CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Suzuki Type Contractions

by

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Declaration of Authorship

I, Tayyba Arooj, declared that this thesis titled, "Suzuki Type Contraction" and the work presented in it are my own. I confirm that:

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"Pure mathematics is, in its way, the poetry of logical ideas."

Albert Einstein

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Abstract

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In this present dissertation, we introduce some new type of contraction mapping by introducing F-contraction in b-metric spaces. F-contraction played an important role in the extension and generalization of Banach Contraction principle. We have extended the notion of F-contraction in complete b-metric space. In this way, we have proved some fixed point results in complete b-metric space by using F-contraction. Further more, we have extended the fixed point results for b-metric space using F-Suzuki contractions that is the generalization of the work of Wardowski's result in F-contraction.

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Tayyba Arooj

Contents

Dedicated to my Mother...

Chapter 1

Introduction

Mathematics is an important branch of scientific knowledge which has a lot of applications for humanity and in every sphere of life. Mathematics is further divided into many branches which have their own significance according to their applications. One of the important branch of mathematics is named as functional analysis, which has a lot of applications in different areas, such as it has a wide applicability in solving the problems like linear and non-linear partial differential equations. It has numerous applications in the field of numerical analysis, error estimation of polynomial interpolation and finite difference method. Combination of analysis and geometry is the important and beautiful outcome in the form of functional analysis.

In the functional analysis, fixed point theory is an important and valueable concept. This certainly enhance the importance and significance of functional analysis due to its wide use in solving the different types of linear and non-linear problems. The concept of fixed point has a lot of applications in various fields of science, such as mathematical economics, game theory, optimization theory, approximation theory and in variational inequalities *etc.*

The first person who had worked on fixed point theory was Poincare [34] in 1886. Afterward, the equation $f(a) = a$ was taken into consideration by Brouwer [12] and he found solution of this equation by proving a fixed point theorem, in 1912. He also contributed to prove fixed point results for the shapes like a square and a sphere *etc.* Kuktani [25] extended and generalized further this work for n-dimensional counter parts of a sphere and a square.

In the same time, an important concept was appeared in the field of fixed point theory that is, Banach contraction named as Banach Contraction Principle. Which

played very important role in solving non-linear problems. This famous and well known result was introduced by Banach [9], in 1922. He proved that every contraction mapping on a complete metric space always has a unique fixed point.
We can observe that the successive approximation method for finding the existence and uniqueness of solution of differential equations is infact the start and origin of contraction principle and fixed point theory. Picard introduced the iterative process that was used in the proof of Banach Contraction theorem.
Due to importance and wide application of Banach Contraction Principle the existence of fixed point and its uniqueness has become very common and intresting phenomena for authors. Kannan [26] further worked on it and developed some new results in this field. Many authors worked on different generalized metric spaces and proved Banach Contraction Principle using different contractions. The work in this direction is further divided into two categories. In the first category, the fixed point theorems are obtained by extending the contraction conditions and hence generalizing the Banach Contraction Principle. In the second category, researchers established fixed point theorems for more general from abstract spaces. The extensions in this field are very vast and here it is not possible to discuss all of these generalizations but we shortly discuss only some of these.
Generalization of metric space named as Partial metric space was introduced by Matthews [28] in 1994. He proved Banach Contraction results on this space. Other authors who extended work on partial metric space are in [4], [5], [31], [32]. In 2007, Huang [23] introduced another space known as Cone Metric Space. In this space many fixed point results are proved by different authors in, *e.g.* [35] and [43].
In 1989, the concept of b-metric was introduced by Bakhtin [8] first time. Due to its importance, b-metric space is used for generalizing contraction mapping and for proving some new results. Czerwik [16] established different results for b-metric space. These results was further extended and generalized in single and multi-valued mappings for diffent purposes. In b-metric space Khamsi and Hussain [24] proved some new results.
The concept of rectangular metric space was introduced by Branciari [11]. He changed the third axiom of metric space and proved different fixed point results in this space. After this, many authors worked in this space using different contractions see [[6],[7],[13],[17],[21]].
A lot of contraction conditions have been established after the Banach Contraction Principle, but we discuss only those which are used in our work.

The concept of a new contraction mapping known as F-contraction was introduced by Wardowski [46], in 2012. He gave some new fixed point results and proved these results for such type of contraction. He produced these results in as a different way rather than traditional ways as done by many authors. After this, fixed point results for F-contraction were produced by Secelean [37] using iterated function. Later on, Piri *et. al.*[33] generalized the fixed point theorey of Wardwoski for F-suzuki contraction by making the condition of a complete metric space. Further, Abbas [1] extended the work of Wardwoski and established some new fixed point theorems using F-contraction mapping. Batra et al.[10] worked on proving Fixed point theorems and also contributed in explaining graphs using the idea of F-contraction. He also discussed altered distance. In complete metric space, Cosentino et al. proved some new results for self contraction mappings. On complete metric space and copmlete ordered metric space Vetro [45] proved some important results of fixed point using F-contraction.

In this dissertation, we review the paper of Hossein Piri and Poom Kuman [33]. we extend the results presented in [33] in the setting of b-metric spaces. we obtained new fixed point theorems in b-metric spaces for new contractive conditions like F-contraction and F-Suzuki contraction.

Following are the details of work, which I have done throughout this dissertation.

- In Chapter 2, we throw light on basic concepts and definitions of Metric spaces, F-contraction in complete metric space and presented few examples.
- In Chapter 3, the paper **"Some Fixed point theorem concerning F-contraction in complete metric spaces"** [**33**] is reviewed comprehensively.
- In Chapter 4, we focused on the generalization of the theorem which is reviewed in chapter 3. A brief conclusion of our work is also presented in this chapter.

Chapter 2

Preliminaries

Throughout in this work we use the notations of $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ for set of real numbers, set of the all positive real numbers and set of all natural numbers respectively.
In this chapter we will review the basic definitions and some examples of various abstract spaces which are related to our research.

2.1 Metric space

We start with the more general concept of "distance" between two elements of a set. That is, the notion of a metric.

Definition 2.1.1. (**Metric Space**)
"Let X be a nonempty set. A Mapping $d\colon X \times X \to \mathbb{R}$ is said to be metric on X if it satisfies the following conditions:

M1 - $d(x, y) \geq 0 \quad \forall\, x, y \in X$ (Non negative)

M2 - $d(x, y) = 0 \quad iff \quad x = 0$

M3 - $d(x, y) = d(y, x)\ \forall\, x, y \in X$ (Symmetry)

M4 - $d(x, z) \leq d(x, y) + d(y, z) \quad \forall\, x, y, z \in X$ (Triangular property)

The pair (X, d) is called metric space".

We clear the above definition with the following examples of metric spaces.

1. **Usual Metric Space:**
   Let $\mathbb{X} = \mathbb{R}$ and define,
   $$d : X \times X \to \mathbb{R}$$
   as
   $$d(x, y) = |x - y|$$
   then $(\mathbb{R}, d)$ is a metric space and d is called usual metric on $\mathbb{R}$.

2. **Euclidean Plane:**
   Let $X = \mathbb{R}^2$, define
   $$d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$
   by
   $$d(x, y) = [(x_1 - y_1)^2 + (x_2 - y_2)]^{1/2}$$
   Then d is a metric on $\mathbb{R}^2$ and $(\mathbb{R}^2, d)$ is a metric space.

3. **Space of Bounded Sequences(ℓ^∞):**
   Let X be the set of all bounded sequences of complex numbers,
   i.e.,
   $$x = \{x_n\}_{n\in\mathbb{N}} \quad \text{or} \quad x = (x_1, x_2, \cdots)$$
   $$\text{and} \quad |x_n| \leq c_x \quad \forall\, n \in \mathbb{N}.$$
   Define $d : X \times X \to \mathbb{R}$ by
   $$d(x, y) = \sup_{n\in\mathbb{N}} |x_n - y_n|$$
   Where $x, y \in X$, $x = x_n$, $y = y_n$ and sup denotes the supremum (least upper bound). This space is denoted by ℓ^∞ and is called sequence space.
   $$x \in \ell^\infty \quad \Rightarrow \quad \sup_{n\in\mathbb{N}} |x_n| < \infty.$$

4. **Function Space ($C[a, b]$):**
   Let $X = C[a, b]$ be the set of all real-valued continuous functions defined on

a closed interval $[a, b]$. The function $d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \max_{t \in [a,b]} |f(t) - g(t)| \quad x, y \in C[a, b]$$

is a metric on X and (X, d) is a metric space denoted by $C[a, b]$.

5. **The space of bounded functions ($B(A)$):**
Let $X = B(A)$ be the set of all bounded functions defined on the set A then $d : B(A) \times B(A) \to \mathbb{R}$ given by

$$d(x, y) = \sup_{t \in A} |x(t) - y(t)|$$

is a metric on $B(A)$. For a set $A = [a, b] \subseteq \mathbb{R}$; $B(A)$ is denoted as $B[a, b]$

6. **Space (ℓ^p):**
The space of real or complex number sequences $x = \{x_n\}_{n=1}^{\infty}$ such that for some $p \geq 1$ the infinite series

$$\sum_{n=1}^{\infty} |x_n|^p$$

Converges. The space is denoted by l^p
The metric $d : \ell^p \times \ell^p \to \mathbb{R}$ is given by:

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p} \quad \forall\ x, y \in \ell^p$$

i.e., both $\sum |x_n|^p, \sum |y_n|^p < \infty$.
For $p = 2$, we get the Hilbert sequence space ℓ^2 with metric given by

$$d(x, y) = \sqrt{\sum_{n=1}^{\infty} |x_n - y_n|^2} \qquad \forall\ x, y \in \ell^2$$

The following definitions are taken from [27].

Definition 2.1.2. (Continuous Mapping)
"Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be continuous at a point x_0 if for each $\epsilon > 0$, $\exists\ \delta > 0$, such that

$$d(Tx, Tx_0) \leq \epsilon$$

whenever,

$$d(x, x_0) < \delta\text{”}$$

Example 2.1.3.

Let us consider a mapping $f : X \to X$ defined on a usual metric space $(\mathbb{R}, d)$ as follows

$$T(a) = a^3 \quad a \in X$$

Then f is continuous mapping.

Definition 2.1.4. (Convergence of Sequence)

A sequence $\{a_n\}$ in a metric space (X, d) is said to be convergent to a point, $a \in X$ if

$$\lim_{n\to\infty} d(a_n, a) = 0$$

Alternatively; by $a_n \to a$ we mean, $\forall\ \epsilon > 0$ there exist a natural number $\mathbb{N} = \mathbb{N}(\epsilon)$ such that,

$$d(a_n, a) < \epsilon \quad \forall\ n > \mathbb{N}$$

Definition 2.1.5. (Cauchy Sequence)

“A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if for every $\epsilon > 0$ there exist a positive number $\mathbb{N} = \mathbb{N}(\epsilon)$ such that

$$d(x_n, x_m) < \epsilon \quad \forall\ m, n > \mathbb{N}\text{”}$$

Definition 2.1.6. (Complete Metric Space)

“If every Cauchy sequence in a metric space (X, d) converges to a point $x \in X$ then X is called complete metric space”.

Definition 2.1.7. (Compact Metric Space)

“A metric space X is called compact if every sequence in X has a convergent sub-sequence”.

2.2 b-metric space

Definition 2.2.1.

Let X be a non-empty set. A real-valued mapping $d_b \colon X \times X \to \mathbb{R}$ that satisfies the following conditions :

1. $d_b(a,b) \geq 0$ and $d_b(a,b) = 0 \Leftrightarrow a = b \quad \forall\ a,b \in X$
2. $d_b(a,b) = d(a,b) \quad \forall\ a,b \in X$ (symmetry)
3. There exist a real number $b \geq 1$, such that :

$$d_b(a,c) \leq b\,[d_b(a,b) + d_b(b,c)] \quad \forall\ a,b,c \in X$$

is called b-metric on X. The pair (X, d_b) is called b-metric space with coefficient b.

We illustrate the above definition with the following examples of $b-$metric spaces.

Example 2.2.2.
Let (X,d) be a metric space . Then for a real number $m > 1$, we define a function $d_1(a,b) = (d(a,b))^m$, then d_1 is a b-metric with $b = 2^{m-1}$.

Proof.
It is very easy to verify first two conditions. We only prove third condition.
Define $f(x) = x^m$ for $x > 0$,

$$\left(\frac{a+b}{2}\right)^m \leq \frac{a^m + b^m}{2}$$
$$\frac{(a+b)^m}{2^m} \leq \frac{a^m + b^m}{2}$$
$$(a+b)^m \leq 2^{m-1}(a^m + b^m)$$

So for $a,b,c \in X$, we have

$$\begin{aligned} d_1(a,c) = (d(a,c))^m &\leq [d(a,b) + d(b,c)]^m \\ &\leq 2^{m-1}[d(a,b)^m + d(b,c)^m] \\ \Rightarrow d_1(a,c) &\leq 2^{m-1}[d_1(a,b) + d_1(b,c)] \end{aligned}$$

Hence (X, d_1) is a b-metric space. □

Remark 2.2.3.
A b-metric d_b need not continuous. This is illustrated by the following example.

Example 2.2.4.
Let $X = \mathbb{N} \cup \{\infty\}$ and define $d_b \colon X \times X \to \{0, +\infty\}$ by

$$d_b(m,n) = \begin{cases} 0 & \text{if } \ m = n, \\ \mid \frac{1}{m} - \frac{1}{n} \mid & \text{if } \ m,n \ \text{ are even or } \ mn = \infty, \\ 5 & \text{if } m \text{ and } n \text{ are odd and } \ m \neq n, \\ 2 & \text{otherwise} \end{cases}$$

It can be checked that for all $m, n, p \in X$, we have

$$d_b(m,p) \leq \frac{5}{2}[d_b(m,n) + d_b(n,p)]$$

Thus (X, d_b) is a b-metric space with $b = \frac{5}{2}$.
Let $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$d_b(2n, \infty) = \frac{1}{2n} \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty$$

that is, $x_n \longrightarrow \infty$, but $d_b(x_n, 1) = 2 \nrightarrow 5 = d_b(\infty, 1)$ as $n \longrightarrow \infty$.

Definition 2.2.5. (Fixed Point)
Let $T \colon X \to X$ be a mapping on a set X. A point $a_0 \in X$ is said to be fixed point of T if

$$Ta_0 = a_0$$

i.e. point is mapped onto itself.

Definition 2.2.6. (Contraction)
Let (X, d) be a metric space. A mapping $T \colon X \to X$ is said to be a contraction if for $a \neq b$ there exist $o \leq \alpha < 1$ such that

$$d(Ta, Tb) \leq \alpha d(a, b) \quad \forall \ a, b \in X$$

Definition 2.2.7. (Contractive Mapping)
Let (X, d) be a metric space. A mapping $T \colon (X, d) \to (X, d)$ is called contractive if for $a \neq b$

$$d(Ta, Tb) < d(a, b) \quad \forall \ a, b \in X$$

Example 2.2.8.
Consider usual metric space $(\mathbb{R}, d)$, *i.e.*

$$d(x, y) = |x - y|$$

Then the function defined as, $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \frac{x}{a} + b$$

is a contraction if $a > 1$ and its fixed point is $x = \frac{ab}{a-1}$.

Example 2.2.9.
Consider the Euclidean metric space $(\mathbb{R}^2, d)$, *i.e.* f: $\mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = (\frac{x}{a} + b, \frac{y}{c} + b)$$

is a contraction for $(a, c) > 1$ and its fixed point is $x = \frac{ab}{a-1}$ and $y = \frac{cd}{c-1}$.

2.3 Banach Contraction Principle

Banach [9], a polish mathematician have proved a most valueable result about a contraction mapping, which is also known as Banach contraction principle after its kind name, in 1922. In the theory of fixed point concepts, it is considered as one of the basic and frequently applied result. Many extensions and generalizations on Banach contraction principle are made by a lot of authors for taking as it has a wide and simple applications. [for instance see [33],[45],[8],[16]]. Banach contraction principle is as follows: "Every contraction mapped on a complete metric space has a unique fixed point *i.e.*
If (X, d) is a complete metric space and $T\colon X \to X$ is a mapping such that $\forall\ a, b \in X$, $\exists\ \alpha \in [o, 1)$ such that

$$d(Ta, Tb) \leq \alpha\ d(a, b), \quad a \neq b$$

Then T has a unique fixed point $a_0 \in X$ *i.e.* $Ta_0 = a_0$ " .

Chapter 3

A Review on F-Contraction in Complete metric Spaces

In this chapter it is our aim to review the concepts regarding the F-contraction mappings in complete Metric sapces which were considered and defined by Wardowski [46]. We also like to review the results of fixed point for F-Suzuki contractions that is the generalization of Wardowski's work result in F-contraction.
M. Edelstein [20] established some new results as a different version of the the well known Banach Contraction Principlein in 1962.
We have reviewed the following theorem of Edelstein.

3.1 Contractive mapping on compact metric space:

Theorem 3.1.1.
Let $S\colon X \to X$ be a self-mapping on a compact metric space (X, d). Assume S is a contractive mapping *i.e.,*

$$d(Sa, Sb) < d(a, b) \quad \forall\, a, b \in X \quad \text{with} \quad a \neq b$$

Then fixed point of S is a unique.

Proof. : Let $S\colon X \to [0, \infty)$ be a function defined as $a \mapsto d(a, Sa)$. Which measures the distance between each point and its S-value.

Since X is compact, so the function $d(a, Sa)$ takes on its minimum value, then there is an $p \in X$ such that

$$d(p, Sp) \leq d(a, Sa) \quad \forall\, a \in X.$$

We will show by contraction that s is a fixed point of S.
If $Sp \neq p$ then according to the given condition (Taking $a = p$ and $a' = Sp$)

$$d(Sp), S(Sp) < d(p, Sp).$$

Which is contradiction because $d(p, Sp)$ is minimum among all numbers $d(a, Sp)$.
So our supposition is wrong. Therefore $Sp = p$.
Hence existence of unique point is proved.
Uniqueness:
Now our desire is to prove that S has exactly one fixed point.
Suppose S has two fixed points $Sp = p \neq p' = (Sp')$, then

$$d(s, s') = d(Ss, Ss') < d(s, s')$$

This is impossible, so $s = s'$
Hence S has a unique fixed point. □

Let us consider some basic concept and theorem about the following F-contraction:

3.2 F-contraction:

In 2012, Wardowski [46] gave the concept of a new type of contraction known as F-contraction and he proved some new results about the fixed point theorem using F-contraction.
Wardowski generalized the Banach contraction principle in some different way and he defined the F-contraction in [33] as follows:

Definition 3.2.1.
"Let (X, d) be a metric space. A mapping $T\colon X \to X$ is said to be an F-contraction if there exist $\tau > 0$ such that $d(Tx, Ty) > 0$

$$\Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \forall\, x, y \in X \tag{3.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions

(F1): F is strictly increasing that is $\forall\ x, y \in \mathbb{R}^+$ such that for $x < y$, $F(x) < F(y)$

(F2): For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{k\to\infty} F(\alpha_n) = -\infty$

(F3): There exist $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$"

Symbol **F** is used for a set containing all those functions that satisfy $(F1), (F2)$ and $(F3)$.

Remark 3.2.2.
From $(F1)$ and (3.1) we can conclude that every F-contraction is necessarily continuous.

Wardowski stated a modified version of the Banach contraction principle as follows

Theorem 3.2.3.
Let $S\colon X \to X$ be an F-contraction on a complete metric space (X, d). Then for every $a \in X$ the sequence $\{Sn\}_{n\in\mathbb{N}}$ converges to a point a_0 and fixed point of S is unique.

Proof.
For the purpose to prove that S has a fixed point. Let us consider that an arbitrary point $a_0 \in X$ is fixed point of S. Take a sequence $\{a_n\}_{n\in N} \subset X$ as

$$a_{n+1} = Sa_n\ , \quad n = 0, 1, 2, ...$$

Let us consider,

$$\alpha_n = d(a_{n+1}, a_n)\ , \quad n = 0, 1, 2...$$

If there exist $n_0 \in N$ for which $a_{n_{0+1}} = a_{n_0}$ then

$$Sa_{n_0} = a_{n_0}$$

and the theorem is proved.
Now suppose that $a_{n+1} \neq a_n$ for every $n \in N$.
Then $\alpha_n > 0 \quad \forall\, n \in N$.

Now using (3.1), we see that

$$\begin{aligned}\mu + F(d(Sa_{n+1}, Sa_n)) &\leq F(d(a_{n+1}, a_n) \\ &\leq F(d(a_{n+1}, a_n)) - \mu \\ &= F(d(Sa_n, Sa_{n-1})) - \mu \\ &\leq F(d(a_n, a_{n-1})) - 2\mu \\ &= F(d(Sa_{n-1}, Sa_{n-2})) - 2\mu \\ &\leq F(d(a_{n-1}, a_{n-2})) - 3\mu \\ &\vdots \\ &\leq F(d(a_1, a_0)) - n\mu\end{aligned} \tag{3.2}$$

So we have,

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \mu \leq F(\alpha_{n-2}) - 2\mu \leq .... \leq F(\alpha_0) - n\mu \tag{3.3}$$

Taking limit when $n \to \infty$, we get

$$\lim_{n\to\infty} F(\alpha_n) = -\infty$$

Using (F2) we get,

$$\lim_{n\to\infty} \alpha_n = 0 \tag{3.4}$$

According to condition (F3) there must exist $k \in (0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n^k F(\alpha_n) = 0 \tag{3.5}$$

Multiplying (3.3) throughout by α_n^k, we get

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq \alpha_n^k (F(\alpha_0) - n\mu) - \alpha_n^k F(\alpha_0) = -\alpha_n^k n\mu \leq 0 \tag{3.6}$$

Taking $n \to \infty$ and using (3.2) and (3.3), we get

$$\lim_{n\to\infty} n\alpha_n^k = 0 \tag{3.7}$$

Now from (3.7), we observe that there exist $n_1 \in N$ such that

$$n\alpha_n^k \leq 1 \quad \forall\ n \geq n_1$$

and consequently we have

$$\alpha_n \leq \frac{1}{n^{\frac{1}{k}}} \quad \forall\, n \geq n_1 \tag{3.8}$$

Now we will prove that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
Let us consider $p, q \in \mathbb{N}$ such that $\quad p \geq q \geq n_1$
then from the definition of the metric and from (3.8) we get

$$d(x_p, x_q) \leq \alpha_{p-1} + \alpha_{p-2} + ... + \alpha_q < \sum_{n=1}^{\infty} \alpha_n \leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \tag{3.9}$$

From the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$, it is clear that $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.
As X is complete then there exist $a_0 \in X$ such that $\lim_{n\to\infty} a_n = a_0$.
From continuity of S, we get

$$d(Sa_0, a_0) = \lim_{n\to\infty} d(Sa_n, a_n) = \lim_{n\to\infty} d(a_{n+1}, a_n) = 0 \tag{3.10}$$

Which implies that,

$$Sa_0 = a_0$$

That is, a_0 is fixed point of S.
Uniqueness:
Suppose a_1, a_2 be two fixed points in X then, $Sa_1 = a_1 \neq a_2 = Sa_2$
then by definition of F-contraction we get

$$\mu \leq F(d(a_1, a_2)) - F(d(Sa_1, Sa_2)) = 0 \tag{3.11}$$

Which is contradiction because $\mu > 0$, so our supposition is wrong.
Hence S has a unique fixed point. □

Lemma 3.2.4. [37]
If $\{t_k\}_{k=1}^{\infty}$ is a bounded sequence of real numbers such that all its convergent sub-sequences have the same limit l, then $\{t_k\}_{k=1}^{\infty}$ is convergent and $\lim_{k\to\infty} t_k = l$

Lemma 3.2.5.
Let an increasing mapping $F\colon \mathbb{R}^+ \to \mathbb{R}$ and a sequence of positive real numbers $\{t_k\}_{k=1}^{\infty}$. Then the following hold:

1. If $\lim_{k\to\infty} F(t_k) = -\infty$, then $\lim_{k\to\infty} t_k = 0$;

2. If $\inf F = -\infty$ and $\lim_{k\to\infty} t_k = 0$, then $\lim_{k\to\infty} F(t_k) = -\infty$.

Proof. :
(1) First of all, we observe that $\{t_k\}_{k=1}^{\infty}$ is bounded.
Indeed if sequence is unbounded above, then we can find a sub-sequence $\{t_{k(p)}\}_{p=1}^{\infty}$ such that $\lim_{k\to\infty} t_{k(p)} = \infty$.
Then for every $\epsilon > 0$, there is $p(\epsilon) \in \mathbb{N}$ such that $t_{k(p)} \geq \epsilon$ for any $p \geq p(\epsilon)$.
So by $(F1)$,

$$F(\epsilon) \leq F(t_{k(p)})$$

i.e,

$$F(\epsilon) \leq \lim_{p\to\infty} F(t_{k(p)}) = -\infty$$

Which is a contradiction.
Therefore $\{t_k\}_{k=1}^{\infty}$ is bounded, hence it has a convergent sub-sequence.
Let $\{t_{k(n)}\}_{n=1}^{\infty}$ be such a sub-sequence and $\lim_{n\to\infty} t_{k(n)} = \alpha$, where $\alpha \geq 0$
Now choose $\epsilon > 0$ and $\epsilon < \alpha$. Then there exist $n(\epsilon) \in \mathbb{N}$ such that

$$t_{k(n)} \in (\alpha - \epsilon, \alpha + \epsilon) \quad \forall\, n \geq n(\epsilon)$$

As F is increasing therefore,

$$F(\alpha - \epsilon) \leq \lim_{n\to\infty} F(t_{k(n)}) = -\infty.$$

Which contradicts that $F(\alpha - \epsilon)$ is an element of $\mathbb{R}$.
As $\lim_{n\to\infty} t_{k(n)} = 0$. Then from Lemma 3.2.4 it follows that

$$\lim_{k\to\infty} t_k = 0$$

Now we prove condition (2)
Assume that $\inf F = -\infty$ and $\lim_{k\to\infty} t_k = 0$.
Now, choose $\epsilon > 0$ and $\alpha > 0$ such that $F(\alpha) < -\epsilon$
Then there exist $k_\alpha \in \mathbb{N}$ such that $t_k < \alpha, \forall\, k \geq k_\alpha$
So,
$F(t_k) < F(\alpha) < -\epsilon \quad \forall\, k \geq k_\alpha$.
Thus,
$\lim_{k\to\infty} F(t_k) = -\infty$ □

After proving Lemma 3.2.5, Secelean [37] replaced condition $(F2)$ in definition of F-contraction by an equivalent condition,
$(F2')$ $\inf F = -\infty$
Also, $(F2'')$ there exist a sequence of positive numbers $\{t_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} F(t_k) = -\infty$$

Here instead of condition $(F3)$ in definition of F-contraction we use $(F3')$ as follows
$(F3')$ F is continuous on $(0, \infty)$.
A set containing all those functions in which conditions $(F1)$, $(F2')$ and $(F3')$ are satisfied is usually denoted by the symbol $\Im$

Example 3.2.6.
Let, $F_1(\beta) = -\frac{1}{\beta}$
then

1. $(F1)$ is satisfied because for every $\beta_1 < \beta_2$, $F(\beta_1) < F(\beta_2)$
2. $(F2')$ is satisfied because $\inf F = -\infty$

3 $(F3')$ also satisfied because $F_1(\beta)$ is continuous on $(0, \infty)$

Similarly, if

$$F_2\beta) = -\frac{1}{\beta} + \beta, \quad F_3(\beta) = \frac{1}{e^{\beta} + e^{-\beta}}, \quad F_4(\beta) = \frac{1}{e^{-\beta}}$$

Then F_1, F_2, F_3 and $F_4 \in \Im$

Remark 3.2.7.
Note that if,

$$F(\beta) = \frac{-1}{\beta^n}$$

then, $(F1)$, $(F2)$ and $(F3')$ conditions are satisfied for $n \geq 1$. But $(F3)$ is not satisfied here.
Now If,

$$F(\beta) = \frac{-1}{(\beta + [\beta])^n}$$

then for $\beta > 1$ and $n \in (0, \frac{1}{\beta})$, F_1 and F_2 conditions are satisfied. Also for some $k \in (\frac{1}{\beta}, 1)$ F_3 is satisfied. But $(F3')$ is not satisfied in this case.
So we can conclude that $(F3)$ and $(F3')$ does not depend on each other.

Theorem 3.2.8.
Let (X, d) be a complete metric space and $S\colon X \to X$ be a self mapping. If $F \in \Im$ and there exist $\mu > 0$, such that $\forall\ a, b \in X, \quad d(Sa, Sb) > 0$

$$\Rightarrow \mu + F(d(Sa, Sb)) \leq F(d(a, b))]$$

holds. Then the sequence $\{S^k a_0\}_{k=1}^{\infty}$ converges to a unique fixed point a^* of S for every $a_0 \in X$.

Proof.
Let us choose $a_0 \in X$ and define a sequence $\{a_k\}_{k=1}^{\infty}$ by

$$a_1 = Sa_0, \quad a_2 = Sa_1 = S^2 a_0, \quad ....., a_{k+1} = Sa_k = S^{k+1} a_0, \quad \forall\ k \in N \tag{3.12}$$

If $d(a_k, Sa_k) = 0$ for some $k \in N$, then there is nothing to prove.
We consider that,

$$0 < d(a_k, Sa_k) = d(Sa_{k-1}, Sa_k), \quad \forall\ k \in \mathbb{N} \tag{3.13}$$

For some $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \mu + F(d(Sa_{k-1}, Sa_k)) &\leq F(d(a_{k-1}, a_k)) \\ F(d(Sa_{k-1}, Sa_k)) &\leq F(d(a_{k-1}, a_k)) - \mu \end{aligned} \tag{3.14}$$

Continuing the same procedure, we obtain

$$\begin{aligned} F(d(Sa_{k-1}, Sa_k)) &\leq F(d(a_{k-1}, a_k)) - \mu \\ &= F(d(Sa_{k-2}, Sa_{k-1})) - \mu \\ &\leq F(d(a_{k-2}, a_{k-1})) - 2\mu \\ &= F(d(Sa_{k-3}, Sa_{k-2})) - 2\mu \\ &\leq F(d(a_{k-3}, a_{k-2})) - 3\mu \\ &\vdots \\ &\leq F(d(a_0, a_1)) - n\mu \end{aligned} \tag{3.15}$$

Applying $\lim_{k\to\infty}$ on both the sides obtain,

$$\lim_{k\to\infty} F(d(Sa_{k-1}, Sa_k)) = -\infty$$

Using $(F2)$ we get,

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0 \tag{3.16}$$

Now, we have to show that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence.
By using contradictory argument, suppose that for a sequences of natural numbers $\{b(k)\}_{k=1}^{\infty}$ and $\{c(k)\}_{k=1}^{\infty}$ there exist $\delta > 0$ such that

$$b(k) > c(k) > k, \quad d(a_{b(k)}, a_{c(k)}) \geq \delta, \quad d(a_{b(k)-1}, a_{c(k)}) < \delta, \quad \forall\ k \in N \tag{3.17}$$

then, we have

$$\begin{aligned} \delta \leq d(a_{b(k)}, a_{c(k)}) &\leq d(a_{b(k)}, a_{b(k)-1}) + d(a_{b(k)-1}, a_{c(k)}) \\ &< d(a_{b(k)}, a_{b(k)-1}) + \delta \\ &= d(a_{b(k)-1}, Sa_{b(k)-1}) + \delta \\ \delta \leq d(a_{b(k)}, a_{c(k)}) &< d(a_{b(k)-1}, Sa_{b(k)-1}) + \delta \end{aligned} \tag{3.18}$$

Letting $\lim_{k\to\infty}$ and using (3.16) in above expression we get

$$\lim_{k\to\infty} d(a_{b(k)}, a_{c(k)}) = \delta \tag{3.19}$$

As,

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0$$

Then there exist $N \in \mathbb{N}$, such that

$$d(a_{b(k)}, Sa_{b(k)}) < \frac{\delta}{4} \quad \text{and} \quad d(a_{c(k)}, Sa_{c(k)}) < \frac{\delta}{4}, \quad \forall\ k \geq N \tag{3.20}$$

Now, we claim that

$$d(Sa_{b(k)}, Sa_{c(k)}) = d(a_{b(k)+1}, a_{c(k)+1}) > 0, \quad \forall\ k \in N \tag{3.21}$$

By using contradictory argument, there exist $l \geq \mathbb{N}$ for which

$$d(a_{b(l)+1}, a_{c(l)+1}) = 0 \tag{3.22}$$

By combining (3.17),(3.20) and (3.22) we get

$$\begin{aligned}\delta \leq d(a_{b(l)}, a_{c(l)}) &\leq d(a_{b(l)}, a_{b(l)+1}) + d(a_{b(l)+1}, a_{c(l)}) \\ &\leq d(a_{b(l)}, a_{b(l)+1}) + d(a_{b(l)+1}, a_{c(l)+1}) + d(a_{c(l)+1}, a_{c(l)}) \\ &= d(a_{b(l)}, Sa_{b(l)}) + d(a_{b(l)+1}, a_{c(l)+1}) + d(a_{c(l)}, Sa_{c(l)}) \\ &< \frac{\delta}{4} + 0 + \frac{\delta}{4} = \frac{\delta}{2}\end{aligned}$$

Which is contradiction, so there does not exist such l.
From (3.21) and conditions of the theorem, we get

$$\mu + F(d(Sa_{b(k)}, Sa_{c(k)})) \leq F(d(a_{b(k)}, a_{c(k)})), \quad \forall\ k \in \mathbb{N} \tag{3.23}$$

From $(F3')$, (3.19) and (3.23), we get

$$\mu + F(\delta) \leq F(\delta)$$

which is contradiction. So our supposition is wrong and hence $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence. As (X, d) is complete metric space then there must exist some $a \in X$ which is convergent point of $\{a_k\}_{k=1}^{\infty}$. As S is continuous then,

$$\begin{aligned}d(Sa, a) &= \lim_{k\to\infty} d(Sa_k, a_k) \\ &= \lim_{k\to\infty} d(a_{k+1}, a_k) \\ &= d(a^*, a^*) \\ &= 0\end{aligned}$$

This implies S has a unique fixed point.
Uniqueness:
Now we will show that S has exactly one fixed point.
Suppose $a_1, a_2 \in X$ are two different fixed points of S, where $a_1 \neq a_2$, *i.e.*

$$Sa_1 = a_1 \neq a_2 = Sa_2$$

Then,

$$d(Sa_1, Sa_2) = d(a_1, a_2) > 0$$

which gives,

$$\begin{aligned} F(d(a_1, a_2)) &= F(d(Sa_1, Sa_2)) \\ &< \mu + F(d(Sa_1, Sa_2)) \\ &\leq F(d(a_1, a_2)) \end{aligned}$$

which is contradiction. So our supposition of two fixed point is wrong.
Hence S has a unique fixed point. □

3.3 F-Suzuki Contraction

Now we discuss fixed point theorem of F-Suzuki contraction. Firstly by defining F-Suzuki contaction as follows:

Definition 3.3.1.
" A mapping $T \colon X \to X$ on a metric space (X, d) is called an F-Suzuki contraction if for all $a, b \in X$ and $Ta \neq Tb$ there exist $\mu > 0$, such that

$$\frac{1}{2} d(a, Ta) < d(a, b)$$

$$\Rightarrow \mu + F(d(Ta, Tb)) \leq F(d(a, b))$$

where F mapping denotes F-contraction".

Theorem 3.3.2.
Let $S \colon X \to X$ be an F-Suzuki contraction on a complete metric space (X, d). Then S has a unique fixed point and the sequence $\{Sa_0\}_{k=1}^{\infty}$ converge the point a^*.

Proof.
Let us suppose $a_0 \in X$ and take a sequence $\{a_k\}_{k=1}^{\infty}$ as

$$a_1 = Sa_0, \quad a_2 = Sa_1 = S^2 a_0, \quad .... \quad a_{k+1} = Sa_k = S^{k+1} a_0, \quad \forall\ k \in \mathbb{N} \tag{3.24}$$

If there exist $k \in \mathbb{N}$ for which $d(a_k, Sa_k) = 0$, then there is nothing to prove.
We assume that, $0 < d(a_k, Sa_k)$, $\forall\ k \in \mathbb{N}$
Therefore,

$$\frac{1}{2} d(a_n, Sa_k) < d(a_k, Sa_k) \quad \forall\ k \in \mathbb{N} \tag{3.25}$$

For some $k \in \mathbb{N}$,

$$\mu + F(d(Sa_k, S^2a_k)) \leq F(d(a_k, Sa_k))$$

i.e.

$$F(d(a_{k+1}, Sa_{k+1})) \leq F(d(a_k, Sa_k)) - \mu$$

Continuing the same process, we obtain

$$\begin{aligned} F(d(a_k, Sa_k)) &\leq F(d(a_{k-1}, Sa_{k-1})) - \mu \\ &\leq F(d(a_{k-2}, Sa_{k-2})) - 2\mu \\ &\leq F(d(a_{k-3}, Sa_{k-3})) - 3\mu \\ &\vdots \\ &\leq F(d(a_0, Sa_0)) - k\mu \end{aligned} \tag{3.26}$$

Taking Limit when $n \to \infty$, we obtain

$$\lim_{k\to\infty} F(d(a_k, Sa_k)) = -\infty$$

Using $(F2)$, we get

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0 \tag{3.27}$$

Now, we have to show that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence.
Contrary suppose that, there exist $\delta > 0$ and sequences of natural numbers $\{b(k)\}_{k=1}^{\infty}$ and $\{c(k)\}_{k=1}^{\infty}$ such that

$$a(k) > b(k) > k, \quad d(a_{b(k)}, a_{c(k)}) \geq \delta, \quad d(a_{b(k)-1}, a_{c(k)}) < \delta, \quad \forall\, k \in \mathbb{N} \tag{3.28}$$

then, we have

$$\begin{aligned} \delta \leq d(a_{b(k)}, a_{c(k)}) &\leq d(a_{b(k)}, a_{b(k)-1}) + d(a_{b(k)-1}, a_{c(k)}) \\ &\leq d(a_{b(k)}, a_{b(k)-1}) + \delta \\ &= d(a_{b(k)-1}, Sa_{b(k)-1}) + \delta \\ \delta \leq d(a_{b(k)}, a_{c(k)}) &\leq d(a_{b(k)-1}, Sa_{b(k)-1}) + \delta \end{aligned} \tag{3.29}$$

Letting $\lim_{k\to\infty}$ and using (3.27) in above expression we get

$$\lim_{k\to\infty} d(a_{b(k)}, a_{c(k)}) = \delta \tag{3.30}$$

Let us choose an integer $N \in \mathbb{N}$ from (3.27) and (3.30), such that

$$\frac{1}{2}d(a_{b(k)}, Sa_{b(k)}) < \frac{1}{2}\delta < d(a_{b(k)}, a_{c(k)}) \quad \forall\ k \geq N$$

As S is F-Suzuki type, then we have

$$\mu + F(d(Sa_{b(k)}, Sa_{c(k)})) \leq F(d(a_{b(k)}, a_{c(k)})) \quad \forall\ k \in N$$

Using (3.24), we see that

$$\mu + F(d(a_{b(k)+1}, a_{c(k)+1})) \leq F(d(a_{b(k)}, a_{c(k)})) \quad \forall\ k \in N \tag{3.31}$$

From $(F3')$, (3.27) and (3.31), we obtain

$$\mu + F(\delta) \leq F(\delta)$$

Which is contradiction. This implies $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence.
As (X, d) is complete therefore the sequence $\{a_k\}_{k=1}^{\infty}$ converge to a point $a^* \in X$. *i.e.*

$$\lim_{k\to\infty} d(a_k, a^*) = 0 \tag{3.32}$$

Let us claim that,

$$\frac{1}{2}d(a_k, Sa_k) < d(a_k, a^*) \quad \textit{and} \quad \frac{1}{2}d(Sa_k, S^2a_k) < d(Sa_k, a^*), \quad \forall\ k \in \mathbb{N} \tag{3.33}$$

Now suppose that there is some $p \in \mathbb{N}$ for which

$$\frac{1}{2}d(a_p, Sa_p) \geq d(a_p, a^*) \quad \textit{and} \quad \frac{1}{2}d(Sa_p, S^2a_p) \geq d(a_p, a^*) \tag{3.34}$$

Therefore,

$$\begin{aligned} 2d(a_p, a^*) &\leq d(a_p, Sa_p) \leq d(a_p, a^*) + d(a^*, Sa_p) \\ d(a_p, a^*) + d(a_p, a^*) &\leq d(a_p, a^*) + d(a^*, Sa_p) \end{aligned}$$

Which implies that,

$$d(a_p, a^*) \leq d(a^*, Sa_p). \tag{3.35}$$

Using (3.34) and (3.35), we get

$$d(a_p, a^*) \leq d(a^*, Sa_p) \leq \frac{1}{2}d(Sa_p, S^2a_p). \tag{3.36}$$

Since,

$$\frac{1}{2}d(a_p, Sa_p) \leq d(a_p, Sa_p)$$

and S is F-Suzuki contraction so, we have

$$\mu + F(d(Sa_p, S^2a_p)) \leq F(d(a_p, Sa_p))$$

So from $(F1)$, we get

$$d(Sa_p, S^2a_p) < d(a_p, Sa_p) \tag{3.37}$$

Using (3.34), (3.36) and (3.37), we get

$$\begin{aligned} d(Sa_p, S^2a_p) &< d(a_p, Sa_p) \\ &\leq d(a_p.a^*) + d(a^*, Sa_p) \\ &\leq \frac{1}{2}d(Sa_p, S^2a_p) + \frac{1}{2}d(Sa_p, S^2a_p) \\ &= d(Sa_p, S^2a_p). \end{aligned} \tag{3.38}$$

Which is contradiction. So, (3.33) holds.
Therefore we can say that, either

$$\mu + F(d(Sa_k, Sa^*)) \leq F(d(a_k, a^*))$$

or

$$\mu + F(d(S^2a_k, Sa^*)) \leq F(d(Sa_k, a^*)) = F(d(a_{k+1}, a^*))$$

holds for all $n \in \mathbb{N}$.
In case first, from (3.32), $(F2')$ and Lemma (3.2.5), we get

$$\lim_{k\to\infty} F(d(Sa_k, Sa^*)) = -\infty$$

Using $(F2')$ and Lemma(3.2.5), we get

$$\lim_{k\to\infty} d(Sa_k, Sa_*) = 0$$

So,

$$d(a^*, Sa^*) = \lim_{k\to\infty} d(a_{k+2}, Sa^*) = \lim_{k\to\infty} d(S^2a_k, Sa^*) = 0$$

Hence a^* is a fixed pointof S.
Now we prove that S has a unique fixed point.

Uniqueness:

Let us consider $a^*, b^* \in X$ be two different fixed points of S *i.e.,* $a^* \neq b^*$ then

$$Sa^* = a^* \neq b^* = Sb^*$$

then,

$$d(a^*, b^*) > 0$$

So,

$$0 = \frac{1}{2}d(a^*, Sa^*) < d(a^*, b^*)$$

According to the definition of F-contraction, we get

$$\begin{aligned} F(d(a^*, b^*)) &= F(d(Sa^*, Sb^*)) \\ &< \mu + F(d(Sa^*, Sb^*)) \\ &\leq F(d(a^*, b^*)) \end{aligned} \tag{3.39}$$

Which is contradiction, so our supposition of two fixed point is wrong.

Hence the uniqueness of fixed point of S is proved. □

Example 3.3.3.

Let us take a sequence $\{T_k\}_{k\in\mathbb{N}}$ defined by:

$T_1 = 1 \times 2, \qquad T_2 = 1 \times 2 + 2 \times 3, \ldots$

$T_k = 1 \times 2 + 2 \times 3 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3} \ldots$

Let $S\colon X \to X$ be a mapping defined by $S(T_1) = T_1$ and $S(T_k) = T_{k-1}$ for every $k > 1$

where, $X = \{T_k : k \in \mathbb{N}\}$ and $d(a, b) = |a - b|$.

Clearly, (X, d) is complete.

Since,

$$\begin{aligned} \lim_{k\to\infty} \frac{d(S(T_k), S(T_1))}{d(T_k, T_1)} &= \lim_{k\to\infty} \frac{|S(T_k) - S(T_1)|}{|T_k - T_1|} \\ &= \lim_{k\to\infty} \frac{|T_{k-1} - T_1|}{|T_k - T1|} \\ &= \lim_{k\to\infty} \frac{\frac{(k-1)k(k+1)}{3} - 2}{\frac{k(k+1)(k+2)}{3} - 2} \\ &= \lim_{k\to\infty} \frac{(k-1)k(k+1) - 6}{k(k+1)(k+2) - 6} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \frac{k^3-k-6}{k^3+3k^2+3k-6} \\
&= \lim_{k\to\infty} \frac{k^3(1-\frac{1}{k^2}-\frac{6}{k^3})}{k^3(1+\frac{3}{k}+\frac{3}{k^2})} \\
&= 1
\end{aligned}
$$

This is the example of mapping S which is a Suzuki contraction. But here S is not Banach.

While, if we take $F \in \Im$ such that,

$F(\gamma) = -\frac{1}{\gamma} + \gamma,$

and by taking $\mu = 6$ then S is F-contraction.

Now we consider the calculation given below.

$$
\begin{aligned}
&\frac{1}{2} d(T_k, ST_k) < d(T_k, T_p) \\
&\Leftrightarrow \quad [(1 = k < p) \vee (1 \le p < k) \vee (1 < k < p)]
\end{aligned}
$$

Now for $(1 = k < p)$, we have

$$
\begin{aligned}
|S(T_p) - S(T_1)| &= |T_{p-1} - T_1| \\
&= 2\times 3 + 3\times 4 + ... + (p-1)p \\
|T_p - T_1| &= 2\times 3 + 3\times 4 + ... + p(p+1)
\end{aligned} \tag{3.40}
$$

Since $p > 1$ and

$$
\frac{-1}{2\times 3 + 3\times 4 + \ldots + (p-1)p} < \frac{-1}{2\times 3 + 3\times 4 + \ldots + p(p+1)}
$$

then we have,

$$
\begin{aligned}
& 6 - \frac{-1}{2\times 3 + 3\times 4 + ... + (p-1)p} + [2\times 3 + 3\times 4 + ... + (p-1)p] \\
& < 6 - \frac{-1}{2\times 3 + 3\times 4 + ... + p(p+1)} + [2\times 3 + 3\times 4 + ... + (p-1)p] \\
& \le \frac{-1}{2+3+....+p} + [2\times 3 + 3\times 4 + ... + (p-1)p] + p(p+1) \\
& = \frac{-1}{2+3+....+p} + 2\times 3 + 3\times 4 + ... + (p-1)p + p(p+1)
\end{aligned}
$$

From (3.40), we get

$$
6 - \frac{1}{|S(T_p) - S(T_1)|} + |S(T_p) - S(T_1)| < -\frac{1}{|T_p - T_1|} + |T_p - T_1|
$$

For $1 \leq p < k$, similar to $1 = k < p$, we get

$$6 - \frac{1}{|S(T_p) - S(T_1)|} + |S(T_p) - S(T_1)| < -\frac{1}{|T_p - T_1|} + |T_p - T_1|$$

For $1 < k < p$, we have

$$|S(T_p) - S(T_k)| = k(k+1) + (k+1)(k+2) + ... + (p-1)P \tag{3.41}$$

$$|T_p - T_k| = (k+1)(k+2) + (k+2)(k+3) + ... + p(p+1) \tag{3.42}$$

Since $p > k > 1$, we have

$$(p+1)p \geq (k+2)(k+1) = k(k+1) + 2(k+1) \geq k(k+1) + 6$$

We know that,

$$\frac{-1}{k(k+1) + (k+1)(k+2) + ... + (p-1)p} < \frac{-1}{(k+1)(k+2) + (k+2)(k+3) + ... + p(p+1)}$$

Therefore,

$$\begin{aligned}
6 - &\frac{1}{k(k+1) + (k+1)(k+2) + ... + (p-1)p} \\
&+ [k(k+1) + (k+1)(k+2) + ... + (p-1)p] \\
&< 6 - \frac{1}{(k+1)(k+2) + (k+2)(k+3) + ... + p(p+1)} \\
&+ [k(k+1) + (k+1)(k+2) + ... + (p-1)p] \\
&= -\frac{1}{(k+1)(k+2) + (k+2)(k+3) + ... + p(p+1)} + 6 + k(k+1) \\
&+ [(k+1)(k+2) + ... + (p-1)p] \\
&\leq -\frac{1}{(k+1)(k+2) + (k+2)(k+3) + ... + p(p+1)} \\
&+ p(p+1) + [(k+1)(k+2) + ... + (p-1)p] \\
&= -\frac{1}{(k+1)(k+2) + (k+2)(k+3) + ... + p(p+1)} \\
&+ [(k+1)(k+2) + ... + (p-1)p]
\end{aligned}$$

So From (3.42), we get

$$6 - \frac{1}{|S(T_p) - S(T_k)|} + |S(T_p) - S(T_k)|$$

$$< -\frac{1}{|T_p - T_k|} + |T_p - T_k|$$

Therefore,

$$\mu + F(d(S(T_p), S(T_K))) \leq d(T_p, T_k) \quad \forall\ p, k \in N$$

Hence S is an F-contraction and $S(T_1) = T_1$.

Chapter 4

F-contraction and b-metric Spaces

In this chapter it is our aim to set up some new concepts and results using the F-contraction mappings in complete b-Metric sapces that were considered and defined by Czerwik [16]. We also extended the fixed point results for b-metric space using F-Suzuki [41] contractions that is the generalization of the work of Wardowski's result in F-contraction.

4.1 F-contraction in b-Metric

We start first by defining F-contraction as follows:

Definition 4.1.1.
"Let (X, d_b) be a b-metric space. A mapping $S\colon X \to X$ is said to be an F-contraction if there exist $\mu > 0$ such that $\forall\ a, b \in X$

$$d(S_a, S_b) > 0 \Rightarrow \mu + F(d(S_a, S_b)) \leq F(d(a, b))$$

Where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions :

F-1 F is strictly increasing that is $\forall a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$

F-2 for each sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive numbers, $\lim_{k\to\infty} \alpha_k = 0$ if and only if $\lim F(\alpha_k) = -\infty$

F-3 There exist $n \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^n F(\alpha) = 0$"

Theorem 4.1.2.
Let (X, d_b) be a complete b-metric space with continuous b-metric d and $b \geq 1$. Let $S\colon X \to X$ be a self mapping on X. If $F \in \Im$ and there exist $\mu > 0$ such that for all $a, b \in X$, $\quad d(Sa, Sb) > 0$

$$\Rightarrow \mu + F(d(Sa, Sb)) \leq F(d((a, b)) \tag{4.1}$$

holds. Then the sequence $\{S^k a_0\}_{k=1}^{\infty}$ converges to a unique fixed point a^* of S for every $a_0 \in X$.

Proof.
We select $a_0 \in X$ and define a sequence $\{a_k\}_{k=1}^{\infty}$ by

$$a_1 = Sa_0, \quad a_2 = Sa_1 = S^2 a_0, \quad ....., a_{k+1} = Sa_k = S^{k+1} a_0, \quad \forall\, k \in \mathbb{N} \tag{4.2}$$

If $d(a_k, Sa_k) = 0$ for some $k \in \mathbb{N}$, then there is nothing to prove.
So, we suppose that

$$0 < d(a_k, Sa_k) = d(Sa_{k-1}, Sa_k), \quad \forall\, k \in \mathbb{N} \tag{4.3}$$

For any $k \in \mathbb{N}$, we get

$$\begin{aligned} \mu + F(d(Sa_{k-1}, Sa_k)) &\leq F(d(a_{k-1}, a_k)) \\ F(d(Sa_{k-1}, Sa_k)) &\leq F(d(a_{k-1}, a_k)) - \mu \end{aligned} \tag{4.4}$$

In the same way,

$$\begin{aligned} F(d(Sa_{k-1}, Sa_k)) &\leq F(d(a_{k-1}, a_k)) - \mu \\ &= F(d(Sa_{k-2}, Sa_{k-1})) - \mu \\ &\leq F(d(a_{k-2}, a_{k-1})) - 2\mu \\ &= F(d(Sa_{k-3}, Sa_{k-2})) - 2\mu \\ &\leq F(d(a_{k-3}, a_{k-2})) - 3\mu \\ &\vdots \\ &\leq F(d(a_0, a_1)) - n\mu \end{aligned} \tag{4.5}$$

Applying $\lim_{k\to\infty}$ on both the sides, we get

$$\lim_{k\to\infty} F(d(Sa_{k-1}, Sa_k)) = -\infty$$

Using the condition $(F2)$, we have

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0 \tag{4.6}$$

By claiming that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence.
In contention, suppose that there exist $\delta > 0$ and sequences of natural numbers $\{b(k)\}_{k=1}^{\infty}$ and $\{c(k)\}_{k=1}^{\infty}$ such that

$$b(k) > c(k) > k, \quad d(a_{b(k)}, a_{c(k)}) \geq b\delta, \quad d(a_{b(k)-1}, a_{c(k)}) < b\delta, \quad \forall\ k \in N, \quad b \geq 1 \tag{4.7}$$

then, we have

$$\begin{aligned} b\delta \leq d(a_{b(k)}, a_{c(k)}) &\leq b[d(a_{b(k)}, a_{b(k)-1}) + d(a_{b(k)-1}, a_{c(k)})] \\ &< bd(a_{b(k)}, a_{b(k)-1}) + b\delta \\ &= bd(a_{b(k)-1}, Sa_{b(k)-1}) + b\delta \\ b\delta \leq d(a_{b(k)}, a_{c(k)}) &< bd(a_{b(k)-1}, Sa_{b(k)-1}) + b\delta \end{aligned} \tag{4.8}$$

Letting $\lim_{k\to\infty}$ and using (4.6) in above expression we get,

$$\lim_{k\to\infty} d(a_{b(k)}, a_{c(k)}) = b\delta \tag{4.9}$$

As,

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0$$

then for $\delta > 0$ there exist $k \in \mathbb{N}$, such that

$$d(a_{b(k)}, Sa_{b(k)}) < \frac{\delta}{4} \quad \text{and} \quad d(a_{c(k)}, Sa_{c(k)}) < \frac{\delta}{4}, \quad \forall\ k \geq \mathbb{N} \tag{4.10}$$

By claiming,

$$d(Sa_{b(k)}, Sa_{c(k)}) = d(a_{b(k)+1}, a_{c(k)+1}) > 0, \quad \forall\ k \in \mathbb{N} \tag{4.11}$$

In conflict, there exist $l \geq \mathbb{N}$ such that

$$d(a_{b(l)+1}, a_{c(l)+1}) = 0 \tag{4.12}$$

By combining (4.7), (4.10) and (4.12) we get

$$\begin{aligned} b\delta \leq d(a_{b(l)}, a_{c(l)}) &\leq b[d(a_{b(l)}, a_{b(l)+1}) + d(a_{b(l)+1}, a_{c(l)})] \\ &\leq bd(a_{b(l)}, a_{b(l)+1}) + bd(a_{b(l)+1}, a_{c(l)+1}) + bd(a_{c(l)+1}, a_{c(l)}) \\ &= bd(a_{b(l)}, Sa_{b(l)}) + bd(a_{b(l)+1}, a_{c(l)+1}) + bd(a_{c(l)}, Sa_{c(l)}) \\ &< \frac{b\delta}{4} + 0 + \frac{b\delta}{4} = \frac{b\delta}{2} \end{aligned} \tag{4.13}$$

Which is contradiction, so there does not exist such l.
From (4.11) and supposition of the theorem, we get

$$\mu + F(d(Sa_{b(k)}, Sa_{c(k)})) \leq F(d(a_{b(k)}, a_{c(k)})), \quad \forall\ k \in \mathbb{N} \tag{4.14}$$

From $(F3')$, (4.9) and (4.14), we get

$$\mu + F(\delta) \leq F(\delta)$$

which is contradiction. So our supposition is wrong and hence $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence. As X is complete then according to continuity of S there exist $a \in X$, such that

$$\begin{aligned} d(Sa, a) &= \lim_{k\to\infty} d(Sa_k, a_k) \\ &= \lim_{k\to\infty} d(a_{k+1}, a_k) \\ &= d(a^*, a^*) \\ &= 0 \end{aligned}$$

This implies S has a unique point.
Uniqueness:
Now we will show that S has exactly one fixed point.
Suppose $a_1, a_2 \in X$ are two different fixed points of S, where $a_1 \neq a_2$, that is

$$Sa_1 = a_1 \neq a_2 = Sa_2$$

then,

$$d(Sa_1, Sa_2) = d(a_1, a_2) > 0$$

that gives,

$$\begin{aligned} F(d(a_1, a_2)) &= F(d(Sa_1, Sa_2)) \\ &< \mu + F(d(Sa_1, Sa_2)) \\ &\leq F(d(a_1, a_2)) \end{aligned} \tag{4.15}$$

which is contradiction. So our supposition of two fixed point is wrong.
Hence S has a unique fixed point. □

4.2 F-Suzuki Contraction in b-metric

Theorem 4.2.1.
Let $S\colon X \to X$ be an F-Suzuki contraction on a complete b-metric space (X, d) with continuous b-metric d and $b \geq 1$. Then the sequence $\{S^k a_0\}_{k=1}^{\infty}$ converges to a unique fixed point a^* of S for every $a_0 \in X$.

Proof.
Let us choose $a_0 \in X$ and take a sequence $\{a_k\}_{k=1}^{\infty}$ as

$$a_1 = Sa_0, \quad a_2 = Sa_1 = S^2 a_0, \quad .... \quad a_{k+1} = Sa_k = S^{k+1} a_0, \quad \forall\ k \in \mathbb{N} \tag{4.16}$$

If there exist $k \in N$ for which $d(a_k, Sa_k) = 0$, then there is nothing to prove.
We assume that,

$$0 < d(a_k, Sa_k), \quad \forall\ k \in \mathbb{N} \tag{4.17}$$

Therefore,

$$\frac{1}{2b} d(a_k, Sa_k) < d(a_k, Sa_k) \quad \forall\ k \in \mathbb{N} \tag{4.18}$$

As F is Suzuki type contraction, then we have

$$\begin{aligned} \mu + F(d(Sa_k, S^2 a_k)) &\leq F(d(a_k, Sa_k)), \\ F(d(Sa_k, S^2 a_k)) &\leq F(d(a_k, Sa_k)) - \mu \\ &< F(d(a_k, Sa_k)) \end{aligned} \tag{4.19}$$

As F is strictly increasing, then we get

$$d(a_{k+1}, Sa_{k+1}) = d(Sa_k, S^2 a_k) < d(a_k, Sa_k) \quad \forall\ k \in \mathbb{N} \tag{4.20}$$

This implies $\{d(a_k, Sa_k)\}_{k=1}^{\infty}$ is decreasing sequence of real numbers bounded from below. Then $\{d(a_k, Sa_k)\}_{k=1}^{\infty}$ converges to limit l

$$\lim_{k\to\infty} d(a_k, Sa_k) = l = \inf\{d(a_k, Sa_k) : k \in \mathbb{N}\} \tag{4.21}$$

Now we have to show that $l = 0$. Assume that $l > 0$, then for every $\delta > 0$ there exist $p \in \mathbb{N}$, such that

$$d(a_p, Sa_p) < l + \delta \tag{4.22}$$

Using $(F1)$, we get

$$F(d(a_p, Sa_p)) < F(l + \delta) \tag{4.23}$$

But we have,

$$\frac{1}{2b} d(a_p, Sa_p) < d(a_p, Sa_p) \tag{4.24}$$

As S is F-Suzuki type contraction, then we get

$$\begin{aligned} \mu + F(d(Sa_p, S^2a_p)) &\le F(d(a_p, Sa_p)), \\ F(d(Sa_p, S^2a_p)) &\le F(d(a_p, Sa_p)) - \mu \end{aligned} \tag{4.25}$$

Similarly,

$$\begin{aligned} \mu + F(d(S^2a_p, S^3a_p)) &\le d(Sa_p, S^2a_p)), \\ F(d(Sa_p, S^2a_p)) &\le F(d(a_p, Sa_p)) - 2\mu \end{aligned} \tag{4.26}$$

Continuing the same process, we obtain

$$\begin{aligned} F(d(S^k a_p, S^{k+1}a_p)) &\le F(d(S^k a_p, S^{k-1}a_p)) - \mu \\ &\le F(d(S^{k-1}a_p, S^{k-2}a_p)) - 2\mu \\ &\vdots \\ &\le F(d(Sa_p, a_p)) - k\mu \\ &< F(l + \delta) - k\mu \end{aligned} \tag{4.27}$$

Taking $\lim_{k\to\infty}$, we get

$$\lim_{k\to\infty} F(d(S^k a_p, S^{k+1}a_p)) = -\infty \tag{4.28}$$

Using $(F2)$, we get

$$\lim_{k\to\infty} d(S^k a_p, S^{k+1}a_p) = 0$$

Then there exist $N_1 \in \mathbb{N}$ such that

$$d(S^k a_p, S^{k+1}a_p) < l, \quad \forall\ k \in N_1 \tag{4.29}$$

From sequence defined in (4.16), we get

$$d(a_{p+k}, Sa_{p+k}) < l, \quad \forall\ k \in N_1$$

Which is opposite to the definition of l.
Therefore,

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0 \tag{4.30}$$

Now we have to show that $\lim_{k,p\to\infty} d(a_k, a_p) = 0$
Oppositely, Suppose that there exist $\delta > 0$ and sequences of natural numbers $\{b(k)\}_{k=1}^{\infty}$ and $\{c(k)\}_{k=1}^{\infty}$ such that

$$a(k) > b(k) > k, \quad d(a_{b(k)}, a_{c(k)}) \geq \delta, \quad d(a_{b(k)-1}, a_{c(k)}) < \delta, \quad \forall\ k \in \mathbb{N} \tag{4.31}$$

As X is b-metric space, then for b-coefficient

$$\begin{aligned} d(a_{b(k)}, a_{c(k)}) &\leq b[d(a_{b(k)}, a_{b(k)-1}) + d(a_{b(k)-1}, a_{c(k)})] \\ &< bd(a_{b(k)}, a_{b(k)-1}) + b\delta \\ &= bd(a_{b(k)-1}, Sa_{b(k)-1}) + b\delta \end{aligned} \tag{4.32}$$

As $\lim_{\to\infty} d(a_k, Sa_k) = 0$, then there exist $N_2 \in \mathbb{N}$ such that

$$d(a_{b(k)}, Sa_{b(k)}) < \delta, \quad \forall\ k > N_2. \tag{4.33}$$

Using (4.33), (4.32) takes the form

$$d(a_{b(k)}, a_{c(k)}) < 2b\delta \quad \forall\ k > N_2 \tag{4.34}$$

Using $(F2)$, we get

$$F(d(a_{b(k)}, a_{c(k)})) < F(2b\delta) \quad \forall\ k > N_2 \tag{4.35}$$

We can also get,

$$\frac{1}{2b}d(a_{b(k)}, Sa_{b(k)}) < \frac{\delta}{2b} < \delta < d(a_{b(k)}, a_{c(k)}) \quad \forall\ k > N_2$$

As S is F-Suzuki type contraction, then we get

$$\mu + F(d(Sa_{b(k)}, Sa_{c(k)})) \leq F(d(a_{b(k)}, a_{c(k)})) \quad \forall\ k \in N_2 \tag{4.36}$$

Putting (4.35) in (4.36) we get,

$$\mu + F(d(Sa_{b(k)}, Sa_{c(k)})) < F(2b\delta)$$

Using and $(F2)$, we get

$$\lim_{k\to\infty} F(d(Sa_{b(k)}, Sa_{c(k)})) = -\infty$$

From $(F2)$, we get

$$\lim_{k\to\infty} d(Sa_{b(k)}, Sa_{c(k)}) = 0 \Leftrightarrow \lim_{k\to\infty} d(a_{b(k)+1}, a_{c(k)+1}) = 0$$

Which contradicts (4.31). Hence $\lim_{k,p\to\infty} d(a_k, a_p) = 0$ this implies that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X.
As (X, d) is complete, then there exist $t \in X$ such that

$$\lim_{k\to\infty} d(a_k, t) = 0 \tag{4.37}$$

Now we claim that, for every $k \in \mathbb{N}$ we have,

$$\begin{aligned} &\frac{1}{2b}d(a_k, Sa_k) < d(a_k, t) \quad \forall\ k \in \mathbb{N} \\ &\frac{1}{2b}d(Sa_k, S_k^2) < d(Sa_k, t) \quad \forall\ k \in \mathbb{N} \end{aligned} \tag{4.38}$$

On contrary suppose that, there exists $p \in \mathbb{N}$ such that

$$\begin{aligned} &\frac{1}{2b}d(a_p, Sa_p) \geq d(a_p, t) \\ &or \\ &\frac{1}{2b}d(Sa_p, S_p^2) \geq d(Sa_p, t) \end{aligned} \tag{4.39}$$

Using $(F1)$ and (4.20) we get,

$$d(Sa_p, S_p^2) < d(a_p, Sa_p) \tag{4.40}$$

From (4.39) and (4.40), we get

$$\begin{aligned} d(a_p, Sa_p) &\leq bd(a_p, t) + bd(t, Sa_p) \\ &\leq \frac{1}{2}d(a_p, Sa_p) + \frac{1}{2}d(Sa_p, S_p^2) \\ &< \frac{1}{2}d(a_p, Sa_p) + \frac{1}{2}d(a_p, Sa_p) \\ &= d(a_p, Sa_p) \end{aligned} \tag{4.41}$$

Which contradicts our supposition. This implies (4.38) holds. As S is F-Suzuki type contraction, then for every $k \in N$ (4.38) gives,

$$\mu + F(d(Sa_k, St)) \leq F(d(a_k, t)) \tag{4.42}$$

Taking $\lim_{k\to\infty}$ of (4.30), (4.31) and then applying $(F2)$,we get

$$\lim_{k\to\infty} F(d(a_k, t)) = -\infty, \quad \lim_{k\to\infty} F(d(a_k, Sa_k)) = -\infty \tag{4.43}$$

$$\lim_{k\to\infty} F(d(Sa_k, St)) = -\infty \tag{4.44}$$

Using $(F2)$, we get

$$\lim_{k\to\infty} d(Sa_k, St)) = 0 \tag{4.45}$$

Now using triangular inequality, we get

$$\begin{aligned} d(t, St) &\leq b[d(t, Sa_k) + d(Sa_k, St)] \\ &= bd(t, a_{k+1}) + bd(Sa_k, St) \end{aligned} \tag{4.46}$$

Letting $k \to \infty$ and using (4.37) and (4.45), we get

$$d(t, St) = 0 \quad \Rightarrow \quad t = St$$

Hence t is fixed point of S.

Uniqueness:

Let us consider $a^*, b^* \in X$ be two different fixed points of S *i.e.,* $a^* \neq b^*$ then

$$Sa^* = a^* \neq b^* = Sb^*$$

then,

$$d(a^*, b^*) > 0$$

So,

$$0 = \frac{1}{2b} d(a^*, Sa^*) < d(a^*, b^*)$$

According to the definition of F-Suzuki contraction, we get

$$\begin{aligned} F(d(a^*, b^*)) &= F(d(Sa^*, Sb^*)) \\ &< \mu + F(d(Sa^*, Sb^*)) \\ &\leq F(d(a^*, b^*)) \end{aligned} \tag{4.47}$$

Which is contradiction, so our supposition of two fixed point is wrong.
Hence the uniqueness of fixed point of S is proved. □

Now we discuss the fixed point theorem of generalized F-Suzuki type contraction mapped on complete b-metric space.
Firstly definition of generalized F-Suzuki type contraction is as follows.

Definition 4.2.2.
"Let (X, d) ba a b-metric space with constant $s \geq 1$. A mapping $T\colon X \to X$ is said to be a generalized F-Suzuki type contraction if there exist $\tau > 0$ such that for all $a, b \in X$ with $Ta \neq Tb$

$$\begin{aligned} \frac{1}{2s} d(T_a, T_b) &< d(a, b) \\ &\Rightarrow \tau + F(d(T_a, T_b)) \\ &\leq \lambda F(d(a, b)) + \mu F(d(a, T_a)) + \nu F(d(b, T_b)) \end{aligned} \tag{4.48}$$

Where $\nu \in [0, 1)$ and $\lambda, \mu \in [0, 1]$ are real numbers with $\lambda + \mu + \nu = 1$ and $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions :

- F-1 F is strictly increasing that is $\forall\ a, b \in \mathbb{R}^+$ such that $a < b$, $F(a) < F(b)$
- F-2 For each sequence $\{\alpha_k\}_{k=1}^{\infty}$ of positive numbers, $\lim_{k\to\infty} \alpha_k = 0$ if and only if $\lim F(\alpha_k) = -\infty$"

Theorem 4.2.3.
Let $S\colon X \to X$ be generalized F-Suzuki contraction on a complete b-metric space (X, d). Then S has a unique fixed point and the sequence $\{Sa_0\}_{k=1}^{\infty}$ converge to a point a^*.

Proof.
Let us consider a point $a_0 \in X$ and take a sequence $\{a_k\}_{k=1}^{\infty}$ as

$$a_1 = Sa_0, \quad a_2 = Sa_1 = S^2 a_0, \quad .... \quad a_{k+1} = Sa_k = S^{k+1} a_0, \quad \forall\ k \in \mathbb{N} \tag{4.49}$$

If there exist $k \in N$ for which $d(a_k, Sa_k) = 0$, then there is nothing to prove. We assume that,

$$0 < d(a_k, Sa_k), \quad \forall\ k \in \mathbb{N} \tag{4.50}$$

Therefore,

$$\frac{1}{2s} d(a_n, Sa_k) < d(a_k, Sa_k) \quad \forall\ k \in \mathbb{N} \tag{4.51}$$

As F is generalized Suzuki type contraction, then we have

$$\begin{aligned} \tau + F(d(Sa_k, S^2 a_k)) &\leq \lambda F(d(a_k, Sa_k)) + \mu F(d(a_k, Sa_k)) \\ &\quad + \nu F(d(Sa_k, S^2 a_k)) \\ \tau + (1-\nu) F(d(Sa_k, S^2 a_k)) &\leq (\lambda + \mu) F(d(a_k, Sa_k)) \end{aligned} \tag{4.52}$$

Since $\lambda + \mu + \nu = 1$, then (4.52) takes the form

$$\begin{aligned} F(d(Sa_k, S^2 a_k)) &\leq F(d(a_k, Sa_k)) - \frac{\tau}{\lambda + \mu} \\ &\leq F(d(a_k, Sa_k)) \end{aligned} \tag{4.53}$$

As F is strictly increasing then we get

$$d(a_{k+1}, Sa_{k+1}) = d(Sa_k, S^2 a_k) < d(a_k, Sa_k) \quad \forall\ k \in \mathbb{N} \tag{4.54}$$

This implies $\{d(a_k, Sa_k)\}_{k=1}^{\infty}$ is decreasing sequence of real numbers bonded from below. Then $\{d(a_k, Sa_k)\}_{k=1}^{\infty}$ converges to limit l

$$\lim_{k \to \infty} d(a_k, Sa_k) = l = \inf\{d(a_k, Sa_k) : k \in \mathbb{N}\} \tag{4.55}$$

Now we have to show that $l = 0$. Assume that $l > 0$, then for every $\delta > 0$ there exist $p \in \mathbb{N}$, such that

$$d(a_p, Sa_p) < l + \delta \tag{4.56}$$

Using $(F1)$, we get

$$F(d(a_p, Sa_p)) < F(l + \delta) \tag{4.57}$$

But we have,

$$\frac{1}{2s}d(a_p, Sa_p) < d(a_p, Sa_p) \tag{4.58}$$

As S is generalized F-Suzuki type contraction, then we get

$$\begin{aligned}\tau + F(d(Sa_p, S^2a_p)) &\leq \lambda F(d(a_p, Sa_p)) + \mu F(d(a_p, Sa_p)) \\ &\quad + \nu F(d(Sa_p, S^2a_p)) \\ \tau + (1-\nu)F(d(Sa_p, S^2a_p)) &\leq (\lambda+\mu)F(d(a_p, Sa_p))\end{aligned} \tag{4.59}$$

As $\lambda + \mu + \nu = 1 \Rightarrow 1 - \nu = \lambda + \mu$

$$F(d(Sa_p, S^2a_p)) \leq F(d(a_p, Sa_p)) - \frac{\tau}{\lambda+\mu} \tag{4.60}$$

Similarly,

$$\begin{aligned}\tau + F(d(S^2a_p, S^3a_p)) &\leq \lambda F(d(Sa_p, S^2a_p)) + \mu F(d(Sa_p, S^2a_p)) \\ &\quad + \nu F(d(S^2a_p, S^3a_p)) \\ F(d(S^2a_p, S^3a_p)) &\leq F(d(Sa_p, S^2a_p)) - \frac{\tau}{\lambda+\mu}\end{aligned} \tag{4.61}$$

Combining (4.60) and (4.61) we get,

$$\begin{aligned}F(d(S^2a_p, S^3a_p)) &\leq F(d(Sa_p, S^2a_p)) - \frac{\tau}{\lambda+\mu} \\ &\leq F(d(a_p, Sa_p)) - \frac{2\tau}{\lambda+\mu}\end{aligned} \tag{4.62}$$

Continuing this procedure, we obtain

$$\begin{aligned}F(d(S^ka_p, S^{k+1}a_p)) &\leq F(d(S^ka_p, S^{k-1}a_p)) - \frac{\tau}{\lambda+\mu} \\ &\leq F(d(S^{k-1}a_p, S^{k-2}a_p)) - \frac{2\tau}{\lambda+\mu} \\ &\vdots \\ &\leq F(d(Sa_p, a_p)) - \frac{k\tau}{\lambda+\mu} \\ &< F(l+\delta) - \frac{k\tau}{\lambda+\mu}\end{aligned} \tag{4.63}$$

Taking $\lim_{k\to\infty}$, we get

$$\lim_{k\to\infty} F(d(S^ka_p, S^{k+1}a_p)) = -\infty \tag{4.64}$$

Using $(F2)$, we get

$$\lim_{k\to\infty} d(S^k a_p, S^{k+1} a_p) = 0$$

Then there exist $N_1 \in \mathbb{N}$ such that

$$d(S^k a_p, S^{k+1} a_p) < l, \quad \forall\ k \in N_1 \tag{4.65}$$

From sequence defined in (4.48), we get

$$d(a_{p+k}, Sa_{p+k}) < l, \quad \forall\ k \in N_1$$

Which is opposite to the definition of l.

Therefore,

$$\lim_{k\to\infty} d(a_k, Sa_k) = 0 \tag{4.66}$$

Now, we have to show that $\lim_{k,p\to\infty} d(a_k, a_p) = 0$

Oppositely, suppose that there exist $\delta > 0$ and sequences of natural numbers $\{b(k)\}_{k=1}^{\infty}$ and $\{c(k)\}_{k=1}^{\infty}$ such that

$$a(k) > b(k) > k, \quad d(a_{b(k)}, a_{c(k)}) \geq \delta, \quad d(a_{b(k)-1}, a_{c(k)}) < \delta, \quad \forall\ k \in \mathbb{N} \tag{4.67}$$

As X is b-metric space, then

$$\begin{aligned} d(a_{b(k)}, a_{c(k)}) &\leq s[d(a_{b(k)}, a_{b(k)-1}) + d(a_{b(k)-1}, a_{c(k)})] \\ &< sd(a_{b(k)}, a_{b(k)-1}) + s\delta \\ &= sd(a_{b(k)-1}, Sa_{b(k)-1}) + s\delta \end{aligned} \tag{4.68}$$

As $\lim_{k\to\infty} d(a_k, Sa_k) = 0$, then there exist $N_2 \in \mathbb{N}$ such that

$$d(a_{b(k)}, Sa_{b(k)}) < \delta, \quad \forall\ k > N_2. \tag{4.69}$$

Using (4.69), (4.68) takes the form

$$d(a_{b(k)}, a_{c(k)}) < 2s\delta \quad \forall\ k > N_2 \tag{4.70}$$

Using $(F2)$, we get

$$F(d(a_{b(k)}, a_{c(k)})) < F(2s\delta) \quad \forall\ k > N_2 \tag{4.71}$$

We can also get,

$$\frac{1}{2s}d(a_{b(k)}, Sa_{b(k)}) < \frac{\delta}{2s} < \delta < d(a_{b(k)}, a_{c(k)}) \quad \forall\ k > N_2$$

So S is F-Suzuki type, then we have

$$\begin{aligned}\tau + F(d(Sa_{b(k)}, Sa_{c(k)})) \leq& \lambda F(d(a_{b(k)}, a_{c(k)})) + \mu F(da_{b(k)}, Sa_{b(k)})) \\ &+ \nu F(d(a_{c(k)}, Sa_{c(k)})) \quad \forall\ k > N_2\end{aligned} \tag{4.72}$$

Putting (4.71) in (4.72) we get,

$$\begin{aligned}\tau + F(d(Sa_{b(k)}, Sa_{c(k)})) \leq& \lambda F(2s\delta) + \mu F(da_{b(k)}, Sa_{b(k)})) \\ &+ \nu F(d(a_{c(k)}, Sa_{c(k)}))\end{aligned} \tag{4.73}$$

Letting $\lim_{k\to\infty}$ in (4.73) and using (4.66) we get,

$$\lim_{k\to\infty} F(d(Sa_{b(k)}, Sa_{c(k)})) = -\infty$$

From $(F2)$ we get,

$$\lim_{k\to\infty}(d(Sa_{b(k)}, Sa_{c(k)}) = 0 \Leftrightarrow \lim_{k\to\infty}(d(a_{b(k)+1}, a_{c(k)+1}) = 0$$

Which contradicts (4.67). Hence $\lim_{k,p\to\infty} d(a_k, a_p) = 0$ this implies that $\{a_k\}_{k=1}^{\infty}$ is a Cauchy sequence in X.
As (X, d) is complete, then there exist $t \in X$ such that

$$\lim_{k\to\infty} d(a_k, t) = 0 \tag{4.74}$$

Now we claim that, for every $k \in \mathbb{N}$ we have,

$$\begin{aligned}&\frac{1}{2s}d(a_k, Sa_k) < d(a_k, t) \quad \forall\ k \in \mathbb{N} \\ &\frac{1}{2s}d(Sa_k, S_k^2) < d(Sa_k, t) \quad \forall\ k \in \mathbb{N}\end{aligned} \tag{4.75}$$

On contrary suppose that, there is some $p \in \mathbb{N}$ which gives

$$\begin{aligned}&\frac{1}{2s}d(a_p, Sa_p) \geq d(a_p, t) \\ &or \\ &\frac{1}{2s}d(Sa_p, S_p^2) \geq d(Sa_p, t)\end{aligned} \tag{4.76}$$

Using $(F1)$ and (4.54) we get,

$$d(Sa_p, S_p^2) < d(a_p, Sa_p) \tag{4.77}$$

From (4.76) and (4.77), we get

$$\begin{aligned} d(a_p, Sa_p) &\leq sd(a_p, t) + sd(t, Sa_p) \\ &\leq \frac{1}{2}d(a_p, Sa_p) + \frac{1}{2}d(Sa_p, S_p^2) \\ &< \frac{1}{2}d(a_p, Sa_p) + \frac{1}{2}d(a_p, Sa_p) \\ &= d(a_p, Sa_p) \end{aligned} \tag{4.78}$$

Which contradicts our supposition. This implies (4.75) holds. As S is F-Suzuki type contraction,then for every $k \in \mathbb{N}$ (4.75) gives,

$$\tau + F(d(Sa_k, St)) \leq \lambda F(d(a_k, t)) + \mu F(d(a_k, Sa_k)) + \nu F(d(t, St)) \tag{4.79}$$

or

$$\tau + F(d(S^2a_k, St)) \leq \lambda F(d(Sa_k, t)) + \mu F(d(Sa_k, S^2a_k)) + \nu F(d(t, St)) \tag{4.80}$$

Applying $(F2)$ on (4.66) and (4.74), we get

$$\lim_{k\to\infty} F(d(a_k, t)) = -\infty, \quad \lim_{k\to\infty} F(d(a_k, Sa_k)) = -\infty \tag{4.81}$$

Taking $\lim_{k\to\infty}$ in (4.79), we get

$$\lim_{k\to\infty} F(d(Sa_k, St)) = -\infty \tag{4.82}$$

Applying $(F2)$, we get

$$\lim_{k\to\infty} d(Sa_k, St)) = 0 \tag{4.83}$$

Now using triangular inequality, we get

$$\begin{aligned} d(t, St) &\leq s[d(t, Sa_k) + d(Sa_k, St)] \\ &= sd(t, a_{k+1}) + sd(Sa_k, St) \end{aligned} \tag{4.84}$$

Taking $k \to \infty$ in (4.84) and using (4.74) and (4.83) in it, we get

$$d(t, St) = 0 \quad \Rightarrow \quad t = St$$

Hence t is fixed point of S.
Now, we discuss the second possibility of (4.79).

$$\begin{aligned} F(d(S^2a_k, St)) &< \tau + F(d(S^2a_k, St)) \\ &\leq \lambda F(d(Sa_k, t)) + \mu F(d(Sa_k, S^2a_k)) + \nu F(d(t, St)) \\ &= \lambda F(d(a_{k+1}, t)) + \mu F(d(a_{k+1}, Sa_{k+1})) + \nu F(d(t, St)) \end{aligned} \tag{4.85}$$

Using (4.66), we have

$$\lim_{k\to\infty} F(d(S^2a_k, St)) = -\infty \tag{4.86}$$

Applying $(F2)$, we have

$$\lim_{k\to\infty} d(S^2a_k, St) = 0 \tag{4.87}$$

Using triangular inequality, we get

$$\begin{aligned} d(t, St) &\leq s[d(t, S^2a_k) + d(S^2a_k, St)] \\ &= sd(t, a_{k+2}) + sd(S^2a_k, St) \end{aligned} \tag{4.88}$$

Taking $k \to \infty$ in (4.84) and using (4.74) and (4.83) we get

$$d(t, St) = 0 \quad \Rightarrow \quad t = St$$

Hence t is fixed point of S.

□

4.3 Conclusion:

We have reconsidered the ideas and concepts of F-contraction and b-metric space introduced by Wardwoski [46] and Bakhtin [8] respectively. We have extended some fixed point results on metric spaces. We extended these results in the setting of b-metric space which is the generalized of a metric space. For these extended results, we have introduced the notion of F-contraction and F-Suzuki contraction for b-metric space and then established certain fixed point results for such contraction. Our results generalized the results given in [33].
The results in this study may proved to be useful in solving different problems in the complete b-metric spaces.

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