\Big)$
- $\frac{\partial^2 v}{\partial x^2}=0$

- $\dfrac{\partial v}{\partial t} = \dfrac{-\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}} f(\eta) - \dfrac{\alpha l}{2\sqrt{1-\alpha t}} \dfrac{\partial \eta}{\partial t} f'(\eta)$
- $\dfrac{\partial v}{\partial y} = \dfrac{-\alpha}{2(1-\alpha t} f'(\eta)$
- $\dfrac{\partial^2 v}{\partial y^2} = \dfrac{-\alpha l}{2\sqrt{1-\alpha t}} \dfrac{\partial^2 \eta}{\partial y^2} f'(\eta) - \dfrac{\alpha l}{2\sqrt{1-\alpha t}} \left(\dfrac{\partial \eta}{\partial y}\right)^2 f''(\eta)$
- $\dfrac{\mu v}{pK_p} = \dfrac{-\alpha \nu}{2Dal\rho(1-\alpha t)^{\frac{3}{2}}} f$

The dimensionless form of (4.3) can be written as;

$$\frac{-\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}} f - \frac{\alpha l}{2(1-\alpha t)^{\frac{1}{2}}} \frac{\partial \eta}{\partial t} f' + \frac{\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}} f f' = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$
$$+ \frac{\sigma B_o^2}{2\rho(1-\alpha t)} \cos\gamma\left(\frac{\alpha x}{(1-\alpha t)} f'(\eta)\sin\gamma + \frac{\alpha l}{\sqrt{1-\alpha t}}\cos\gamma f\right) - \frac{-\alpha\nu}{2Dal\rho(1-\alpha t)^{\frac{3}{2}}} f$$

Dimensionless form of L.H.S of (4.3) is;

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{-\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}} f - \frac{\alpha l}{2(1-\alpha t)^{\frac{1}{2}}} \frac{\partial \eta}{\partial t} f' + \frac{\alpha^2 l}{4(1-\alpha t)^{\frac{3}{2}}} f f' \tag{4.12}$$

Taking derivative w.r.t 'x' of equation (4.12)

$$\Longrightarrow \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}\right)v = 0 \tag{4.13}$$

Dimensionless form of R.H.S of (4.3) is;

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\sigma B_m^2}{\rho}(u\sin\gamma - v\cos\gamma)\cos\gamma - \frac{-\alpha\nu}{2Dal\rho(1-\alpha t)^{\frac{3}{2}}} f$$
$$= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\nu + \frac{\sigma B_o^2}{2\rho(1-\alpha t)}\cos\gamma\left[\frac{\alpha x}{(1-\alpha t)} f'(\eta)\sin\gamma\right.$$
$$\left. + \frac{\alpha l}{\sqrt{1-\alpha t}}\cos\gamma f\right] - \frac{-\alpha\nu}{2Dal\rho(1-\alpha t)^{\frac{3}{2}}} f \tag{4.14}$$

Taking derivative w.r.t 'x' of equation (4.14).

$$\frac{\partial}{\partial x}\left[-\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\sigma B_m^2}{\rho}(u\sin\gamma - v\cos\gamma)\cos\gamma\right] = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}$$
$$+ \frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f' \tag{4.15}$$

Combining equation (4.13) and (4.15).

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}+\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\,\alpha\cos\gamma\sin\gamma f'=0$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}=\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\,\alpha\cos\gamma\sin\gamma f' \tag{4.16}$$

Putting (4.16) in equation (4.11) we get,

$$\begin{aligned}
&\frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}[3f''+\eta f'''+f'f''-ff''']+\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x\partial y}-\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\frac{\nu}{l^3}\Big[f''''\\
&+\frac{-\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)\Big]=\frac{\alpha^2 x}{4l(1-\alpha t)^{\frac{5}{2}}}[3f''+\eta f'''\\
&+f'f''-ff''']+\frac{\sigma B_o^2}{2\rho(1-\alpha t)^2}\alpha\cos\gamma\sin\gamma f'-\frac{\alpha x}{2(1-\alpha t)^{\frac{5}{2}}}\frac{\nu}{l^3}\Big[\Big(1+\frac{1}{\beta}\Big)f''''\\
&-\frac{\sigma l^2 B_o^2}{\rho\nu}\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)-\frac{1}{Da}f''(\eta)\Big]
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
\Longrightarrow\ &\frac{\alpha}{2}\,(3f''+\eta f'''+f'f''-ff''')-\frac{\nu}{l^2}\,\Big[\Big(1+\frac{1}{\beta}\Big)f''''-\frac{\sigma l^2 B_o^2}{\rho\nu}\,\sin\gamma(\delta f'\cos\gamma\\
&+f''\sin\gamma)\Big]+\frac{\sigma B_o^2\delta}{\rho}\,\cos\gamma\sin\gamma f'+\frac{1}{Da}f''(\eta)=0,\\
\Longrightarrow\ &\frac{\alpha l^2}{2\nu}\,(3f''+\eta f'''+f'f''-ff''')-\frac{\nu l^2}{l^2\nu}\,\Big[\Big(1+\frac{1}{\beta}\Big)f''''-\frac{\sigma l^2 B_o^2 l^2}{\rho\nu}\\
&\sin\gamma(\delta f'\cos\gamma+f''\sin\gamma)\Big]+\frac{\sigma B_o^2 l^2\delta}{\rho\nu}\,\cos\gamma\sin\gamma f'+\frac{1}{Da}f''(\eta)=0,
\end{aligned}$$

where $\quad M^2=\frac{\sigma B_o^2 l^2}{\rho\nu}\,,\quad S=\frac{\alpha l^2}{2\nu}\quad$ and $\quad\delta=\frac{H}{x}=\frac{l\sqrt{1-\alpha t}}{x}$.

$$\begin{aligned}
\Longrightarrow\ &\frac{\alpha l^2}{2\nu}\,(3f''+\eta f'''+f'f''-ff''')-\Big(1+\frac{1}{\beta}\Big)f''''-\frac{\sigma B_o^2 l^2}{\rho\nu}\\
&(\delta\cos\gamma f'+\sin\gamma f'')\sin\gamma+\frac{\sigma B_o^2 l^2\delta}{\rho\nu}\,\cos\gamma sin\gamma f'+\frac{1}{Da}f''(\eta)=0,\\
\Longrightarrow\ &S(3f''+\eta f'''+f'f''-ff''')-\Big(1+\frac{1}{\beta}\Big)f''''+M^2 sin\gamma\delta f'\cos\gamma\\
&+M^2 f''\sin\gamma+M^2\cos\gamma\sin\gamma f'+\frac{1}{Da}f''(\eta)=0,
\end{aligned} \tag{4.18}$$

Final form of ordinary differential equation is:

$$\Big(1+\frac{1}{\beta}\Big)f'''' - S(3f''+\eta f'''+f''f'-f'''f) - M^2(2\delta\cos\gamma f'+\sin\gamma f'')\sin\gamma \\ -\frac{1}{Da}f''(\eta)=0. \tag{4.19}$$

Subject to the boundary conditions:

$$f(0)=S_b,\quad f'(0)=R,\quad \theta(0)=0,\ f(1)=1,\quad f'(1)=0,\quad \theta(1)=1. \tag{4.20}$$

where the squeezing number is S, the prandtl number is Pr, the magnetic parameter is M, the Eckert number is Ec and the lower-plate stretching parameter is R. S_b reflects a function of the lower-plate suction/injuction with $S_b<0$ for damage and $S_b>0$ for suction. The following formulation is available for various parameters used in the above equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\alpha x}{2l(1-\alpha t)^{\frac{3}{2}}}\,f''(\eta), & \frac{\partial T}{\partial y} &= \frac{T_o}{l(1-\alpha t)^{\frac{3}{2}}}\,\theta', \\ v_H &= \frac{-\alpha l}{2\sqrt{1-\alpha t}}, & T_H - T_O &= \frac{T_o}{(1-\alpha t)}, \\ Re_x &= \frac{u_s x}{\nu}, & u_s &= \frac{bx}{(1-\alpha t)}. \end{aligned}\right\} \tag{4.21}$$

Before going towards the mathematical solution the skin friction coefficient C_f or the shear stress and the Nusselt number Nu or heat transfer coefficient on the lower plate surface are represented as:

$$C_f = \frac{\mu\left(\frac{\partial u}{\partial y}\right)_{y=H(t)}}{\rho v_H^2}$$

$$Nu = \frac{l\left(\frac{\partial T}{\partial y}\right)_{y=H(t)}}{T_H - T_o}$$

From equation (4.6),

$$C_f^* = \frac{\alpha l^3(1-\alpha t)^{\frac{3}{2}}}{bx^3}\,Re_x C_f = f''(1),$$
$$Nu^* = \left(\frac{\nu}{b}\right)^{\frac{1}{2}} x^{-1}(Re_x)^{\frac{-1}{2}}\,Nu = \theta'(1),$$

where $Re_x = \frac{u_s x}{\nu}$ represents the local Reynolds number.

Hence

$$C_f = \frac{\mu\left(\frac{\partial u}{\partial y}\right)_{y=H(t)}}{\rho v_H^2}, \quad N_u = \frac{1}{T_H - T_O}\left(\frac{\partial T}{\partial y}\right)_{y=H(t)} \tag{4.22}$$

$$\Rightarrow C_f = -\frac{\mu x v_H}{\rho v_H^2 l^2(1-\alpha t)} f''(\eta) = -\frac{\mu x}{\rho v_H l^2(1-\alpha t)} f''(\eta)$$

$$\Rightarrow C_f = -\frac{\mu x}{\rho\left(\frac{-\alpha l}{2\sqrt{1-\alpha t}}\right) l^2(1-\alpha t)} f''(\eta)$$

$$\Rightarrow C_f = \frac{2\mu x}{\rho\alpha l^3\sqrt{1-\alpha t}} f''(\eta)$$

$$\Rightarrow f''(\eta) = \frac{\rho\alpha l^3\sqrt{1-\alpha t}}{\mu x} C_f, \quad \text{where} \quad y = H(t) \quad \text{and} \quad \eta = 1.$$

$$\Rightarrow f''(1) = \frac{\rho\alpha l^3\sqrt{1-\alpha t}}{\nu\rho x} C_f = \frac{\alpha l^3\sqrt{1-\alpha t}}{\frac{u_s x}{Re_x} x} C_f$$

$$\Rightarrow f''(1) = \frac{\alpha l^3\sqrt{1-\alpha t}}{u_s x^2} Re_x C_f$$

$$\Rightarrow f''(1) = \frac{\alpha l^3\sqrt{1-\alpha t}}{\frac{bx}{(1-\alpha t)} x^2} Re_x C_f$$

$$\Rightarrow f''(1) = \frac{\alpha l^3(1-\alpha t)^{\frac{3}{2}}}{bx^3} Re_x C_f$$

The local number Nusselt is defined as follows:

- $$Nu = \frac{1}{T_H - T_O}\left(\frac{\partial T}{\partial y}\right)_{y=H(t)}$$

$$\Rightarrow Nu = \frac{1}{T_H - T_O}\frac{T_o}{l(1-\alpha t)^{\frac{3}{2}}}\theta'(\eta)$$

$$\Rightarrow Nu = \frac{1}{T_H - T_O}(T_H - T_O)\frac{1}{l(1-\alpha t)^{\frac{1}{2}}}\theta'(\eta)$$

$$\Rightarrow Nu = \frac{1}{l(1-\alpha t)^{\frac{1}{2}}}\theta'(\eta), \quad \text{where} \quad y = H(t) \quad \text{and} \quad \eta = 1$$

$$\Rightarrow \theta'(1) = l(1-\alpha t)^{\frac{1}{2}} Nu$$

$$\Rightarrow \theta'(1) = \frac{l(bx)^{\frac{1}{2}}}{(u_s)^{\frac{1}{2}}} Nu, \ (1-\alpha t)^{\frac{1}{2}} = \frac{(bx)^{\frac{1}{2}}}{(u_s)^{\frac{1}{2}}}$$

$$\Rightarrow \theta'(1) = \frac{l(bx)^{\frac{1}{2}}}{(\frac{Re_x \nu}{x})^{\frac{1}{2}}} N_u$$

4.4 Numerical Treatment

This section is focused on the implementation of the shooting method to solve the transformed ODE (4.19) subject to the Boundary Conditions (4.21). For this purpose, we first transform the system of higher order ODEs into first order ODEs. Let us use the notations:

$$f = y,\ f' = y',\ f'' = y'',\ f''' = y''',\ f'''' = y''''. \tag{4.23}$$

Further denote

$$y = y_1,\ y' = y_1' = y_2,\ y'' = y_2' = y_3,\ y''' = y_3' = y_4,\ y'''' = y_4'.$$

Equations are,

$$\left.\begin{aligned}
&y_1' = y_2; && y_1(0) = S_b = \frac{2v_o}{\alpha l}\ ,\\
&y_2' = y_3; && y_2(0) = R = \frac{u_s \delta}{v_H}\ ,\\
&y_3' = y_4; && y_3(0) = \alpha_1,\\
&y_4' = S(3y_3 + \eta y_4 + y_2 y_3 - y_1 y_4) + M^2 sin\gamma(2\delta cos\gamma y_2 + sin\gamma y_2); &&\\
& && y_4(0) = \alpha_2.
\end{aligned}\right\} \tag{4.24}$$

In the above system of equations the missing conditions α_1 and α_2 are to be chosen such that

$$y_4(\eta_\infty, \alpha_1, \alpha_2) = 0,$$
$$y_5(\eta_\infty, \alpha_1, \alpha_2) = 0.$$

Now

$$y_3(0) = y''(0) = \alpha_1, \qquad y_4(0) = y'''(0) = \alpha_2.$$

To solve the system of algebraic equations we use the Newtons method which has the following iterative scheme:

$$\begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = \begin{pmatrix} u^{n} \\ v^{n} \end{pmatrix} - \begin{pmatrix} \frac{\partial y_1}{\partial \alpha_1} & \frac{\partial y_1}{\partial \alpha_2} \\ \frac{\partial y_2}{\partial \alpha_1} & \frac{\partial y_2}{\partial \alpha_2} \end{pmatrix}^{-1} \begin{pmatrix} y_1(1) - 1 \\ y_2(1) - 0 \end{pmatrix} \tag{4.25}$$

Now use the following notations:

$$\begin{aligned} \frac{\partial y_1}{\partial \alpha_1} &= y_5, & \frac{\partial y_2}{\partial \alpha_1} &= y_6, & \frac{\partial y_3}{\partial \alpha_1} &= y_7, & \frac{\partial y_4}{\partial \alpha_1} &= y_8, \\ \frac{\partial y_1}{\partial \alpha_2} &= y_9, & \frac{\partial y_2}{\partial \alpha_2} &= y_{10}, & \frac{\partial y_3}{\partial \alpha_2} &= y_{11}, & \frac{\partial y_4}{\partial \alpha_2} &= y_{12}. \end{aligned}$$

As the result of these new notations, the Newton's iterative scheme gets the form:

$$\begin{pmatrix} u^{n+1} \\ v^{n+1} \end{pmatrix} = \begin{pmatrix} u^{n} \\ v^{n} \end{pmatrix} - \begin{pmatrix} y_5 & y_9 \\ y_6 & y_{10} \end{pmatrix}^{-1} \begin{pmatrix} y_1(1) - 1 \\ y_2(1) - 0 \end{pmatrix} \tag{4.26}$$

Now differentiate the above system of four first order ODEs with respect to each of the variables α_1 and α_2 to have another system of eight ODEs. Writing all these twelve ODEs together, we have the following IVP:

$$\begin{aligned}
y_5' &= y_6; & y_5(0) &= 0, \\
y_6' &= y_7; & y_6(0) &= 0, \\
y_7' &= y_8; & y_7(0) &= 0, \\
y_8' &= \frac{\beta}{(1+\beta)}\big[S(3y_7 + \eta y_8 + y_5 y_3 - y_5 y_4 + y_2 y_7 - y_1 y_8) \\
&\quad + M^2 \sin\gamma(2\delta\cos\gamma y_6 + \sin\gamma y_7) + \frac{1}{Da} y_7\big]; & y_8(0) &= 0, \\
y_9' &= y_{10}; & y_9(0) &= 0, \\
y_{10}' &= y_{11}; & y_{10}(0) &= 0, \\
y_{11}' &= y_{12}; & y_{11}(0) &= 0, \\
y_{12}' &= \frac{\beta}{(1+\beta)}\Big[S(3y_{11} + \eta y_{12} + y_{10} y_3 - y_9 y_4 + y_2 y_{11} - y_1 y_{12}) \\
&\quad + M^2 \sin\gamma(2\delta\cos\gamma y_{10} + \sin\gamma y_{11}) + \frac{1}{Da} y_{11}\Big]; & y_{12}(0) &= 0.
\end{aligned}$$

The fourth Runge-Kutta method is used to solve the twelve equations system above with α_1 and α_2 initial guess. Such estimates are modified by the scheme of the Newton. The iterative method is performed before the conditions here are met:

$$max\left[\mid {\alpha_1}^{n+1} - {\alpha_1}^{n} \mid, \quad \mid {\alpha_2}^{n+1} - {\alpha_2}^{n} \mid\right] < \epsilon$$

for an arbitrarily small positive value of ϵ . Throughout this chapter ϵ has been taken as $(10)^{-6}$.

4.5 Results with discussion

The numerical effects are presented in this section in the form of graphs. Using curves, variation is measured in the linear velocity and temperature curves. Figure (4.2)-(4.15) contains variables such as the squeezing number, magnetic parameter, lower-plate stretching parameter, magnetic tilt angle, Darcy number, Eckert number, Casson fluid parameter and Lower-plate suction/injuction parameter.

Figure 4.2 and Figure 4.3 display the influence of the squeeze number on the temperature $\theta(\eta)$ and the velocity profile $f'(\eta)$. Figure 4.2 shows the influence of S on the profile of velocity. Remember that the velocity of the fluid decreases by growing the squeezing parameter values. Figure 4.3 indicates that decreased in S causes a decrease in fluid temperature across parallel plates.

Figure 4.4 and Figure 4.5 display the velocity and temperature profile of the fluid with different magnetic parameter values. It is observed that Figure 4.4, it is noticed that a particular time, a rising magnetic parameter causes the fluid's velocity to increase in regions similar to the upper or lower plates, while the fluid's velocity in the central region indicates an obvious decrease. In the central area the fluid has a higher velocity as compared to the surface fluid. The fluid in the central area has a higher velocity relative to the viscous fluid plates and it is found from

Figure 4.5 that temperature rises with magnetic parameter change. Moreover, the temperature of the fluid decreases from the lower to the upper plate area while the parameter of the magnetic field is small. The fluid temperature offers maximum values for larger magnetic parameter values not on the upper plate area but in the middle area between the walls. However, stronger magnetic fields naturally influence the temperature distribution.

Figure 4.6 and Figure 4.7 represent the velocity and temperature behaviours by rising the inclination angle of the magnetic field applied, respectively. The angles of magnetic inclination vary between 0 and $\pi/2$. Related velocity and temperature patterns were obtained from the two estimates as applied to the corresponding velocity and temperature profiles with specific magnetic parameter values. The influencing of the inclination angle on both fluid temperature and velocity profile is close to the magnetic parameters.

Figure 4.8 and Figure 4.9 illustrate the impact on dimensionless temperature of the lower-surface and velocity of stretching parameter, respectively. Figure 4.8 indicates that the fluid velocity near the lower plate rises in order to raise the magnitude of the lower-plate stretching function while the velocity of the fluid near the upper plate falls with the fluid. Since the parameter for lower-plate stretching is increasing gradually, the fluid with the velocity of maximum value does not appear in the central area between the plates, but on the lower-plate side. Figure 4.9 shows that the increasing lower-plate stretching velocity is initially decreases fluid temperature below the lower-plate.

Figure 4.10 and Figure 4.11 depicts the lower-plate suction/injection affect on temperature profile and velocity profile, respectively. Figure 4.10 indicates that for the lower-plate injection/suction function, the velocity profile is rising. The fluid with peak velocity will not occur in the central area when extending the lower-plate for better suction across the lower-plate, and the velocity of the fluid monotonically decreases from the lower-plate area to the upper-plate. Temperature profiles increase as the injection/suction parameter decreases. In particular, it is found out that when the injection/suction parameter S_b falls in the central area, the fluid has high temperature could not occur at the upper-plate area between the two plates.

In Figure 4.12 by the increment of Eckert number their is rise in temperature profile. It is obvious the temperature rises to raise Eckert number values. Joule heating and viscous dissipation is due to Ec, which significantly increases the fluid temperature between two surfaces. Figure 4.12 also indicates that the maximum fluid temperature for the larger Eckert number occurs in the central area between the two plates and for the smaller Eckert number it exists on the upper plate side.

In Figure 4.13 the magnetic inclination angle on the skin friction coefficient, the squeeze number and the Nusselt number are seen as the magnetic inclination angle ranges between 0 and $\frac{\pi}{2}$, respectively. It can be seen that as Nusselt number decreases its function, the absolute value of the skin friction coefficient increases. In addition , the number of squeezes allows the absolute value of the skin friction coefficient as well as the number of Nusselt to increase for the fixed magnetic inclination angle.

Figure 4.14 demonstrates the velocity profile behaviour, illustrated by increasing Casson fluid parameter values on the velocity. The parameter in Casson fluid ranges from 0.1 to 1.0. From Figure 4.14 shows that the velocity increases with growing of Casson fluid parameter, the velocity of fluid increases when the values of Casson fluid is large.
Figure 4.15 display the velocity behaviour by increasing Darcy number values. The Darcy number ranges from 2.0 to 8.0. From Figure 4.15 velocity profile is observed to increase with Da increase, the fluid velocity increases when the values of Da is large.

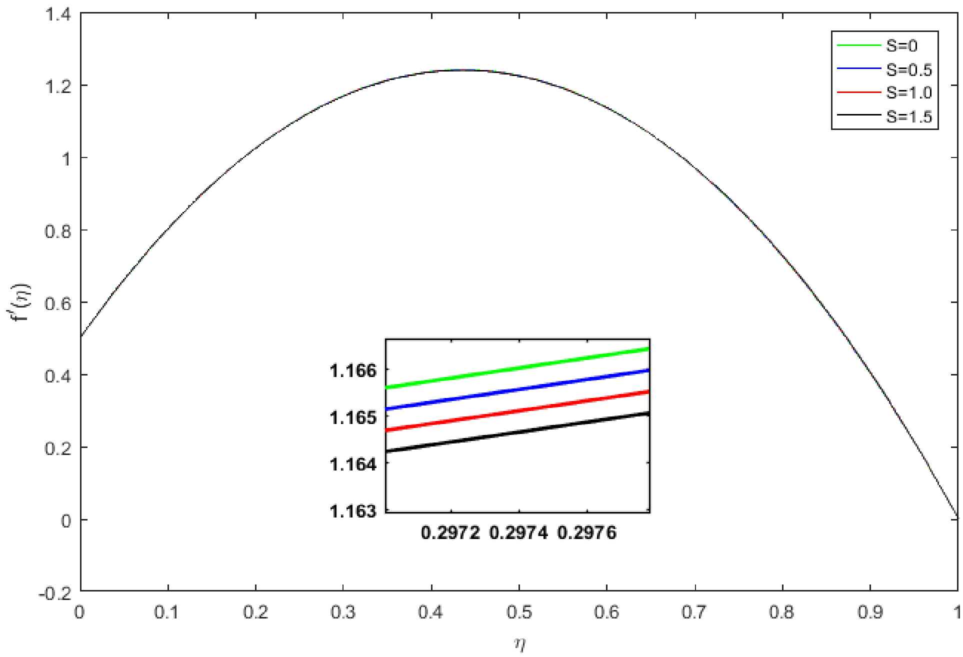

FIGURE 4.2: Impact of S on $f'(\eta)$.
$R = M = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{6}$, $Da = 2.0$ and $\beta = \delta = S_b = Ec = 0.1$

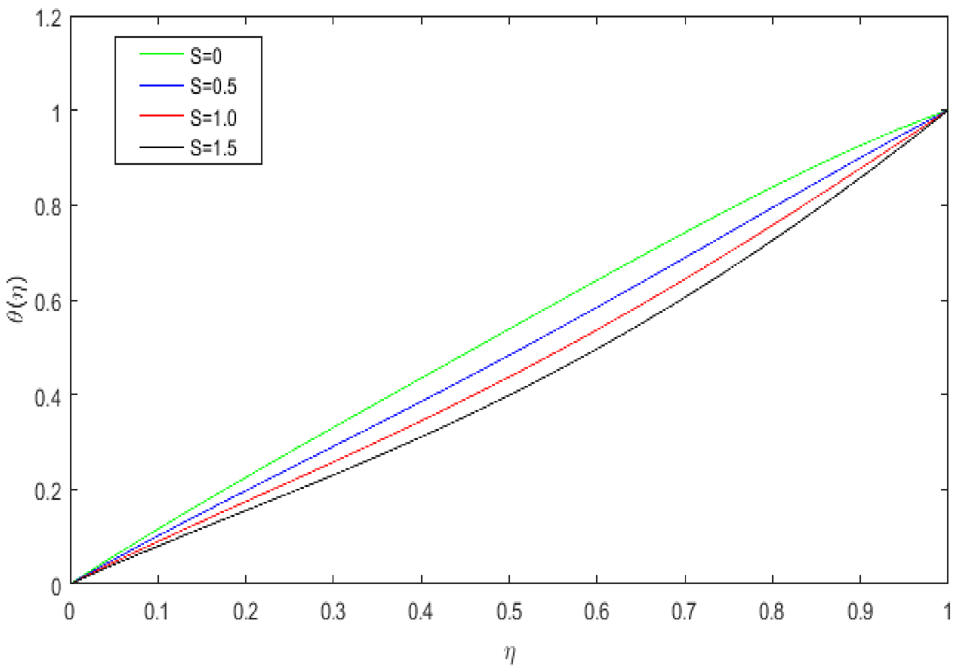

FIGURE 4.3: Impact of S on $\theta(\eta)$.
$R = M = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{6}$, $Da = 2.0$, $\beta = 0.1$ and $\beta = \delta = S_b = Ec = 0.1$

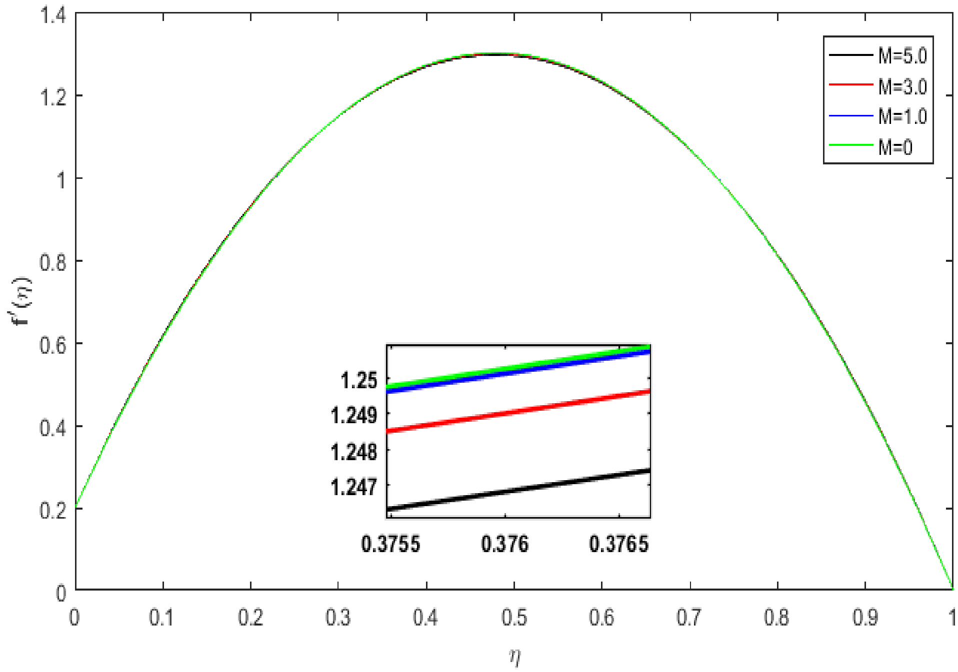

FIGURE 4.4: Impact of M on $f'(\eta)$. when
$R = 0.2$, $S = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.3$, $Da = 2.0$ and $S_b = \beta = \delta = 0.1$

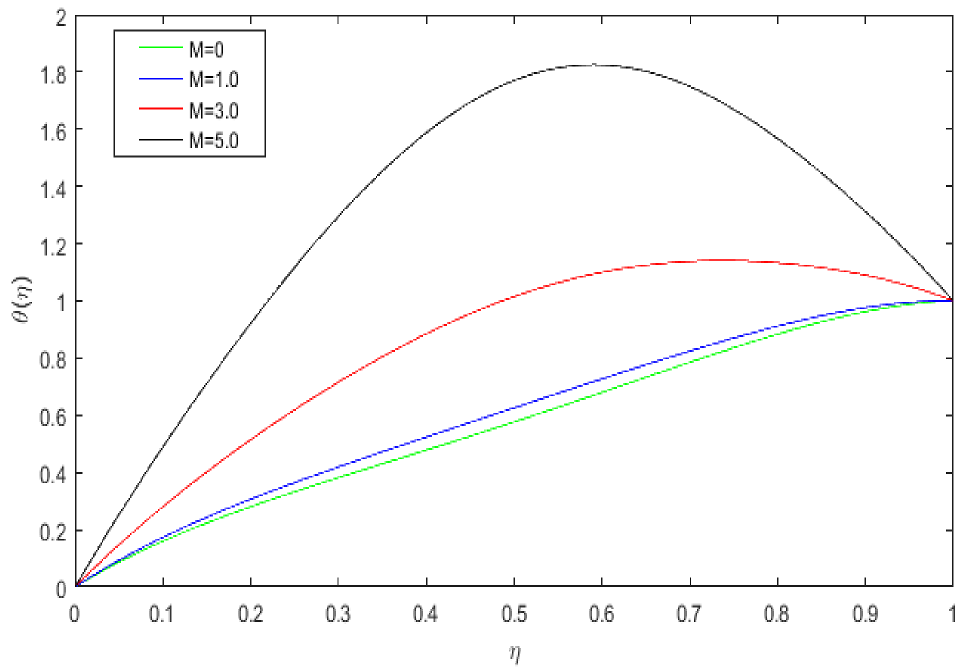

FIGURE 4.5: Influence of M on $\theta(\eta)$. when
$R = 0.2$, $S = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.3$, $Da = 2.0$ and $S_b = \beta = \delta = 0.1$

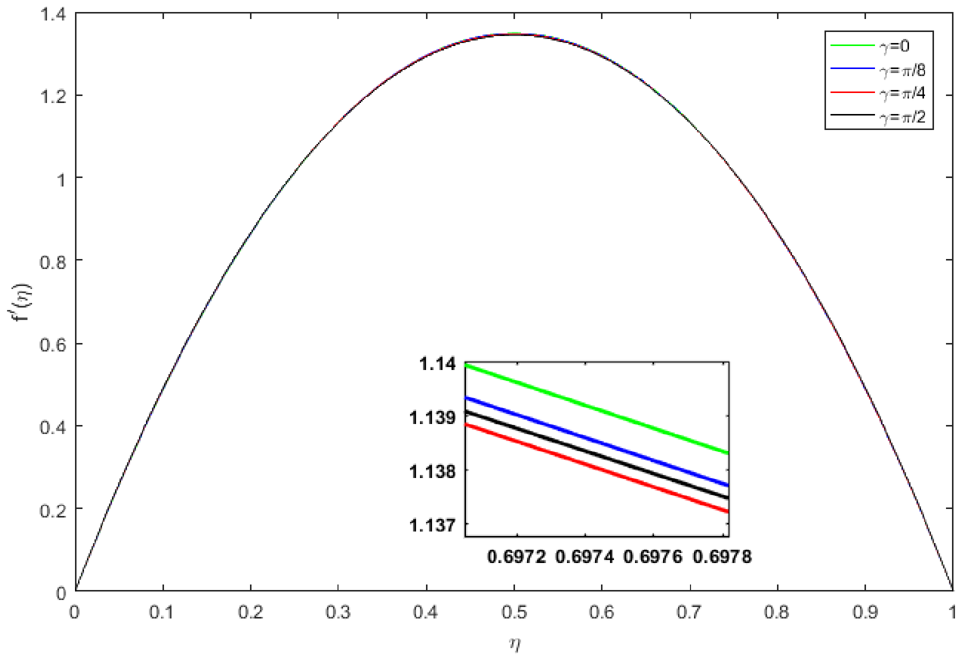

FIGURE 4.6: Impact of γ on $\theta(\eta)$. when
$R = 0$, $S = 0.5$, $Pr = 1.0$, $M = 3.0$, $Ec = 0.3$, $Da = 2.0$ and $S_b = \beta = \delta = 0.1$

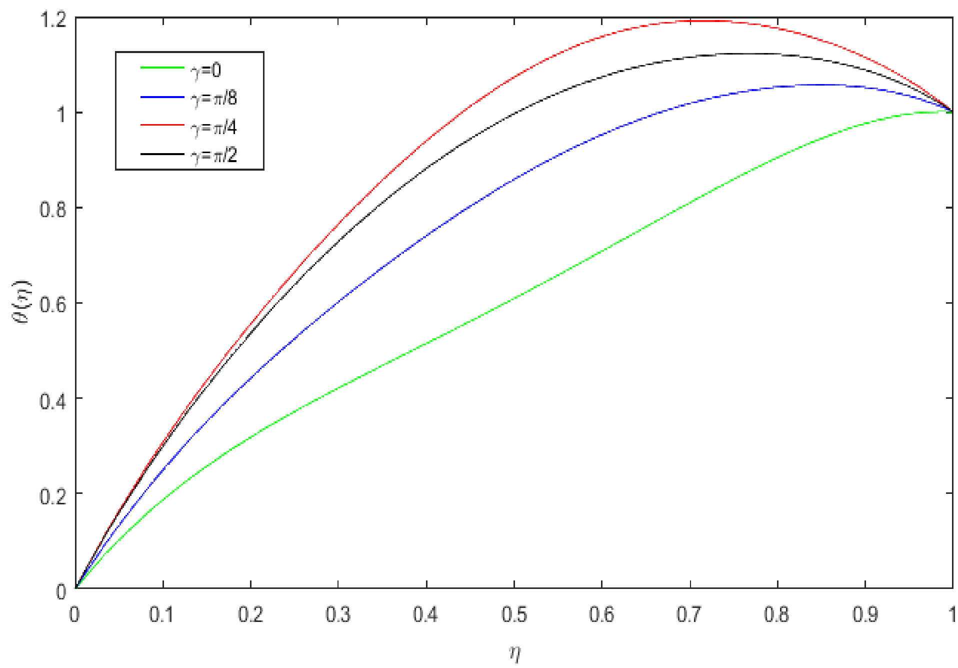

FIGURE 4.7: Influence of γ on $\theta(\eta)$. when
$R = 0$, $S = 0.5$, $Pr = 1.0$, $M = 3.0$, $Ec = 0.3$, $Da = 2.0$ and $S_b = \beta = \delta = 0.1$

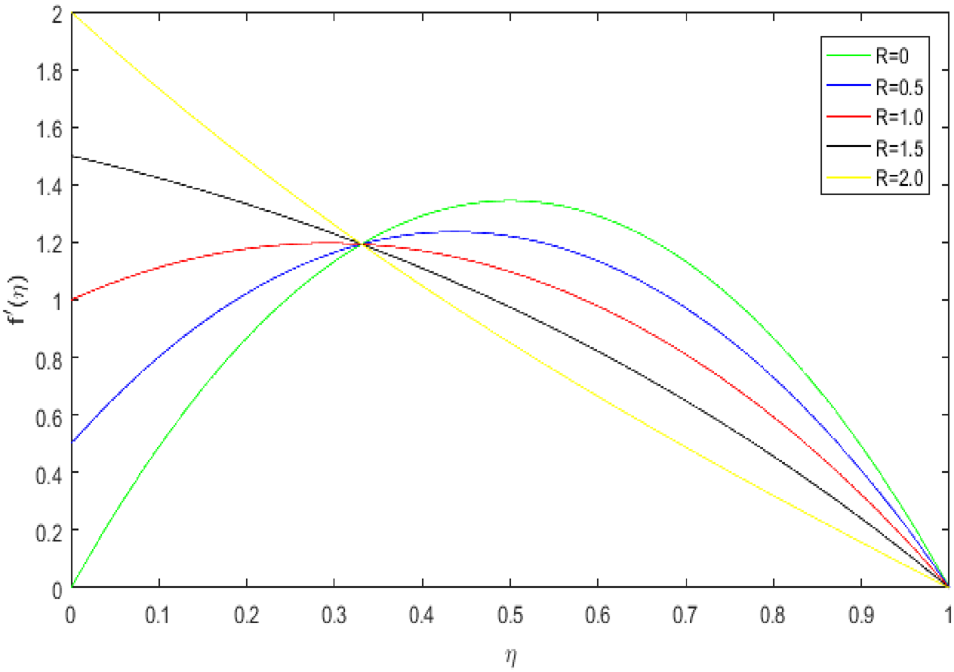

FIGURE 4.8: Influence of R on $f'(\eta)$.
$S = 0.5$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.6$, $Da = 2.0$ and $S_b = \beta = \delta = 0.1$

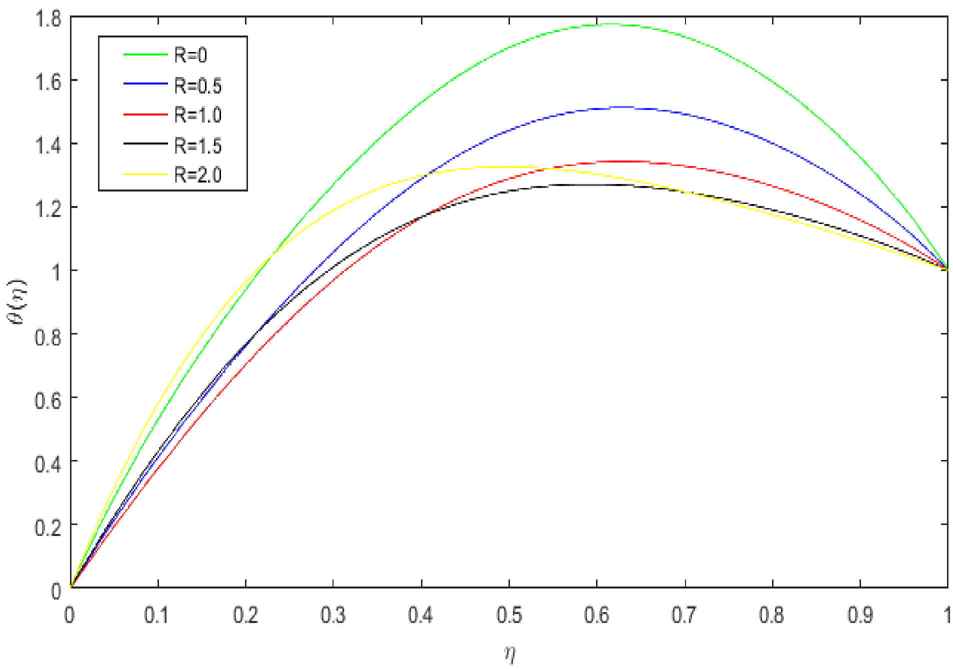

FIGURE 4.9: Impact of R on $\theta(\eta)$. when
$M = 3.0$, $S = 0.5$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Ec = 0.3$, $Da = 2.0$ and $S_b = \beta = \delta = 0.1$

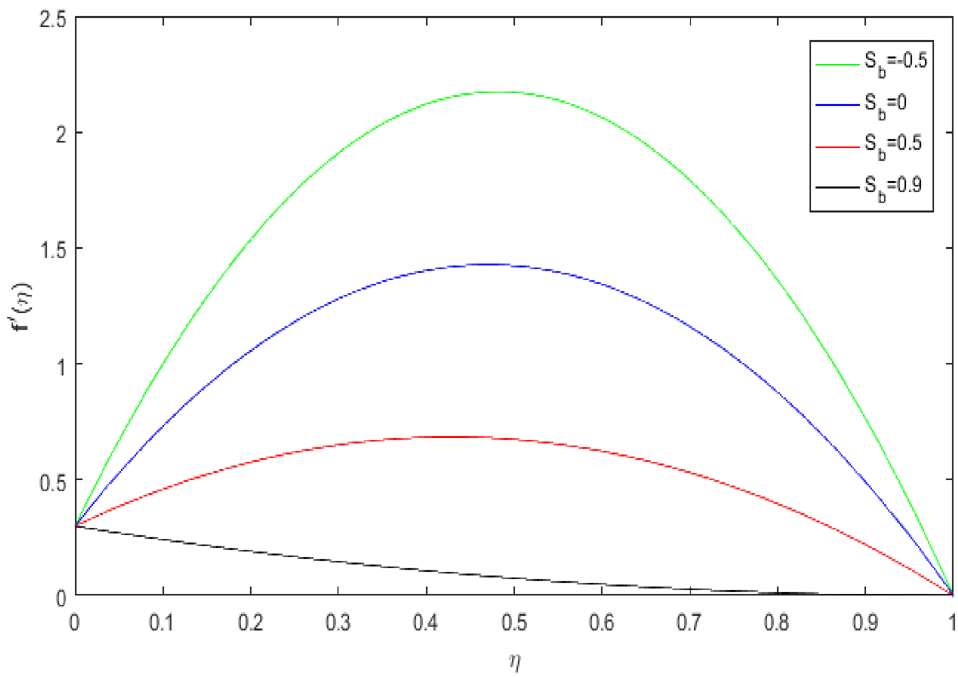

FIGURE 4.10: Influence of S_b on $f'(\eta)$.
$R = Ec = 0.3$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $Da = 2.0$, $\beta = \delta = 0.1$ and $S = 0.5$

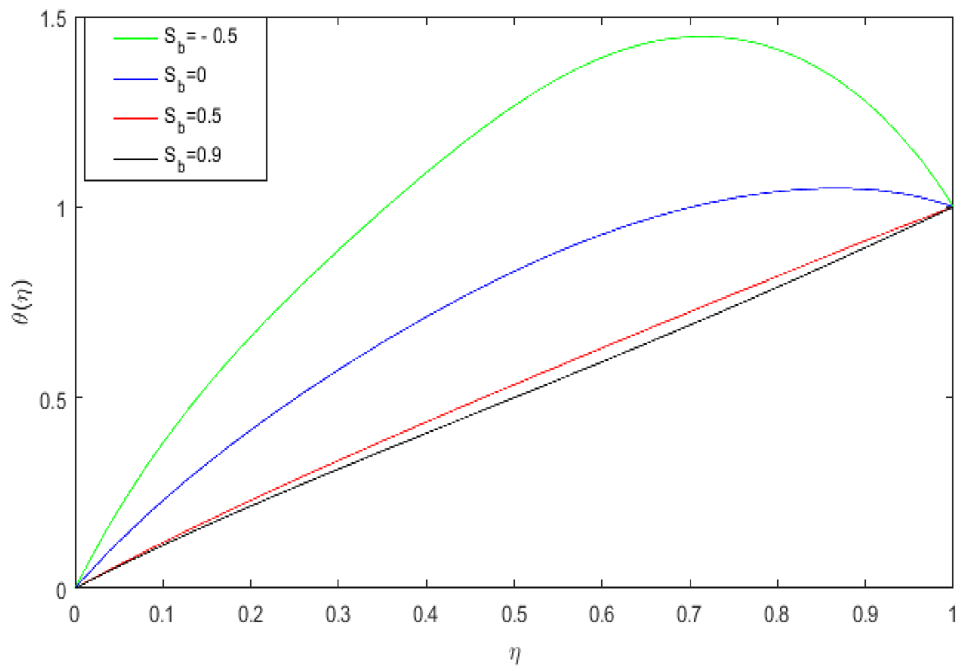

FIGURE 4.11: Influence of S_b on $\theta(\eta)$.
$R = Ec = 0.3$, $M = 2.0 = Da$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $\beta = \delta = 0.1$ and $S = 0.5$

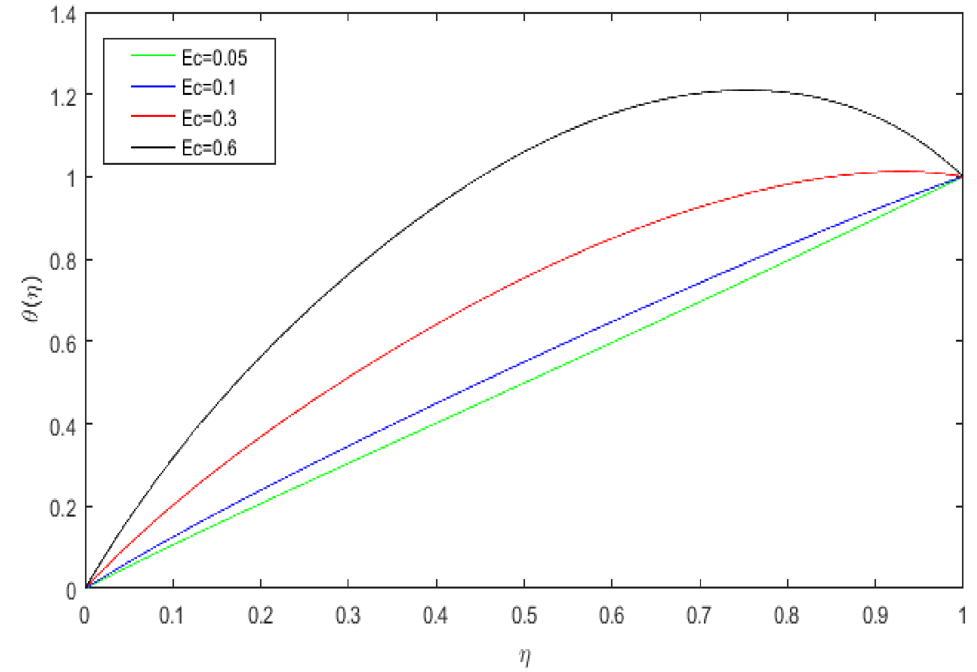

FIGURE 4.12: Influence of Ec on $\theta(\eta)$.
$R = 0.3,\ M = 2.0 = Da,\ Pr = 1.0,\ \gamma = \frac{\pi}{4},\ S = 0.5$ and $\beta = S_b = \delta = Ec = 0.1$

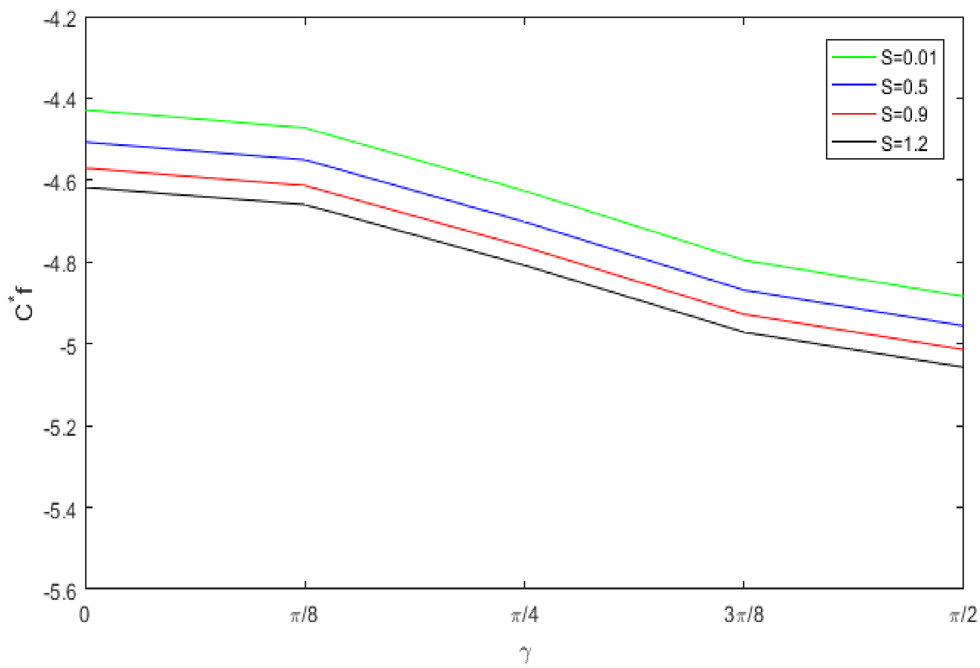

FIGURE 4.13: Effect of S on C^*f.
$R = 0.5,\ M = 3.0,\ Pr = 1.0,\ \gamma = \frac{\pi}{2},\ Da = 2.0$ and $\beta = S_b = \delta = Ec = 0.1$

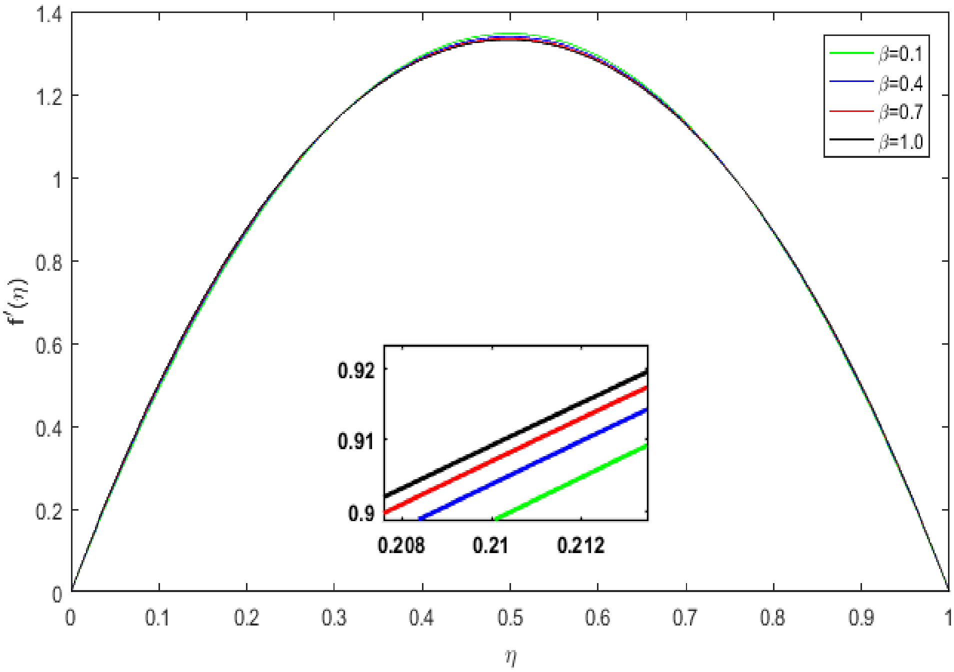

FIGURE 4.14: Impact of β parameter on $f'(\eta)$.
$R = 0.2$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $S = 0.5$ and $\beta = \delta = S_b = Ec = 0.1$

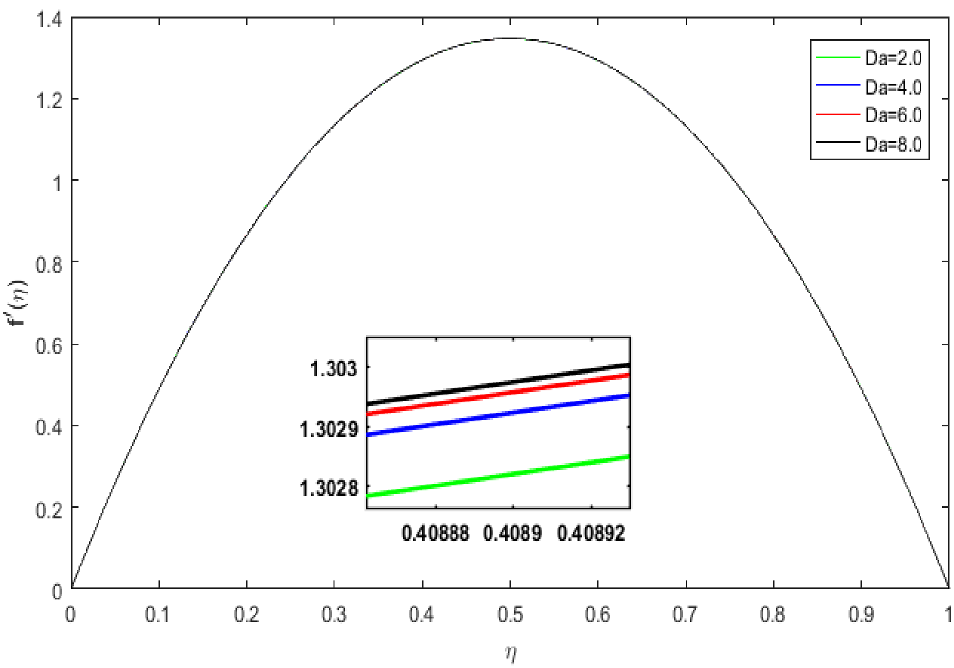

FIGURE 4.15: Impact of Da on $f'(\eta)$.
$R = 0.2$, $M = 3.0$, $Pr = 1.0$, $\gamma = \frac{\pi}{4}$, $S = 0.5$ and $\beta = \delta = S_b = Ec = 0.1$

Chapter 5

Conclusion

Firstly, the research and examination of the impact of magnetic fields on squeezing flow was performed to investigate the effects of inclination angle γ and secondly Darcy number and Casson flow is investigated by considering the inclined magnetic field effect in the velocity equations. u_s is stretching velocity lower plate, v_c denote mass flux velocity of lower plate, v_H represent velocity of upper plate, T_o denotes lower plate surface temperature, T_H is the upper plate surface temperature, Nu denote Nusselt number and C_f is the skin friction. The non-linear PDEs of mass transfer, momentum and energy changed into ODEs by utilizing a proper similarity transformation. By using the shooting method, numerical solution of these modeled ODEs is obtained. The fundamental points are described below.

• There is enhancement of fluid velocity in the regions close to the inner boundaries of plates, while in the centre of plates the velocity decreases due to increase in squeeze number.

• The squeeze number increases or the suction/injection parameter decreases the temperature of fluid.

• As magnetic parameter M, Eckert number Ec or γ increases their is enhancement in temperature profile.

• The temperature and velocity profile of the fluid close to the lower plate shows an opposite behaviour as compared to the fluid near the upper plate.

• By raising the Casson fluid parameter the velocity profile increases.

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