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Some Applications of Fixed-Point Results for Monotone Multivalued and Integral Type Contractive Mappings

by

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Dedicated to my ***Parents*** *and* ***Teachers***

CERTIFICATE OF APPROVAL

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Abstract

In this research study, fixed point results for monotone multivalued mappings in partially ordered complete $\mathcal{D}_b^{\star}$-metric sapces, where the partial ordered set $(\mathcal{W}, \preccurlyeq)$ is obtained via a pair of functions (κ, Θ) are established. Moreover, several existence and uniqueness coupled fixed point theorems of mappings satisfying contractive conditions have been investigated and verified in the setting of partially ordered complete $\mathcal{D}_b^{\star}$-metric sapces by using the concept of integral type contractions with respect to partially ordered $\mathcal{D}_b^{\star}$-metric sapces. Some corollaries being the spacial cases of main results are also presented. To strengthen the validity of our results some supportive examples are provided.

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Abbreviations

b**MS**	b-metric space
BCP	Banach Contraction Principle
b.above	bounded above
b.below	bounded below
CS	Cauchy Sequence
CF	Continuous Function
C.condition	Contraction Condition
CM	Contraction mapping
Con.	Convergent
$\mathcal{D}$**MS**	$\mathcal{D}$-metric space
$\mathcal{D}_b$**MS**	$\mathcal{D}_b$-metric space
$\mathcal{D}^\star$**MS**	$\mathcal{D}^\star$-metric space
$\mathcal{D}_b^\star$**MS**	$\mathcal{D}_b^\star$-metric space
FP	Fixed Point
LI	Lebesgue Integrable
LSC	Lower semi continuous
MS	Metric Space
MM	mixed monotone
MV	multivalued
PO	Partially Ordered
s.t.	such that
USC	Upper semi continuous

Symbols

$\rightarrow$	Approaches to
$(\mathcal{W}, \zeta_b)$	b-Metric space
$\in$	Belongs to
$(\mathcal{W}, \mathcal{D})$	$\mathcal{D}$-Metric space
$(\mathcal{W}, \mathcal{D}_b)$	$\mathcal{D}_b$-Metric space
$(\mathcal{W}, \mathcal{D}^\star)$	$\mathcal{D}^\star$-Metric space
$(\mathcal{W}, \mathcal{D}_b^\star)$	$\mathcal{D}_b^\star$-metric space
$\forall$	For all
$\implies$	Implies
∞	Infinity
lim	Limit
ζ	Metric function
$(\mathcal{W}, \zeta)$	Metric space
$\preccurlyeq$	Partial Order
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^+$	The set of non-negative real numbers
$\mathbb{N}$	The set of natural numbers
$\exists$	There exists

Chapter 1

Introduction

1.1 Background

Functional analysis emerged in the early 20th century as a key part of a broader shift towards abstraction in mathematics, often referred to as the "arithmetization" of analysis. This movement also laid the groundwork for modern abstract linear algebra, geometry, and topology. Today, functional analysis is a vast and diverse field, encompassing a significant portion of modern mathematical analysis, making it challenging to define precisely.

The concept of metric space (MS) was first introduced by M. Frechet [1, 2] in 1905, and later extended by his student D. Kurepa [3] in 1934 to more abstract spaces where the metric takes values in an ordered vector space. These spaces have been referred to by various names in the literature, including generalized MSs, cone valued MSs, vector valued MSs, and cone MSs. Since 2007, numerous authors have explored the properties of cone MSs over Banach spaces, and have investigated the existence of FPs in such spaces, as seen in works such as [4–6].

In various branches of mathematics and applied sciences, the existence of solutions to numerous mathematical models is tantamount to solving fixed point (FP) problems for specific maps. As a result, the study of FPs plays a pivotal role in many applied disciplines. At the heart of FP theory lies the existence of solutions to operator equations, subject to certain conditions, such as Fredholm and Volterra integral equations, two-point boundary value problems in differential equations, and certain eigenvalue problems. The foundation of FP theory is rooted in a elegant fusion of analytical, topological, and geometric concepts.

Consider a non-empty set $\mathcal{W}$ and a self-map $\mho : \mathcal{W} \to \mathcal{W}$. The FP problem seeks to answer the following questions:

- Does there exist an element $u \in \mathcal{W}$ such that $\mho(u) = u$?
- If such an element exists, is it unique?

In other words, the FP problem investigates whether the map $\mho$ has a FP u, and if so, whether that FP is the only one.

The solution to the FP problem relies on both the properties of the mapping $\mho$ and the structure of the set $\mathcal{W}$. Many researchers have investigated these factors to address this problem. A seminal answer was provided by Stefan Banach in his 1922 PhD thesis, which led to the renowned Banach Fixed Point Theorem [7]. This theorem establishes sufficient conditions for the existence and uniqueness of a FP, stating that if $\mathcal{W}$ is a complete MS and $\mho$ is a contraction mapping (CM) on $\mathcal{W}$, such that

$$\zeta(\mho(u), \mho(v)) \leq \alpha\zeta(u, v) \ \forall \ u, v \in \mathcal{W}, \tag{1.1}$$

where $\alpha \in [0, 1)$ then $\mho$ has a unique FP u in $\mathcal{W}$.

The Banach Fixed Point Theorem, also known as the Banach Contraction Principle (BCP), has become a fundamental tool in the theory of FPs, particularly in addressing non-linear problems in physics and engineering sciences.

Over the years, the BCP has proven to be a crucial instrument in establishing the existence of solutions to various non-linear issues. Moreover, it offers a straightforward and efficient method for finding the unique FP, providing both theoretical and practical significance.

Following the BCP, researchers have extensively developed the theory of FPs, branching out in two main directions. The first direction involves establishing conditions on the mapping $\mho$, while the second direction focuses on generalizing the set $\mathcal{W}$ to more abstract structures, leading to a richer understanding of FP theory and its applications.

The first generalization of the BCP was introduced by M. Edelstein [8], who relaxed the Banach condition (1.1) by considering distinct points in $\mathcal{W}$ and allowing the constant α to equal 1.

Later, Rakotch [9] further extended this generalization by replacing the constant α with a monotone decreasing function $\alpha : [0, \infty) \to [0, 1]$, leading to the contractive condition:

$$\zeta(\mho(u), \mho(v)) \leq \alpha(t)\zeta(u, v) \; \forall \; u, v \in \mathcal{W}. \tag{1.2}$$

For additional insights and extensions of contractive conditions, we recommend consulting the works cited in [10–12], among others. A thorough and comparative analysis of various CMs can be found in [13], providing a comprehensive overview of the topic.

Given that contractions are always continuous, a natural question arises: Can we find contraction-like conditions that don't necessarily imply continuity?

In 1968, Kannan [14] provided the first response to this inquiry, substituting the contraction condition (C.condition) with a new criterion:

$$\zeta(\mho(u), \mho(v)) \leq a\mho(u, \mho(u)) + \zeta(v, \mho(v) \text{ where } 0 < a < \frac{1}{2}, \tag{1.3}$$

along with additional requirements. Building upon Kannan's work, Chatterjea [15] later established a FP theorem for operators that satisfy the condition:

$$\zeta(\mho(u), \mho(v)) \leq b[\zeta(u, \mho(v)) + \zeta(v, \mho(u))] \; \forall \; u, v \in \mathcal{W}, \text{ where } 0 < b < 1. \tag{1.4}$$

In the second category of generalizations, researchers explored the structure of the underlying space on which the mapping $\mho$ is defined. Notable examples include the extension of FP theory to pseudo-MSs [16], metric-like spaces [17], 2-MS [18–20], partially ordered (PO) MSs [21, 22], cone MSs [23, 24], and b-metric spaces (bMSs) [25, 26], significantly broadening the scope of the theory.

In the late 1960s, Nadler [27] and Markin [28] significantly expanded the FP theory by extending it from single-valued to multivalued (MV) mappings. A MV mapping is defined as a mapping $\mho$ from a non-empty set $\mathcal{W}$ to a collection of non-empty subsets of $\mathcal{W}$, denoted as $\mho(u)$. A point u is considered a FP of $\mho$ if it belongs to the subset $\mho(u)$. Numerous authors have investigated the existence of FPs for MV mappings under various conditions [29, 30].

Nadler's Theorem has been extensively generalized and extended by relaxing the contractive requirements, but with additional constraints, such as considering compact-valued mappings [30–33].

Theorems related to FPs in partially ordered (PO) MSs are essential tools for establishing the existence and singularity of solutions to various differential and integral equations. In recent decades, the study of MV mappings has developed into a crucial and vibrant branch of mathematics, garnering significant attention and interest, with far-reaching applications in fields like convex optimization, and differential inclusions, attracting significant attention and interest.

The successful application of the BCP in establishing the occurrence and singularity of solutions for numerous integral and differential equations inspired researchers to explore generalizations and extensions of this fundamental theorem. Edelstein [34] made a notable breakthrough by extending the BCP, introducing a more relaxed set of conditions. By examining subsequences within the sequence of iterates, Edelstein established a weaker condition, which has since become famously known as Edelstein's FP theorem.

Simultaneously, Boyd and Wong [35] extended the existing framework by introducing a continuous function (CF), $\kappa : [0, \infty) \to [0, \infty)$, which substitutes $\alpha\zeta(u, v)$ with $\sigma(\zeta(u, v))$ where $\sigma : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, thereby generalizing BCP. In 1976, Caristi [36] made a significant contribution by developing a set of comprehensive FP theorems, leveraging the characterization of weakly inward mappings, thereby expanding the existing framework for FP theory. Furthermore, an equivalent result of the BCP in the context of PO sets was proven in [37], now known as the Ran-Reurings FP theorem, offering a significant extension to the original theorem.

Despite the successes in generalizing the BCP, there was an unsuccessful attempt by Dhage et al. [38] to extend it to $\mathcal{D}$-metric space ($\mathcal{D}$MS) topology, which was later shown to have limitations [39–41].

However, in 2007, Sedghi et al. [42] corrected this by introducing the concept of $\mathcal{D}^{\star}$-metric spaces ($\mathcal{D}^{\star}$MS), a revised version of $\mathcal{D}$MS. This correction paved the way for various authors [43, 44] to establish FP theorems in these spaces, making significant progress in the field.

Researchers have also explored coupled FP results for mixed monotone mappings in ordered MSs [45, 46], and for further insights into coupled FP and n-tuples FP theorems, readers can refer to [47] and its cited references. Furthermore, FP problems have been broadly studied in the setting of PO complete $\mathcal{D}^{\star}$MSs.

Notably, Al. Jumaili [48] utilized the concept of $\mathcal{D}^{\star}$MSs to establish coincidence FP theorems for functions that satisfy certain contractive properties involving monotone increasing η-mappings in these spaces, contributing to the advancement of the field.

Ghasab et al. [49] used the idea of integral-type contractions to establish coupled FP theorems in ordered G-MSs. Majid et al. [50] developed and investigated new FP theorems for MV functions in PO complete $\mathcal{D}^{\star}$MSs, where the order is defined by a pair of functions (κ, Θ). Majid et al. investigated the existence and uniqueness of coupled FPs for mappings that satisfy contractive conditions in these spaces, employing integral-type contractions in their approach.

This thesis presents the concept of $\mathcal{D}_b^{\star}$MSs and develops significant results within this framework, which are rigorously proven. To illustrate the applicability of these results, a supporting example is also provided.

The thesis is further divided into four chapters, which are organized in the following manner:

Chapter 2 provides the fundamental definitions and concepts, including MSs, bMSs, $\mathcal{D}$MSs, $\mathcal{D}^{\star}$MSs, FPs, CMs and partial ordered relations. Illustrative examples related to these concepts are also given.

Chapter 3 provides the review of the article [50]. Some FP results in $\mathcal{D}^{\star}$MSs endowed with monotone MV and integral type contractive mappings are presented. In the end, an example is presented to show the validity of obtained results.

Chapter 4 investigates the existence of FPs in $\mathcal{D}_b^{\star}$MSs, focusing on monotone MV and integral-type contractive mappings. The chapter culminates with a illustrative example that demonstrates the efficacy and validity of the obtained results.

Chapter 5 provides the conclusion of the thesis.

Chapter 2

Preliminaries

This chapter lays the foundation for future chapters by introducing essential concepts from functional analysis. We define and illustrate key terms such as:

- Metric spaces
- b-metric spaces
- $\mathcal{D}$-metric spaces
- $\mathcal{D}^{\star}$-metric spaces

Additionally, various types of CMs are explained through examples.

2.1 Metric Space

A MS is a set of points where the distance between every pair of points has a defined and numerical distance between them, enabling comparisons of point locations and distances. This metric adheres to specific axioms, enabling the development of fundamental notions like convergence, continuity, and related concepts.

Maurice Fréchet [1], a French mathematician, introduced the idea of metric spaces in 1906.

Definition 2.1. "Let $\mathcal{W}$ be a non-empty set. A function $\zeta : \mathcal{W} \times \mathcal{W} \to \mathbb{R}$ is called a metric on $\mathcal{W}$, if $\forall\ u, v, w \in \mathcal{W}$, the following axioms are satisfied:

(M_1) $\zeta(u,v) \geq 0$;

(M_2) $\zeta(u,v) = 0 \iff u = v$;

(M_3) $\zeta(u,v) = \zeta(v,u)$; (Symmetry)

(M_4) $\zeta(u,w) \leq \zeta(u,v) + \zeta(v,w)$. (Triangle Inequality)

The pair $(\mathcal{W},\zeta)$ is called a metric space" [51].

Example 2.1.1. Let $\mathcal{W} = \mathbb{R}$. The mapping $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as:

$$\zeta(u,v) = \frac{|u-v|}{1+|u-v|} \quad \forall\, u,v \in \mathcal{W},$$

is a metric on $\mathbb{R}$ and $(\mathbb{R},\zeta)$ is a MS.

Example 2.1.2. Let $\mathcal{W} = \mathbb{R}^2$, define $\zeta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\zeta(u,v) = max\{|u_2 - u_1|, |v_2 - v_1|\} \quad \forall\, u = (u_1,v_1), \quad v = (u_2,v_2).$$

The metric ζ satisfies all the axioms of MS. Hence, $(\mathcal{W},\zeta)$ is a metric space.

Example 2.1.3. Let $\mathcal{W} = C[0,1]$ is the set of CFs on $[0,1]$. Define $\zeta : C[0,1] \times C[0,1] \to \mathbb{R}$ by

$$\zeta(f,g) = \int_0^1 |\mathtt{f}(u) - \mathtt{g}(u)|du,$$

Then is a metric on $C[0,1]$.

Definition 2.2. "Let $(\mathcal{W},\zeta)$ be a MS. A sequence (u_s) in $\mathcal{W}$ is said to be convergent to $u \in \mathcal{W}$ if

$$\lim_{s\to\infty} \zeta(u_s,u) = 0$$

or, given any $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$$\zeta(u_s,u) < \epsilon \quad \forall s \geq N\text{" [51].}$$

Example 2.1.4. The sequence $(\frac{1}{n})$ in $\mathbb{R}$ is convergent in $\mathbb{R}$ with metric defined as $\zeta(u,v) = |u-v|$.

Definition 2.3. "A sequence (u_s) in a MS $(\mathcal{W}, \zeta)$ is called a Cauchy sequence (CS), if given any $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t.

$$\zeta(u_s, u_t) < \epsilon, \quad \forall s, t \geq N\text{" [51].}$$

Remark: Every convergent sequence in a MS is CS.

Definition 2.4. "A MS $(\mathcal{W}, \zeta)$ is called complete if every Cauchy sequence (CS) in $\mathcal{W}$ converges in $\mathcal{W}$" [51].

Example 2.1.5. Every finite dimensional MS is complete.

Example 2.1.6. The space $C[0, 1]$ is a complete MS.

2.2 b-Metric Space

In 1989, I. Bakhtin [26] pioneered the concept of bMSs, introducing a new mathematical framework that has since been widely studied and applied.

Definition 2.5. "Let $\mathcal{W} \neq \emptyset$. A mapping $\zeta_b : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+$ is called a b-metric if $\forall\ u, v, w \in \mathcal{W}$ the following conditions are satisfied:

(B_1) $\zeta_b(u, v) = 0 \iff u = v$;

(B_2) $\zeta_b(u, v) = \zeta_b(v, u)$;

(B_3) $\zeta_b(u, w) \leq l\{\zeta_b(u, v) + \zeta_b(v, w)\}$, where $l \geq 1$.

The pair $(\mathcal{W}, \zeta_b)$ is called bMS" [26].

Remark: Class of bMS is bigger than class of MS as for $l = 1$ (B_3) become same as triangular inequality of MS.

Remark: The convergence, completeness and Cauchyness in bMS can be generalized from MS.

Example 2.2.1. Let $\mathcal{W} = \mathbb{R}$. The mapping $\zeta_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by:

$$\zeta_b(u, v) = (u - v)^2, \quad \forall\, u, v \in \mathcal{W},$$

is b-metric with $l = 2$.

Example 2.2.2. Let $\mathcal{W} = C[0, 1]$ be a space of CF on $[0, 1]$. Define:

$$\zeta_b(f, g) = \int_0^1 |f(u) - g(u)|^3 du.$$

Then $(\mathcal{W}, \zeta_b)$ is a bMS with $l = 3$.

2.3 $\mathcal{D}$-Metric Space

The concept of $\mathcal{D}$MS was first introduced by B.C. Dhage [38], who also developed the theory of convergence of sequences in such spaces.

Definition 2.6. "Let $\mathcal{W} \neq \emptyset$, a real valued function $\mathcal{D}$ on $\mathcal{W} \times \mathcal{W} \times \mathcal{W}$ is said to be a $\mathcal{D}$-metric on $\mathcal{W}$ if the following conditions are satisfied:

(D_1) $\mathcal{D}(u, v, w) \geq 0 \quad for\ all\ u, v, w \in \mathcal{W}$. (non-negativity);

(D_2) $\mathcal{D}(u, v, w) = 0 \iff u = v = w$ (coincidence);

(D_3) $\mathcal{D}(u, v, w) = \mathcal{D}(p(u, v, w))$ for any permutation $p(u, v, w)$ of u, v, w (symmetry);

(D_4) $\mathcal{D}(u, v, w) \leq \mathcal{D}(u, v, x) + \mathcal{D}(u, x, w) + \mathcal{D}(x, v, w)$ $\forall\, u, v, w, x \in \mathcal{W}$. (Tetrahedral inequality).

Then $(\mathcal{W}, \mathcal{D})$ is called $\mathcal{D}$MS" [38].

Example 2.3.1. Let $\mathcal{W} = \mathbb{R}$, define $\mathcal{D} : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by,

$$\mathcal{D}(u, v, w) = |u - v| + |u - w| + |v - w| \quad \forall\, u, v, w \in \mathcal{W}.$$

Then $(\mathbb{R}, \mathcal{D})$ is a $\mathcal{D}$MS.

Example 2.3.2. For $u, v, w \in \mathbb{R}$ define,

$$\mathcal{D}_\infty(u, v, w) = max\{|u - v|, |u - w|, |v - w|\} \quad \forall\, u, v, w \in \mathcal{W},$$

then $(\mathcal{W}, \mathcal{D}_\infty)$ is a $\mathcal{D}$MS.

Example 2.3.3. Define a function $\mathcal{D}$ on $\mathcal{W} \times \mathcal{W} \times \mathcal{W}$ by

$$\mathcal{D}(u, v, w) = \begin{cases} 0, & \text{if } u = v = w \\ 1, & \text{otherwise,} \end{cases}$$

then $(\mathcal{W}, \mathcal{D})$ is a $\mathcal{D}$MS.

Definition 2.7. "A sequence (u_s) in a $\mathcal{D}$MS $(\mathcal{W}, \mathcal{D})$ is said to be $\mathcal{D}$-convergent (or convergent) if there exists an element $u \in \mathcal{W}$ such that for given $\epsilon > 0$ there exists a positive integer t_0 such that $\mathcal{D}(u_s, u_t, u) < \epsilon \quad \forall\, t \geq t_0$ *and* $s \geq t_0$" [38].

Definition 2.8. "A sequence (u_s) in $\mathcal{D}$MS is said to be CS if for all $\epsilon > 0$ there exists an $s_0 \in \mathbb{N}$ such that for all $s,\ t,\ p \geq s_0, \mathcal{D}(u_s, u_t, u_p) < \epsilon$" [38].

Definition 2.9. "A $\mathcal{D}$MS $(\mathcal{W}, \mathcal{D})$ is said to be complete if every CS in $(\mathcal{W}, \mathcal{D})$ converges" [38].

2.4 $\mathcal{D}^\star$-Metric Space

In 2007 Sedghi [42] introduced the notion of $\mathcal{D}^\star$MS.

Definition 2.10. "Let $\mathcal{W} \neq \emptyset$. A function $\mathcal{D}^\star : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to [0, \infty)$ is called $\mathcal{D}^\star$-metric on $\mathcal{W}$ if for all $u, v, w, x \in \mathcal{W}$, the following conditions are satisfied:

$(D_1^\star)$ $\mathcal{D}^\star(u, v, w) = 0 \iff u = v = w$;

$(D_2^\star)$ $\mathcal{D}^\star(u, v, w) = \mathcal{D}^\star(p\{u, v, w\})$, where p is a permutation function;

$(D_3^\star)$ $\mathcal{D}^\star(u, v, w) \leq \mathcal{D}^\star(u, v, x) + \mathcal{D}^\star(x, w, w)$.

Then $(\mathcal{W}, \mathcal{D}^\star)$ is called $\mathcal{D}^\star$MS" [42].

Example 2.4.1. Let $\mathcal{W} = \mathbb{R}$. The function $\mathcal{D}^\star : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\mathcal{D}^\star(u, v, w) = max\{\zeta(u, v), \zeta(u, w), \zeta(v, w)\},$$

is a $\mathcal{D}^\star$-metric on $\mathcal{W}$, where $\zeta(u, v) = |u - v|$.

Example 2.4.2. If $\mathcal{W} = \mathbb{R}$, define

$$\mathcal{D}^\star(u, v, w) = \begin{cases} 0, & \text{if } u = v = w, \\ max\{u, v, w\}, & \text{otherwise.} \end{cases}$$

Then $\mathcal{D}^\star$ is $\mathcal{D}^\star$-metric on $\mathcal{W}$.

Example 2.4.3. If $\mathcal{W} = \mathbb{R}^n$, define

$$\mathcal{D}^\star(u, v, w) = (\|u - v\|^p + \|u - w\|^p + \|v - w\|^p)^{\frac{1}{p}},$$

for every $p \in \mathbb{R}^+$. Then $(\mathcal{D}^\star, \mathbb{R}^n)$ is $\mathcal{D}^\star$MS.

Remark: In a $\mathcal{D}^\star$MS, we show that the equality $\mathcal{D}^\star(u, u, v) = \mathcal{D}^\star(u, v, v)$ holds for all $u, v \in \mathcal{W}$.

$$\mathcal{D}^\star(u, u, v) \leq \mathcal{D}^\star(u, u, u) + \mathcal{D}^\star(u, v, v) = \mathcal{D}^\star(u, v, v), \tag{2.1}$$

and similarly

$$\mathcal{D}^\star(v, v, u) \leq \mathcal{D}^\star(v, v, v) + \mathcal{D}^\star(v, u, u) = \mathcal{D}^\star(v, u, u). \tag{2.2}$$

Hence, by (2.1) and (2.2), $\mathcal{D}^\star(u, u, v) = \mathcal{D}^\star(u, v, v)$ [42].

Definition 2.11. "Suppose that $(\mathcal{W}, \mathcal{D}^\star)$ is a $\mathcal{D}^\star$MS. A sequence (u_s) in $\mathcal{W}$ is said to convergent to $u \in \mathcal{W}$ if and only if $\mathcal{D}^\star(u_s, u_s, u) = \mathcal{D}^\star(u, u, u_s) \to 0$ as $s \to \infty$.

That is, for each $\epsilon > 0$, there exist a positive integer s_0 such that for all $s \geq s_0 \Rightarrow \mathcal{D}^\star(u, u, u_s) < \epsilon$. Equivalently: for each $\epsilon > 0$, there exist a positive integer s_0 such that $\mathcal{D}^\star(u, u_s, u_r) < \epsilon \quad \forall\ s, r \geq s_0$" [42].

Definition 2.12. "A sequence $(u_s) \in \mathcal{W}$ is called a CS if for given any $\epsilon > 0$ there exists s_0 (a positive integer) such that, for each $s, r \geq s_0$,

$$\mathcal{D}^\star(u_s, u_s, u_r) < \epsilon\text{"[42]}.$$

Definition 2.13. "A $\mathcal{D}^\star$MS $(\mathcal{W}, \mathcal{D}^\star)$ is a complete $\mathcal{D}^\star$-metric if every CS in $(\mathcal{W}, \mathcal{D}^\star)$ converges in $(\mathcal{W}, \mathcal{D}^\star)$" [42].

Lemma 2.4.4. "Let $(\mathcal{W}, \mathcal{D}^\star)$ be a $\mathcal{D}^\star$MS. If (u_s) is convergent sequence then it is also CS" [42].

Proof. Let (u_s) is convergent i.e, $\exists\ \epsilon > 0$ such that

$\mathcal{D}^\star(u_s, u_s, u) < \frac{\epsilon}{2},\ \forall\ s \geq s_0.$

Now for all $s, r \geq s_0$

$$\begin{aligned}\mathcal{D}^\star(u_s, u_s, u_r) &\leq \mathcal{D}^\star(u_s, u_s, u) + \mathcal{D}^\star(u, u_r, u_r)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ &= \epsilon.\end{aligned}$$

$\Rightarrow (u_s)$ is a CS. □

Lemma 2.4.5. "Let $(\mathcal{W}, \mathcal{D}^\star)$ be a $\mathcal{D}^\star$MS. Then $\mathcal{D}^\star$ is a CF on $\mathcal{W}^3$, that is,

$$\lim_{s \to \infty} \mathcal{D}^\star(u_s, v_s, w_s) = \mathcal{D}^\star(u, v, w),$$

whenever a sequence (u_s, v_s, w_s) in $\mathcal{W}^3$ converges to a point $(u, v, w) \in \mathcal{W}^3$" [42].

Proof. Suppose the sequence (u_s, v_s, w_s) in $\mathcal{W}^3$ converges to a point $(u, v, w) \in \mathcal{W}^3$, that is,

$$\lim_{s \to \infty} u_s = u,\ \lim_{s \to \infty} v_s = v,\ \lim_{s \to \infty} w_s = w.$$

Then for each $\epsilon > 0\ \exists\ s_1,\ s_2,\ s_3 \in \mathbb{N}$ such that

$\mathcal{D}^\star(u, u, u_s) < \frac{\epsilon}{3}\ \forall\ s \geq s_1, \mathcal{D}^\star(v, v, v_s) < \frac{\epsilon}{3}\ \forall\ s \geq s_2$ *and*

$\mathcal{D}^\star(w, w, w_s) < \frac{\epsilon}{3}\ \forall\ s \geq s_3.$

If we set $s_0 = max\{s_1, s_2, s_3\}$ then $\forall s \geq s_0$, by triangular inequality

$$\begin{aligned}\mathcal{D}^\star(u_s, v_s, w_s) &\leq \mathcal{D}^\star(u_s, v_s, w) + \mathcal{D}^\star(w, w_s, w_s)\\ &\leq \mathcal{D}^\star(u_s, w, v) + \mathcal{D}^\star(v, v_s, v_s) + \mathcal{D}^\star(w, w_s, w_s)\\ &\leq \mathcal{D}^\star(w, v, u) + \mathcal{D}^\star(u, u_s, u_s) + \mathcal{D}^\star(v, v_s, v_s) + \mathcal{D}^\star(w, w_s, w_s)\\ &< \mathcal{D}^\star(u, v, w) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\\ &= \mathcal{D}^\star(u, v, w) + \epsilon.\end{aligned}$$

$\Rightarrow \mathcal{D}^\star(u_s, v_s, w_s) - \mathcal{D}^\star(u, v, w) < \epsilon.$

Now

$$\begin{aligned}\mathcal{D}^\star(u, v, w) &\leq \mathcal{D}^\star(u, v, w_s) + \mathcal{D}^\star(w_s, w, w)\\ &\leq \mathcal{D}^\star(u, w_s, v_s) + \mathcal{D}^\star(v_s, v, v) + \mathcal{D}^\star(w_s, w, w)\\ &\leq \mathcal{D}^\star(w_s, v_s, u_s) + \mathcal{D}^\star(u_s, u, u) + \mathcal{D}^\star(v_s, v, v) + \mathcal{D}^\star(w_s, w, w)\\ &< \mathcal{D}^\star(u_s, v_s, w_s) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\\ &= \mathcal{D}^\star(u_s, v_s, w_s) + \epsilon.\end{aligned}$$

$\Rightarrow \mathcal{D}^\star(u, v, w) - \mathcal{D}^\star(u_s, v_s, w_s) < \epsilon.$

Therefore

$$|\mathcal{D}^\star(u_s, v_s, w_s) - \mathcal{D}^\star(u, v, w)| < \epsilon \ \forall \ s \geq s_0.$$

$\Rightarrow \lim_{s\to\infty} \mathcal{D}^\star(u_s, v_s, w_s) = \mathcal{D}^\star(u, v, w).$ □

2.5 Banach Contraction Principle

Definition 2.14. "Let $(\mathcal{W}, \zeta)$ be a MS and $\mho$ be a self mapping i.e, $\mho : \mathcal{W} \to \mathcal{W}$. Then $\mho$ is called a CM if there exist a constant $0 \leq \alpha < 1$ such that $\forall u, v \in \mathcal{W}$

$$\zeta(\mho(u), \mho(v)) \leq \alpha\zeta(u, v).$$

This α is called contraction constant" [51].

Example 2.5.1. Let $\mathcal{W} = \mathbb{R}$ with usual metric on $\mathbb{R}$. The function $\mho : \mathbb{R} \to \mathbb{R}$ defined as $\mho(u) = \frac{1}{2}u$ is a contraction with $\alpha = \frac{1}{2}$.

Example 2.5.2. Let the function $\mho : \mathbb{R} \to \mathbb{R}$ defined as:

$$\mho(u) = \sin(\sin u).$$

The derivative of $\mho$ is

$$\mho'(u) = \cos(\sin u)[\cos u].$$

The derivative is bounded by

$$|\mho'(u)| \leq |\cos(\sin u)||\cos u| \leq 1.$$

Since the derivative is bounded by 1, the function $\mho(u) = \sin(\sin u)$ is a CM on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Definition 2.15. "A FP of a function $\mho : \mathcal{W} \to \mathcal{W}$ is a point u s.t.

$$\mho(u) = u\text{" [51].}$$

Geometrically:

Let $\mathbf{y} = \mho(u)$ be a real valued function then FP is the point of intersection of the curve $\mathbf{y} = \mho(u)$ and the line $\mathbf{y} = u$.

In general a mapping may or may not have FPs, and a FP may or may not be unique. The existence and uniqueness of FPs vary depending on the specific mapping.

Example 2.5.3. Consider $\mho : \mathbb{R} \to \mathbb{R}$ defined by:

$$\mho(u) = \frac{u+1}{2} + 7.$$

Then $\mho$ has a unique FP.

Example 2.5.4. Let $\mathcal{W} = \mathbb{R}$. Define a mapping $\mho : \mathbb{R} \to \mathbb{R}$ by:

$$\mho(u) = \frac{u + \sqrt{u^2+1}}{2}.$$

Then $\mho$ has no FP.

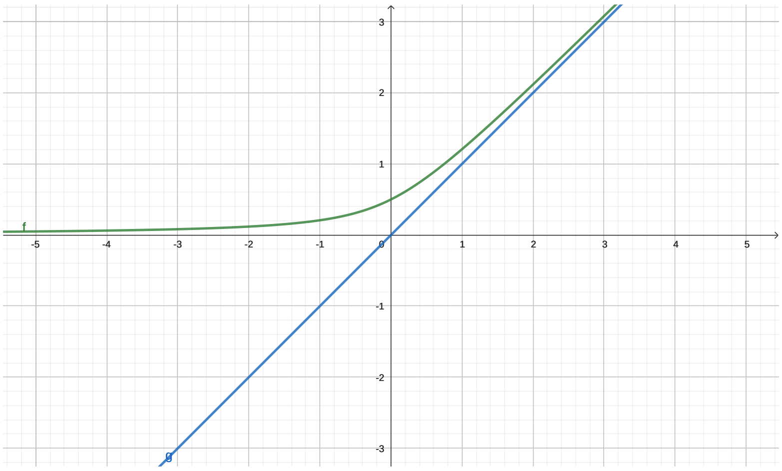

FIGURE 2.1: Mapping having no FP

Example 2.5.5. Let $\mathcal{W} = \mathbb{R}$. The mapping

$$\mho(u) = 4u^3,$$

has three FPs $u = 0, \pm\frac{1}{2}$.

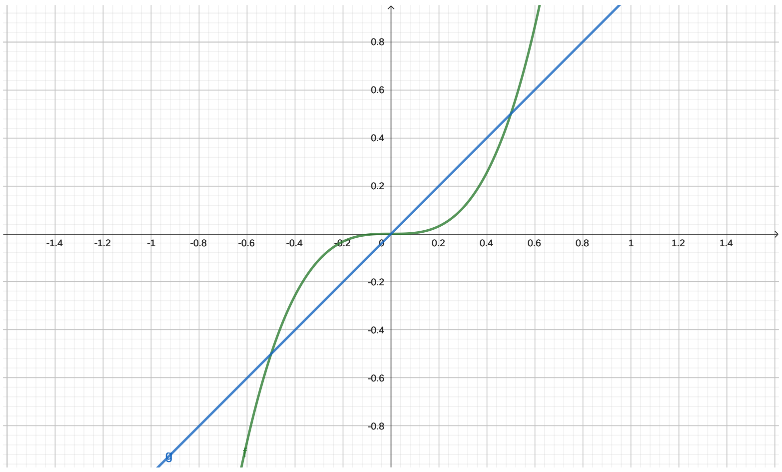

FIGURE 2.2: Mapping having more than one FPs

Example 2.5.6. Let $\mathcal{W} = \mathbb{R}$. The transcendental function

$$\mho(u) = u + cos(u),$$

has infinitely many FPs.

The BCP [7], a seminal result in mathematical analysis, was first introduced by renowned Polish mathematician Stefan Banach in 1922.

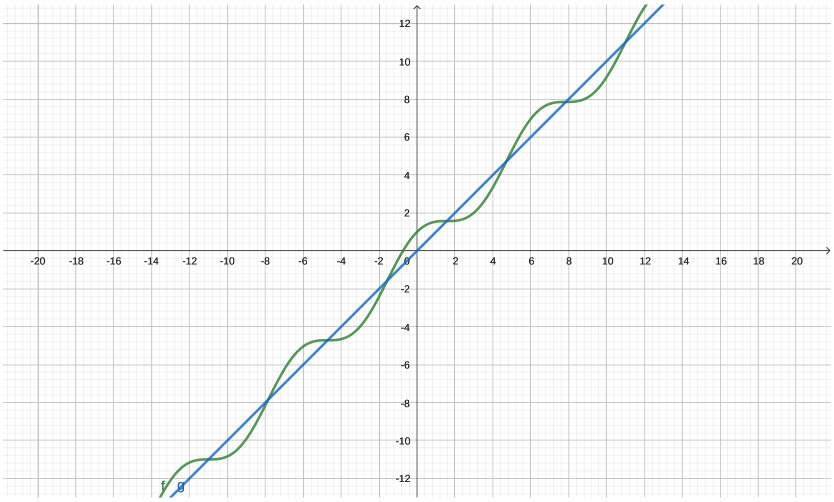

FIGURE 2.3: Mapping having infinitely many FPs

Theorem 2.5.7. "Assume $(\mathcal{W}, \zeta)$ is a MS where $\mathcal{W} \neq \emptyset$. Suppose that $\mathcal{W}$ is complete and let $\mho : \mathcal{W} \to \mathcal{W}$ be a CM on $\mathcal{W}$. Then $\mho$ has precisely one FP" [51].

Example 2.5.8. Let $(\mathbb{R}, \zeta)$ be a MS, where $\zeta(u, v) = |u - v|$. The mapping $\mho : \mathbb{R} \to \mathbb{R}$ is defined by:

$$\mho(u) = \frac{u+3}{9},$$

is a contraction with $\alpha = \frac{1}{9}$, and $\mho$ has only one FP i.e, $u = \frac{3}{8}$.

2.6 Partial Order Relation

Definition 2.16. "Let $\mathcal{W}$ be a non-empty set. The relation $\preccurlyeq$ on $\mathcal{W}$ is said to be partial order if the following conditions are satisfied for each $u, v, w \in \mathcal{W}$:

(i) $u \preccurlyeq u$; (Reflexive)

(ii) $u \preccurlyeq v \textit{ and } v \preccurlyeq u \iff u = v$; (Anti-symmetric)

(iii) $u \preccurlyeq v \textit{ and } v \preccurlyeq w \Rightarrow u \preccurlyeq w$. (Transitive)

The set $\mathcal{W}$ with partial order $\preccurlyeq$ is called partially ordered (PO) set" [52].

Example 2.6.1. The set of real numbers, $\mathbb{R}$, is PO by the standard "less than" $(<)$ and "greater than" $(>)$ relations.

Example 2.6.2. Let $\mathcal{W} = \mathbb{R}^2$. Define a partial order relation $\preccurlyeq$ on $\mathcal{W}$ by

$$(u_1, u_2) \preccurlyeq (v_1, v_2) \iff u_1 \leq v_1 \text{ and } u_2 \leq v_2,$$

where $\leq$ is the standard ordering on $\mathbb{R}$. Then clearly $\preccurlyeq$ is a PO on $\mathcal{W} = \mathbb{R}^2$ and $\mathbb{R}^2$ is a PO set.

Definition 2.17. "Let $(\mathcal{W}, \zeta)$ be a MS and $\Theta : \mathcal{W} \to [0, \infty)$ be a functional. Define the relation $\preccurlyeq$ on $\mathcal{W}$ by $u \preccurlyeq v$ if and only if $\zeta(u, v) \leq \Theta(u) - \Theta(v)$. Then $\preccurlyeq$ is a partial order relation on $\mathcal{W}$ induced by Θ and $(\mathcal{W}, \preccurlyeq)$ is called an ordered MS introduced by Θ" [36].

Definition 2.18. "Suppose that $(\mathcal{W}, \preccurlyeq)$ is an ordered partial MS. If relation $\preccurlyeq$ is defined on $\mathcal{W}^2$ by $(a, b) \preccurlyeq (u, v) \iff a \preccurlyeq u,\ b \preccurlyeq v$. Then $(\mathcal{W}^2, \preccurlyeq)$ is an ordered partial MS" [53].

Definition 2.19. "Suppose that $(\mathcal{W}, \preccurlyeq)$ is a PO set. The mapping $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ is said to have mixed monotone property if $\mathcal{H}(u, v)$ is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument; i.e, for all $u_1, u_2 \in \mathcal{W}, u_1 \preccurlyeq u_2 \Rightarrow \mathcal{H}(u_1, v) \preccurlyeq \mathcal{H}(u_2, v)\ \forall\ v \in \mathcal{W}$ and for all $v_1, v_2 \in \mathcal{W}, v_1 \preccurlyeq v_2 \Rightarrow \mathcal{H}(u, v_1) \succcurlyeq \mathcal{H}(u, v_2)\ \forall\ u \in \mathcal{H}$" [53].

Definition 2.20. "An element $(a, b) \in \mathcal{W}^2$ is said to be a coupled FP of a mapping $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ if $\mathcal{H}(a, b) = a$ *and* $\mathcal{H}(b, a) = b$" [53].

Example 2.6.3. Consider the mapping $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{H}(u_1, u_2) = u_1 + u_2, \qquad \forall\ (u_1, u_2) \in \mathbb{R}^2.$$

Then the point $(0, 0)$ is a coupled FP of the mapping $\mathcal{H}$.

Example 2.6.4. Consider the mapping $\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{H}(u, v) = u^3, \qquad \forall (u, v) \in \mathbb{R}^2. \tag{2.3}$$

Then the points $(-1, 0), (0, -1), (0, 0), (1, 0), (0, 1)$ and $(1, 1)$ are all the coupled FPs of the mapping $\mathcal{H}$.

Definition 2.21. "A function $\mho : \mathcal{W} \to \mathbb{R}$ is called a lower semi-continuous (LSC) if for any $\{u_n\} \subset \mathcal{W}$ *and* $u \in \mathcal{W}$

$$u_n \to u \Rightarrow \mho(u) \leq \lim_{n\to\infty} \inf \mho(u_n)\text{" [54].}$$

Example 2.6.5. Define a function $\mho : \mathbb{R} \to \mathbb{R}$ as:

$$\mho(u) = \begin{cases} -1, & \text{if } u = 0 \\ \sin\frac{1}{u}, & \text{if } u \neq 0. \end{cases}$$

Since the left and right limits do not exist, so $\mho$ is not left or right-continuous at $u = 0$. It is LSC at $u = 0$, as the limit inferior of $\mho$ as u approaches 0 is greater than or equal to $\mho(0) = -1$.

Definition 2.22. "Suppose $\kappa : [0, \infty) \to [0, \infty)$ be a mapping which satisfies the following properties:

(i) κ is continuous:

(ii) κ is non-decreasing with $\kappa(\tau) = 0 \iff \tau = 0$.

The collection of all mappings satisfying above properties are named as σ" [36].

Definition 2.23. "Define $\Theta : [0, \infty) \to [0, \infty)$ satisfying:

(i) Θ is LSC;

(ii) $\Theta(\tau) > 0\ \forall\ \tau = 0$ and $\Theta(0) = 0$.

The collection of mappings having the above properties are named as η" [36].

Definition 2.24. "Suppose $\mathcal{A}$ and $\mathcal{B}$ are non-empty sets. A MV mapping from $\mathcal{A}$ to $\mathcal{P}(\mathcal{B})$ is denoted by $\mathcal{H} : \mathcal{A} \to 2^{\mathcal{B}}$, where $\mathcal{H}$ is a function that maps elements from $\mathcal{A}$ to subsets of $\mathcal{B}$" [55].

Example 2.6.6. Let

$$\mathcal{A} = \{u_1, u_2, u_3, u_4, u_5\}$$
$$\mathcal{B} = \{1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, \ldots, 4\}$$

Define $\mathcal{H} : \mathcal{A} \to \mathcal{P}(\mathcal{B})$, by

$$\mathcal{H}(u_1) = \{1, 2, 2.75\} \quad \mathcal{H}(u_2) = \{1.5, 2.5, 3\} \quad \mathcal{H}(u_3) = \{2, 2.5, 3.25\}$$
$$\mathcal{H}(u_4) = \{2.5, 3.5, 3.75\} \quad \mathcal{H}(u_5) = \{4\} \quad \mathcal{H}(u_6) = \{1, 2, 3, 3.5\}$$

Then $\mathcal{H}$ is a MV mapping.

Chapter 3

Some Results in $\mathcal{D}^{\star}$-Metric Spaces

This chapter provides an in-depth examination of the paper [50], focusing specifically on the investigation of MV and integral type contractive mappings.

3.1 MV Functions and $\mathcal{D}^{\star}$MSs

This part focuses on introducing and examining new FP theorems for monotone MV functions, specifically within the context of PO complete $\mathcal{D}^{\star}$MSs.

Majid et al. [50] defined the class of functions as follows:

Definition 3.1. Let $(\mathcal{W}, \mathcal{D}^{\star}, \preccurlyeq)$ be a $\mathcal{D}^{\star}$MS, and $\Theta : \mathcal{W} \to [0, \infty)$ is a functional. We define a relation $\preccurlyeq$ between elements u and v in $\mathcal{W}$ as follows:

$$u \preccurlyeq v \iff \kappa(\mathcal{D}^{\star}(u, u, v)) \leq \Theta(u) - \Theta(v) \ \forall \ u, v \in \mathcal{W}.$$

where $\kappa : [0, \infty) \to [0, \infty)$ be a mapping which satisfies

(i) κ is increasing and CF.

(ii) $\kappa^{-1}(\{0\}) = \{0\}$.

(iii) $\kappa(\mathfrak{a} + \mathfrak{b}) \leq \kappa(\mathfrak{a}) + \kappa(\mathfrak{b}) \ \forall \ \mathfrak{a}, \mathfrak{b} \in [0, \infty)$ [50].

Proposition 3.1.1. Assume that $(\mathcal{W}, \mathcal{D}^\star)$ be a $\mathcal{D}^\star$MS, the relation $\preccurlyeq$ in Definition 3.1 is a PO on $\mathcal{W}$ and consequently $(\mathcal{W}, \preccurlyeq)$ forms a PO set [50].

Proof. First, we demonstrate that the relation $\preccurlyeq$ satisfies the reflexive property. Since $\kappa(\mathcal{D}^\star(u,u,u)) = \Theta(u) - \Theta(u)$ for all $u \in \mathcal{W}$, it thus follows that $\preccurlyeq$ possesses reflexivity.

Secondly, we demonstrate the antisymmetry of $\preccurlyeq$.

If $u, v \in \mathcal{W}$ s.t. $u \preccurlyeq v$ and $v \preccurlyeq u$, then $\kappa(\mathcal{D}^\star(u,u,v)) \leq \Theta(u) - \Theta(v)$ and $\kappa(\mathcal{D}^\star(v,v,u)) \leq \Theta(v) - \Theta(u)$.

$\Rightarrow \kappa(\mathcal{D}^\star(u,u,v)) + \kappa(\mathcal{D}^\star(v,v,u)) = 0.$

$\Rightarrow \kappa(\mathcal{D}^\star(u,u,v)) = \kappa(\mathcal{D}^\star(v,v,u)) = 0.$

$\Rightarrow \kappa(\mathcal{D}^\star(u,u,v)) = 0.$

$\Rightarrow u = v,$

which shows that $\preccurlyeq$ is antisymmetric.

Thirdly, we prove that $\preccurlyeq$ is transitive.

If $u, v, w \in \mathcal{W}$ s.t. $u \preccurlyeq v$ and $v \preccurlyeq w$, then

$\kappa(\mathcal{D}^\star(u,u,v)) \leq \Theta(u) - \Theta(v),$

and

$\kappa(\mathcal{D}^\star(v,v,w)) \leq \Theta(v) - \Theta(w).$

Hence, $\kappa(\mathcal{D}^\star(u,u,v)) + \kappa(\mathcal{D}^\star(v,v,w)) \leq \Theta(u) - \Theta(w)$. Utilizing the definition of $\mathcal{D}^\star$MS and behaviour of the function κ, we deduce

$$\begin{aligned}\kappa(\mathcal{D}^\star(u,u,w)) &\leq \kappa\left(\mathcal{D}^\star(u,u,v) + \mathcal{D}^\star(v,w,w)\right)\\ &\leq \kappa\left(\mathcal{D}^\star(u,u,v)\right) + \kappa\left(\mathcal{D}^\star(v,w,w)\right)\\ &\leq \Theta(u) - \Theta(w).\end{aligned}$$

Thus, we get $u \preccurlyeq w$. □

Definition 3.2. Let $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ be an ordered $\mathcal{D}^\star$MS induced by the functions κ and Θ. The following defines the ordered intervals in $\mathcal{W}$:

(i) Closed interval:$[u,v] = \{w \in \mathcal{W} : u \preccurlyeq w \preccurlyeq v\}$.

(ii) Right-unbounded interval: $[u, \infty) = \{w \in \mathcal{W} : u \preccurlyeq w\}$.

(iii) Left-unbounded interval: $(-\infty, u] = \{w \in \mathcal{W} : w \preccurlyeq u\}$ [50].

Definition 3.3. Suppose that $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ is a MV mapping. We say that $\mathcal{H}$ is USC if whenever $(u_s) \in \mathcal{W}$ and $(v_s) \in \mathcal{H}(u_s)$ with $u_s \to m \in \mathcal{W}$ and $v_s \to e \in \mathcal{W}$, then $e \in \mathcal{H}(m)$ [54].

Example 3.1.2. The function $\mho : \mathbb{R} \to \mathbb{R}$ defined as

$$\mho(u) = \begin{cases} 2u - u^2 + \frac{1}{2}, \ if \ u < 1 \\ 2 \ if \ u = 1 \\ \frac{1}{2} \ if \ u > 1. \end{cases}$$

is USC at $u = 1$, but it is neither left-continuous nor right-continuous at that point.

Definition 3.4. Let $u \in \mathcal{W}$, then u is said to be a FP of a MV mapping

$$\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$$

if $u \in \mathcal{H}(u)$ [54].

Example 3.1.3. The MV mapping $\mathcal{H} : \mathbb{R} \to 2^{\mathbb{R}}$, defined by

$$\mathcal{H}(u) = [-|u|, |u|],$$

has FPs at $u = 0$ and $u = 1$.

Example 3.1.4. Consider a MV mapping $\mathcal{H} : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$\mathcal{H}(u, v) = \{(u, v), (v, u)\},$$

Then FPs of the MV mapping $\mathcal{H}$ are the points $(u, v) \in \mathbb{R}^2$ that satisfy $(u, v) = (v, u)$.

Theorem 3.1.5. Suppose $(\mathcal{W}, \mathcal{D}^{\star}, \preccurlyeq)$ is a PO complete $\mathcal{D}^{\star}$MS generated by (κ, Θ), where $\Theta : \mathcal{W} \to [0, \infty)$ is a mapping which is b.below. Let $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ be a MV mapping and $\mathcal{M} = \{u \in \mathcal{W} : \mathcal{H}(u) \cap [u, \infty) \neq \emptyset\}$. Assume that

(i) $\mathcal{H}$ is USC.

(ii) If $u \in \mathcal{M}$, then $v \in \mathcal{M} \ \forall \ v \in \mathcal{H}(u) \cap [u, \infty)$.

(iii) $\mathcal{H}(m) \cap [m, \infty) \neq \emptyset$ for some $m \in \mathcal{W}$.

Then there exists a sequence (u_s) s.t. $u_{s-1} \preccurlyeq u_s \in \mathcal{H}(u_{s-1})$ for all $s \in \mathbb{N}$, and H has a FP u_0 s.t. $u_s \to u_0$. In addition, if Θ is LSC, then $u_s \preccurlyeq u_0$ for all s [50].

Proof. By condition (iii), the set $\mathcal{W}$ contains an element m s.t. $m \in \mathcal{M}$. Thus choosing $n \in \mathcal{H}(m) \cap [m, \infty)$ yields $m \preccurlyeq n$. By (ii), $n \in \mathcal{M}$.
Choose $\tau \in \mathcal{H}(n) \cap [n, \infty)$, and we have $n \preccurlyeq \tau$. Proceeding in this way, we generate a sequence $(u_s) \in \mathcal{W}$ satisfying
$u_{s-1} \preccurlyeq u_s \in \mathcal{H}(u_{s-1}) \ \forall \ s \in \mathbb{N}$.
As $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ is a PO $\mathcal{D}^\star$MS generated by (κ, Θ), therefore
$\kappa(\mathcal{D}^\star(u_{s-1}, u_{s-1}, u_s)) \leq \Theta(u_{s-1}) - \Theta(u_s)$.
Since κ is non-negative mapping, from above we get
$\Theta(u_{s-1}) - \Theta(u_s) \geq 0 \ \forall \ s \in \mathbb{N}$.
$\Rightarrow \Theta(u_{s-1}) \geq \Theta(u_s) \ \forall \ s \in \mathbb{N}$.

Since Θ is b.below, we get $\Theta(u_s)$ is a decreasing sequence and has a lower bound. Hence, by completeness property of $\mathbb{R}$, $\lim_{s \to \infty} \Theta(u_s) = \inf\{u_s : s \in \mathbb{N}\}$. Thus,

$$\lim_{s,r \to \infty} \kappa(\mathcal{D}^\star(u_s, u_s, u_r)) \leq \lim_{s \to \infty} \Theta(u_s) - \lim_{r \to \infty} \Theta(u_r).$$

Therefore, $\lim_{s,r \to \infty} \kappa(\mathcal{D}^\star(u_s, u_s, u_r)) = 0$.
Now, using the continuity of κ and the property that $\kappa^{-1}(\{0\}) = \{0\}$,

$$\lim_{s,r \to \infty} \mathcal{D}^\star(u_s, u_s, u_r) = 0.$$

Therefore, (u_s) is a CS in $\mathcal{W}$.
By the completeness of $\mathcal{W}$, there is $u_0 \in \mathcal{W}$ s.t. (u_s) is $\mathcal{D}^\star$-Con. to u_0. Since $u_{s-1} \in \mathcal{W}, u_s \in \mathcal{H}(u_{s-1}), u_{s-1} \to u_0$, and $u_s \to u_0$, via the definition of upper semi-continuity of $\mathcal{H}$, we have $u_0 \in \mathcal{H}(u_0)$.

Now, assuming the lower semi-continuity of Θ, for all $s \in \mathbb{N}$,

$$\begin{aligned}\kappa(\mathcal{D}^\star(u_s, u_s, u_0)) &= \lim_{r\to\infty} \kappa(\mathcal{D}^\star(u_s, u_s, u_r)) \\ &\leq \lim_{r\to\infty} \{\inf \Theta(u_s) - \Theta(u_r)\} \\ &= \Theta(u_s) - \lim_{r\to\infty} \Theta(u_r) \\ &\leq \Theta(u_s) - \Theta(u_0).\end{aligned}$$

Thus, $u_s \preccurlyeq u_0$ for all $s \in \mathbb{N}$. □

Corollary 3.1.6. Let $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}^\star$MS generated by (κ, Θ), where $\Theta : \mathcal{W} \to [0, \infty)$ b.below, and let $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ be a MV mapping where

(i) $\mathcal{H}$ is USC.

(ii) $\mathcal{H}$ satisfies the condition of monotonic sequence i.e. $\forall\ u, v \in \mathcal{W}$ and $u \preccurlyeq v$ and for every $\alpha \preccurlyeq \mathcal{H}(u)$, there exist $\beta \preccurlyeq \mathcal{H}(v)$ s.t. $\alpha \preccurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $\mathcal{H}(m) \cap [0, \infty) \neq \emptyset$.

Then there exists a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \preccurlyeq u_s \in \mathcal{H}(u_{s-1})$ for all $s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Furthermore, if Θ is LSC, then $u_s \preccurlyeq u_0\ \forall\ s$ [50].

Proof. By condition (ii), we have $m \in \mathcal{M}$. Let $v \in \mathcal{H}(m) \cap [0, \infty)$, then the monotonicity of $\mathcal{H}$ implies the existence of $w \in \mathcal{H}(v)$ s.t. $v \preccurlyeq w$. In other words,

$$w \in \mathcal{H}(v) \cap [0, \infty) \neq \emptyset.$$

Hence $v \in \mathcal{M}$ and then by applying the Theorem 3.1.5, the result follows. □

Corollary 3.1.7. Let $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}^\star$MS generated by (κ, Θ) s.t. $\Theta : \mathcal{W} \to [0, \infty)$ is b.below, and let $\mathtt{f} : \mathcal{W} \to \mathcal{W}$ satisfies:

(i) $\mathtt{f}$ is CF.

(ii) For any $\alpha \in \mathtt{f}(u)$, there is $\beta \in \mathtt{f}(v)$ s.t. $\alpha \preccurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $m \preccurlyeq \mathtt{f}(m)$.

Then there is a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \preccurlyeq u_s \in \mathtt{f}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Also if Θ is LSC, then $u_s \preccurlyeq u_0\ \forall\ s$ [50].

Proof. Define $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ by $\mathcal{H}(u) = \{\mathtt{f}(u)\}\ \forall\ u \in \mathcal{W}$, then $\mathcal{H}$ and $\mathcal{W}$ satisfy all the assumptions of Theorem 3.1.5 which yields the result. □

The following results are similar to the previous ones, with the main difference being that the conditions of being bounded from below are replaced by conditions of being bounded from above.

Theorem 3.1.8. Let $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}^\star$MS generated by (κ, Θ), where $\Theta : \mathcal{W} \to (-\infty, 0]$ is a mapping that is b.above. Suppose that $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ is a MV mapping and $\mathcal{M} = \{u \in \mathcal{W} : \mathcal{H}(u) \cap (-\infty, u] \neq \emptyset\}$. Assume that

(i) $\mathcal{H}$ is USC.

(ii) $\forall\ u \in \mathcal{M}, \mathcal{H}(u) \cap \mathcal{M} \cap (-\infty, u] \neq \emptyset$.

Then there is a sequence (u_s) s.t. $u_{s-1} \succcurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Also, if Θ is LSC, then $u_s \succcurlyeq u_0$ for all s [50].

Proof. According to (ii), there is an element $m \in \mathcal{W}$ s.t. $m \in \mathcal{M}$. Thus choosing $n \in \mathcal{H}(m) \cap (-\infty, m]$ implies $m \succcurlyeq n$. By (ii), $n \in \mathcal{M}$. Choose $\tau \in \mathcal{H}(n) \cap (-\infty, n]$, so $n \succcurlyeq \tau$. Proceeding in this manner, we generate a sequence $(u_s) \in \mathcal{W}$ s.t. $u_{s-1} \succcurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$.
Since $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ is a PO $\mathcal{D}^\star$MS induced via (κ, Θ), therefore

$$\kappa(\mathcal{D}^\star(u_{s-1}, u_{s-1}, u_s)) \leq \Theta(u_{s-1}) - \Theta(u_s). \tag{3.1}$$

The non-negativity of κ implies that
$\Theta(u_{s-1}) - \Theta(u_s) \geq 0\ \forall\ s \in \mathbb{N}$.
$\Rightarrow \Theta(u_{s-1}) \geq \Theta(u_s)\ \forall\ s \in \mathbb{N}$.
Since Θ is b.above, we get $\Theta(u_s)$ is an increasing sequence that is b.above. So by completeness property of $\mathbb{R}$,

$$\lim_{s \to -\infty} \Theta(u_s) = \inf\{u_s : s \in \mathbb{N}\}.$$

Hence by (3.1) we get

$$\lim_{s,r\to-\infty} \kappa(\mathcal{D}^\star(u_s,u_s,u_r)) \leq \lim_{s\to-\infty} \Theta(u_s) - \lim_{r\to-\infty} \Theta(u_r).$$

Therefore, $\lim_{s,r\to-\infty} \kappa(\mathcal{D}^\star(u_s,u_s,u_r)) = 0$.
Now, by using the continuity of κ and the fact that $\kappa^{-1}(\{0\}) = \{0\}$, we have

$$\lim_{s,r\to-\infty} \mathcal{D}^\star(u_s,u_s,u_r) = 0.$$

This implies that (u_s) is a CS in $\mathcal{W}$. Since $\mathcal{W}$ is complete, there exists $u_0 \in \mathcal{W}$ s.t. (u_s) is convergent to u_0. Since $u_{s-1} \in \mathcal{W}, u_s \in \mathcal{H}(u_{s-1}), u_{s-1} \to u_0$, and $u_s \to u_0$, via the definition of upper semi-continuity of $\mathcal{H}$, we have $u_0 \in \mathcal{H}(u_0)$.
Now, since Θ is LSC, then for all $s \in \mathbb{N}$,

$$\begin{aligned}\kappa(\mathcal{D}^\star(u_s,u_s,u_0)) &= \lim_{r\to\infty} \kappa(\mathcal{D}^\star(u_s,u_s,u_r))\\ &\leq \lim_{r\to\infty}\{\inf \Theta(u_s) - \Theta(u_r)\}\\ &= \Theta(u_s) - \lim_{r\to\infty} \Theta(u_r)\\ &\leq \Theta(u_s) - \Theta(u_0).\end{aligned}$$

Thus, $u_s \succcurlyeq u_0$ for all $s \in \mathbb{N}$. □

Corollary 3.1.9. Suppose that $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}^\star$MS generated by (κ, Θ), where $\Theta : \mathcal{W} \to (-\infty, 0]$ is b.above, and let $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ be a MV mapping where

(i) $\mathcal{H}$ is USC.

(ii) $\forall\ u, v \in \mathcal{W}$ and $u \succcurlyeq v$ and every $\alpha \in \mathcal{H}(u)$, there exist $\beta \in \mathcal{H}(v)$ s.t. $\alpha \succcurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $\mathcal{H}(m) \cap (-\infty, m] \neq \emptyset$.

Then there exists a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \succcurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Furthermore, if Θ is LSC, then $u_s \succcurlyeq u_0\ \forall\ s$ [50].

Proof. By condition (ii), $m \in \mathcal{M}$. Now, consider any v in the intersection of $\mathcal{H}(m)$ and the non-positive real numbers $(-\infty, 0]$. Then, by the definition of $\mathcal{H}$, there exists some w in $\mathcal{H}(v)$ s.t. v is contained in $\mathcal{W}$.

$\Rightarrow w \in \mathcal{H}(v) \cap (-\infty, v] \neq \emptyset$.

Hence $v \in \mathcal{M}$ and then the result follows by Theorem 3.1.8. □

Corollary 3.1.10. Suppose that $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}^\star$MS generated by (κ, Θ) s.t. $\Theta : \mathcal{W} \to (-\infty, 0]$ is b.above mapping, and let $\mathtt{f} : \mathcal{W} \to \mathcal{W}$ satisfies the following:

(i) $\mathtt{f}$ is CF.

(ii) For any $\alpha \in \mathtt{f}(u)$, there exists $\beta \in \mathtt{f}(v)$ s.t. $\alpha \succcurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $m \succcurlyeq \mathtt{f}(m)$.

Then there is a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \succcurlyeq u_s \in \mathtt{f}(u_{s-1})$ $\forall$ $s \in \mathbb{N}$, and f has a FP u_0 s.t. $u_s \to u_0$. Also if Θ is LSC, then $u_s \succcurlyeq u_0$ $\forall$ s [50].

Proof. Define a MV mapping $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ via $\mathcal{H}(u) = \{\mathtt{f}(u)\}$ for all $u \in \mathcal{W}$, then $\mathcal{H}$ and $\mathcal{W}$ satisfy all the hypothesis of Theorem 3.1.8. So the proof follows from Theorem 3.1.8. □

3.2 Coupled FP Theorems and $\mathcal{D}^\star$MSs

In this section various results related to coupled FPs that satisfy C.conditions in the context of PO complete $\mathcal{D}^\star$MS, utilizing the concept of integral-type contractions has been discussed.

Throughout the thesis by σ we mean a mapping $\sigma : [0, \infty) \to [0, \infty)$ which satisfies the following properties:

(i) σ is monotonically increasing on the interval $[0, \infty)$.

(ii) $\sigma(t) \leq t$ $\forall$ $t > 0$.

(iii) σ has the property of additivity i.e., $\sigma(u + v) = \sigma(u) + \sigma(v)$.

(iv) $\sum_{s=1}^{\infty} s\sigma^s(t) < \infty \ \forall \ t > 0,$

and $\eta : [0,\infty) \to [0,\infty)$ be a mapping which satisfies the following properties:

(i) η is decreasing function on $[0,\infty)$.

(ii) η is LI.

(iii) For every $\epsilon > 0$, $\int_0^{\epsilon} \eta(t)dt > 0$.

(iv) η is a continuous mapping.

The main result of [50] is embodied in the following theorem:

Theorem 3.2.1. Let $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}^\star$MS, and let $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CM with the MM property on $\mathcal{W}$ s.t.

$$\int_0^{\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \leq \sigma\left(\int_0^{\mathcal{D}^\star(u,m,f)+\mathcal{D}^\star(v,n,w)} \eta(t)dt\right), \tag{3.2}$$

where $u, v, w, m, n, f \in \mathcal{W}$ and $\eta : [0,\infty) \to [0,\infty)$ is a LI mapping (i.e., with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0, v_0)$ and $\mathcal{H}(v_0, u_0) \preccurlyeq v_0$, then $\mathcal{H}$ has a coupled FP in $\mathcal{W}$.

Proof. By hypothesis, there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0, v_0)$ and $\mathcal{H}(v_0, u_0) \preccurlyeq v_0$. Define $u_1, v_1 \in \mathcal{W}$ as follows:

$$u_0 \preccurlyeq \mathcal{H}(u_0, v_0) = u_1 \ and \ v_1 = \mathcal{H}(v_0, u_0) \preccurlyeq v_0.$$

Suppose that $u_2 = \mathcal{H}(u_1, v_1)$ and $v_2 = \mathcal{H}(v_1, u_1)$. Therefore
$u_2 = \mathcal{H}(u_1, v_1) = \mathcal{H}(\mathcal{H}(u_0, v_0), \mathcal{H}(v_0, u_0)) = \mathcal{H}^2(u_0, v_0),$
$v_2 = \mathcal{H}(v_1, u_1) = \mathcal{H}(\mathcal{H}(v_0, u_0), \mathcal{H}(u_0, v_0)) = \mathcal{H}^2(v_0, u_0).$
Utilizing the mixed monotonicity property of the mapping $\mathcal{H}$,
$u_2 = \mathcal{H}^2(u_0, v_0) = \mathcal{H}(u_1, v_1) \succcurlyeq \mathcal{H}(u_0, v_0) = u_1 \succcurlyeq u_0,$
$v_2 = \mathcal{H}^2(v_0, u_0) = \mathcal{H}(v_1, u_1) \preccurlyeq \mathcal{H}(v_0, u_0) = v_1 \preccurlyeq v_0.$
Repeatedly applying the above process for all $s \geq 0$ leads to the following:

$$u_0 \preccurlyeq u_1 \preccurlyeq u_2 \preccurlyeq \ldots \preccurlyeq u_{s+1} \preccurlyeq \ldots,$$
$$v_0 \succcurlyeq v_1 \succcurlyeq v_2 \succcurlyeq \ldots \succcurlyeq v_{s+1} \succcurlyeq \ldots$$

such that

$$u_{s+1} = \mathcal{H}^{s+1}(u_0, v_0) = \mathcal{H}(\mathcal{H}^s(u_0, v_0), \mathcal{H}^s(v_0, u_0)),$$
$$v_{s+1} = \mathcal{H}^{s+1}(v_0, u_0) = \mathcal{H}(\mathcal{H}^s(v_0, u_0), \mathcal{H}^s(u_0, v_0)).$$

If $(u_{s+1}, v_{s+1}) = (u_0, v_0)$, i.e.,

$$u_s = \mathcal{H}(u_s, v_s) \text{ and } v_s = \mathcal{H}(v_s, u_s).$$

Therefore, a coupled FP exists for the mapping $\mathcal{H}$.

Now we assume that $(u_{s+1}, v_{s+1}) \neq (u_s, v_s)$ for all $s \geq 0$, that is, let either

$u_{s+1} = \mathcal{H}(u_s, v_s) \neq u_s$ or $v_{s+1} = \mathcal{H}(v_0, u_0) \neq v_s$. By using equation (3.2),

$$\begin{aligned} \int_0^{\mathcal{D}^\star(u_s,u_s,u_{s+1})} \eta(t)dt &= \int_0^{\mathcal{D}^\star(\mathcal{H}(u_{s-1},v_{s-1}),\mathcal{H}(u_{s-1},v_{s-1}),\mathcal{H}(u_s,v_s))} \eta(t)dt \\ &\leq \sigma\left(\int_0^{\mathcal{D}^\star(u_{s-1},u_{s-1},u_s),\mathcal{D}^\star(v_{s-1},v_{s-1},v_s)} \eta(t)dt\right). \end{aligned} \tag{3.3}$$

Preceding in the same way

$$\begin{aligned} \int_0^{\mathcal{D}^\star(v_s,v_s,v_{s+1})} \eta(t)dt &= \int_0^{\mathcal{D}^\star(\mathcal{H}(v_{s-1},u_{s-1}),\mathcal{H}(v_{s-1},u_{s-1}),\mathcal{H}(v_s,u_s))} \eta(t)dt \\ &\leq \sigma\left(\int_0^{\mathcal{D}^\star(u_{s-1},u_{s-1},u_s),\mathcal{D}^\star(v_{s-1},v_{s-1},v_s)} \eta(t)dt\right). \end{aligned} \tag{3.4}$$

Since η is non-increasing, for every $a, b \geq 0$,

$$\int_0^{a+b} \eta(t)dt \leq \int_0^a \eta(t)dt + \int_0^b \eta(t)dt. \tag{3.5}$$

Additionally, since σ is a linear and monotonically increasing mapping, by combining (3.2), (3.3) and (3.5) for all $s \geq 0$

$$
\begin{aligned}
\int_0^{\mathcal{D}^{\star}(u_s,u_s,u_{s+1})} \eta(t)dt &= \int_0^{\mathcal{D}^{\star}(\mathcal{H}(u_{s-1},v_{s-1}),\mathcal{H}(u_{s-1},v_{s-1}),\mathcal{H}(u_s,v_s))} \eta(t)dt \\
&\leq \sigma\left(\int_0^{\mathcal{D}^{\star}(u_{s-1},u_{s-1},u_s)+\mathcal{D}^{\star}(v_{s-1},v_{s-1},v_s)} \eta(t)dt\right) \\
&\leq \sigma\left(\int_0^{\mathcal{D}^{\star}(u_{s-1},u_{s-1},u_s)} \eta(t)dt\right) + \sigma\left(\int_0^{\mathcal{D}^{\star}(v_{s-1},v_{s-1},v_s)} \eta(t)dt\right) \\
&= \sigma\left(\int_0^{\mathcal{D}^{\star}(\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-1},v_{s-1}))} \eta(t)dt\right) \\
&+ \sigma\left(\int_0^{\mathcal{D}^{\star}(\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-1},u_{s-1}))} \eta(t)dt\right) \\
&\leq \sigma\left(\sigma\left(\int_0^{\mathcal{D}^{\star}(u_{s-2},u_{s-2},u_{s-1})+\mathcal{D}^{\star}(v_{s-2},v_{s-2},v_{s-1})} \eta(t)dt\right)\right) \\
&+ \sigma\left(\sigma\left(\int_0^{\mathcal{D}^{\star}(v_{s-2},v_{s-2},v_{s-1})+\mathcal{D}^{\star}(u_{s-2},u_{s-2},u_{s-1})} \eta(t)dt\right)\right) \\
&\leq \sigma^2\left(\int_0^{\mathcal{D}^{\star}(u_{s-2},u_{s-2},u_{s-1})} \eta(t)dt\right) \\
&+ \sigma^2\left(\int_0^{\mathcal{D}^{\star}(v_{s-2},v_{s-2},v_{s-1})} \eta(t)dt\right) \\
&+ \sigma^2\left(\int_0^{\mathcal{D}^{\star}(v_{s-2},v_{s-2},v_{s-1})} \eta(t)dt\right) \\
&+ \sigma^2\left(\int_0^{\mathcal{D}^{\star}(u_{s-2},u_{s-2},u_{s-1})} \eta(t)dt\right) \\
&= 2\sigma^2\left(\int_0^{\mathcal{D}^{\star}(u_{s-2},u_{s-2},u_{s-1})} \eta(t)dt\right) \\
&+ 2\sigma^2\left(\int_0^{\mathcal{D}^{\star}(v_{s-2},v_{s-2},v_{s-1})} \eta(t)dt\right) \\
&= 2\sigma^2\left(\int_0^{\mathcal{D}^{\star}(u_{s-2},u_{s-2},u_{s-1})} \eta(t)dt + \int_0^{\mathcal{D}^{\star}(v_{s-2},v_{s-2},v_{s-1})} \eta(t)dt\right)
\end{aligned}
$$

$$\begin{aligned}
&\leq 2\sigma^2\left(\int_0^{\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})+\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&\vdots \qquad (3.6)\\
&\leq s\sigma^s\left(\int_0^{\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)}\eta(t)dt\right).
\end{aligned}$$

Repeating the above procedure, we get

$$\begin{aligned}
\int_0^{\mathcal{D}^\star(v_s,v_s,v_{s+1})}\eta(t)dt &= \int_0^{\mathcal{D}^\star(\mathcal{H}(v_{s-1},u_{s-1}),\mathcal{H}(v_{s-1},u_{s-1}),\mathcal{H}(v_s,u_s))}\eta(t)dt\\
&\leq \sigma\left(\int_0^{\mathcal{D}^\star(v_{s-1},v_{s-1},v_s)+\mathcal{D}^\star(u_{s-1},u_{s-1},u_s)}\eta(t)dt\right)\\
&\leq \sigma\left(\int_0^{\mathcal{D}^\star(v_{s-1},v_{s-1},v_s)}\eta(t)dt\right)+\sigma\left(\int_0^{\mathcal{D}^\star(u_{s-1},u_{s-1},u_s)}\eta(t)dt\right)\\
&= \sigma\left(\int_0^{\mathcal{D}^\star(\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-1},u_{s-1}))}\eta(t)dt\right)\\
&+ \sigma\left(\int_0^{\mathcal{D}^\star(\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-1},v_{s-1}))}\eta(t)dt\right)\\
&\leq \sigma\left(\sigma\left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})+\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\right)\\
&+ \sigma\left(\sigma\left(\int_0^{\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})+\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\right)\\
&\leq \sigma^2\left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&+ \sigma^2\left(\int_0^{\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\\
&+ \sigma^2\left(\int_0^{\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\\
&+ \sigma^2\left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&= 2\sigma^2\left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&+ 2\sigma^2\left(\int_0^{\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\\
&= 2\sigma^2\left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt+\int_0^{\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\sigma^2 \left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})+\mathcal{D}^\star(u_{s-2},u_{s-2},u_{s-1})} \eta(t)dt \right) \\
&\vdots \\
&\leq s\sigma^s \left(\int_0^{\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)} \eta(t)dt \right).
\end{aligned} \tag{3.7}
$$

Assuming $r, s \in \mathbb{N}$ with $r > s$, then by the definition of $\mathcal{D}^\star$MS

$$
\int_0^{\mathcal{D}^\star(u_s,u_s,u_r)} \eta(t)dt \leq \int_0^{l[\mathcal{D}^\star(u_s,u_s,u_{s+1})+\mathcal{D}^\star(u_{s+1},u_{s+1},u_r)]} \eta(t)dt
$$

$$
\begin{aligned}
&= \int_0^{l\mathcal{D}^\star(u_s,u_s,u_{s+1})+l\mathcal{D}^\star(u_{s+1},u_{s+1},u_r)} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}^\star(u_s,u_s,u_{s+1})} \eta(t)dt + \int_0^{l\mathcal{D}^\star(u_{s+1},u_{s+1},u_r)} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}^\star(u_s,u_s,u_{s+1})} \eta(t)dt \\
&+ \int_0^{l[l(\mathcal{D}^\star(u_{s+1},u_{s+1},u_{s+2})+\mathcal{D}^\star(u_{s+2},u_{s+2},u_r))]} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}^\star(u_s,u_s,u_{s+1})} \eta(t)dt + \int_0^{l^2\mathcal{D}^\star(u_{s+1},u_{s+1},u_{s+2})} \eta(t)dt \\
&+ \int_0^{l^2\mathcal{D}^\star(u_{s+2},u_{s+2},u_r)} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}^\star(u_s,u_s,u_{s+1})} \eta(t)dt + \int_0^{l^2\mathcal{D}^\star(u_{s+1},u_{s+1},u_{s+2})} \eta(t)dt \\
&+ \int_0^{l^3\mathcal{D}^\star(u_{s+2},u_{s+2},u_{s+3})} \eta(t)dt + \ldots + \int_0^{l^{r-s}\mathcal{D}^\star(u_{r-2},u_{r-2},u_{r-1})} \eta(t)dt \\
&+ \int_0^{l^{r-s}\mathcal{D}^\star(u_{r-1},u_{r-1},u_r)} \eta(t)dt \\
&\leq s\sigma^s \left(\int_0^{l[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ (s+1)\sigma^{s+1} \left(\int_0^{l^2[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ (s+2)\sigma^{s+2} \left(\int_o^{l^3[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ \ldots + (r-2)\sigma^{r-2} \left(\int_0^{l^{r-s}[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ (r-1)\sigma^{r-1} \left(\int_0^{l^{r-s}[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right)
\end{aligned}
$$

$$= \sum_{i=s}^{i=r-1} i\sigma^i \left(\int_0^{l^{i-s+1}[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right)$$
$$\leq \sum_{i=s}^{\infty} i\sigma^i \left(\int_0^{l^{i-s+1}[\mathcal{D}^\star(u_0,u_0,u_1)+\mathcal{D}^\star(v_0,v_0,v_1)]} \eta(t)dt \right).$$

Since $\sum_{i=s}^{\infty} i\sigma^i(t) < \infty \ \forall \ t \in [0, +\infty)$, it follows that $\lim_{s,r\to\infty} \mathcal{D}^\star(u_s, u_s, u_r) = 0$ and the sequence (u_s) is a CS in $\mathcal{W}$.

Similarly, for (v_s), it can be proved that

$$\int_0^{\mathcal{D}^\star(v_s,v_s,v_r)} \eta(t)dt \leq \int_0^{l[\mathcal{D}^\star(v_s,v_s,v_{s+1})+\mathcal{D}^\star(v_{s+1},v_{s+1},v_r)]} \eta(t)dt$$
$$= \int_0^{l\mathcal{D}^\star(v_s,v_s,v_{s+1})+l\mathcal{D}^\star(v_{s+1},v_{s+1},v_r)} \eta(t)dt$$

$$\leq \int_0^{l\mathcal{D}^\star(v_s,v_s,v_{s+1})} \eta(t)dt + \int_0^{l\mathcal{D}^\star(v_{s+1},v_{s+1},v_r)} \eta(t)dt$$
$$\leq \int_0^{l\mathcal{D}^\star(v_s,v_s,v_{s+1})} \eta(t)dt$$
$$+ \int_0^{l[l(\mathcal{D}^\star(v_{s+1},v_{s+1},v_{s+2})+\mathcal{D}^\star(v_{s+2},v_{s+2},v_r))]} \eta(t)dt$$
$$\leq \int_0^{l\mathcal{D}^\star(v_s,v_s,v_{s+1})} \eta(t)dt + \int_0^{l^2\mathcal{D}^\star(v_{s+1},v_{s+1},v_{s+2})} \eta(t)dt$$
$$+ \int_0^{l^2\mathcal{D}^\star(v_{s+2},v_{s+2},v_r)} \eta(t)dt$$
$$\leq \int_0^{l\mathcal{D}^\star(v_s,v_s,v_{s+1})} \eta(t)dt + \int_0^{l^2\mathcal{D}^\star(v_{s+1},v_{s+1},v_{s+2})} \eta(t)dt$$
$$+ \int_0^{l^3\mathcal{D}^\star(v_{s+2},v_{s+2},v_{s+3})} \eta(t)dt + \ldots + \int_0^{l^{r-s}\mathcal{D}^\star(v_{r-2},v_{r-2},v_{r-1})} \eta(t)dt$$
$$+ \int_0^{l^{r-s}\mathcal{D}^\star(v_{r-1},v_{r-1},v_r)} \eta(t)dt$$
$$\leq s\sigma^s \left(\int_0^{l[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]} \eta(t)dt \right)$$
$$+ (s+1)\sigma^{s+1} \left(\int_0^{l^2[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]} \eta(t)dt \right)$$
$$+ (s+2)\sigma^{s+2} \left(\int_o^{l^3[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]} \eta(t)dt \right)$$
$$+ \ldots + (r-2)\sigma^{r-2} \left(\int_0^{l^{r-s}[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]} \eta(t)dt \right)$$

$$+ (r-1)\sigma^{r-1}\left(\int_0^{l^{r-s}[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]}\eta(t)dt\right)$$
$$= \sum_{i=s}^{i=r-1} i\sigma^i\left(\int_0^{l^{i-s+1}[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]}\eta(t)dt\right)$$
$$\leq \sum_{i=s}^{\infty} i\sigma^i\left(\int_0^{l^{i-s+1}[\mathcal{D}^\star(v_0,v_0,v_1)+\mathcal{D}^\star(u_0,u_0,u_1)]}\eta(t)dt\right).$$

Since $\sum_{i=s}^{\infty} i\sigma^i(t) < \infty \ \forall \ t \in [0,+\infty)$, it follows that $\lim_{s,r\to\infty} \mathcal{D}^\star(v_s, v_s, v_r) = 0$ and the sequence (v_s) is a CS in $\mathcal{W}$.

Since $\mathcal{W}$ is complete $\mathcal{D}^\star$MS, there exist $u, v \in \mathcal{W}$ s.t. $\lim_{s\to\infty} u_s = u$ and $\lim_{s\to\infty} v_s = v$. Since $\mathcal{H}$ is a CF, it follows that $\mathcal{H}(u,v) = u$ and $\mathcal{H}(v,u) = v$; that is, (u,v) is a a coupled FP of the mapping $\mathcal{H}$. □

Theorem 3.2.2. Suppose $(\mathcal{W}, \mathcal{D}^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}^\star$MS subject to the following conditions:

(i) If (u_s) is an increasing sequence which converges to $u \in \mathcal{W}$, then $u_s \preccurlyeq u \ \forall \ s$.

(ii) If v_s is a decreasing sequence which converges to $v \in \mathcal{W}$, then $v_s \succcurlyeq v \ \forall \ s$.

Also suppose that $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a mapping having the MM property on $\mathcal{W}$ s.t.

$$\int_0^{\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))}\eta(t)dt \leq \sigma\left(\int_0^{\mathcal{D}^\star(u,m,f)+\mathcal{D}^\star(v,n,w)}\eta(t)dt\right), \tag{3.8}$$

where $u, v, w, m, n, f \in \mathcal{W}$ and $\eta : [0,\infty) \to [0,\infty)$ is a LI mapping (i.e., with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq H(u_0, v_0)$ and $(v_0, u_0) \preccurlyeq v_0$, then $\mathcal{H}$ has a coupled FP in $\mathcal{W}$.

Proof. Using the similar approach to that used in the proof of Theorem 3.2.1, we obtain two CSs (u_s) and $(v_s) \in \mathcal{W}$.

According to conditions (i) and (ii) there exists $u, v \in \mathcal{W}$ s.t. $u_s \preccurlyeq u$ and $v_s \succcurlyeq v$ for all $s \geq 0$.

If $u_s = u$ and $v_s = v$ for some s, then $u_{s+1} = u$ and $v_{s+1} = v$; that is, (u,v) is a coupled FP. Without loss of generality, let either $u_s \neq u$ or $v_s \neq v$. By utilizing equation (3.8)

$$\begin{aligned}
\int_0^{\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),u)} \eta(t)dt &\leq \int_0^{\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),\mathcal{H}(u_s,v_s))+\mathcal{D}^\star(\mathcal{H}(u_s,v_s),\mathcal{H}(u_s,v_s),u)} \eta(t)dt \\
&\leq \int_0^{\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),\mathcal{H}(u_s,v_s))} \eta(t)dt \\
&+ \int_0^{\mathcal{D}^\star(\mathcal{H}(u_s,v_s),\mathcal{H}(u_s,v_s),u)} \eta(t)dt \\
&\leq \sigma\left(\int_0^{\mathcal{D}^\star(u,u,u_s)+\mathcal{D}^\star(v,v,v_s)} \eta(t)dt\right) \\
&+ \int_0^{\mathcal{D}^\star(u_{s+1},u_{s+1},u)} \eta(t)dt.
\end{aligned} \tag{3.9}$$

Hence, by equation (3.9) with as $s \to \infty$, we get $\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),u) = 0$, which gives that $\mathcal{H}(u,v) = u$. In the same way, it can be proved that

$$\begin{aligned}
\int_0^{\mathcal{D}^\star(\mathcal{H}(v,u),\mathcal{H}(v,u),v)} \eta(t)dt &\leq \int_0^{\mathcal{D}^\star(\mathcal{H}(v,u),\mathcal{H}(v,u),\mathcal{H}(v_s,u_s))+\mathcal{D}^\star(\mathcal{H}(v_s,u_s),\mathcal{H}(v_s,u_s),v)} \eta(t)dt \\
&\leq \int_0^{\mathcal{D}^\star(\mathsf{H}(v,u),\mathsf{H}(v,u),\mathsf{H}(v_s,u_s))} \eta(t)dt \\
&+ \int_0^{\mathcal{D}^\star(\mathcal{H}(v_s,u_s),\mathcal{H}(v_s,u_s),v)} \eta(t)dt \\
&\leq \sigma\left(\int_0^{\mathcal{D}^\star(v,v,v_s)+\mathcal{D}^\star(u,u,u_s)} \eta(t)dt\right) \\
&+ \int_0^{\mathcal{D}^\star(v_{s+1},v_{s+1},v)} \eta(t)dt.
\end{aligned} \tag{3.10}$$

Hence, via equation (3.10) with $s \to \infty$, $\mathcal{D}^\star(\mathcal{H}(v,u),H(v,u),v) = 0$ and thus $H(v,u) = v$. Therefore, (u,v) is a coupled FP of the mapping $\mathcal{H}$. □

In the next theorem, we establish that the coupled FP of $\mathcal{H}$ is unique.

Theorem 3.2.3. Let $(\mathcal{W},\mathcal{D}^\star,\preccurlyeq)$ be a PO complete $\mathcal{D}^\star$MS with the following properties:

(i) If (u_s) is a increasing sequence which converges to u $\in \mathcal{W}$, then $u_s \preccurlyeq u\ \forall\ s$.

(ii) If (v_s) is a decreasing sequence which converges to $v \in \mathcal{W}$, then $v_s \succcurlyeq v\ \forall\ s$.

(iii) For all (u,v), $(u_1,v_1) \in \mathcal{W}^2$, there exists $(w_1,w_2) \in \mathcal{W}^2$ which is comparable with (u, v) and (u_1, v_1).

Let $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CM having the MM property on $\mathcal{W}$ such that

$$\int_0^{\mathcal{D}^{\star}(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \leq \sigma \left(\int_0^{\mathcal{D}^{\star}(u,m,f)+\mathcal{D}^{\star}(v,n,w)} \eta(t)dt \right), \tag{3.11}$$

where $u, v, w, m, n, f \in \mathcal{W}$ and $\eta : [0, \infty) \to [0, \infty)$ is a LI mapping (i.e. with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0, v_0)$ and $\mathcal{H}(v_0, u_0) \preccurlyeq v_0$, then $\mathcal{H}$ has a unique coupled FP in $(\mathcal{W}, \mathcal{D}^{\star})$.

Proof. Let (u_1, v_1) is another FP of $\mathcal{H}$. We now consider the following scenarios.

Case 1: Suppose (u, v) and (u_1, v_1) are comparable with respect to the PO $\preccurlyeq$ in $\mathcal{W}^2$. Assuming $(u, v) \preccurlyeq (u_1, v_1)$ i.e. $u \preccurlyeq u_1$ and $v \preccurlyeq v_1$. Now by using Theorem 3.2.1, we obtain

$$\int_0^{\mathcal{D}^{\star}(\mathcal{H}^s(u,v),\mathcal{H}^s(u_1,v_1),\mathcal{H}^s(u_1,v_1))} \eta(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}^{\star}(u,u_1,u_1)+\mathcal{D}^{\star}(v,v_1,v_1)} \eta(t)dt \right). \tag{3.12}$$

Suppose that $s \to \infty$, therefore via (3.12) we get $u = u_1$.

Similarly, it can be proved that

$$\int_0^{\mathcal{D}^{\star}(\mathcal{H}^s(v,u),\mathcal{H}^s(v_1,u_1),\mathcal{H}^s(v_1,u_1))} \eta(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}^{\star}(v,v_1,v_1)+\mathcal{D}^{\star}(u,u_1,u_1)} \eta(t)dt \right). \tag{3.13}$$

If $s \to \infty$, then via (3.13) we get $v = v_1$.

Case 2: Let (u, v) be not comparable with (u_1, v_1). So by condition (iii) there exists $(w_1, w_2) \in \mathcal{W}^2$, which is comparable to (u, v) and (u_1, v_1). We can assume that $w_1 \preccurlyeq u, w_2 \preccurlyeq v, w_1 \preccurlyeq u_1$ and $w_2 \preccurlyeq v_1$.

Again by using Theorem 3.2.1, we get

$$\int_0^{\mathcal{D}^{\star}(\mathcal{H}^s(u,v),\mathcal{H}^s(w_1,w_2),\mathcal{H}^s(w_1,w_2))} \eta(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}^{\star}(u,w_1,w_1)+\mathcal{D}^{\star}(v,w_2,w_2)} \eta(t)dt \right). \tag{3.14}$$

As $s \to \infty$, by applying (3.14)

$$\mathcal{D}^{\star}(\mathcal{H}^s(u, v), \mathcal{H}^s(w_1, w_2), \mathcal{H}^s(w_1, w_2)) = 0.$$

Thus, $\lim_{s\to\infty} \mathcal{H}^s(u,v) = \lim_{s\to\infty} \mathcal{H}^s(w_1,w_2) = u.$

$$\int_0^{\mathcal{D}^\star(\mathcal{H}^s(u_1,v_1),\mathcal{H}^s(w_1,w_2),\mathcal{H}^s(w_1,w_2))} \eta(t)dt \le \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}^\star(u_1,w_1,w_1)+\mathcal{D}^\star(v_1,w_2,w_2)} \eta(t)dt\right). \tag{3.15}$$

From above $\lim_{s\to\infty} \mathcal{H}^s(u_1,v_1) = \lim_{s\to\infty} \mathcal{H}^s(w_1,w_2) = u_1$. Thus $u = u_1$.
In the same way, by using (w_2,w_1)

$$\int_0^{\mathcal{D}^\star(\mathcal{H}^s(v,u),\mathcal{H}^s(w_2,w_1),\mathcal{H}^s(w_2,w_1))} \eta(t)dt \le \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}^\star(v,w_2,w_2)+\mathcal{D}^\star(u,w_1,w_1)} \eta(t)dt\right). \tag{3.16}$$

Taking $s \to \infty$, (3.16) yields $\mathcal{D}^\star(\mathcal{H}^s(v,u),\mathcal{H}^s(w_2,w_1),\mathcal{H}^s(w_2,w_1)) = 0$. Thus, $\lim_{s\to\infty} \mathcal{H}^s(v,u) = \lim_{s\to\infty} \mathcal{H}^s(w_2,w_1) = v.$

$$\int_0^{\mathcal{D}^\star(\mathcal{H}^s(v_1,u_1),\mathcal{H}^s(w_2,w_1),\mathcal{H}^s(w_2,w_1))} \eta(t)dt \le \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}^\star(v_1,w_2,w_2)+\mathcal{D}^\star(u_1,w_1,w_1)} \eta(t)dt\right). \tag{3.17}$$

From above we get $\lim_{s\to\infty} \mathcal{H}^s(v_1,u_1) = \lim_{s\to\infty} \mathcal{H}^s(w_2,w_1) = v_1$. Thus $v = v_1$.
Then, in all cases, we have $(u,v) = (u_1,v_1)$, i.e, the mapping $\mathcal{H}$ has a unique coupled FP. □

Theorem 3.2.4. Let $(\mathcal{W},\mathcal{D}^\star,\preccurlyeq)$ be a PO complete $\mathcal{D}^\star$MS subject to the following conditions:

(i) If (u_s) is a non-decreasing sequence converges to $u \in \mathcal{W}$, then $u_s \preccurlyeq u \;\forall\; s$.

(ii) If (v_s) is a non-increasing sequence converges to $v \in \mathcal{W}$, then $v_s \succcurlyeq v \;\forall\; s$.

(iii) Every pair of the element $\mathcal{W}$ has an upper bound and a lower bound in $\mathcal{W}$.

Let $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CF having the MM property on $\mathcal{W}$ such that:

$$\int_0^{\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \le \sigma \left(\int_0^{\mathcal{D}^\star(u,m,f)+\mathcal{D}^\star(v,n,w)} \eta(t)dt\right), \tag{3.18}$$

where $u,v,w,m,n,f \in \mathcal{W}$ and $\eta : [0,\infty) \to [0,\infty)$ is a LI mapping (i.e. with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \ne f$ or $n \ne w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0,v_0)$ and $\mathcal{H}(v_0,u_0) \preccurlyeq v_0$, then $u = v$.

Proof. Initially, we suppose that u and v are comparable with respect to the PO $\preccurlyeq$ in $\mathcal{W}$. Assuming $u \preccurlyeq v$ and $v \preccurlyeq v$ without loss of generality. Following the similar argument as in Theorem 3.2.3, we conclude $u = v$.

Next we consider the case where u and v are not comparable. Then there is an upper bound and a lower bound of u and v; that is, there is $w \in \mathcal{W}$ which is comparable with both u and v.

$\Rightarrow u \preccurlyeq w$ *and* $v \preccurlyeq w$.

By utilizing Theorem 3.2.3, we obtain $(u, v) = (w, w)$. Thus we have $u = v$. $\square$

Example 3.2.5. Consider the set $\mathcal{W} = [0, 1]$ and define a mapping $\mathcal{D}^{\star} : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to \mathbb{R}^{+}$ by $\mathcal{D}^{\star}(u, v, w) = |u - v| + |u - w| + |v - w| \; \forall \; u, v, w \in \mathcal{W}$. Therefore $(\mathcal{W}, \mathcal{D}^{\star})$ is a complete $\mathcal{D}^{\star}$MS.

Now, let $\sigma(t) = \frac{t}{2} \; \forall \; t \in [0, \infty)$, and let $\mathcal{H} : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ be a mapping defined by $\mathcal{H}(\mathsf{g}, \mathsf{h}) = \frac{\mathsf{g}+\mathsf{h}}{16}$. Since $|\mathsf{g}+\mathsf{h}-(m+n)| \leq |\mathsf{g}-m|+|\mathsf{h}-n|$ holds for all $\mathsf{g}, \mathsf{h}, m, n \in \mathcal{W}$. Thus the conditions of Theorem 3.2.1 are satisfied. So,

$$\begin{aligned}
\int_0^{\mathcal{D}^{\star}(\mathcal{H}(\mathsf{g},\mathsf{h}),\mathcal{H}(m,n),\mathcal{H}(c,k))} \eta(t)dt &= \int_0^{|\mathcal{H}(\mathsf{g},\mathsf{h})-\mathcal{H}(m,n)|+|\mathcal{H}(\mathsf{g},\mathsf{h})-\mathcal{H}(c,k)|+|\mathcal{H}(m,n)-\mathcal{H}(c,k)|} \eta(t)dt \\
&= \int_0^{|\frac{\mathsf{g}+\mathsf{h}}{16}-\frac{m+n}{16}|+|\frac{\mathsf{g}+\mathsf{h}}{16}-\frac{c+l}{16}|+|\frac{m+n}{16}-\frac{c+k}{16}|} \eta(t)dt \\
&= \int_0^{\frac{1}{16}(|\mathsf{g}-m|+|\mathsf{h}-n|+|\mathsf{g}-c|+|\mathsf{h}-k|+|m-c|+|n-k|)} \eta(t)dt \\
&\leq \frac{1}{16}\int_0^{(|\mathsf{g}-m|+|\mathsf{h}-n|+|\mathsf{g}-c|+|\mathsf{h}-k|+|m-c|+|n-k|)} \eta(t)dt \\
&\leq \sigma(\int_0^{\mathcal{D}^{\star}(\mathsf{g},m,c)+\mathcal{D}^{\star}(\mathsf{h},n,k)} \eta(t)dt)
\end{aligned}$$

where $\mathsf{g}, \mathsf{h}, c, m, n, k \in \mathcal{W}$. Clearly $\mathcal{F}$ meets all the requirements of Theorem 3.2.1. Hence $\mathcal{H}$ has a coupled FP.

Chapter 4

Fixed Point Theorems in $D_b^\star$-Metric Spaces

This segment introduces and investigates innovative FP theorems for monotone MV functions, specially in the environment of PO complete $\mathcal{D}_b^\star$MS.

4.1 FP Results for MV Mappings in $\mathcal{D}_b^\star$MSs

Definition 4.1. A function $\mathcal{D}_b^\star : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to [0, \infty)$ is a $\mathcal{D}_b^\star$-metric on $\mathcal{W}$ if $\forall\ u, v, w, x \in \mathcal{W}$ the following conditions are satisfied:

($\mathcal{D}_{b1}^\star$) $\mathcal{D}_b^\star(u, v, w) = 0 \iff u = v = w.$

($\mathcal{D}_{b2}^\star$) $\mathcal{D}_b^\star(u, v, w)$ is invariant under permutations of its arguments, i.e., $\mathcal{D}_b^\star(u, v, w) = \mathcal{D}_b^\star(p\{u, v, w\})$ for any permutation p.

($\mathcal{D}_{b3}^\star$) $\mathcal{D}_b^\star(u, v, w) \leq l[\mathcal{D}_b^\star(u, v, x) + \mathcal{D}_b^\star(x, w, w)]$ for some constant $l \geq 1$.

The pair $(\mathcal{W}, \mathcal{D}_b^\star)$ is called a $\mathcal{D}_b^\star$MS.

Definition 4.2. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star)$ is a $\mathcal{D}_b^\star$MS. A sequence (u_s) in $\mathcal{W}$ is said to converge to $u \in \mathcal{W}$ if and only if

$$\mathcal{D}_b^\star(u_s, u_s, u) = \mathcal{D}_b^\star(u, u, u_s) \to 0 \ as\ s \to \infty.$$

Definition 4.3. A sequence $(u_s) \in \mathcal{W}$ is called a CS if for given any $\epsilon > 0$ there exists a positive integer s_0 such that, $\forall\ s, r \geq s_0, \mathcal{D}_b^\star(u_s, u_s, u_r) < \epsilon$.

Definition 4.4. A $\mathcal{D}_b^\star$MS $(\mathcal{W}, \mathcal{D}_b^\star)$ is a complete $\mathcal{D}_b^\star$-metric if every CS in $(\mathcal{W}, \mathcal{D}_b^\star)$ converges in $(\mathcal{W}, \mathcal{D}_b^\star)$.

Example 4.1.1. Let $\mathcal{W} = [0, \infty)$ and $\mathcal{D}_b^\star : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+$ be a mapping defined by:

$$\mathcal{D}_b^\star(u, v, w) = |u - v|^q + |u - w|^q + |v - w|^q,\ \textit{where}\ q \geq 1.$$

(i) $\mathcal{D}_b^\star(u, v, w) \geq 0$, obvious.

(ii) $\mathcal{D}_b^\star(u, v, w) = 0 \iff |u - v|^q + |u - w|^q + |v - w|^q = 0 \iff |u - v|^q = 0,\ |u - w|^q = 0,\ |v - w|^q = 0 \iff |u - v| = 0 \Rightarrow u = v,\ |u - w| = 0 \Rightarrow u = w,\ |v - w| = 0 \Rightarrow v = w \iff u = v = w$.

(iii) Trivial.

(iv) Using the facts that

$$(a + b)^q \leq 2^{q-1}(a^q + b^q)\ \textit{where}\ q \geq 1.$$

$$2^n(a + b) + c \leq 2^n(a + b + c)\ \textit{where}\ n \geq 1.$$

Now,

$$\begin{aligned}
\mathcal{D}_b^\star(u, v, w) &= |u - v|^q + |u - w|^q + |v - w|^q \\
&= |u - v|^q + |v - x + x - w|^q + |w - x + x - u|^q \\
&\leq |u - v|^q + 2^{q-1}(|v - x|^q + |x - w|^q) + 2^{q-1}(|w - x|^q + |x - u|^q) \\
&\leq 2^{q-1}(|u - v|^q + |v - x|^q + |x - w|^q + |w - x|^q + |x - u|^q) \\
&= 2^{q-1}(\mathcal{D}_b^\star(u, v, x) + \mathcal{D}_b^\star(x, w, w)) \\
&= l(\mathcal{D}_b^\star(u, v, x) + \mathcal{D}_b^\star(x, w, w)).
\end{aligned}$$

Since all conditions are satisfied, therefore $(\mathcal{W}, \mathcal{D}_b^\star)$ is a $\mathcal{D}_b^\star$MS.

Definition 4.5. Let $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ be a $\mathcal{D}_b^\star$MS, and $\Theta : \mathcal{W} \to [0, \infty)$ be a functional. We define the relation $\preccurlyeq$ as follows:

$$u \preccurlyeq v \iff \kappa(\mathcal{D}_b^\star(u, u, v)) \leq \Theta(u) - \Theta(v) \;\forall\; u, v \in \mathcal{W},$$

where $\kappa : [0, \infty) \to [0, \infty)$ be a mapping which satisfies

(i) κ is increasing and CF.

(ii) $\kappa^{-1}(\{0\}) = \{0\}$.

(iii) $\kappa(l(a + b)) \leq \kappa(a) + \kappa(b) \;\forall\; a, b \in [0, \infty)$.

$(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ with this partial order is called an ordered $\mathcal{D}_b^\star$MS induced via (κ, Θ).

Proposition 4.1.2. Let $(\mathcal{W}, \mathcal{D}_b^\star)$ be a $\mathcal{D}_b^\star$MS, then $\preccurlyeq$ is a PO on $\mathcal{W}$ and $(\mathcal{W}, \preccurlyeq)$ is a PO set.

Proof. Let start by showing that the relation $\preccurlyeq$ is reflexive, meaning that every element is $\preccurlyeq$ itself.
Since $\kappa(\mathcal{D}_b^\star(u, u, u)) = \Theta(u) - \Theta(u) \;\forall\; u \in \mathcal{W}$, this implies that $\preccurlyeq$ is reflexive.
Next to show that the relation $\preccurlyeq$ is antisymmetric.

If $u, v \in \mathcal{W}$ with $u \preccurlyeq v$ and $v \preccurlyeq u$, then $\kappa(\mathcal{D}_b^\star(u, u, v)) \leq \Theta(u) - \Theta(v)$ and $\kappa(\mathcal{D}_b^\star(v, v, u)) \leq \Theta(v) - \Theta(u)$.
$\Rightarrow \kappa(\mathcal{D}_b^\star(u, u, v)) + \kappa(\mathcal{D}_b^\star(v, v, u)) = 0$.
Thus $\kappa(\mathcal{D}_b^\star(u, u, v)) = \kappa(\mathcal{D}_b^\star(v, v, u)) = 0$. So $\kappa(\mathcal{D}_b^\star(u, u, v)) = 0$, and so $u = v$, which shows that $\preccurlyeq$ is antisymmetric.
Lastly, we prove that $\preccurlyeq$ is transitive.
If $u, v, w \in \mathcal{W}$ s.t. $u \preccurlyeq v$ and $v \preccurlyeq w$, then

$$\kappa(\mathcal{D}_b^\star(u, u, v)) \leq \Theta(u) - \Theta(v). \tag{4.1}$$

Also

$$\kappa(\mathcal{D}_b^\star(v, v, u)) \leq \Theta(v) - \Theta(u). \tag{4.2}$$

Hence, combining (4.1) and (4.2)

$\kappa(\mathcal{D}_b^\star(u,u,v)) + \kappa(\mathcal{D}_b^\star(v,v,u)) \leq \Theta(u) - \Theta(w)$.

By applying the definition of $\mathcal{D}_b^\star$MS and property of the function κ

$$\begin{aligned}
\kappa(\mathcal{D}_b^\star(u,u,w)) &= \kappa[l\,(\mathcal{D}_b^\star(u,u,v) + \mathcal{D}_b^\star(v,v,w))] \\
&\leq \kappa\,(\mathcal{D}_b^\star(u,u,v)) + \kappa\,(\mathcal{D}_b^\star(v,v,w)) \\
&= \Theta(u) - \Theta(v) + \Theta(v) - \Theta(w) \\
&\leq \Theta(u) - \Theta(w).
\end{aligned}$$

Thus, $u \preccurlyeq w$. □

$(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is called PO $\mathcal{D}_b^\star$MS induced via (κ, Θ).

Definition 4.6. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is an ordered $\mathcal{D}_b^\star$MS induced via (κ, Θ), the following defines the ordered intervals in $\mathcal{W}$:

(i) $[u,v] = \{w \in \mathcal{W} : u \preccurlyeq w \preccurlyeq v\}$.

(ii) $[u,\infty) = \{w \in \mathcal{W} : u \preccurlyeq w\}$.

(iii) $(-\infty,u] = \{w \in \mathcal{W} : w \preccurlyeq u\}$.

Our first main result is stated in the following theorem.

Theorem 4.1.3. Let $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}_b^\star$MS generated by (κ, Θ), where $\Theta : \mathcal{W} \to [0,\infty)$ is a mapping which is b.below. Let $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ be a MV mapping and $\mathcal{M} = \{u \in \mathcal{W} : \mathcal{H}(u) \cap [u,\infty) \neq \emptyset\}$. Let

(i) $\mathcal{H}$ is USC.

(ii) If $u \in \mathcal{M}$, then $v \in \mathcal{M}\ \forall\ v \in \mathcal{H}(u) \cap [u,\infty)$.

(iii) $\mathcal{H}(m) \cap [m,\infty) \neq \emptyset$ for some $m \in \mathcal{W}$.

Then there is a sequence (u_s) s.t. $u_{s-1} \preccurlyeq u_s \in \mathcal{H}(u_{s-1})$ for all $s \in \mathbb{N}$, and H has a FP u_0 s.t. $u_s \to u_0$. In addition, if Θ is LSC, then $u_s \preccurlyeq u_0\ \forall\ s$.

Proof. By using (iii), there is an $m \in \mathcal{W}$ that belongs to $\mathcal{M}$. Then choose $n \in \mathcal{H}(m) \cap [m, \infty)$, and we have $m \preccurlyeq n$. By condition (ii), $n \in \mathcal{M}$. Choose $\tau \in \mathcal{H}(n) \cap [n, \infty)$ s.t. $n \preccurlyeq \tau$. By repeating the process, we get a sequence $(u_s) \in \mathcal{W}$ s.t. $u_{s-1} \preccurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$.

Since $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO $\mathcal{D}_b^\star$MS induced via (κ, Θ)

$$\Rightarrow \kappa(\mathcal{D}_b^\star(u_{s-1}, u_{s-1}, u_s)) \leq \Theta(u_{s-1}) - \Theta(u_s). \tag{4.3}$$

Given that κ is non-negative mapping,

$\Rightarrow \Theta(u_{s-1}) - \Theta(u_s) \geq 0\ \forall\ s \in \mathbb{N}$.

$\Rightarrow \Theta(u_{s-1}) \geq \Theta(u_s)\ \forall\ s \in \mathbb{N}$.

Since the mapping Θ is b.below, the sequence $\Theta(u_s)$ is both decreasing and b.below. Therefore, by the completeness property of $\mathbb{R}$, $\lim_{s\to\infty} \Theta(u_s) = \inf\{u_s : s \in \mathbb{N}\}$. Thus, by equation (4.3)

$$\lim_{s,r\to\infty} \kappa(\mathcal{D}_b^\star(u_s, u_s, u_r)) \leq \lim_{s\to\infty} \Theta(u_s) - \lim_{r\to\infty} \Theta(u_r).$$

Therefore, $\lim_{s,r\to\infty} \kappa(\mathcal{D}_b^\star(u_s, u_s, u_r)) = 0$.

By exploiting the continuity of κ and the fact that $\kappa^{-1}(\{0\}) = \{0\}$, it follows that $\lim_{s,r\to\infty} \mathcal{D}_b^\star(u_s, u_s, u_r) = 0$.

Therefore, (u_s) is a CS in $\mathcal{W}$. Since $\mathcal{W}$ is complete $\mathcal{D}_b^\star$MS, there exists $u_0 \in \mathcal{W}$ s.t. (u_s) is $\mathcal{D}_b^\star$-Con. to u_0. Since $u_{s-1} \in \mathcal{W}, u_s \in \mathcal{H}(u_{s-1}), u_{s-1} \to u_0$, and $u_s \to u_0$, via the definition of upper semi-continuity of $\mathcal{H}$, we have $u_0 \in \mathcal{H}(u_0)$.

Now assuming Θ is LSC, then for each $s \in \mathbb{N}$,

$$\begin{aligned}
\kappa(\mathcal{D}_b^\star(u_s, u_s, u_0)) &= \lim_{r\to\infty} \kappa(\mathcal{D}_b^\star(u_s, u_s, u_r)) \\
&\leq \lim_{r\to\infty} \inf\{\Theta(u_s) - \Theta(u_r)\} \\
&= \Theta(u_s) - \lim_{r\to\infty} \inf \Theta(u_r) \\
&\leq \Theta(u_s) - \Theta(u_0).
\end{aligned}$$

Thus, $u_s \leq u_0\ \forall\ s \in \mathbb{N}$. □

Corollary 4.1.4. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}_b^\star$MS induced via (κ, Θ), where $\Theta : \mathcal{W} \to [0, \infty)$ is a b.below mapping, and let $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ be a MV mapping which satisfies

(i) $\mathcal{H}$ is USC.

(ii) $\mathcal{H}$ satisfies the condition of monotonic sequence: $\forall\ u, v \in \mathcal{W}$ and $u \preccurlyeq v$ and every $\alpha \preccurlyeq \mathcal{H}(u)$, there exist $\beta \preccurlyeq \mathcal{H}(v)$ s.t. $\alpha \preccurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $\mathcal{H}(m) \cap [0, \infty) \neq \emptyset$.

Then there exists a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \preccurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Furthermore, if Θ is LSC, then $u_s \preccurlyeq u_0\ \forall\ s$.

Proof. By property (ii), $m \in \mathcal{M}$. Now, consider $v \in \mathcal{H}(m) \cap [0, \infty)$, then by the condition of $\mathcal{H}$, there exists $w \in \mathcal{H}(v)$ s.t. $v \preccurlyeq w$. Equivalently, $w \in \mathcal{H}(v) \cap [0, \infty) \neq \emptyset$. This implies that $v \in \mathcal{M}$ and then by Theorem 4.1.3 proof is completed. □

Corollary 4.1.5. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}_b^\star$MS induced via (κ, Θ) s.t. $\Theta : \mathcal{W} \to [0, \infty)$ is a b.below mapping, and let $\mathtt{f} : \mathcal{W} \to \mathcal{W}$ satisfies the following:

(i) $\mathtt{f}$ is a CF.

(ii) For any $\alpha \in \mathtt{f}(u)$, there exists $\beta \in \mathtt{f}(v)$ s.t. $\alpha \preccurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $m \preccurlyeq \mathtt{f}(m)$.

Then there is a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \preccurlyeq u_s \in \mathtt{f}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathtt{f}$ has a FP u_0 s.t. $u_s \to u_0$. Also if Θ is LSC, then $u_s \preccurlyeq u_0\ \forall\ s$.

Proof. Define a MV mapping, $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ via $\mathcal{H}(u) = \{\mathtt{f}(u)\}\ \forall\ u \in \mathcal{W}$, then $\mathcal{H}$ and $\mathcal{W}$ satisfy all the conditions of Theorem 4.1.3. Therefore the proof follows from Theorem 4.1.3. □

By replacing the conditions of b.below with the conditions of b.above, we obtain the following results.

Theorem 4.1.6. Let $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}_b^\star$MS induced via (κ, Θ), where $\Theta : \mathcal{W} \to (-\infty, 0]$ is a mapping that is b.above. Suppose that $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ is a MV mapping and $\mathcal{M} = \{u \in \mathcal{W} : \mathcal{H}(u) \cap (-\infty, u] \neq \emptyset\}$. Assume that

(i) $\mathcal{H}$ is USC.

(ii) For all $u \in \mathcal{M}, \mathcal{H}(u) \cap \mathcal{M} \cap (-\infty, u] \neq \emptyset$.

Then there is a sequence (u_s) s.t. $u_{s-1} \succcurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Also, if Θ is LSC, then $u_s \succcurlyeq u_0\ \forall\ s$.

Proof. By using condition *(ii)*, there exists $m \in \mathcal{W}$ s.t. $m \in \mathcal{M}$. By choosing $n \in \mathcal{H}(m) \cap (-\infty, m]$, and we get $m \succcurlyeq n$. By condition *(ii)*, $n \in \mathcal{M}$. Choose $\tau \in \mathcal{H}(n) \cap (-\infty, n]$,
$\Rightarrow n \succcurlyeq \tau$.
By proceeding in this way, there is a sequence $(u_s) \in \mathcal{W}$ s.t. $u_{s-1} \succcurlyeq u_s \in \mathcal{H}(u_{s-1})$ for all $s \in \mathbb{N}$.
Since $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO $\mathcal{D}_b^\star$MS induced via (κ, Θ),

$$\Rightarrow \kappa(\mathcal{D}_b^\star(u_{s-1}, u_{s-1}, u_s)) \leq \Theta(u_{s-1}) - \Theta(u_s).$$

Given that κ is non-negative mapping, $\Theta(u_{s-1}) - \Theta(u_s) \geq 0\ \forall\ s \in \mathbb{N}$.
$\Rightarrow \Theta(u_{s-1}) \geq \Theta(u_s)\ \forall\ s \in \mathbb{N}$.
As Θ is b.above, we get $\Theta(u_s)$ is an increasing sequence which is b.above. By the completeness of $\mathbb{R}$, $\lim_{s \to -\infty} \Theta(u_s) = \inf\{u_s : s \in \mathbb{N}\}$, thus

$$\lim_{s,r \to -\infty} \kappa(\mathcal{D}_b^\star(u_s, u_s, u_r)) \leq \lim_{s \to -\infty} \Theta(u_s) - \lim_{r \to -\infty} \Theta(u_r).$$

Therefore, $\lim_{s,r \to -\infty} \kappa(\mathcal{D}_b^\star(u_s, u_s, u_r)) = 0$.
Now, since κ is continuous $\kappa^{-1}(\{0\}) = \{0\}$, we get $\lim_{s,r \to -\infty} \mathcal{D}_b^\star(u_s, u_s, u_r) = 0$. Therefore, (u_s) is a CS in $\mathcal{W}$. Since $\mathcal{W}$ is complete, there exists $u_0 \in \mathcal{W}$ s.t. (u_s) is $\mathcal{D}^\star$-Con. to u_0. Since $u_{s-1} \in \mathcal{W}, u_s \in \mathcal{H}(u_{s-1}), u_{s-1} \to u_0$, and $u_s \to u_0$, via the definition of upper semi-continuity of $\mathcal{H}$, we have $u_0 \in \mathcal{H}(u_0)$.

Now, if Θ is LSC, then for all $s \in \mathbb{N}$,

$$\begin{aligned}\kappa(\mathcal{D}_b^\star(u_s, u_s, u_0)) &= \lim_{r\to\infty} \kappa(\mathcal{D}_b^\star(u_s, u_s, u_r)) \\ &\leq \lim_{r\to\infty} \{\inf \Theta(u_s) - \Theta(u_r)\} \\ &= \Theta(u_s) - \lim_{r\to\infty} \Theta(u_r) \\ &\leq \Theta(u_s) - \Theta(u_0).\end{aligned}$$

Thus, $u_s \succcurlyeq u_0$ for all $s \in \mathbb{N}$. □

Corollary 4.1.7. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}_b^\star$MS induced via (κ, Θ), where $\Theta : \mathcal{W} \to (-\infty, 0]$ is b.above, and let $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ be a MV mapping where

(i) $\mathcal{H}$ is USC.

(ii) For all $u, v \in \mathcal{W}$ and $u \succcurlyeq v$ and every $\alpha \in \mathcal{H}(u)$, there exist $\beta \in \mathcal{H}(v)$ s.t. $\alpha \succcurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $\mathcal{H}(m) \cap [0, \infty) \neq \emptyset$.

Then there exists a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \succcurlyeq u_s \in \mathcal{H}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathcal{H}$ has a FP u_0 s.t. $u_s \to u_0$. Furthermore, if Θ is LSC, then $u_s \succcurlyeq u_0\ \forall\ s$.

Proof. By condition (ii), we know that $m \in \mathcal{M}$. Now consider $v \in \mathcal{H}(m) \cap (-\infty, 0]$. Then by the condition (ii) of $\mathcal{H}$ there exists $w \in \mathcal{H}(v)$ s.t. $v \succcurlyeq w$. That means, $w \in \mathcal{H}(v) \cap (-\infty, v] \neq \emptyset$. Hence $v \in \mathcal{M}$ and then by Theorem 4.1.6, the result follows. □

Corollary 4.1.8. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}_b^\star$MS induced via (κ, Θ) s.t. $\Theta : \mathcal{W} \to (-\infty, 0]$ is b.above, and let $\mathtt{f} : \mathcal{W} \to \mathcal{W}$ satisfies the following:

(i) $\mathtt{f}$ is CF.

(ii) For any $\alpha \in \mathtt{f}(u)$, there exists $\beta \in \mathtt{f}(v)$ s.t. $\alpha \succcurlyeq \beta$.

(iii) There exists $m \in \mathcal{W}$ s.t. $m \succcurlyeq \mathtt{f}(m)$.

Then there is a sequence $(u_s) \in \mathcal{W}$ with $u_{s-1} \succcurlyeq u_s \in \mathtt{f}(u_{s-1})\ \forall\ s \in \mathbb{N}$, and $\mathtt{f}$ has a FP u_0 s.t. $u_s \to u_0$. Also if Θ is LSC, then $u_s \succcurlyeq u_0\ \forall\ s$.

Proof. Define a multivalued mapping, $\mathcal{H} : \mathcal{W} \to 2^{\mathcal{W}}$ via $\mathcal{H}(u) = \{\mathtt{f}(u)\}$ for all $u \in \mathcal{W}$, then $\mathcal{H}$ and $\mathcal{W}$ satisfy all the conditions of Theorem 4.1.6. So the proof follows from Theorem 4.1.6. □

4.2 Coupled FP Theorems and $\mathcal{D}_b^\star$MSs

The main contribution of our work is summarized in the following theorem:

Theorem 4.2.1. Suppose that $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}_b^\star$MS, and let $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CF with the MM property on $\mathcal{W}$ s.t.

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \le \sigma\left(\int_0^{\mathcal{D}_b^\star(u,m,f)+\mathcal{D}_b^\star(v,n,w)} \eta(t)dt\right), \tag{4.4}$$

where $u, v, w, m, n, f \in \mathcal{W}$ and $\eta : [0,\infty) \to [0,\infty)$ is a LI mapping (i.e., with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq H(u_0, v_0)$ and $H(v_0, u_0) \preccurlyeq v_0$, then $\mathcal{H}$ has a coupled FP in $\mathcal{W}$.

Proof. By hypothesis, there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0, v_0)$ and $\mathcal{H}(v_0, u_0) \preccurlyeq v_0$.

Define $u_1, v_1 \in \mathcal{W}$ as follows:

$$u_0 \preccurlyeq \mathcal{H}(u_0, v_0) = u_1 \textit{ and } v_1 = \mathcal{H}(v_0, u_0) \preccurlyeq v_0.$$

Suppose that $u_2 = \mathcal{H}(u_1, v_1)$ and $v_2 = \mathcal{H}(v_1, u_1)$, therefore

$$u_2 = \mathcal{H}(u_1, v_1) = \mathcal{H}(\mathcal{H}(u_0, v_0), \mathcal{H}(v_0, u_0)) = \mathcal{H}^2(u_0, v_0).$$

$$v_2 = \mathcal{H}(v_1, u_1) = \mathcal{H}(\mathcal{H}(v_0, u_0), \mathcal{H}(u_0, v_0)) = \mathcal{H}^2(v_0, u_0).$$

Utilizing the mixed monotonicity property of the mapping $\mathcal{H}$,

$$u_2 = \mathcal{H}^2(u_0, v_0) = \mathcal{H}(u_1, v_1) \succcurlyeq \mathcal{H}(u_0, v_0) = u_1 \succcurlyeq u_0,$$
$$v_2 = \mathcal{H}^2(v_0, u_0) = \mathcal{H}(v_1, u_1) \preccurlyeq \mathcal{H}(v_0, u_0) = v_1 \preccurlyeq v_0.$$

Repeatedly applying the above process for all $s \geq 0$ leads to the following:

$$u_0 \leq u_1 \preccurlyeq u_2 \preccurlyeq \ldots \preccurlyeq u_{s+1} \preccurlyeq \ldots,$$
$$v_0 \succcurlyeq v_1 \succcurlyeq v_2 \succcurlyeq \ldots \succcurlyeq v_{s+1} \succcurlyeq \ldots$$

s.t.

$$u_{s+1} = \mathcal{H}^{s+1}(u_0, v_0) = \mathcal{H}(\mathcal{H}^s(u_0, v_0), \mathcal{H}^s(v_0, u_0)),$$
$$v_{s+1} = \mathcal{H}^{s+1}(v_0, u_0) = \mathcal{H}(\mathcal{H}^s(v_0, u_0), \mathcal{H}^s(u_0, v_0)).$$

If $(u_{s+1}, v_{s+1}) = (u_0, v_0)$, then a coupled FP exists for the mapping $\mathcal{H}$.

Now we assume that $(u_{s+1}, v_{s+1}) \neq (u_s, v_s)$ for all $s \geq 0$, that is, let either $u_{s+1} = \mathcal{H}(u_s, v_s) \neq u_s$ or $v_{s+1} = \mathcal{H}(v_0, u_0) \neq v_s$. By equation (4.4), it follows that

$$\begin{aligned} \int_0^{\mathcal{D}_b^\star(u_s, u_s, u_{s+1})} \eta(t)dt &= \int_0^{\mathcal{D}_b^\star(\mathcal{H}(u_{s-1}, v_{s-1}), \mathcal{H}(u_{s-1}, v_{s-1}), \mathcal{H}(u_s, v_s))} \eta(t)dt \\ &\leq \sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-1}, u_{s-1}, u_s), \mathcal{D}_b^\star(v_{s-1}, v_{s-1}, v_s)} \eta(t)dt\right). \end{aligned} \tag{4.5}$$

In the same way, it can be proved that

$$\begin{aligned} \int_0^{\mathcal{D}_b^\star(v_s, v_s, v_{s+1})} \eta(t)dt &= \int_0^{\mathcal{D}_b^\star(\mathcal{H}(v_{s-1}, u_{s-1}), \mathcal{H}(v_{s-1}, u_{s-1}), \mathcal{H}(v_s, u_s))} \eta(t)dt \\ &\leq \sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-1}, u_{s-1}, u_s), \mathcal{D}_b^\star(v_{s-1}, v_{s-1}, v_s)} \eta(t)dt\right). \end{aligned} \tag{4.6}$$

Since η is non-increasing mapping, then for each $a, b \geq 0$,

$$\int_0^{a+b} \eta(t)dt \leq \int_0^a \eta(t)dt + \int_0^b \eta(t)dt. \tag{4.7}$$

Additionally, since σ is a linear and monotonically increasing mapping, it follows from (4.4), (4.5) and (4.7) that for all $s \geq 0$

$$
\begin{aligned}
\int_0^{\mathcal{D}_b^\star(u_s,u_s,u_{s+1})}\eta(t)dt &= \int_0^{\mathcal{D}_b^\star(\mathcal{H}(u_{s-1},v_{s-1}),\mathcal{H}(u_{s-1},v_{s-1}),\mathcal{H}(u_s,v_s))}\eta(t)dt\\
&\le \sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-1},u_{s-1},u_s)+\mathcal{D}_b^\star(v_{s-1},v_{s-1},v_s)}\eta(t)dt\right)\\
&\le \sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-1},u_{s-1},u_s)}\eta(t)dt\right)+\sigma\left(\int_0^{\mathcal{D}_b^\star(v_{s-1},v_{s-1},v_s)}\eta(t)dt\right)\\
&= \sigma\left(\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-1},v_{s-1}))}\eta(t)dt\right)\\
&+ \sigma\left(\int_0^{\mathcal{D}_b^\star(\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-1},u_{s-1}))}\eta(t)dt\right)\\
&\le \sigma\left(\sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})+\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\right)\\
&+ \sigma\left(\sigma\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})+\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\right)\\
&\le \sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\\
&+ \sigma^2\left(\int_0^{\mathcal{D}^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&+ \sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&+ \sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\\
&= 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\\
&+ 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&= 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt+\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&\le 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})+\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\\
&\vdots\\
&\le s\sigma^s\left(\int_0^{\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)}\eta(t)dt\right). \qquad (4.8)
\end{aligned}
$$

Following the same steps, it can be proved that

$$\begin{aligned}
\int_0^{\mathcal{D}_b^\star(v_s,v_s,v_{s+1})}\eta(t)dt &= \int_0^{\mathcal{D}_b^\star(\mathcal{H}(v_{s-1},u_{s-1}),\mathcal{H}(v_{s-1},u_{s-1}),\mathcal{H}(v_s,u_s))}\eta(t)dt \\
&\leq \sigma\left(\int_0^{\mathcal{D}_b^\star(v_{s-1},v_{s-1},v_s)+\mathcal{D}_b^\star(u_{s-1},u_{s-1},u_s)}\eta(t)dt\right) \\
&\leq \sigma\left(\int_0^{\mathcal{D}_b^\star(v_{s-1},v_{s-1},v_s)}\eta(t)dt\right)+\sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-1},u_{s-1},u_s)}\eta(t)dt\right) \\
&= \sigma\left(\int_0^{\mathcal{D}_b^\star(\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-2},u_{s-2}),\mathcal{H}(v_{s-1},u_{s-1}))}\eta(t)dt\right) \\
&+ \sigma\left(\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-2},v_{s-2}),\mathcal{H}(u_{s-1},v_{s-1}))}\eta(t)dt\right) \\
&\leq \sigma\left(\sigma\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})+\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right)\right) \\
&+ \sigma\left(\sigma\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})+\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right)\right) \\
&\leq \sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right) \\
&+ \sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right) \\
&+ \sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right) \\
&+ \sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right) \\
&= 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt\right) \\
&+ 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right) \\
&= 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})}\eta(t)dt+\int_0^{\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right) \\
&\leq 2\sigma^2\left(\int_0^{\mathcal{D}_b^\star(v_{s-2},v_{s-2},v_{s-1})+\mathcal{D}_b^\star(u_{s-2},u_{s-2},u_{s-1})}\eta(t)dt\right) \\
&\;\vdots \\
&\leq s\sigma^s\left(\int_0^{\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)}\eta(t)dt\right). \qquad (4.9)
\end{aligned}$$

Let $r, s \in \mathbb{N}$ s.t. $r > s$, then from the definition of $\mathcal{D}_b^\star$MS

$$
\begin{aligned}
\int_0^{\mathcal{D}_b^\star(u_s,u_s,u_r)} \eta(t)dt &\leq \int_0^{l[\mathcal{D}_b^\star(u_s,u_s,u_{s+1})+\mathcal{D}_b^\star(u_{s+1},u_{s+1},u_r)]} \eta(t)dt \\
&= \int_0^{l\mathcal{D}_b^\star(u_s,u_s,u_{s+1})+l\mathcal{D}_b^\star(u_{s+1},u_{s+1},u_r)} \eta(\mathsf{t})dt \\
&\leq \int_0^{l\mathcal{D}_b^\star(u_s,u_s,u_{s+1})} \eta(t)dt + \int_0^{l\mathcal{D}_b^\star(u_{s+1},u_{s+1},u_r)} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}_b^\star(u_s,u_s,u_{s+1})} \eta(t)dt \\
&+ \int_0^{l[l(\mathcal{D}_b^\star(u_{s+1},u_{s+1},u_{s+2})+\mathcal{D}_b^\star(u_{s+2},u_{s+2},u_r))]} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}_b^\star(u_s,u_s,u_{s+1})} \eta(t)dt + \int_0^{l^2\mathcal{D}_b^\star(u_{s+1},u_{s+1},u_{s+2})} \eta(t)dt \\
&+ \int_0^{l^2\mathcal{D}_b^\star(u_{s+2},u_{s+2},u_r)} \eta(t)dt \\
&\leq \int_0^{l\mathcal{D}_b^\star(u_s,u_s,u_{s+1})} \eta(t)dt + \int_0^{l^2\mathcal{D}_b^\star(u_{s+1},u_{s+1},u_{s+2})} \eta(t)dt \\
&+ \int_0^{l^3\mathcal{D}_b^\star(u_{s+2},u_{s+2},u_{s+3})} \eta(t)dt + \ldots + \int_0^{l^{r-s}\mathcal{D}_b^\star(u_{r-2},u_{r-2},u_{r-1})} \eta(t)dt \\
&+ \int_0^{l^{r-s}\mathcal{D}_b^\star(u_{r-1},u_{r-1},u_r)} \eta(t)dt \\
&\leq s\sigma^s \left(\int_0^{l[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ (s+1)\sigma^{s+1} \left(\int_0^{l^2[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ (s+2)\sigma^{s+2} \left(\int_o^{l^3[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ \ldots + (r-2)\sigma^{r-2} \left(\int_0^{l^{r-s}[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&+ (r-1)\sigma^{r-1} \left(\int_0^{l^{r-s}[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&= \sum_{i=s}^{i=r-1} i\sigma^i \left(\int_0^{l^{i-s+1}[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right) \\
&\leq \sum_{i=s}^{\infty} i\sigma^i \left(\int_0^{l^{i-s+1}[\mathcal{D}_b^\star(u_0,u_0,u_1)+\mathcal{D}_b^\star(v_0,v_0,v_1)]} \eta(t)dt \right).
\end{aligned}
$$

Since $\sum_{i=s}^{\infty} i\sigma^i(t) < \infty \;\forall\; t \in [0, +\infty)$, this implies that $\lim_{s,r\to\infty} \mathsf{D}_b^\star(u_s, u_s, u_r) = 0$ and the sequence (u_s) is a CS in $\mathcal{W}$. In a similar manner, the following result can be obtained

$$\begin{aligned}
\int_0^{\mathcal{D}_b^\star(v_s,v_s,v_r)}\eta(t)dt &\leq \int_0^{l[\mathcal{D}_b^\star(v_s,v_s,v_{s+1})+\mathcal{D}_b^\star(v_{s+1},v_{s+1},v_r)]}\eta(t)dt\\
&= \int_0^{l\mathcal{D}_b^\star(v_s,v_s,v_{s+1})+l\mathcal{D}_b^\star(v_{s+1},v_{s+1},v_r)}\eta(t)dt\\
&\leq \int_0^{l\mathcal{D}_b^\star(v_s,v_s,v_{s+1})}\eta(t)dt+\int_0^{l\mathcal{D}_b^\star(v_{s+1},v_{s+1},v_r)}\eta(t)dt\\
&\leq \int_0^{l\mathcal{D}_b^\star(v_s,v_s,v_{s+1})}\eta(t)dt\\
&+\int_0^{l[l(\mathcal{D}_b^\star(v_{s+1},v_{s+1},v_{s+2})+\mathcal{D}_b^\star(v_{s+2},v_{s+2},v_r))]}\eta(t)dt\\
&\leq \int_0^{l\mathcal{D}_b^\star(v_s,v_s,v_{s+1})}\eta(t)dt+\int_0^{l^2\mathcal{D}_b^\star(v_{s+1},v_{s+1},v_{s+2})}\eta(t)dt\\
&+\int_0^{l^2\mathcal{D}_b^\star(v_{s+2},v_{s+2},v_r)}\eta(t)dt\\
&\leq \int_0^{l\mathcal{D}_b^\star(v_s,v_s,v_{s+1})}\eta(t)dt+\int_0^{l^2\mathcal{D}_b^\star(v_{s+1},v_{s+1},v_{s+2})}\eta(t)dt\\
&+\int_0^{l^3\mathcal{D}_b^\star(v_{s+2},v_{s+2},v_{s+3})}\eta(t)dt+\ldots+\int_0^{l^{r-s}\mathcal{D}_b^\star(v_{r-2},v_{r-2},v_{r-1})}\eta(t)dt\\
&+\int_0^{l^{r-s}\mathcal{D}_b^\star(v_{r-1},v_{r-1},v_r)}\eta(t)dt\\
&\leq s\sigma^s\left(\int_0^{l[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right)\\
&+(s+1)\sigma^{s+1}\left(\int_0^{l^2[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right)\\
&+(s+2)\sigma^{s+2}\left(\int_o^{l^3[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right)\\
&+\ldots+(r-2)\sigma^{r-2}\left(\int_0^{l^{r-s}[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right)\\
&+(r-1)\sigma^{r-1}\left(\int_0^{l^{r-s}[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right)\\
&=\sum_{i=s}^{i=r-1} i\sigma^i\left(\int_0^{l^{i-s+1}[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right)\\
&\leq\sum_{i=s}^{\infty} i\sigma^i\left(\int_0^{l^{i-s+1}[\mathcal{D}_b^\star(v_0,v_0,v_1)+\mathcal{D}_b^\star(u_0,u_0,u_1)]}\eta(t)dt\right).
\end{aligned}$$

Since $\sum_{i=s}^{\infty} i\sigma^i(t) < \infty \; \forall \; t \in [0, +\infty)$, then $\lim_{s,r\to\infty} \mathcal{D}_b^\star(v_s, v_s, v_r) = 0$ and the sequence (v_s) is a CS in $\mathcal{W}$.

Since $\mathcal{W}$ is complete $\mathcal{D}_b^\star$MS, there exist $u, v \in \mathcal{W}$ s.t.

$\lim_{s\to\infty} u_s = u$ and $\lim_{s\to\infty} v_s = v$.

Since $\mathcal{H}$ is continuous, it follows that $\mathcal{H}(u, v) = u$ and $\mathcal{H}(v, u) = v$, that is, (u, v) is a a coupled FP of $\mathcal{H}$. □

Theorem 4.2.2. Let $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ is a PO complete $\mathcal{D}_b^\star$MS satisfying the following conditions:

(i) If (u_s) is non-decreasing sequence which converges to $u \in \mathcal{W}$, then $u_s \preccurlyeq u \; \forall \; s$.

(ii) If v_s is non-increasing sequence which converges to $v \in \mathcal{W}$, then $v_s \succcurlyeq v \; \forall \; s$.

Also suppose that $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CF having the MM property on $\mathcal{W}$ s.t.

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \le \sigma\left(\int_0^{\mathcal{D}_b^\star(u,m,f)+\mathcal{D}_b^\star(v,n,w)} \eta(t)dt\right), \tag{4.10}$$

where $u, v, w, m, n, f \in \mathcal{W}$ and $\eta : [0, \infty) \to [0, \infty)$ is a LI mapping (i.e., with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$.

If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq H(u_0, v_0)$ and $(v_0, u_0) \preccurlyeq v_0$, then $\mathcal{H}$ has a coupled FP in $\mathcal{W}$.

Proof. Using the similar approach to that in the proof of Theorem 4.2.1, gives two CSs (u_s) and $(v_s) \in \mathcal{W}$.

Conditions (i) and (ii) implies that there exists $u, v \in \mathcal{W}$ s.t.

$u_s \preccurlyeq u$ and $v_s \succcurlyeq v$ for all $s \geq 0$.

If $u_s = u$ and $v_s = v$ for some s, then $u_{s+1} = u$ and $v_{s+1} = v$; that is, (u, v) is a coupled FP.

Now, without loss of generality, let either $u_s \neq u$ or $v_s \neq v$.

By using (4.10),

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),u)} \eta(t)dt \le \int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),\mathcal{H}(u_s,v_s))+\mathcal{D}_b^\star(\mathcal{H}(u_s,v_s),\mathcal{H}(u_s,v_s),u)} \eta(t)dt$$
$$\le \int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),\mathcal{H}(u_s,v_s))} \eta(t)dt$$
$$+ \int_0^{\mathcal{D}_b^\star(\mathcal{H}(u_s,v_s),\mathcal{H}(u_s,v_s),u)} \eta(t)dt$$
$$\le \sigma\left(\int_0^{\mathcal{D}_b^\star(u,u,u_s)+\mathcal{D}_b^\star(v,v,v_s)} \eta(t)dt\right)$$
$$+ \int_0^{\mathcal{D}_b^\star(u_{s+1},u_{s+1},u)} \eta(t)dt. \tag{4.11}$$

Hence, via (4.11) with $s \to \infty$, we get $\mathcal{D}^\star(\mathcal{H}(u,v),\mathcal{H}(u,v),u) = 0$, which gives that $\mathcal{H}(u,v) = u$. Likewise, the same approach can be applied to write

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}(v,u),\mathcal{H}(v,u),v)} \eta(t)dt \le \int_0^{\mathcal{D}_b^\star(\mathcal{H}(v,u),\mathcal{H}(v,u),\mathcal{H}(v_s,u_s))+\mathcal{D}_b^\star(\mathcal{H}(v_s,u_s),\mathcal{H}(v_s,u_s),v)} \eta(t)dt$$
$$\le \int_0^{\mathcal{D}_b^\star(\mathsf{H}(v,u),\mathsf{H}(v,u),\mathsf{H}(v_s,u_s))} \eta(t)dt$$
$$+ \int_0^{\mathcal{D}_b^\star(\mathcal{H}(v_s,u_s),\mathcal{H}(v_s,u_s),v)} \eta(t)dt$$
$$\le \sigma\left(\int_0^{\mathcal{D}_b^\star(v,v,v_s)+\mathcal{D}_b^\star(u,u,u_s)} \eta(t)dt\right)$$
$$+ \int_0^{\mathcal{D}_b^\star(v_{s+1},v_{s+1},v)} \eta(t)dt. \tag{4.12}$$

Hence, via (4.12) with $s \to \infty$, we get $\mathcal{D}_b^\star(\mathcal{H}(v,u),H(v,u),v) = 0$ and thus $H(v,u) = v$. Therefore, (u,v) is a coupled FP of the mapping $\mathcal{H}$. □

The following theorem shows that the coupled FP of $\mathcal{H}$ can be unique.

Theorem 4.2.3. Let $(\mathcal{W}, \mathcal{D}_b^\star, \preccurlyeq)$ be a PO complete $\mathcal{D}_b^\star$MS which satisfies the following conditions:

(i) If (u_s) is a non-decreasing sequence which converges to u $\in \mathcal{W}$, then $u_s \preccurlyeq u \; \forall \; s$.

(ii) If (v_s) is a non-increasing sequence which converges to $v \in \mathcal{W}$, then $v_s \succcurlyeq v \; \forall \; s$.

(iii) For all (u,v), $(u_1,v_1) \in \mathcal{W}^2$, there exists $(w_1,w_2) \in \mathcal{W}^2$ which is comparable with (u,v) and (u_1,v_1).

Suppose that $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CF having the MM property on $\mathcal{W}$ s.t.

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \le \sigma \left(\int_0^{\mathcal{D}_b^\star(u,m,f)+\mathcal{D}_b^\star(v,n,w)} \eta(t)dt \right), \tag{4.13}$$

where $u, v, w, m, n, f \in \mathcal{W}$ and $\eta : [0, \infty) \to [0, \infty)$ is a LI mapping (i.e., with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0, v_0)$ and $\mathcal{H}(v_0, u_0) \preccurlyeq v_0$, then $\mathcal{H}$ has a unique coupled FP in $(\mathcal{W}, \mathcal{D}^\star)$.

Proof. Suppose that (u_1, v_1) is another FP of $\mathcal{H}$. The following cases are now considered.

Case 1: Let (u, v) and (u_1, v_1) be elements in $\mathcal{W}^2$ that are comparable that $(u, v) \preccurlyeq (u_1, v_1)$ i.e. $u \preccurlyeq u_1$ and $v \preccurlyeq v_1$. Now by using (4.13),

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}^s(u,v),\mathcal{H}^s(u_1,v_1),\mathcal{H}^s(u_1,v_1))} \eta(t)dt \le \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}_b^\star(u,u_1,u_1)+\mathcal{D}_b^\star(v,v_1,v_1)} \eta(t)dt \right). \tag{4.14}$$

Taking $\lim\limits_{s\to\infty}$ and by the use of (4.14), $u = u_1$.
Following a similar pattern, it is clear that

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}^s(v,u),\mathcal{H}^s(v_1,u_1),\mathcal{H}^s(v_1,u_1))} \eta(t)dt \le \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}_b^\star(v,v_1,v_1)+\mathcal{D}_b^\star(u,u_1,u_1)} \eta(t)dt \right). \tag{4.15}$$

Taking $\lim\limits_{s\to\infty}$ and by the use of (4.15), we obtain $u = u_1$.

Case 2: Let (u, v) be not comparable with (u_1, v_1). So by condition (iii) there exists $(w_1, w_2) \in \mathcal{W}^2$, which is comparable to (u, v) and (u_1, v_1). We can assume that $w_1 \preccurlyeq u, w_2 \preccurlyeq v, w_1 \preccurlyeq u_1$ and $w_2 \preccurlyeq v_1$. Again by using (4.13),

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}^s(u,v),\mathcal{H}^s(w_1,w_2),\mathcal{H}^s(w_1,w_2))} \eta(t)dt \le \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}_b^\star(u,w_1,w_1)+\mathcal{D}_b^\star(v,w_2,w_2)} \eta(t)dt \right). \tag{4.16}$$

Taking $\lim\limits_{s\to\infty}$ and by (4.16),
$\mathcal{D}_b^\star(\mathcal{H}^s(u, v), \mathcal{H}^s(w_1, w_2), \mathcal{H}^s(w_1, w_2)) = 0.$

$$\Rightarrow \lim_{s\to\infty} \mathcal{H}^s(u, v) = \lim_{s\to\infty} \mathcal{H}^s(w_1, w_2) = u.$$

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}^s(u_1,v_1),\mathcal{H}^s(w_1,w_2),\mathcal{H}^s(w_1,w_2))} \eta(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}_b^\star(u_1,w_1,w_1)+\mathcal{D}_b^\star(v_1,w_2,w_2)} \eta(t)dt \right). \tag{4.17}$$

From (4.17) $\lim_{s\to\infty} \mathcal{H}^s(u_1,v_1) = \lim_{s\to\infty} \mathcal{H}^s(w_1,w_2) = u_1$.

$\Rightarrow u = u_1$.

Preceding in the same way

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}^s(v,u),\mathcal{H}^s(w_2,w_1),\mathcal{H}^s(w_2,w_1))} \eta(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}_b^\star(v,w_2,w_2)+\mathcal{D}_b^\star(u,w_1,w_1)} \eta(t)dt \right). \tag{4.18}$$

Assume that $s\to\infty$, then (4.18) gives $\mathcal{D}_b^\star(\mathcal{H}^s(v,u),\mathcal{H}^s(w_2,w_1),\mathcal{H}^s(w_2,w_1)) = 0$.

$\Rightarrow \lim_{s\to\infty} \mathcal{H}^s(v,u) = \lim_{s\to\infty} \mathcal{H}^s(w_2,w_1) = v$.

Similarly, it can be proved that

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}^s(v_1,u_1),\mathcal{H}^s(w_2,w_1),\mathcal{H}^s(w_2,w_1))} \eta(t)dt \leq \sum_{s=0}^{\infty} s\sigma^s \left(\int_0^{\mathcal{D}_b^\star(v_1,w_2,w_2)+\mathcal{D}_b^\star(u_1,w_1,w_1)} \eta(t)dt \right). \tag{4.19}$$

Also $\lim_{s\to\infty} \mathcal{H}^s(v_1,u_1) = \lim_{s\to\infty} \mathcal{H}^s(w_2,w_1) = v_1$ by using (4.19).

$\Rightarrow v = v_1$.

Hence, in all cases, $(u,v) = (u_1,v_1)$, which means that the coupled FP of the mapping $\mathcal{H}$ unique. □

Theorem 4.2.4. Let $(\mathcal{W},\mathcal{D}_b^\star,\preccurlyeq)$ be a PO complete $\mathcal{D}_b^\star$MS satisfying the following conditions:

(i) If (u_s) is a non-decreasing sequence which converges to $u \in \mathcal{W}$, then $u_s \preccurlyeq u \ \forall\ s$.

(ii) If (v_s) is a non-increasing sequence which converges to $v \in \mathcal{W}$, then $v_s \succcurlyeq v \ \forall\ s$.

(iii) Every pair of the element $\mathcal{W}$ has an upper bound and a lower bound in $\mathcal{W}$.

Also, let $\mathcal{H} : \mathcal{W}^2 \to \mathcal{W}$ be a CF having the MM property on $\mathcal{W}$ s.t.

$$\int_0^{\mathcal{D}_b^\star(\mathcal{H}(u,v),\mathcal{H}(m,n),\mathcal{H}(f,w))} \eta(t)dt \leq \sigma \left(\int_0^{\mathcal{D}_b^\star(u,m,f)+\mathcal{D}_b^\star(v,n,w)} \eta(t)dt \right), \tag{4.20}$$

where $u,v,w,m,n,f \in \mathcal{W}$ and $\eta : [0,\infty) \to [0,\infty)$ is a LI mapping (i.e., with finite integral) with $f \preccurlyeq m \preccurlyeq u$ and $v \preccurlyeq n \preccurlyeq w$, where either $m \neq f$ or $n \neq w$. If there exist $u_0, v_0 \in \mathcal{W}$ s.t. $u_0 \preccurlyeq \mathcal{H}(u_0,v_0)$ and $\mathcal{H}(v_0,u_0) \preccurlyeq v_0$, then $u = v$.

Proof. Assume that u and v are comparable under the partial ordering $\preccurlyeq$ in $\mathcal{W}$

allowing us to assume that $u \preccurlyeq v$ and $v \preccurlyeq v$. Using the same argument as in Theorem 4.6, we arrived at $u = v$.

Next assume that u and v are incomparable. Then there is a common upper bound $w \in \mathcal{W}$ that is comparable with both u and v. So suppose that $u \preccurlyeq w$ and $v \preccurlyeq w$. By applying Theorem 4.6, $(u, v) = (w, w)$. Thus $w = v$. $\square$

The following examples validate our result.

Example 4.2.5. Consider the set $\mathcal{W} = [0, 1]$ and define a mapping $\mathcal{D}_b^\star : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+$ by $\mathcal{D}_b^\star(u, v, w) = |u - v|^2 + |u - w|^2 + |v - w|^2 \ \forall \ \ u, v, w \in \mathcal{W}$. Therefore $(\mathcal{W}, \mathcal{D}_b^\star)$ is a complete $\mathcal{D}_b^\star$MS.

Now, assume that $\sigma(t) = \frac{t}{2} \ \forall \ t \in [0, \infty)$, and let $\mathcal{H} : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ be a mapping defined by $\mathcal{H}(\mathsf{g}, \mathsf{h}) = \frac{3(\mathsf{g}+\mathsf{h})}{16}$. Thus the conditions of Theorem 4.2.1 are satisfied. That is,

$$\begin{aligned}
\int_0^{\mathcal{D}_b^\star(\mathcal{H}(\mathsf{g},\mathsf{h}),\mathcal{H}(m,n),\mathcal{H}(c,k))} \eta(t)dt &= \int_0^{|\mathcal{H}(\mathsf{g},\mathsf{h})-\mathcal{H}(m,n)|^2+|\mathcal{H}(\mathsf{g},\mathsf{h})-\mathcal{H}(c,k)|^2+|\mathcal{H}(m,n)-\mathcal{H}(c,k)|^2} \eta(t)dt \\
&= \int_0^{|\frac{3(\mathsf{g}+\mathsf{h})}{16}-\frac{3(m+n)}{16}|^2+|\frac{3(\mathsf{g}+\mathsf{h})}{16}-\frac{3(c+l)}{16}|^2+|\frac{3(m+n)}{16}-\frac{3(c+k)}{16}|^2} \eta(t)dt \\
&\le \int_0^{\frac{9(2)}{256}(|\mathsf{g}-m|^2+|\mathsf{h}-n|^2+|\mathsf{g}-c|^2+|\mathsf{h}-k|^2+|m-c|^2+|n-k|^2)} \eta(t)dt \\
&\le \frac{1}{256}\int_0^{l[|\mathsf{g}-m|^2+|\mathsf{h}-n|^2+|\mathsf{g}-c|^2+|\mathsf{h}-k|^2+|m-c|^2+|n-k|^2]} \eta(t)dt \\
&\le \sigma\left(\int_0^{l[\mathcal{D}_b^\star(\mathsf{g},m,c)+\mathcal{D}_b^\star(\mathsf{h},n,k)]} \eta(t)dt\right)
\end{aligned}$$

where $\mathsf{g}, \mathsf{h}, c, m, n, k \in \mathcal{W}$. Since $\mathcal{F}$ meets all the requirements of Theorem 4.2.1, so $\mathcal{H}$ has a coupled FP.

Example 4.2.6. Let $\mathcal{W} = [0, \infty)$ and $\mathcal{D}_b^\star : \mathcal{W} \times \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+$ be a mapping defined by:

$$\mathcal{D}_b^\star(u, v, w) = |u - v|^q + |v - w|^q + |w - u|^q.$$

Then $\mathcal{D}_b^\star$ is $\mathcal{D}_b^\star$MS by example 4.1.1. Now, suppose that $\sigma(t) = \frac{1}{2}t \ \forall \ t \in [0, \infty]$, and let $\mathcal{H} : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ be a mapping defined by $\mathcal{H}(\mathsf{g}, \mathsf{h}) = \frac{g+h}{16}$.

$$\begin{aligned}
&\int_0^{\mathcal{D}_b^\star(\mathcal{H}(\mathsf{g},\mathsf{h}),\mathcal{H}(m,n),\mathcal{H}(c,k))} \eta(t)dt \\
&= \int_0^{|\mathcal{H}(\mathsf{g},\mathsf{h})-\mathcal{H}(m,n)|^q+|\mathcal{H}(\mathsf{g},\mathsf{h})-\mathcal{H}(c,k)|^q+|\mathcal{H}(m,n)-\mathcal{H}(c,k)|^q} \eta(t)dt \\
&= \int_0^{|\frac{\mathsf{g}+\mathsf{h}}{16}-\frac{m+n}{16}|^q+|\frac{\mathsf{g}+\mathsf{h}}{16}-\frac{c+k}{16}|^q+|\frac{m+n}{16}-\frac{c+k}{16}|^q} \eta(t)dt \\
&\leq \int_0^{(\frac{1}{16})^q(2)^{q-1}(|\mathsf{g}-m|^q+|\mathsf{h}-n|^q+|\mathsf{g}-c|^q+|\mathsf{h}-k|^q+|m-c|^q+|n-k|^q)} \eta(t)dt \\
&\leq \left(\frac{1}{16}\right)^q \int_0^{l(|\mathsf{g}-m|^q+|\mathsf{h}-n|^q+|\mathsf{g}-c|^q+|\mathsf{h}-k|^q+|m-c|^q+|n-k|^q)} \eta(t)dt \\
&\leq \sigma\left(\int_0^{l[\mathcal{D}`tar_b(\mathsf{g},m,c)+\mathcal{D}_b^\star(\mathsf{h},n,k)]} \eta(t)dt\right)
\end{aligned}$$

where $\mathsf{g}, \mathsf{h}, c, m, n, k \in \mathcal{W}$. Clearly $\mathcal{F}$ satisfies all the conditions of Theorem 4.2.1, so $\mathcal{H}$ possesses a coupled FP.

Chapter 5

Conclusion

The FP results in PO MSs are crucial in developing mathematical methods to tackle various problems in pure and applied mathematics. Moreover, the study of PO MSs has far-reaching implications in numerous fields. In this research,

- A comprehensive review of work by Majid et al. [50] on monotone multivalued and integral type contractive mappings is presented.
- The notions of $\mathcal{D}$MS are extended to $\mathcal{D}_b^\star$MS in coordination with bMSs.
- The concepts of partial order and contractive conditions are redefined within the framework of $\mathcal{D}_b^\star$MS and it is observed that contractive condition of Majid et al. is a special case of this.
- Extending the research presented in [50], the results for monotonic MV mappings and integral type contractive mappings within the broader context of $\mathcal{D}_b^\star$MS are generalized.
- In order to show the applicability of our results, non-trivial examples are provided.

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