Numerical Solution of Boundary Layer Stagnation-point Flow Over a Shrinking Sheet with Joule Heating and Viscous Dissipation

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MASTER OF PHILOSOPHY
IN MATHEMATICS

DEPARTMENT OF MATHEMATICS
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"The difference between the poet and the mathematician is that the poet tries to get his head into the heavens while the mathematician tries to get the heavens into his head"

G.K. Chesterton

Abstract

This dissertation investigates the boundary layer stagnation point flow over a shrinking sheet in the presence of magnetic field and Joule heating along with viscous dissipation. To transform the governing non-linear PDEs into corresponding non-linear coupled ODEs along with boundary conditions using similarity transformation. Resulting non-linear ODEs are solved numerically using the shooting method and the results are validated by the Matlab solver bvp4c. The effects of various emerging parameters such as Magnetic parameter M, heat source S, reaction rate β and Prandtl number P_r, Schmidt number S_c, Eckert number E_c on the fluid velocity $f^{'}(\eta)$, temperature $\theta(\eta)$ and concentration $\phi(\eta)$ are investigated graphically. Also influence of these parameters on skin friction co-efficient $f^{''}(0)$, rate of heat transfer $-\theta'(0)$ and rate of mass transfer $-\phi'(0)$ is presented in the tabular form.

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I start with the name of ALLAH who is the most merciful and beautiful, the best guardian of humanity. Who showed all of us the height of knowledge and made us the crown of universe. I owe to my Holy Prophet(P.B.U.H) who is the best teacher and helper of all mankind.

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Waqar-un-Nisa.

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Abbreviations

ODE	**O**rdinary **D**ifferential **E**quation
PDE	**P**artial **D**ifferential **E**quation
BVP	**B**oundary **V**alue **P**roblem
IVP	**I**nitial **V**alue **P**roblem
BC	**B**oundary **C**ondition
MHD	**M**agnetohydrodynamics

Symbols

M	**M**agnetic parameter
a	**S**training rate parameter($a > 0$)
b/a	**V**elocity ratio parameter
b	**S**hrinking/stretching rate parameter
D	**M**olecular diffusivity
ϕ	**N**on-dimensional species concentration
C	**S**pecies concentration
S	**H**eat source parameter
ρ	**D**ensity of fluid
C_f	**L**ocal skin friction
S_h	**S**herwood number
S_c	**S**chmidt number
K_c	**C**hemical reaction parameter
f'	**V**elocity of boundary layer fluid
T	**T**emperature of the field
θ	**D**imensionless temperature
U	**S**training velocity
β	**R**eaction rate parameter
κ	**T**hermal diffusivity
σ	**E**lectrical conductivity
τ	**S**hear stress
N_u	**L**ocal Nusselt number
P_r	**P**randtl number

E_c	**E**ckert number
T_∞	**A**mbient temperature
T_w	**W**all temperature
C_∞	**A**mbient concentration in the free stream
C_w	**W**all concentration
u	**V**elocity component in x direction
v	**V**elocity component in y direction
x	**H**orizontal co-ordinate
y	**V**ertical co-ordinate
μ	**D**ynamic viscosity
υ	**K**inematic coefficient of viscosity
w	**C**ondition on porous plate
ψ	**S**tream function
η	**S**imilarity variable

Chapter 1

INTRODUCTION

Fluid is serving as a source of life for human beings and human beings have always curiosity for discovering nature and fluid is an important factor of nature, so it attracts human. It also attracted many scientists to understand the patterns of flow of sea to the smallest pond. Archimedes was first who investigated fluid statics and buoyancy, and formulated his famous law known as the Archimedes principle, which was published in his work "On floating bodies" generally considered to be the first major work on fluid mechanics. Rapid advancement in fluid mechanics began in fifteenth century. Leonardo Da Vinci responded to this attraction by observing and recording the phenomenon that we recognize today as a fundamental law of physics; namely the law of mass conservation. In this regard, Da Vinci was the first person who took the task of making sketches of different fields of flow. From the time of Da Vinci, there has been a remarkable change in the studies of fluid dynamics. The study of boundary layer flow of an incompressible viscous fluid over a shrinking sheet has many applications in manufacturing industries, such as the shrinking sheet flows happen in some applicable circumstances like, for glass fiber production, for paper production, also for metal and polymer processing. In both theory and practice, for all at rest, motion and linearly shrinking surfaces, heat transfer effects are important for boundary layer, MHD and stagnation point flows. For manufacturing process, these flows have applications in industries. Boundary layer applications include material handling along conveyers, the aerodynamics, blood flowing problems, extrusion of plastic sheets, in a bath the cooling of metallic plate, in paper and textile industries Ashraf [1]. Boundary layer for incompressible, steady fluid of a viscous flow over a linearly fluctuating stretching sheet considered first time by Crane [2]. Using least square technique to minimize residual of a differentiable equation and the result for the problem approximated for laminar steady flow of electrically conducting a viscous incompressible flow over a stretching sheet analyzed by Chakraborty and Mazumdar [3]. Ishak et al. [4] studied electrical and an incompressible viscous fluid with magnetic field of two dimensional stagnation-point over a stretching vertical sheet.

On a smooth plate, Hiemenz [5] considered the classical stagnation point flow of two dimension and the axisymmetric case was extended by Homann [6]. Viscous flow of magneto hydrodynamic fluid over a shrinking surface analyzed by Noor et al. [7] and the results obtained by Homotopy analysis method and Adomain decomposition was the same. The parametric influence of radiation while resolving the problematic cases of MHD stable and irregular flow of an electrical directing flow on infinite semi inactive surface was discussed numerically by Raptis et al. [8]. The Homotopy analysis method to solve the problem of viscous fluid with stagnation point flow with a stretching sheet was used by Nadeem et al. [9]. The dual solution for stagnation-point of mixed convection flow over a vertical sheet represented by Ishak et al. [10–12].
On a vertical sheet in a porous medium, the mixed convection boundary layer stagnation-point flow with slip condition studied by Harris et al. [13]. Recently, Aziz [14] used local similarity with slip boundary condition over a flat surface with constant heat flux. The MHD slip flow over a flat plate and the steady slip flow with porous medium discussed by Bhattacharyya et al. [15, 16]. Recently, Rohni et al. [17] reported mixed boundary-layer convection of unsteady flow with numerical investigations near stagnation point of two-dimensional flow on a porous surface vertical with thermal slip condition.
The chemical reaction with the diffusion of species for the boundary layer fluid have numerous applications in atmosphere pollution, water, fluids relevant to atmosphere and many other problems of chemical engineering. For boundary layer laminar flow of reactive chemically species with the diffusion which are used by a body over the surface considered by Chamber and Young [18]. For non-Newtonian fluids and their solution for the species of diffusion with chemical reactive in a flow over a stretching sheet with porous medium reported by Akyildiz et al. [19]. Cortell [20] also discuss the two types of viscoelastic fluid over a porous stretching sheet with the chemically reactive species. Hiemenz flow through porous media considered by Chamka and Khaled [21] with the presence of magnetic field. Heat

transfer with steady condition considered by Sriramalu et al. [22] for incompressible viscous fluid with porous type species over a stretching surface. Khan et al. [23] discussed MHD viscoelastic fluid, transfer of mass and heat over a permeable stretching surface with stress work and energy dissipation. The fluid on stretching surface close with stagnation-point discussed by Tripathy et al. [24]. Seddeek and Salem [25] observed that the mass and heat transfer distribution on stretching type surface with thermal diffusivity and variable viscosity.

In engineering processes as it is recognized that allocation of heat is connected and it is attended with heat ganeration/absorption. For the final product in industry, the knowledge of transfer of heat in polymer processing may clue to a preferred worth of that item. Abdelmeguid [26] discussed Newtonian fluid with the effect of thermal radiation on heat transfer in a boundary layer having temperature dependent diffusivity over a stretching surface with variable heat flux. Aspect of Heat transfer without magnetic field and the magnetohydrodynamics flow with stagnation-point in the direction of a stretching surface studied by Mohapatra and Gupta [27, 28]. The convective-radiation impacts on stagnation point flow of nanofluids with a stretching/shrinking sheet with viscous dissipation discussed by Pal et al. [29]. The effects of convection-radiation interaction on stagnation point flow of nanofluids over a stretching/shrinking sheet with viscous dissipation analyzed by Pal and Mandal [30]. The impact of Hall current over a non-linear stretching/shrinking sheet of nanofluids with magnetohydrodynamic heat transfer discussed by Pal and Mandal [31].

In a viscoelastic fluid flow effect of heat transfer over a stretching sheet with internal heat source/sink observed by Khan [32] and Bataller [33]. On temperature regulation, the final product dependent on their quality and property. In controlling momentum, in the boundary layer flow of numerous conducting flows the magnetic filed can play an important role by using its application and transfer of heat . In view of that, for Newtonian and non-Newtonian fluids past stretching surface, the impact of magnetic field on fluid and transfer of heat studied by many researchers. The MHD Newtonian fluid on a stretching surface have considered by Chen [34].
Recently, Bhattacharyya and Wang [35, 36] has studied mass transfer and chemical reaction past a stretching sheet with the dual solutions in boundary layer and he

has deliberated that flow is to be nonconducting electrically. He did not consider all aspects of transfer of heat. In the boundary layer flow we had measured the effect of first order diffusing species with chemical reaction.

1.1 Thesis contribution:

In this thesis, we provide a review study of Dash et al. [37] and then extend the flow analysis with Joule heating and viscous dissipation properties. There are many practical applications for the boundary layer stagnation-point flow over a shrinking sheet in the presence of Joule heating and viscous dissipation effects. The obtained system of PDEs are transmuted into a system of non-linear and coupled ODEs by using a suitable similarity transformation. A numerical solution of the system of ODEs obtained by using the Shooting method and compared the precision of the obtained numerical results with the Matlab bvp4c code. The numerical results are discussed for different physical parameters appearing in the solution affecting the flow and heat transfer.

1.2 Thesis outline:

The thesis is arranged as follows:

In **Chapter 2**, we present basic definitions of fluids, heat transfer, boundary layer flow, basic governing laws, the similarity transform and the shooting method. These basic concepts are used further in describing the flow, heat transfer under the influence of Joule heating and viscous dissipation properties.

Chapter 3 contains a comprehensive review of Dash et al. [37]. A numerical study of boundary layer stagnation point flow past a shrinking sheet in the presence of the magnetic field and Joule heating along with viscous dissipation is analyzed. The constitutive equations of the flow model are solved numerically and the impact of physical parameters concerning the flow model on the dimensionless temperature, velocity and concentration are presented through graphs and tables. Also a comparison of the achieved numerical results by the Shooting method with the published results of Dash et al. [37] has been made and found both in excellent

agreement.
In **Chapter 4**, we discuss the effects of Joule heating along with the viscous dissipation on the boundary layer stagnation point flow past a shrinking sheet in the presence of the magnetic field. The reduced system of ODEs after applying a proper similarity transform are solved numerically. Graphs and tables describe the behavior of physical parameters. Numerical values of Skin friction coefficient and Nusselt number have also been computed and discussed in this work.
Chapter 5 summarizes up the dissertation and gives the major conclusion from the entire research and recommendations for the future work.
All the references used in this dissertation are listed in **Bibliography**.

Chapter 2

BASIC DEFINITIONS AND GOVERNING EQUATIONS

2.1 Basic definitions

2.1.1 Fluid

Fluid is a material that changes regularly by action of shear stress. It does not depend how small the shear stress is and continuously change its shape as long as the shear stress acts

2.1.2 Fluid mechanics

Fluid mechanics is the branch of engineering that deals with the behavior of fluid at rest or in motion. It is divided into two branches contains the discussion of different properties of fluids and the effect of different forces on it.

2.1.3 Fluid statics

In fluid statics we deal with all properties of fluids that are at rest.

2.1.4 Fluid dynamics

In fluid dynamics we deal with all properties of fluids that are in motion.

2.2 Some physical properties of fluid

2.2.1 Density

The ratio between mass and unit volume is called density. It is denoted by ρ and mathematically, it can be written as

$$\rho = \frac{m}{V}, \tag{2.1}$$

where m and V are the mass and volume of the material respectively. Dimension of density is $[ML^{-3}]$ and SI unit is kg/m^3.

2.2.2 Viscosity

Viscosity is an intrinsic (internal) property of fluid that measures the fluid resistance against any deformation when different forces are acting on it.

2.2.3 Dynamic Viscosity

Viscosity is the property of the fluid that measures the fluid resistance against any deformation when different forces are acting on it. In other words, a fluid viscosity is that property which measures the amount of resistance to the shear stress. It is denoted by μ and mathematically, it can be written as

$$\text{viscosity}(\mu) = \frac{\text{shear stress}}{\text{shear strain}}, \tag{2.2}$$

unit of viscosity in SI system is kg/ms.

2.2.4 Kinematic viscosity

The ratio between the dynamic density and viscosity is called kinematic viscosity. Symbolically, it can be written as ν and mathematically, it can be written as

$$\nu = \frac{\mu}{\rho}, \tag{2.3}$$

where ρ and μ denote the density and the dynamic viscosity respectively. SI unit is m^2/s and its dimension is $[L^2/T]$.

2.2.5 Pressure

The ratio of applied force to the unit area is called pressure. It is denoted by P and mathematically, it can be written as

$$P = \frac{F}{A}, \tag{2.4}$$

where F, A denote the applied force and area of the surface, respectively.

2.2.6 Stress

Stress is the force acting on the surface of the unit area within a deformable body. Mathematically, it can be written as

$$\sigma = \frac{F}{A}, \tag{2.5}$$

where A is area and F is the force.

2.2.7 Shear stress

The component of stress in which a force acts parallel to the unit surface area is called shear stress.

2.2.8 Normal stress

Normal stress is the component of stress in which force acts perpendicular to the unit surface area.

2.2.9 Porous medium

A porous medium is a substance containing pores. The skinny part of the substance is usually said to be a matrix or frame. The voids are normally full with a fluid. The skeletal substance is usually a solid, but configuration like foam are often usefully analyzed using the concept of porous media.

2.3 Types of fluids

2.3.1 Fluid

A material that can flow or a substance that deform continuously is call fluid.

2.3.2 Ideal fluid

The fluid which has zero viscosity ($\mu = 0$) is said to be an ideal fluid.

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{2.6}$$

2.3.3 Real fluid or viscous fluid

In a real fluid, viscosity is non zero ($\mu \neq 0$) and the effect of viscosity can not be neglected.

2.3.4 Newton's law of viscosity

The shear stress which distorts the fluid component is directly and linearly proportional to the velocity gradient is said to be the Newton's law of viscosity.

Mathematically, it can be written as

$$\tau_{xy} \propto \left[\frac{du}{dy}\right],$$

$$\tau_{xy} = \mu \frac{du}{dy}, \tag{2.7}$$

where τ_{xy} is the shear stress component of the fluid, u is the component of the velocity along x-axis and μ is viscosity proportionality constant.

2.3.5 Newtonian fluids

The real fluids for which the shear stress of the fluid varies directly and linearly as the deformation rate are called Newtonian fluids. In other words, the fluids which satisfy the Newton's law of viscosity are also called Newtonian fluids. Shear stress of Newtonian fluid is mathematically defined as

$$\tau_{yx} = \mu \frac{du}{dy}, \tag{2.8}$$

where τ_{yx} is the shear stress, x denotes the x-component of velocity and μ denotes the dynamic viscosity. Examples of Newtonian fluids are air, water, oxygen gas and silicone oil etc.

2.3.6 Non-Newtonian fluids

Non-Newtonian fluids are those for which the shear stress is not linearly proportional to the deformation rate. In other words, the fluid which does not satisfy the Newton's law of viscosity are said to be non-Newtonian fluids. Mathematically, it can be written as

$$\tau_{xy} \propto \left[\frac{du}{dy}\right]^m, \; m \neq 1$$

$$\tau_{xy} = \mu \left[\frac{du}{dy}\right]^m, \tag{2.9}$$

where ν denotes the viscosity and m is the index of flow performance. Note that for $m = 1$, the above equation reduces to the Newton's law of viscosity. Examples of non-Newtonian fluids are shampoo, grease, paint, blood and melt polymer etc.

2.3.7 Skin friction

It is the drag force that takes place among the exterior of the sheet and the fluid/liquid.

2.3.8 Nanofluids

Nanofluids are suspensions of nanoparticles in fluids that show important improvement of their properties at simple nanoparticle concentrations. Nanofluids are fluids containing nanometer sized particles of metals, oxides, carbides or nano tubes known as a nanoparticles. Nanofluids are suspension involving a fluid containing nanoparticles i.e., particles of solid with a dimension measured in nanometers.

2.4 Types of flows

2.4.1 Flow

A material goes under deformation when different forces act on it. If the deformation continuously increases without limit is known as flow.

2.4.2 Laminar flow

Type of flow, in which the fluid moves smoothly along well defined path is known as laminar. The flow of high viscosity fluids such as oil at low velocity is typically a laminar flow.

2.4.3 Turbulent flow

A flow moves randomly in any direction and has no specific path. The flow of low viscosity fluids such as air at high velocity is typically turbulent.

2.4.4 Steady flow

The flow which is independent of time is called steady flow. Mathematically, it can be written as

$$\frac{d\xi}{dt} = 0, \tag{2.10}$$

where ξ is fluid property.

2.4.5 Unsteady flow

The flow which depends on time is called unsteady flow. Mathematically, it can be written as

$$\frac{d\xi}{dt} \neq 0, \tag{2.11}$$

where ξ is fluid property.

2.4.6 Uniform flow

If the velocity of flow has same magnitude as well as direction during the motion of fluid, then the flow is called uniform flow. Mathematically, it can be written as

$$\frac{dv}{ds} = 0, \tag{2.12}$$

where s is the displacement in any direction and v is the velocity.

2.4.7 Non-Uniform flow

In non-uniform flow, the velocity is not same at every point in the fluid at a given instant. Mathematically, it can be written as

$$\frac{dv}{ds} \neq 0, \tag{2.13}$$

where s is the displacement and v is the velocity.

2.4.8 Compressible flow

The fluid in which the density with respect to the substance is not constant is called compressible flow(e.g, high speed gas flow). Mathematically, it can be written as

$$\frac{d\rho}{dt} \neq 0,$$

2.4.9 Incompressible flow

If the density of flowing fluid remains nearly constant throughout flow is called incompressible flow(e.g, liquid flow). Mathematically, it can be written as

$$\frac{D\rho}{Dt} = 0,$$

where ρ denotes the fluid density and $\frac{D}{Dt}$ is the material derivative given by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla. \tag{2.14}$$

In Eq. (2.14), $\mathbf{V}$ denotes the velocity of the flow and ∇ is the differential operator. In Cartesian coordinate system ∇ is given as

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

2.4.10 Internal flow

Completely bounded flow by the solid surface is known as internal flow. The flow in a pipe or duct is an example of the internal flow.

2.4.11 External flow

The flow, which is not bounded by the solid surface is called external flow. An example of the external flow is the water flow in the river or in the ocean.

2.5 Some basic definitions of heat transfer

When two different bodies located at different temperature interact then heat transfer usually occurs from higher temperature body to the lower temperate body. It modify the internal energy of both systems involved. The transfer of heat takes place in the following ways.

2.5.1 Conduction

A process in which the heat is transferred between those objects that are in physical contact is called conduction. Mathematically, it can be written as

$$q = -kA\left[\frac{\Delta T}{\Delta n}\right], \tag{2.15}$$

where k and $\frac{\Delta T}{\Delta n}$ denote the constant of the thermal conductivity and gradient of the temperature respectively.

2.5.2 Convection

When the heat is transferred through fluids $(gases\ or\ liquids)$ is known as convection. In this process, heat always transfers from a warmer spot to a cooler spot. Convective heat transfer arises between a fluid and a bounding surface. If there is a difference in the temperature of fluid and bounding surface, then thermal boundary layer is created. Fluid particles which interact with the surface, attain equilibrium at the surface temperature and transfer energy in the next layer and so on. Through this mode, temperature gradients are produced in fluid. The area of fluid containing these temperature gradients are identified as thermal boundary layer. Since the convective heat transfer is by both random molecular motion and the bulk motion of the fluid, the molecular motion is more adjacent to the surface where the fluid velocity is less. Convective heat transfer depends upon the nature of the flow. Convection has three forms: Forced convection, Natural (free) convection, Mixed convection.

2.5.3 Forced convection

Forced convection is a process, or kind of energy transfer in which fluid motion is produced by an external source. In other words, the heat transfer in which fluid motion is originated by an independent source like a pump and fan etc is called forced convection.

2.5.4 Natural convection

Natural convection is a heat transport process, in which the fluid motion is not developed by any external source, but only by density differences in the fluid taking place due to temperature gradients. It exists due to the temperature differences which affect the density of the fluid. It is also known as free convection.

2.5.5 Mixed convection

It is a combination of both forced convection and natural convection and occurs when natural convection and forced convection act collectively to transfer heat. This is also defined as the circumstances where both pressure forces and buoyant forces act together. In other words, when both natural and forced convection processes simultaneously and significantly contribute to heat transfer, mixed convection flow appears.

2.5.6 Radiation

A process in which heat is transferred directly by electromagnetic waves is known as radiation and it occurs when two bodies of different temperature are aligned.

2.6 Thermal conductivity

The property of a substance which measures the ability to conduct heat is called thermal conductivity. Fourier's law of conduction which relates the rate of heat transfer by conduction to the temperature gradient is

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}, \tag{2.16}$$

where A, $\frac{dQ}{dt}$, k, $\frac{dT}{dx}$ are the area, the rate of heat transfer, the thermal conductivity and the temperature gradient respectively. Thermal conductivity of most of the liquids decreases with the increase of temperature except water. The SI unit of thermal conductivity is $\frac{Kg.m}{s^3}$ and the dimension of thermal conductivity is $[\frac{ML}{T^3}]$.

2.7 Thermal diffusivity

Thermal diffusivity is material property for characterizing unsteady heat conduction.
Mathematically,

$$\alpha = \frac{k}{\rho C_p}, \tag{2.17}$$

where k is the thermal conductivity of material, ρ its density and C_p its specific heat.

2.8 Joule heating

When a current flows with finite conductivity through a solid or liquid, electric energy is transformed to heat energy through resistive loses in the substance. This process is known as Joule heating. When conduction electrons shift energy to the conductors atoms through collisions then heat is produced on the micro scale. Many applications depends on the Joule heating such as microvalves, cooking plates and toasters.

2.9 Magnetohydrodynamics(MHD)

The study of the dynamics of the electrically conducting fluids such as plasmas or electrolytes etc, is known as MHD.

2.10 Boundary layer flow

The philosophy of boundary layer flow was given by a German Ludwig Prandlt in 1904. The thin layer close to the wall of container is called the boundary layer. In the boundary layer, flow is energetic for aerodynamic drag and lift of the flying articles. A boundary layer can be laminar or turbulent if the flow takes place in layers such that each layer slides pass the adjacent layers, then this layer is identified as laminar boundary layer, although the turbulent boundary layer is one in which there is an intense agitation.

2.10.1 Hydrodynamic boundary layer

A region near solid surface where the flow configuration is achieved by viscous drag directly from surface wall is known as a hydrodynamic boundary layer.

2.10.2 Thermal boundary layer

Transfer of heat due to thermodynamic relations happens in thermal boundary. The temperature alters of fluid stream happens in this layer and is placed in the region near solid surface where cooling or heating of the surface wall impacts the fluid temperature.

2.11 Dimensionless numbers

2.11.1 Prandtl number

The ratio of the momentum diffusivity to the thermal diffusivity is said to be the Prandtl number. It is denoted by P_r and mathematically, it can be written as

$$P_r = \frac{\nu}{\alpha},$$

$$P_r = \frac{\frac{\mu}{\rho}}{\frac{k}{\rho C_p}},$$

$$P_r = \frac{\mu C_p}{k}, \tag{2.18}$$

where ν, α denote the momentum diffusivity or kinematic diffusivity and the thermal diffusivity respectively. It controls the relative thickness of momentum and temperature function.

2.11.2 Eckert number

It is a dimensionless number defining the ratio between the kinetic energy of flow and eathalpy. Mathematically,

$$E_c = \frac{u_w^2}{C_p(\Delta T)}, \tag{2.19}$$

where u_w^2 is the characteristic flow velocity, C_p the specific heat and ΔT is the temperature.

2.11.3 Skin friction coefficient

Skin friction coefficient occurs between the fluid and the solid surface which leads to slow down the motion of fluid. The skin friction coefficient can be defined as

$$C_f = \frac{2\tau_w}{\rho U^2},$$

where τ_w denotes the wall shear stress, ρ the density and U the free-stream velocity, respectively.

2.11.4 Sherwood number

It is the ratio of total rate of mass transfer to the rate of diffusive mass transport. Mathematically, it can be expressed as

$$S_h = \frac{\hat{k}L}{D} \tag{2.20}$$

where $\hat{k}$, L and D are the mass transfer coefficient, characteristic length and the mass diffusivity, respectively.

2.11.5 Nusselt number

It examines the ratio of convective to the conductive heat transfer through the boundary of the surface. It is the dimensionless number which was introduced by the German mathematician Nusselt. Heat transfer due to conduction is denoted by $\frac{k\Delta T}{\delta}$ and heat transfer due to convection is denoted by $h\Delta T$. It is denoted by N_u and mathematically, it is expressed by

$$N_u = \frac{h\delta}{k}, \tag{2.21}$$

where h, δ, k denote the coefficient of heat transfer, the characteristic length and the thermal conductivity respectively.

2.12 Basic equations

2.12.1 Continuity equation

Continuity equation is derived from the law of conservation of mass and mathematically, it is expressed by

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0, \tag{2.22}$$

where t is the time. If fluid is an incompressible, then the continuity equation is expressed by

$$\nabla.V = 0, \tag{2.23}$$

2.12.2 Law of conservation of momentum

Each particle of fluid obeys Newton's second law of motion which is at rest or in steady state or accelerated motion. This law states that the combination of all applied external forces acting on a body is equal to the time rate of change of linear momentum of the body. In vector notation this law can be represented as

$$\rho \frac{dV}{dt} = \text{div}T + \rho b, \tag{2.24}$$

For Navier-Stokes equation

$$\tau = -pI + \mu A_1, \tag{2.25}$$

where A_1 is the tensor.

$$A_1 = \text{grad}V + (\text{grad}V)^t, \tag{2.26}$$

In the above equations, $\frac{d}{dt}$ denote material time derivative or total derivative, ρ denote density, V denote velocity field, τ the Cauchy stress tensor, b the body forces, p the pressure, μ the dynamic viscosity.
The Cauchy stress tensor is expressed in the matrix form as

$$\tau = \begin{pmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{pmatrix}, \tag{2.27}$$

where σ_{xx}, σ_{yy} and σ_{zz} are normal stresses. For two-dimensional flow, we have $V = [u(x, y, 0), v(x, y, 0), 0]$ and thus

$$\text{grad}V = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.28}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]. \tag{2.29}$$

Similarly, we repeat the above process for Y component as follows:

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]. \tag{2.30}$$

2.12.3 Energy equation

The energy equation for the fluid is

$$\rho C_p\left[\frac{\partial}{\partial t} + V\nabla\right]T = k\nabla^2 T + \tau L + \rho C_p\left[D_B \nabla C.\nabla T + \frac{DT}{Tm}\nabla T\right], \tag{2.31}$$

where the specific heat of the basic fluid and material are denoted by $(C_p)_f$ and $(C_p)_s$, ρ_f the density of basic fluid, T the temperature and L denote the rate of strain tensor of the fluid, D_B the Brownian motion coefficient and D_T the temperature diffusion coefficient and T_m denote the mean temperature. The expression for Cauchy stress tensor τ for viscous incompressible fluid is expressed by

$$\tau = -pI + \mu A_1, \tag{2.32}$$

where A_1 is the tensor, p the pressure and μ the dynamic viscosity.

$$A_1 = \text{grad}V + (\text{grad}V)^t, \tag{2.33}$$

where t represents transpose of the matrix for two dimensional velocity field of the fluid, τ the stain tensor and can be written as

$$\tau = \begin{pmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{pmatrix}. \tag{2.34}$$

Chapter 3

NUMERICAL SOLUTION OF BOUNDARY LAYER STAGNATION-POINT FLOW

In this chapter, we discuss the magnetohydrodynamics flow with heat and mass diffusion of an electrically conducting stagnation point flow past a shrinking/stretching sheet with chemical reaction of diffusing species and internal heat absorption/-generation numerically. Flow equations are modified to a system of non-linear ODEs by using the similarity transformations. Numerical solution of the ODEs is found by using the shooting method. Finally, the results are discussed for different parameters affecting the flow and transfer of heat.

3.1 Mathematical modeling

A steady two dimensional laminar boundary layer stagnation point flow of viscous incompressible electrically conducting fluid towards a stretching/shrinking sheet with chemically reactive species undergoing first order chemical reaction is considered. The flow field is exposed to uniform transverse magnetic field $\vec{B}_0 = (0, B_0, 0)$. It is assumed that the flow is generated by stretching of non-conducting elastic boundary sheet by imposing two opposite and equal forces along x-axis in such a way that the velocity of the boundary sheet is of linear order in the flow direction and the origin remains fixed. A uniform magnetic field of strength B_0 is assumed to be applied in the positive y-direction normal to the plate. The magnetic Reynolds number of the flow is taken small, therefore induced magnetic field is negligible in comparison with the applied one. The level of concentration of foreign mass assumes to be low, there for Soret and Dufour effects are negligible. The model of first order chemical reaction is considered. By usual boundary layer approximation, following Bhattacharyya [35], the governing equations are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}(u - U), \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho C_p}(T - T_\infty), \tag{3.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - R(C - C_\infty), \tag{3.4}$$

where u and v are the velocity components along the x and y axes, U is the straining velocity, C is the concentration, T is the fluid temperature, ν is the kinematic viscosity of fluid, ρ is the density of fluid, C_p is the specific heat at constant pressure, D is the species diffusion coefficient, σ is the electric conductivity of fluid, B_0 is the applied uniform magnetic field normal to the surface of the sheet, Q is the heat source parameter. The boundary conditions for Eqs. $(3.1 - 3.4)$ are

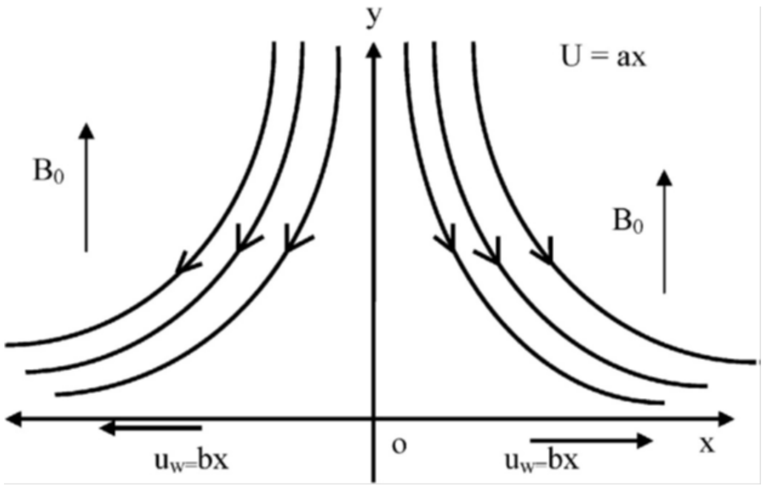

Figure 3.1: Flow geometry.

$$u = bx, v = 0, \quad T = T_w, \quad C = C_w, \quad \text{at} \quad y = 0,$$
$$u \to U(x) = ax, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty. \tag{3.5}$$

We use similarity transformation [21, 22] to solve Eqs. $(3.1 - 3.4)$

$$\psi(x, y) = \sqrt{a\nu}xf(\eta), \quad \theta(\eta) = (T - T_\infty)/(T_w - T_\infty),$$
$$\phi(\eta) = (C - C_\infty)/(C_w - C_\infty), \quad \eta = y\sqrt{a/\nu}, \tag{3.6}$$

the velocity component of stream function which is defined as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{3.7}$$

So, we have

$$u = axf^{'}(\eta), \qquad v = -\sqrt{a\nu}f(\eta), \tag{3.8}$$

where prime shows differentiation with respect to η.
Using Eq. (3.6) in Eq. (3.1) that will be satisfied, also using Eqs. $(3.5 - 3.7)$ in Eqs. $(3.2 - 3.4)$, we will get the followning ordinary differential equations.

$$f^{'''} + ff^{''} - (f^{'})^2 - M(f^{'} - 1) + 1 = 0, \tag{3.9}$$

$$\theta^{''} + P_r f\theta^{'} + P_r S\theta = 0, \tag{3.10}$$

$$\phi^{''} + S_c f\phi^{'} - S_c\beta\phi = 0, \tag{3.11}$$

with the boundary conditions

$$f(0) = 0, \quad f^{'}(0) = b/a, \quad \theta(0) = 1, \quad \phi(0) = 1 \quad \text{at} \quad \eta = 0, \tag{3.12}$$
$$f^{'}(\eta) \to 1, \quad \theta(\eta) \to 0, \quad \phi(\eta) \to 0 \quad \text{as} \quad \eta \to \infty.$$

The dimensionless constants P_r, S_c, M, S, β, represent the Prandtl number, the Schmidt number, the magnetic parameter, the heat source parameter, the reaction rate parameter, which are defined as

$$P_r = \frac{\nu}{k}, S_c = \frac{\nu}{D}, M = \frac{\sigma B_0^2}{a\rho}, S = \frac{Q}{a\rho C_p}, \beta = \frac{R}{a}. \tag{3.13}$$

In this problem the quantities of physical interest are the local Nusselt number N_u, the skin friction coefficient C_f and the local Sherwood number Sh,which are defined as

$$N_u = \frac{xq_w}{k(T_w - T_\infty)}, \quad S_h = \frac{xh_m}{D(C_w - C_\infty)}, \quad C_f = \frac{\tau_w}{\rho U^2/2}. \tag{3.14}$$

where h_m, q_w and τ_w are mass flux from the sheet, heat flux and the skin friction or shear stress, which are given by

$$h_m = -D\left[\frac{\partial C}{\partial y}\right]_{y=0}, \qquad q_w = -k\left[\frac{\partial T}{\partial y}\right]_{y=0}, \qquad \tau_w = \mu\left[\frac{\partial u}{\partial y}\right]_{y=0}. \tag{3.15}$$

where μ is the dynamic viscosity of the fluid and k is thermal diffusivity. Using the similarity variables Eq.(3.6), we get

$$\frac{N_u}{\sqrt{Re_x}} = -\theta^{'}(0), \frac{S_h}{\sqrt{Re_x}} = -\phi^{'}(0), \frac{1}{2}C_f\sqrt{Re_x} = f^{''}(0). \tag{3.16}$$

where $Re_x = \frac{\rho b x^2}{\mu}$.

3.2 Method for solution

Eqs. $(3.9-3.11)$ are non-linear and coupled. We opted to solve the non-linear system consisting of Eqs. $(3.9-3.11)$ with boundary conditions Eq. (3.12) by using the shooting iteration technique together with the fourth order Runge-Kutta integration scheme. While using this technique, boundary value problem is converted into the initial value problem. In this method we have to choose a suitable finite value of $\eta \to \infty$. Let's convert Eqs. $(3.9-3.11)$ by using following substitution:

$$f = y_1, \qquad f' = y_2, \qquad f'' = y_3, \qquad f''' = y_3', \tag{3.17}$$

$$\theta = y_4, \qquad \theta' = y_5, \qquad \theta'' = y_5', \tag{3.18}$$

$$\phi = y_6, \qquad \phi' = y_7, \qquad \phi'' = y_7'. \tag{3.19}$$

Using above notations as a result we get seven first order non linear coupled ODEs with the boundary conditions are also adjust according to the above supposition, written below

$$\left.\begin{aligned}
y_1^{'} &= y_2, \\
y_2^{'} &= y_3, \\
y_3^{'} &= -y_1 y_3 + y_2^2 + M\left(y_2 - 1\right) - 1, \\
y_4^{'} &= y_5, \\
y_5^{'} &= -Pry_1 y_5 - PrSy_4, \\
y_6^{'} &= y_7, \\
y_7^{'} &= Sc\beta y_6 - Scy_1 y_7,
\end{aligned}\right\} \tag{3.20}$$

The associated initial conditions are

$$\begin{aligned}
&y_1(0) = 0, \quad y_2(0) = b/a, \quad y_3(0) = t, \quad y_4(0) = 1, \\
&y_5(0) = q, \quad y_6(0) = 1, \quad y_7(0) = w,
\end{aligned} \tag{3.21}$$

In Eq. (3.21) t, q and w are the three initial guesses. Runge-Kutta method of order four is used to solve the intermediate initial value problem with some suitable initial guess $t = t_0$ $q = q_0$ and $w = w_0$. For the next iteration,the values of t, q and w are updated by the Newton's method as follows

$$\begin{bmatrix} t_{n+1} \\ q_{n+1} \\ w_{n+1} \end{bmatrix} = \begin{bmatrix} t_n \\ q_n \\ w_n \end{bmatrix} - \begin{bmatrix} y_9 & y_{16} & y_{23} \\ y_{11} & y_{18} & y_{25} \\ y_{13} & y_{20} & y_{27} \end{bmatrix}^{-1} \begin{bmatrix} y_2 \\ y_4 \\ y_6 \end{bmatrix}$$

.

where $n = 0, ! \ 1, \ 2, \ 3...$

3.2.1 Results and discussion

The main objective to study the effect of different parameters on velocity profile $f^{'}(\eta)$, temperature profile $\theta(\eta)$ and skin friction profile $\phi(\eta)$. For the conformation of the result, compared with Bhattacharyya [35]. The skin friction, rate of heat

TABLE 3.1: The skin friction $f''(0)$ for different values of b/a and M.

b/a	M	Bhattacharyya [35]	Dash et. al [37]	Present
-1.25	0		0.5971	0.5970
-1.15	0	1.0822	1.0822	1.0822
-2.14	1		1.5629	1.5628
-1	0	1.3288	1.3288	1.3288
-1	1		2.4299	2.4297
-1	2		3.1526	3.1525
-0.5	0	1.4956	1.4956	1.4955
-0.5	1		2.1201	2.1202
-0.5	2		2.5975	2.5975
1	0		0.0001	0.0003
1	1		0.0001	0.0002
1	2		0.0001	0.0002
0	0		1.2325	1.2319
0	1		1.5853	1.5847
0	2		1.8735	1.8725
0.5	0		0.7132	0.7130
0.5	1		0.8696	0.8686
0.5	2		1.0024	1.0021
2.0	1		-2.1326	-2.1365
2.0	0		-1.8873	-1.8886

TABLE 3.2: Rate of heat transfer $-\theta'(0)$ for different values of b/a.

b/a	Pr	M	S	Bhattacharyya [35]	Dash et. al [37]	Present
-1.24	0.1	0	0	0.1282	0.1281	0.1281
-1.24	0.5	0	0	0.0983	0.0958	0.0951
-1.24	0.5	1	-1	0.6537	0.5981	0.5982
-1	0.71	0	0		0.2282	0.2281
-1	0.71	1	0		0.3249	0.3249
-1	0.71	1	0.2		0.1782	0.1781
-0.5	0.71	1	0.2		0.2997	0.2997
0	0.71	1	0.2		0.4028	0.4027
1	0.71	1	0.2		0.5740	0.5740
-1	7	1	0.2		-0.6851	-0.6855
-1	0.71	1	-0.2		0.4483	0.4482
-0.5	0.71	1	-0.2		0.5397	0.5397
-0.5	0.71	2	0.2		0.3296	0.3294
0.5	0.71	1	0.2		0.4931	0.4931
0.5	7	1	0.2		1.3393	1.3392

TABLE 3.3: The rate of mass transfer $-\phi'(0)$ for different values of b/a.

b/a	Sc	M	Kc	Bhattacharyya [35]	Dash et. al [37]	Present
-1.24	0.1	0	0	0.1282	0.1281	0.1284
-1.24	0.5	0	0	0.0983	0.0958	0.0956
-1.24	0.5	1	1	0.6537	0.5981	0.5980
-1	0.22	0	0		0.2432	0.2432
-1	0.22	1	1		0.4895	0.4894
-1	0.6	1	1		0.7603	0.7603
0	0.22	0	0		0.3173	0.3175
0	0.22	1	1		0.5440	0.5440
0	0.6	1	1		0.8741	0.8741
0.5	0.22	0	0		0.3472	0.3471
0.5	0.22	1	1		0.5668	0.5668
0.5	0.6	1	1		0.9241	0.9240
0.5	100	1	1		11.3770	11.3770
0.5	100	0	1		11.3606	11.3609
-1.0	100	1	1		8.1333	8.1334
0.5	0.22	1	-1		0.0521	.0520
-1	0.6	1	-1		-1.1561	-1.1557
-1.24	0.6	1	1		0.7288	0.7288
0.5	0.22	1	-1		-0.1994	-0.1992

transfer and rate of mass diffusion at the surface represented in Table 3.1 − 3.3. Further Bhattacharyya [35] discussed only mass diffusion and momentum cases, but in the present work thermal diffusion also involved. So we have restrained our conversation to a particular solution dependent upon both the parameters b/a and M for the energy, momentum and mass diffusion equations. In Table 3.1, we observed that the magnetic field intensity had direct relation with skin friction with constant shrinking rate. Skin friction coefficient is decreasing with an increase in the shrinking rate ($b/a < 0$) in both absence and presence of the magnetic field. The conclusion is that the skin friction is enhanced with the presence of magnetic field. The skin friction negative when stretching rate more then 1. Its mean that ($b/a > 1$), in case of stretching, the stretching rate (b) greater than the straining rate (a), that may lead to flow instability. Further, it is observed that keeping the shrinking rate constant the skin friction increases with an increase in magnetic field strength. Also it is noted that when ($b/a = 1$) the skin friction disappear. There is no relative motion between plate velocity and free stream velocity and hence shearing stress also disappears. It is observed that skin friction increased due to the increase of magnetic field. It is concluded that presence of magnetic field enhances the skin friction but the rate of shrinking/stretching decreases it in both stretching and shrinking of the sheet. And the change of sign is a sign of the flow reversal when the velocity of the stretching sheet exceeds the free stream velocity. Table 3.2, shows the rate of heat transfer for stretching/shrinking of the plate surface. It is noted that the Nusselt number N_u decreases with an increase in the value of P_r without magnetic field and heat source ($M = 0, S = 0$), but it is increasing with the presence of magnetic field but opposite impact is observed with an increase in the value of source parameter. Further, it is observed that due to an increase in the magnitude of shrinking rate with the presence of both heat source and magnetic field, N_u decreases. Also with the presence of sink ($S = -1$), the Nusselt number remains positive. For high value of $P_r = 7.0$, the Nusselt number N_u takes negative value for shrinking case. It is observed that an increase in Prandtl number and stretching rate both contribute to increase the rate of heat transfer but Prandtl number P_r contributes significantly. Further, in case of the shrinking sheet, high Prandtl number fluid causes a thermal instability at the surface (negative value of N_u) whereas no such case arises for stretching sheet. Table 3.3, shows the value of Sherwood number denoted by S_h. For constructive reaction ($K_c < 0$), the Sherwood number S_h decreases and takes negative value, for destructive reaction ($K_c > 0$) Sherwood number S_h is positive. As Schmidt number S_c increases, Sherwood number S_h decreases in the absence of K_c and M but due to the chemical reaction K_c as well as the magnetic field M, S_h increases. It is also noted that the effect of ($M = 0, 1$) with ($K_c = 1$) and ($S_c = 100$) is same as that of shrinking sheet. In Fig. 3.2, we observe the effect of the velocity ratio (b/a) i.e. stretching rate of the bounding surface (b) and straining rate of the stagnation point flow ($a > 0$). For ($b/a < 1$), i.e. stretching rate at the plate is less than straining of potential flow, the boundary layer thickness decreases. Further, it is observed that in case of shrinking sheet ($b/a = -0.5$), there is a limited back flow for a few layers near the plate. It is also observed that the opposing force due to the magnetic field reduces the boundary layer thickness in the flow region. Hence, it is concluded that the shrinking of the boundary surface is to be controlled to avoid the back flow. In Fig. 3.3, without magnetic field ($M = 0$) as the shrinking rate (b/a) increases, the velocity decreases within boundary layer,but velocity increases with the presence of magnetic field M. When boundary layer decreases then electromagnetic force is high and causes a back flow. If $b/a < 1$, then the magnetic field increases the velocity. Therefore, it is concluded that the stretching ratio (b/a) and the shrinking of the bounding surface change the flow field overriding the effect of magnetic field. In Fig. 3.4, the temperature profile is discussed through the Figs. 3.4 to 3.7 in case of shrinking sheet. Curves I, II, III, IV, and V indicate that the temperature increases with source strength and increment in temperature is noted near the plate when the viscosity and conductivity of the fluid have value ($P_r = 1.0$). In case of sink ($S < 0$), the opposite effect is observed (curves VI and VII). In Fig. 3.5, the temperature gradient is discussed. It is observed that the negative temperature gradient is mostly seen in the flow domain except two profiles which show positive rate of heat transfer at the plate. Figs. 3.6 and 3.7, show the impact of magnetic parameter on temperature and on its gradient. The resistive force generated due to interaction of conducting fluid and magnetic field reduces the temperature at all points with a transverse compression of profiles (curves for $M = 1$) reducing the thermal boundary layer thickness due to transverse magnetic field. Fig. 3.7, indicate the fluctuation of the temperature gradient near the plate. We can observe that the presence of magnetic field achieve maximum of temperature in layers near to the bounding surface. Figs. 3.8 and 3.9, show the profiles of concentration and its gradient without reactive species with impact of shrinking of the sheet and magnetic field. The magnetic field decreases the concentration level of the species near the plate. The high values of S_c increases the concentration level near the plate but later opposite effect is observed. In the absence of magnetic field ($M = 0$), the concentration profile increases. Fig. 3.9, represents the concentration gradient without chemical reaction. The concentration gradient increases near the plate without magnetic field. In Fig. 3.10, it is noted that the chemical reaction parameter has a different impact in decreasing concentration level with magnetic field, a transverse compression resulting the thinner concentration boundary layer. The destructive reaction is responsible to decrease the concentration level at all the layers. Fig. 3.11, show that as shrinking rate increases with and with out reaction parameter, the concentration gradient level decreases.

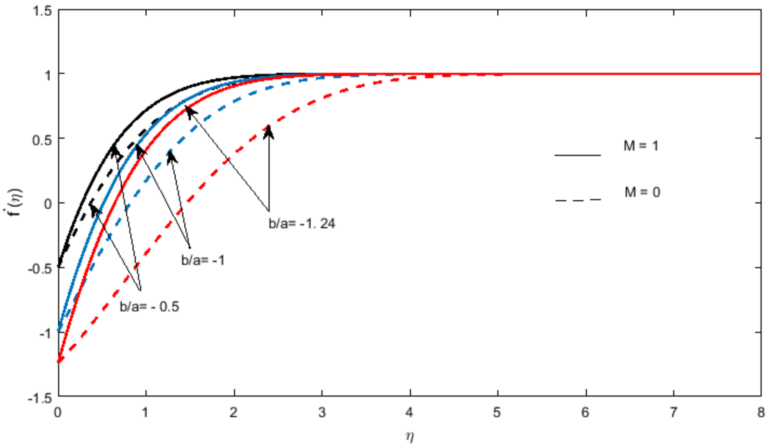

FIGURE 3.2: Velocity profile for different values of b/a.

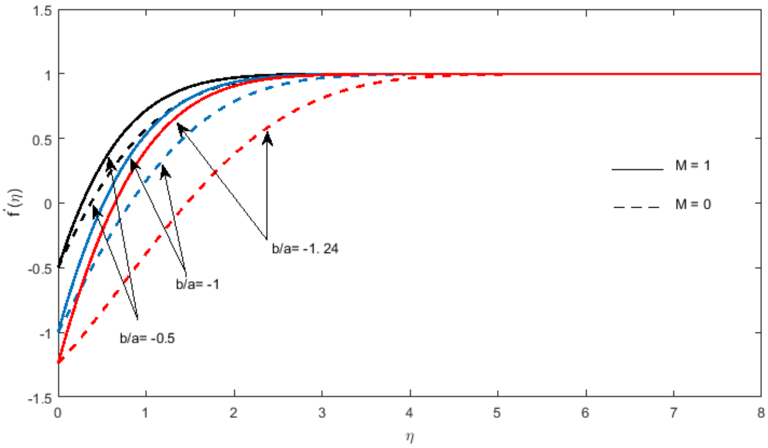

FIGURE 3.3: Velocity profile for shrinking sheet with $S = 0.2$.

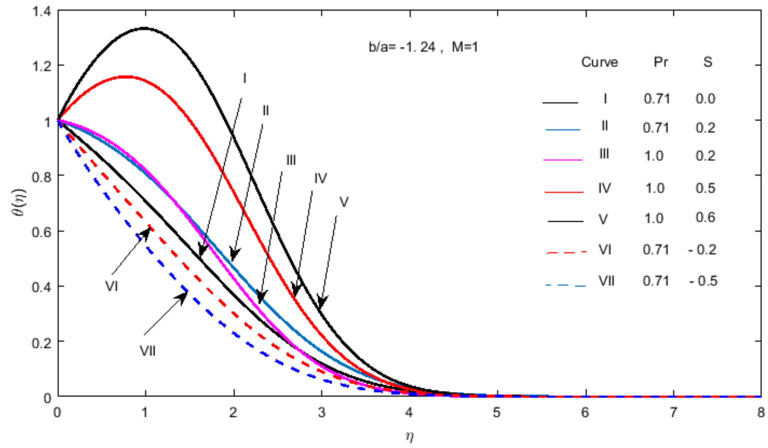

FIGURE 3.4: Impact of P_r and S on the temperature profile.

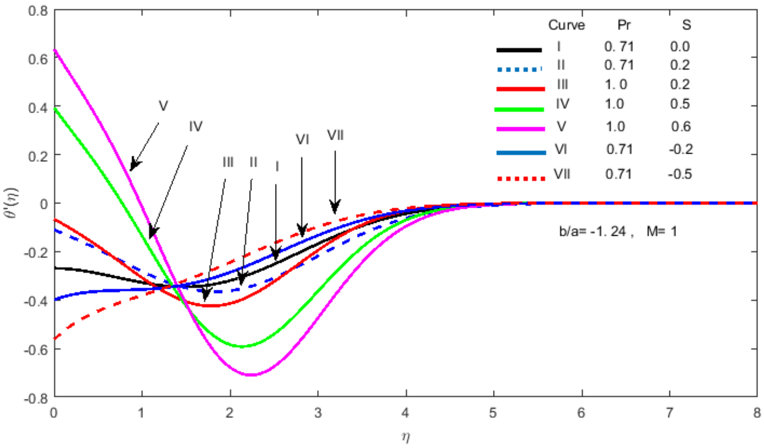

FIGURE 3.5: Impact of P_r and S on the temperature gradient profile.

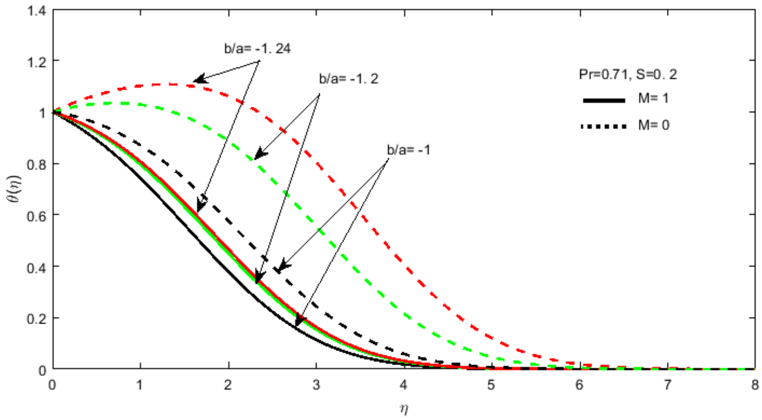

FIGURE 3.6: Impact of b/a and M on the temperature profile.

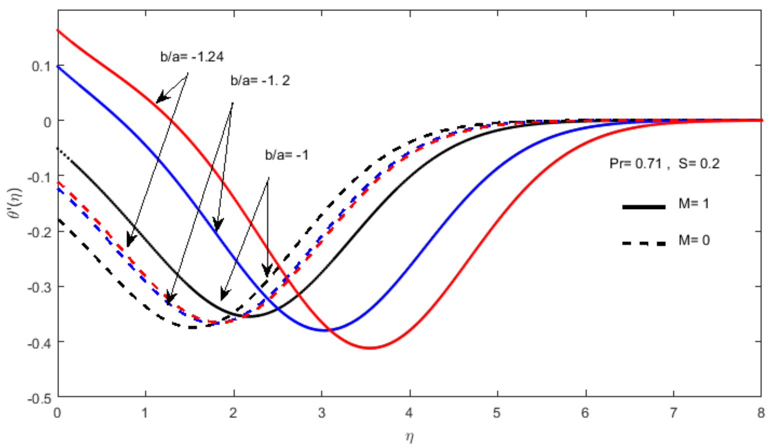

FIGURE 3.7: Impact of b/a and M on the temperature gradient profile.

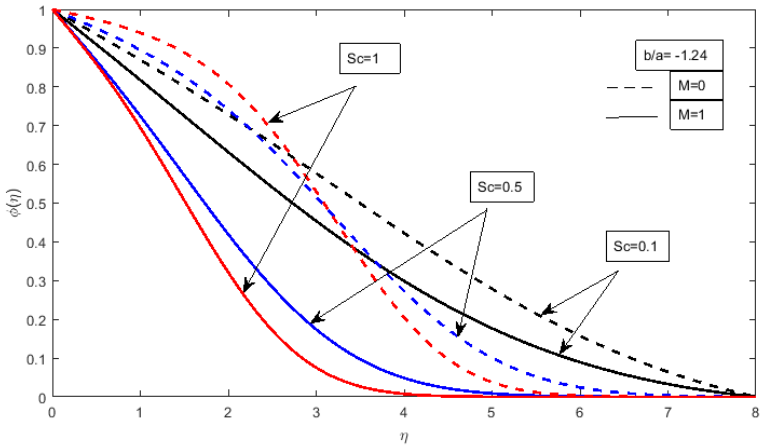

FIGURE 3.8: Impact of S_c and M on concentration profile.

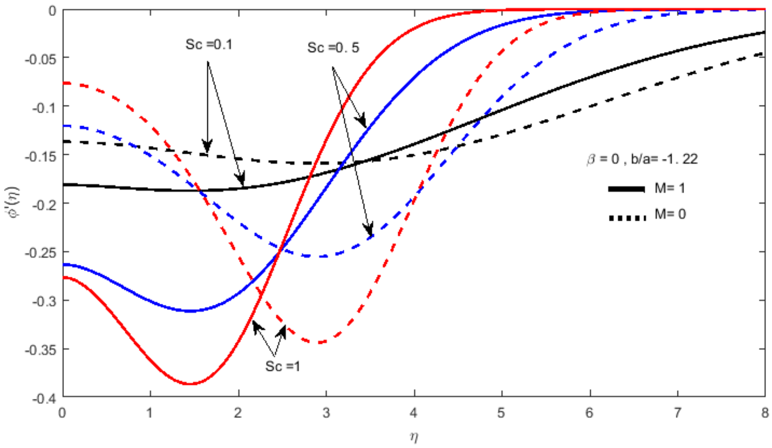

FIGURE 3.9: Impact of S_c and M on concentration gradient profile.

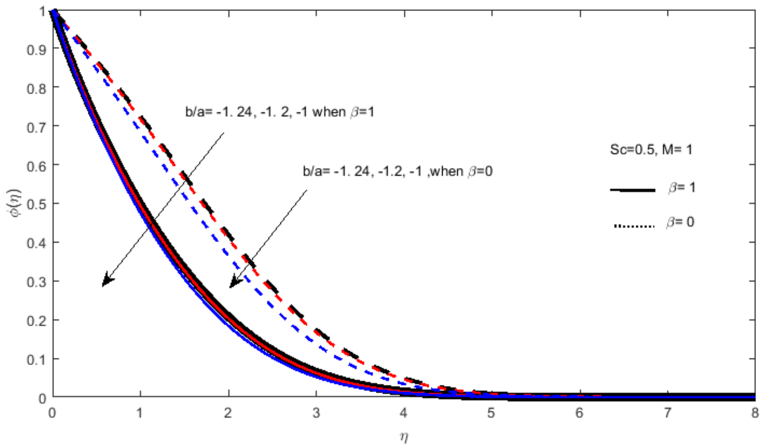

FIGURE 3.10: Impact of b/a and β on concentration profile.

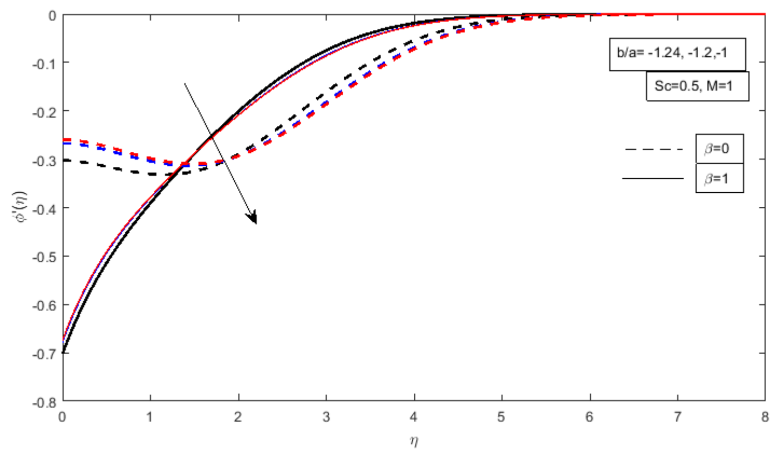

FIGURE 3.11: Impact of b/a and β on concentration gradient profile.

Chapter 4

NUMERICAL SOLUTION OF BOUNDARY LAYER STAGNATION-POINT FLOW OVER A SHRINKING SHEET WITH JOULE HEATING AND VISCOUS DISSIPATION

In the modern industry, the study of magnetohydrodynamics MHD flow has been done by many researchers due to its many practical applications. This chapter consists of the solution for the boundary layer stagnation-point flow over a shrinking sheet in the presence of Joule heating and viscous dissipation effects. The nonlinear partial differential equations of mass, momentum and concentration are transformed to the ordinary differential equations by using similarity transformations. The solution of the present problem is formed by the shooting technique and also to validate the accuracy of the results by using bvp4c MATLAB code. Finally, the results are discussed for different parameters affecting the flow and transfer of heat.

4.1 Formulation of the problem

In this section the boundary layer stagnation-point flow with the steady two dimensional boundary layer is discussed. The flow field is exposed to uniform transverse magnetic field $\vec{B_0} = (0, B_0, 0)$. It is assumed that the flow is generated by stretching of non-conducting elastic boundary sheet by imposing two opposite and equal forces along x-axis in such a way that the velocity of the boundary sheet is of linear order in the flow direction and the origin remains fixed as shown in Fig. 3.1. A uniform magnetic field of strength B_0 is assumed to be applied in the positive y-direction normal to the plate. The governing equations of the problem are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}(u - U), \tag{4.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho C_p}(T - T_\infty) + \frac{\sigma B_0^2 u^2}{\rho C_p} + \frac{\mu}{\rho C_p}(\frac{\partial u}{\partial y})^2, \tag{4.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - R(C - C_\infty), \tag{4.4}$$

where u and v are the velocity components along the x and y axis respectivly, U is the straining velocity, C is the concentration, T is the fluid temperature, ν is the kinematic viscosity of fluid, ρ is the density of fluid, C_p is the specific heat at constant pressure, D is the species diffusion coefficient, σ is the electric conductivity of fluid, B_0 is the applied uniform magnetic field normal to the surface of the sheet, Q is the heat source parameter. The boundary conditions for Eqs. $(4.1 - 4.4)$ are

$$\begin{aligned} &u = bx, \quad v = 0, \quad T = T_w, \quad C = C_w, \quad \text{at} \quad y = 0, \\ &u \to U(x) = ax, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty. \end{aligned} \tag{4.5}$$

We use similarity transformation $[21, 22]$ to solve Eqs. $(4.1 - 4.4)$

$$\begin{aligned} &\psi(x, y) = \sqrt{a\nu}xf(\eta), \quad \theta(\eta) = (T - T_\infty)/(T_w - T_\infty), \\ &\phi(\eta) = (C - C_\infty)/(C_w - C_\infty), \quad \eta = y\sqrt{a/\nu}, \end{aligned} \tag{4.6}$$

the velocity component of stream function which is defined as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{4.7}$$

So, we have

$$u = axf^{'}(\eta), \qquad v = -\sqrt{a\nu} f(\eta), \tag{4.8}$$

where prime shows differentiation with respect to η. Using (4.6) in Eq. (4.1) that will be satisfied, also using $(4.5-4.7)$ in Eqs. $(4.2-4.4)$, we will get the following ordinary differential Eqs.

$$f^{'''} + ff^{''} - (f^{'})^2 - M(f^{'} - 1) + 1 = 0, \tag{4.9}$$

$$\theta^{''} + P_r f \theta^{'} + P_r S\theta + P_r E_c (M f^{'2} + f^{''2}) = 0, \tag{4.10}$$

$$\phi^{''} + S_c f \phi^{'} - S_c \beta \phi = 0, \tag{4.11}$$

with boundary conditions

$$\begin{aligned} f(0) = 0, \quad f^{'}(0) = b/a, \quad \theta(0) = 1, \quad \phi(0) = 1 \quad at \quad \eta = 0, \\ f^{'}(\eta) \to 1, \quad \theta(\eta) \to 0, \quad \phi(\eta) \to 0 \quad as \quad \eta \to \infty, \end{aligned} \tag{4.12}$$

The dimensionless constants P_r, S_c, M, S, β and E_c represent the Prandtl number, the Schmidt number, the magnetic parameter, the heat source parameter, the reaction rate parameter, Eckert number which are defined as

$$P_r = \frac{\nu}{k}, \quad S_c = \frac{\nu}{D}, \quad M = \frac{\sigma B_0^2}{a\rho}, S = \frac{Q}{a\rho C_p}, \quad \beta = \frac{R}{a}, \quad E_c = \frac{a^2x^2}{C_p}, \tag{4.13}$$

In this problem the quantities of physical interest are the local Nusselt number N_u, the local Sherwood number S_h,the skin friction coefficient C_f which are defined as

$$N_u = \frac{xq_w}{k(T_w - T_\infty)}, \quad S_h = \frac{xh_m}{D(C_w - C_\infty)}, \quad C_f = \frac{\tau_w}{\rho U^2/2}, \tag{4.14}$$

where h_m mass flux, q_w heat flux and τ_w the wall shear stress or skin friction, which are given by

$$h_m = -D\left[\frac{\partial C}{\partial y}\right]_{y=0}, \qquad q_w = -k\left[\frac{\partial T}{\partial y}\right]_{y=0}, \qquad \tau_w = \mu\left[\frac{\partial u}{\partial y}\right]_{y=0}, \tag{4.15}$$

where μ is the dynamic viscosity of the fluid and k is thermal diffusivity. Using the similarity variables Eq. (4.6), we get

$$\frac{N_u}{\sqrt{Re_x}} = -\theta^{'}(0), \frac{S_h}{\sqrt{Re_x}} = -\phi^{'}(0), \frac{1}{2}C_f\sqrt{Re_x} = f^{''}(0), \tag{4.16}$$

where $Re_x = \frac{\rho b x^2}{\mu}$.

4.2 Method for solution

As Eqs. (4.9 − 4.11) are non-linear and coupled with boundary conditions in Eq. (4.12). Before solving, these equations are converted from the boundary value problem into the initial value problem. Applying the Shooting technique together with fourth order Runge-Kutta method. Let's convert Eqs. (4.9 − 4.11) by using following substitution:

$$f = y_1, \quad f' = y_2, \quad f'' = y_3, \quad f''' = y_3', \tag{4.17}$$

$$\theta = y_4, \quad \theta' = y_5, \quad \theta'' = y_5', \tag{4.18}$$

$$\phi = y_6, \quad \phi' = y_7, \quad \phi'' = y_7'. \tag{4.19}$$

Using above notations as a result we get seven first order non linear coupled ODEs with the boundary conditions are also adjust according to the above supposition, written below

$$\left.\begin{aligned}
y_1^{'} &= y_2,\\
y_2^{'} &= y_3,\\
y_3^{'} &= -y_1 y_3 + y_2^2 + M(y_2 - 1) - 1,\\
y_4^{'} &= y_5,\\
y_5^{'} &= -P_r y_1 y_5 - P_r S y_4 - P_r E_c (M y_2^2 + y_3^2),\\
y_6^{'} &= y_7,\\
y_7^{'} &= S_c A y_6 - S_c y_1 y_7,
\end{aligned}\right\} \tag{4.20}$$

The associated initial conditions are

$$
\begin{aligned}
&y_1(0) = 0, \quad y_2(0) = b/a, \quad y_3(0) = t, \quad y_4(0) = 1, \\
&y_5(0) = q, \quad y_6(0) = 1, \quad y_7(0) = w,
\end{aligned} \tag{4.21}
$$

In Eq. (4.21) t, q and w are the three initial guesses. Runge-Kutta method of order four is used to solve the intermediate initial value problem with some suitable initial guess $t = t_0$, $q = q_0$ and $w = w_0$. For the next iteration, the values of t, q and w are updated by the Newton's method.

4.2.1 Results and discussion

The main objective of this section is to study the effect of Eckert number E_c on different parameters like skin friction profile $f''(\eta)$, the temperature profile -$\theta'(\eta)$ and concentration profile -$\phi'(\eta)$.

In Table 4.1, we can see that when the Eckert number E_c and the Prandtl number P_r increases then temperature profile increases. By increasing magnetic parameter M, the skin friction and temperature profile also increases but concentration profile decreases. Where as reaction parameter shows its impact only on concentration profile. We can see concentration profile decreases as reaction parameter increases.

Fig. 4.1, show the impact of the magnetic parameter M on the velocity profile. As the magnetic parameter M increasing the velocity profile also increases but due

TABLE 4.1: Values of $f''(0)$, $-\theta'(0)$ and $-\phi'(0)$ by using the Shooting method and bvp4c

				$-f''(0)$		$-\theta'(0)$		$-\phi'(0)$	
E_c	P_r	M	β	Shooting	bvp4c	Shooting	bvp4c	Shooting	bvp4c
1	0.71	0.5	0.5	1.8333	1.8337	1.3156	1.3154	-0.2887	-0.2894
5				1.8339	1.8341	7.6796	7.66896	-0.2894	-0.2897
10				1.8343	1.8351	15.6165	15.6572	-0.2897	-0.2890
	1			1.8330	1.8351	0.7697	0.7697	-0.2890	-0.2887
	5			1.8338	1.8356	3.8651	3.9994	-0.2897	-0.2894
	10			1.8340	1.8361	7.8783	7.8796	-0.2894	-0.2897
		0.5		1.8337	1.8351	0.5185	0.5186	-0.2894	-0.2897
		5		3.6714	3.6741	4.2376	4.2371	-0.3031	-0.3047
		10		4.9737	4.9749	8.6967	8.6955	-0.3101	-0.3149
			1	1.8339	1.8351	0.5184	0.5186	-0.3580	-0.3582
			2	1.8337	1.8346	0.5183	0.5179	-0.4712	-0.4707
			3	1.8338	1.8350	0.5193	0.5186	-0.5625	-0.5634

to the resistive effects the boundary layer thickness decreases. This is due to the fact that the magnetic force enhance the fluid motion in boundary layer. Therefor, it is concluded that the stretching ratio (b/a) and the shrinking of the bounding surface effected by magnetic parameter. In Fig. 4.2, it is observed that the velocity profile decreases with increasing value of the shrinking parameter b/a and consequently the thickness of boundary layer increases. Reverse flow is observed near the surface as the shrinking parameter decreases. In Fig. 4.3, we observe the impact of P_r and S on temperature profile. We have $(b/a = -1.24, M = 1)$, and different values of P_r and S the temperature profile increase. Curves I, II, III, IV, and V indicates the temperature increases with source strength and a hike in temperature is noted near the plate. On the other hand, in case of sink $S < 0$ and $P_r = 0.71$, the opposite effect is observed in curves VI and VII due to the sink $S < 0$ boundary layer decreases. In Fig. 4.4, it indicate that increasing the value of Eckert number E_c has the enhansing effect on temperature profile and increases the thermal boundary layer thickness in the flow field. The temperature increases due to increasing the Eckert number E_c that generate heat in fluid. In Fig. 4.5, the effect of the magnetic parameter M on temperature distribution. An increasing impact is observed in the temperature profile with an increase in the value of the magnetic parameter M. The thermal boundary layer thickness increases by increasing the magnetic parameter M. Physically the magnetic field has a stabilizing effect on fluid flow. This mean that heat transfer from hot surface to the cool fluid. Fig. 4.6, show the impact of the Prandtl number P_r on temperature profile. The temperature in the boundary layer increases due to the increasing the value of the Prandtl number P_r and the boundary layer thickness also increases. If P_r increases, the thermal diffusivity increases and this leads to increase energy ability that increases the thermal boundary layer. In Fig. 4.7, the change of temperature due to heat source parameter S is shown. As we increase the heat source S, the thermal boundary layer thickness also increases. The temperature profile significantly increases as the heat source/sink parameter S increases. In Fig. 4.8, the impact of velocity ratio b/a on the temperature profile is shown for the case of the shrinking sheet. The temperature of fluid increases with decreasing the value of the shrinking parameter b/a. Thus, the thermal boundary layer thickness become thicker and thicker.

In Fig. 4.9, we can see the impact of reaction rate β on concentration profile. When the chemical reaction parameter β increases, the boundary layer thickness decreases. This is due to the fact that the chemical reaction in this system results in consumption of the chemical and results in decrease of concentration profile. Fig. 4.10, show that the concentration of fluid increases with an increase in the value of the shrinking parameter b/a. There for, the concentration boundary layer thickness become thicker and thicker. In Fig. 4.11, the concentration of fluid decreases with increasing the value of the magnetic parameter M. The magnetic parameter M causes to decrease the concentration boundary layer thickness. This means that the heat and mass are transferred from hot surface to the cool fluid. Fig. 4.12, illustrate the effect of the Schmidt number S_c on concentration profile. It is noted that as the Schmidt number S_c increases, the concentration of fluid medium decreases. This happens because of this fact that the molecular diffusion decreases as the Schmidt number S_c increases.

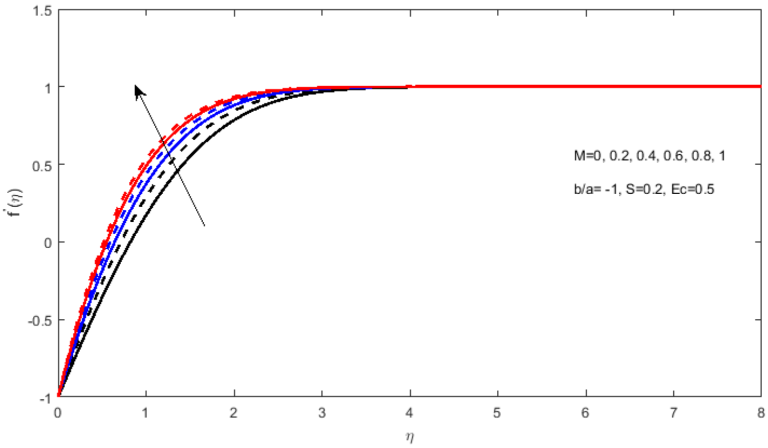

FIGURE 4.1: Impact of M on velocity profile.

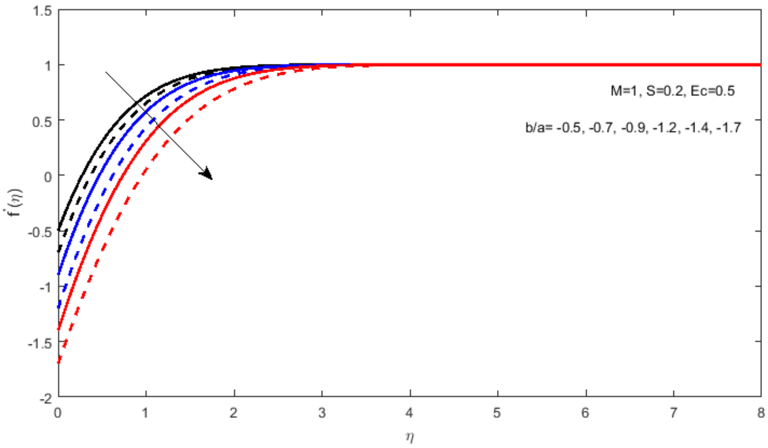

FIGURE 4.2: Impact of b/a on velocity profile.

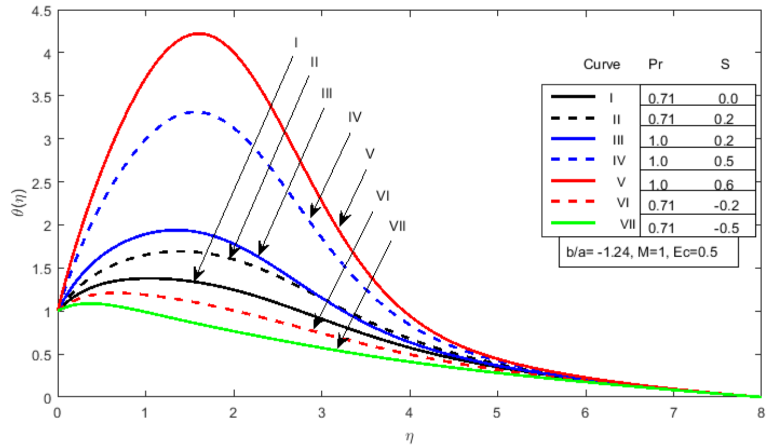

FIGURE 4.3: Impact of P_r and S on temperature profile.

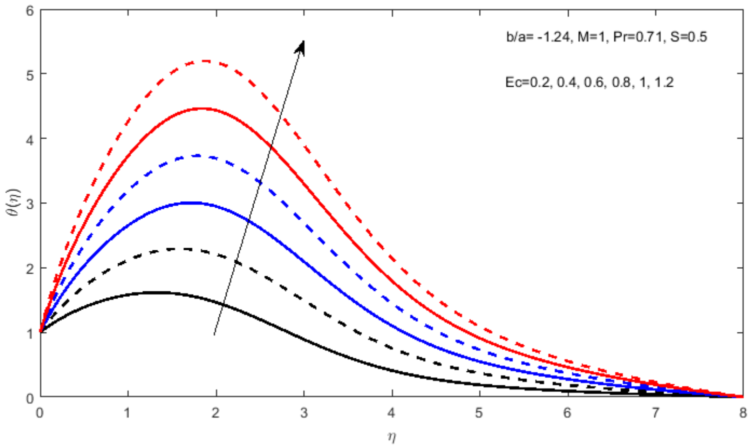

FIGURE 4.4: Impact of E_c on temperature profile.

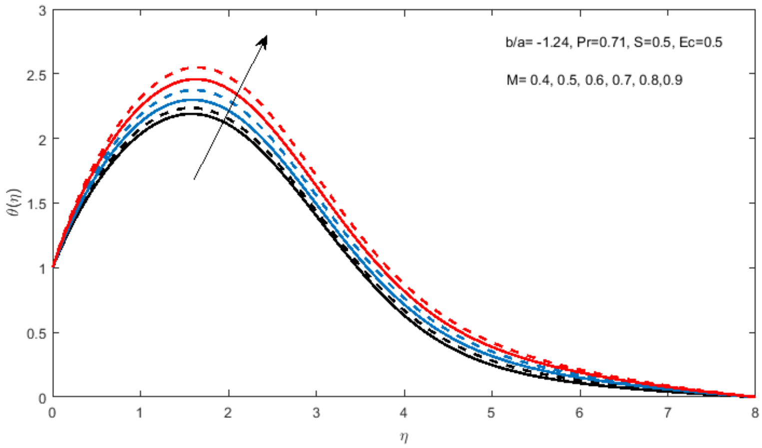

FIGURE 4.5: Impact of M on temperature profile.

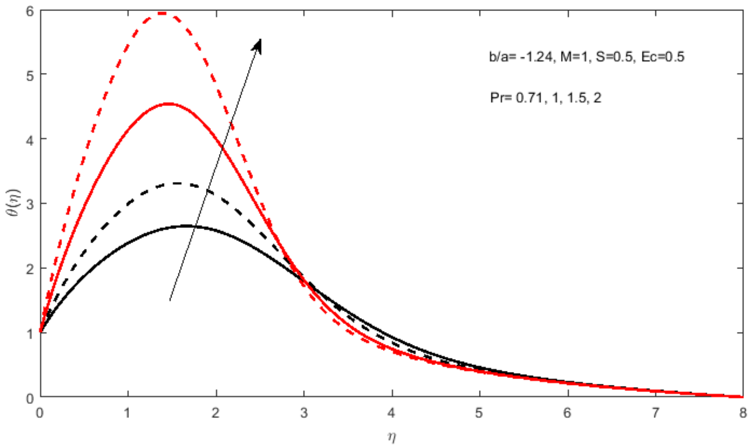

FIGURE 4.6: Impact of P_r on temperature profile.

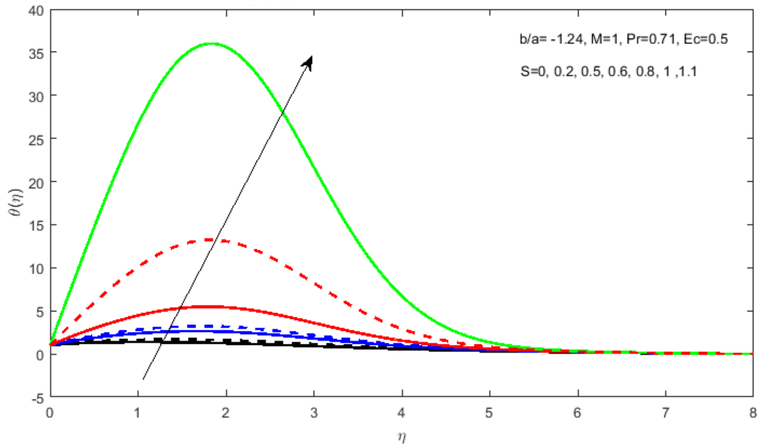

FIGURE 4.7: Impact of S on temperature profile.

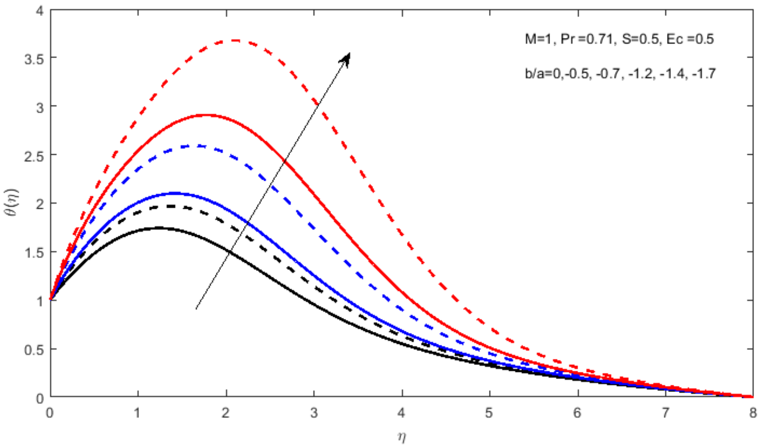

FIGURE 4.8: Impact of b/a on temperature profile.

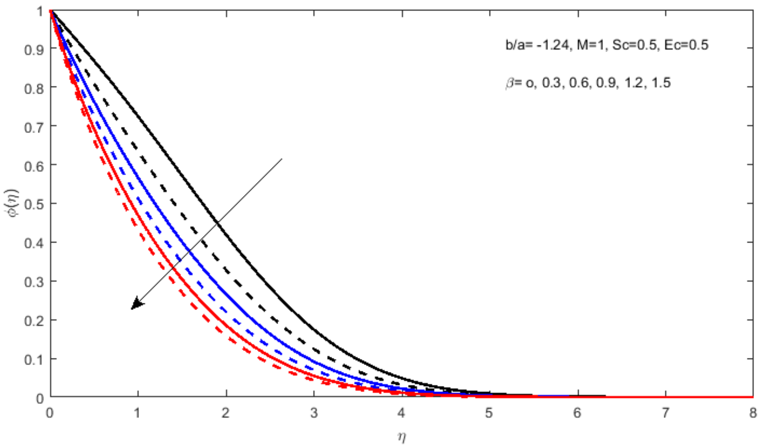

FIGURE 4.9: Impact of β on concentration profile.

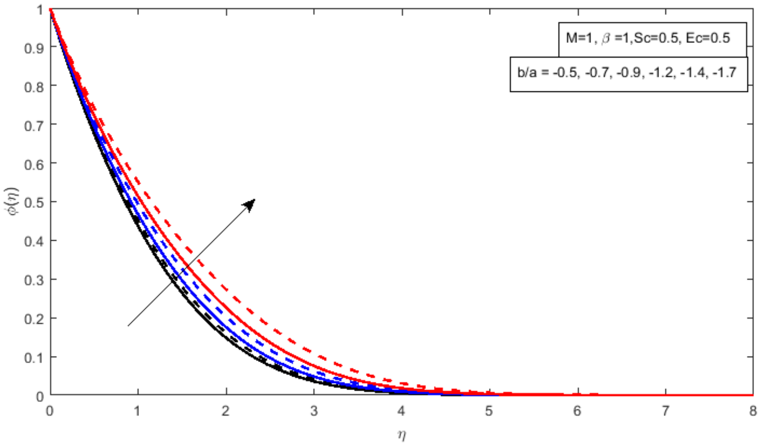

FIGURE 4.10: Impact of b/a on concentration profile.

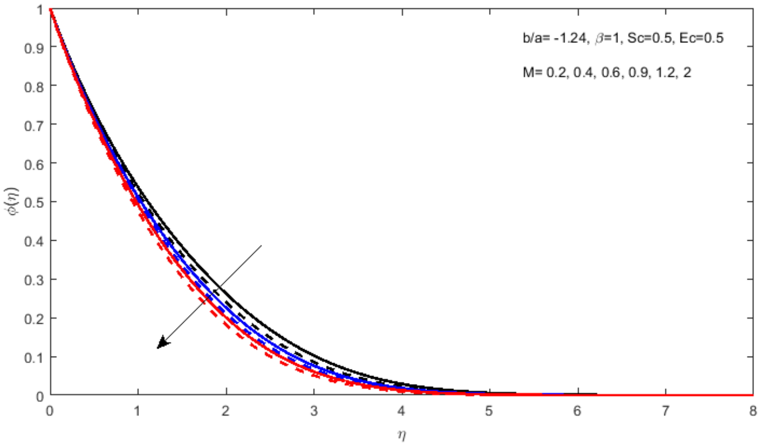

FIGURE 4.11: Impact of M on concentration profile.

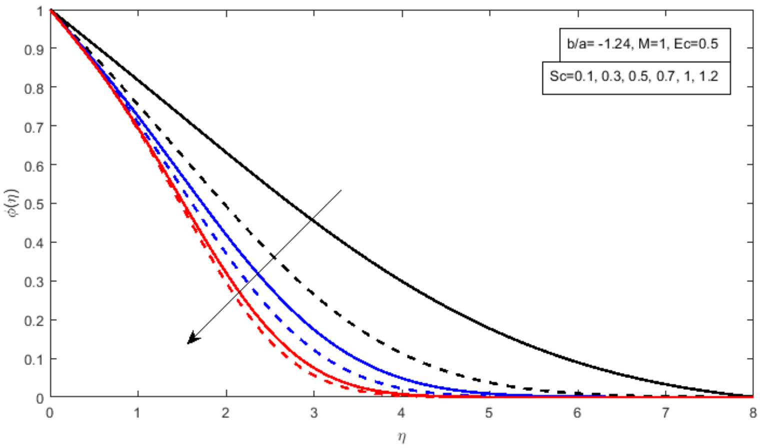

FIGURE 4.12: Impact of S_c on concentration profile.

Chapter 5

CONCLUSION

MHD, incompressible boundary layer stagnation-point flow past a stretching/shrinking sheet is studied numerically. Numerical solution is computed by shooting method using MATLAB. The study shows that velocity, temperature and the solid volume fraction of the nanofluid profiles in the relevant boundary layers depend on seven dimensionless parameters.
Conclusions which are obtained:

- The effect of chemical reaction K_c and the magnetic parameter M is thesame for both streatching/shrinking sheet.

- Diffusion is more sensitive rather than reaction parameter.

- Resistive force is generated by magnetic field and magnetic field increases the velocity which is greater than the plate velocity.

- Shrinking of boundary surface is linked with heat source S but heat source has not much impact on velocity.

- When $P_r = 1$, then heat source S increases the temperature near the plate while magnetic parameter M decrease the temperature.

- The skin friction decreases as the shrinking velocity increases but the magnetic field M increases the skin friction and decreases the rate of heat transfer at the plate.

- The temperature profile increases as the Eckert number increases.

- When the chemical reaction parameter β increases, the boundary layer thickness decreases.

- Magnetic field increase velocity and temperature but reduces the concentration.

5.1 Future recommendations.

The present model focus on Joule heating and viscous dissipation. An interesting area to investigate in future will be the use of thermal radiation, impact of different nano particles, different chemical properties, hydrogenous and homogenous reaction.

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