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Best proximity Point Results for Generalized Θ-Contractions in b-Metric Spaces

by

Shaista Batool

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

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I dedicate my dissertation work to my beloved ***family***

CERTIFICATE OF APPROVAL

Best proximity Point Results for Generalized Θ-Contractions in b-Metric Spaces

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Abstract

The concept of best proximity point in metric space under specific contraction mappings is demonstrated by many researchers . In the present dissertation the notion of Θ-contraction and some best proximity point results for such contractions in the setting of b-metric spaces has been established. The Θ-contraction played an important role in the extension and generalization of Banach contraction principle. Our results will be valuable in solving particular best proximity points and fixed point results in the setting of b-metric spaces.

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Abbreviations

BCP	Banach contraction principle
BPP	Best proximity point
BPPR for GΘC	Best proximity point results for generalized Θ-contractions
WPP	Weak P-property

Symbols

$(\mathbb{W}, \mathcal{D})$	Metric space
$(\mathbb{W}, \mathcal{D}_b)$	b-Metric space
$\mathcal{D}$	Distance function
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\Rightarrow$	Implies that
$\in$	Belong to
$\rightarrow$	Approaches to
$\forall$	For all
$\sum$	Sigma
∞	Infinity
max	Maximum
min	Minimum
inf	Infimum
lim	Limit
ϕ	Non empty set

Chapter 1

Introduction

1.1 Background

Mathematics is an essential subject of scientific knowledge with numerous applications in all aspects of life. Because mathematics plays a vital part in scientific understanding, it is referred to as the mother of sciences. It is further subdivided into various divisions, among which functional analysis is regarded as one of the most significant fields of mathematics. In functional analysis, fixed point theory plays an important role. Fixed point theory establishes the necessary conditions for the existence of a solution of certain problem. The concept of fixed point theory has numerous applications in various fields of science, including optimization theory, mathematical economics, and approximation theory etc. Fixed point theory has become the most exciting and rapidly increasing research area of mathematics in the last five to six decades.

Poincare [1] was one of the first mathematicians who studied the topic of fixed point theory in 1886. Brouwer [2] then examined the fixed point problem and established fixed point theorems for solving the equation $\mathbb{T}(g) = g$. He also demonstrated multiple fixed point results in various dimensions.

Fixed point theory revolves around metric space. Frechet [3], a French mathematician, was the first to propose the notion of metric space. He is considered as the father of modern topology, and he made fundamental contributions to set theory and functional theory.

In 1922, Stefan Banach [4] demonstrated a significant result known as the Banach contraction principle (BCP). The analysis of BCP is often regarded as the most fundamental result in the field of fixed point theory. This idea gives rise to the two key points. The first is that it ensures the existence and uniqueness of the fixed point. The second and most essential is that it evolved a method for determining the fixed point of mapping.

Kannan [5] and Chatterjea [6] examined BCP further by replacing contraction conditions. Many researchers in the literature of fixed point theory used various strategies for extension and generalization of BCP, such as employing different spaces or modifying the contraction conditions.

Bakhtin [7] launched an investigation into one of the most intriguing generalizations of metric spaces known as b-metric spaces, and he extended the BCP in the setting of b-metric spaces. Many researchers extended fixed point theory by using multivalued mappings in b-metric spaces [8–11].

Ma et al. [12] introduced the notion of C*-valued contraction mappings. Batul et al. [13] generalized the notion of C*-valued contraction mappings by weakening the contractive condition introduced by Ma et al. Shehwar et al. [14] present the extension of Caristi's fixed point theorem mappings defined on C*-algebra valued metric spaces and proved the existence of fixed point using the concept of minimal element in C*-algebra valued metric space by introducing the notion of partial order on X.

Prešiȁ [15] established a contractive condition on the finite product of metric spaces and proved a fixed point theorem. The analysis of Prešiȁ is regarded as one of the most important extensions of BCP for operators specified on product spaces. Many additional scholars, like Berinde et al. [16], Khan et al. [17], and Shukla et al. [18, 19], focused on other types of Prešiȁ findings.

Aside from differential equations, some issues in various branches of mathematics, such as optimization theory, can be expressed as a fixed point equation of the type $\mathbb{T}(g) = g$. If $\mathbb{T}$ is self mapping and other requirements are met, the preceding equation finds a solution. However, if $\mathbb{T}$ is not a self mapping and $\mathbb{U}, \mathbb{V}$ two non-empty subsets of a metric space $(\mathbb{W}, \mathcal{D})$, then for the existence of a fixed point it is necessary that $\mathbb{T}(\mathbb{U}) \cap \mathbb{U} \neq \phi$. If this does not hold, $\mathcal{D}(g, \mathbb{T}g) > 0$ for each $g \in \mathbb{U}$ that is $\mathcal{D}(g, \mathbb{T}g)$ cannot be zero. The best approximation theory has been developed in this sense.

In 1969, Fan [20] proposed the idea of best proximity point (BPP) results for non self continuous mappings. Many researchers have used various ways to investigate the presence of the BPP in the literature. The existence and convergence of BPP is attractive feature of optimization theory and it has pulled the consideration of a lot of mathematician. Basha [21] provided the notion and extended BCP for the existence of BPP in 2010, as well as stated certain results for proximal contraction outcomes. Jleli and Samet [22] discussed the nature of BPP using generalized α-ψ-proximial contractions in complete metric spaces. Hussain et al. [23] established the BCP for modified Suzuki α-proximal contractions in the setting of complete metric spaces.

In this thesis we develop a detailed study of best proximity point results for generalized Θ-contraction presented by Ma et al. [24]. After the comprehensive analysis of the paper, results have been extended in setting of b-metric spaces.

The rest of the dissertation is organized as follows:

- **Chapter 2**

  The fundamental concepts, definitions, and examples of metric spaces, b-metric spaces, fixed point, and BPP are covered in this chapter.

- **Chapter 3**

  This chapter is about the study of BPP results for generalized Θ-contraction in metric spaces. In this chapter, we first introduce the notion of weak P-property and α-proximal admissibility. Then we introduced the Ćirić type α-ψ-Θ-contraction and prove some BPP results in complete metric space. This chapter also contain some examples to elaborate the results.

- **Chapter 4**

  This chapter emphasizes on the idea of generalized Θ-contraction in b-metric spaces. We first introduce the notion of weak P-property and α-proximal admissible in the setting of b-metric space. Then we introduced the Ćirić type α-ψ-Θ-contraction in the setting of b-metric space and proved some BPP results in complete metric space.

- **Chapter 5**

  The conclusion is presented in this chapter.

Chapter 2

Preliminaries

In the following chapter, we discuss some fundamental definitions, examples and results that will be used in subsequent chapters. There are three sections in this chapter. The first section contains a few definitions and examples of metric spaces. The second section of this chapter covers various b-metric space concepts and examples. The next section deals with the fixed points, contractions and α-proximal admissible mappings in metric spaces.

2.1 Metric Space

Metric is an extension of the Euclidean distance derived from the four well-known features of the Euclidean distance in mathematics. Euclidean metric determines the distance between two points on a straight line. However, distances other than straight lines, such as taxicab distances, may exist. In 1906 Frechet [3] developed the idea of metric spaces.

Definition 2.1.1. "A metric space is a pair $(\mathbb{W}, \mathcal{D})$, where $\mathbb{W}$ is a set and $\mathcal{D}$ is a metric on $\mathbb{W}$ (or distance function on $\mathbb{W}$), that is, a function define on $\mathbb{W} \times \mathbb{W}$ such that for all $x, y, z \in \mathbb{W}$

(M1) $\mathcal{D}$ is real-valued, finite and nonnegative

(M2) $\mathcal{D}(x,y) = 0$ if and only if $x = y$

(M3) $\mathcal{D}(x,y) = \mathcal{D}(y,x)$ **(Symmetry)**

(M4) $\mathcal{D}(x,y) \leq \mathcal{D}(x,z) + \mathcal{D}(z,y)$ **(Triangular inequality)**" [25]

Example 2.1.2. Consider $\mathbb{W} = \mathbb{R}$ then the mapping $\mathcal{D} : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ defined as

$$\mathcal{D}(g,h) = |g-h| \quad \text{for all } g,h \in \mathbb{W},$$

is a metric on $\mathbb{R}$ and $(\mathbb{R}, \mathcal{D})$ is a metric space.

Example 2.1.3. Suppose $\mathbb{W} = \mathbb{R}^2$ then the mapping $\mathcal{D} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\mathcal{D}(g,h) = \sqrt{(g_1-h_1)^2 + (g_2-h_2)^2} \quad \text{for all } g,h \in \mathbb{W}.$$

is a metric on $\mathbb{R}^2$ and $(\mathbb{R}^2, \mathcal{D})$ is a metric space.

Definition 2.1.4. "Let $\mathbb{W} = (\mathbb{W}, \mathcal{D})$ and $\mathbb{Z} = (\mathbb{Z}, \mathcal{D}_1)$ be metric spaces. A mapping $\mathbb{T} : \mathbb{W} \to \mathbb{Z}$ is said to be continuous at a point $x_0 \in \mathbb{W}$ if for each $\epsilon > 0$ there exist $\delta > 0$ such that

$$\mathcal{D}_1(\mathbb{T}x, \mathbb{T}x_0) \leq \epsilon \quad \text{whenever} \quad \mathcal{D}(x, x_0) < \delta.$$

$\mathbb{T}$ is said to be continuous if it is continuous at every point of $\mathbb{W}$." [25]

Example 2.1.5. Assume $\mathbb{W} = \mathbb{R}$ along with metric $\mathcal{D}(g,h) = |g-h|$. Define a self mapping $\mathbb{T}$, such that

$$\mathbb{T}(h) = h^3 \quad \text{where} \quad h \in \mathbb{W},$$

then $\mathbb{T}$ is continuous mapping.

Definition 2.1.6. "A sequence $\{x_n\}$ in a metric space $\mathbb{W} = (\mathbb{W}, \mathcal{D})$ is said to be converge or to convergent if there is an $x \in \mathbb{W}$ such that

$$\lim_{n\to\infty} \mathcal{D}(x_n, x) = 0,$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x$$

or, simply,

$$x_n \to x.$$

We say that $\{x_n\}$ converges to x or has the limit x. If $\{x_n\}$ is not convergent, it is said to be divergent." [25]

Example 2.1.7. Consider the set of real numbers $\mathbb{R}$ with metric $\mathcal{D}(g, h) = |g - h|$ then the sequence $g_n = \frac{1}{n}$ in $\mathbb{W}$ is a convergent sequence.

Definition 2.1.8. "A metric space $\mathbb{W}$ is called compact if every sequence in $\mathbb{W}$ has a convergent subsequence." [25]

Definition 2.1.9. "A sequence $\{x_n\}$ in a metric space $\mathbb{W} = (\mathbb{W}, \mathcal{D})$ is said to be Cauchy sequence if for every $\epsilon > 0$ there is an $\mathbb{N} = \mathbb{N}(\epsilon)$ such that

$$\mathcal{D}(x_n, x_m) < \epsilon \quad \text{for every } m, n > \mathbb{N}.\text{" [25]}$$

Definition 2.1.10. "If every Cauchy sequence in a metric space $\mathbb{W} = (\mathbb{W}, \mathcal{D})$ converges to a point $x \in \mathbb{W}$ then $\mathbb{W}$ is called complete metric space." [25]

Example 2.1.11. With usual metric on $\mathbb{R}$ the closed interval $[0, 1]$ is complete.

2.2 b-Metric Spaces

Bakhtin [7] proposed the notion of b-metric in 1989, that generalises the concept of metric space.

Definition 2.2.1. "Consider a nonempty set $\mathbb{W}$ with a real number $b \geq 1$. A function $\mathcal{D}_b : \mathbb{W} \times \mathbb{W} \to [0, \infty)$ is called a b-metric if it satisfies the following properties for each $x, y, z \in \mathbb{W}$, (b1) $\mathcal{D}_b(x,y) = 0 \Leftrightarrow x = y$;

(b2) $\mathcal{D}_b(x,y) = \mathcal{D}_b(y,x)$;

(b3) $\mathcal{D}_b(x,y) \leq b[\mathcal{D}_b(x,z) + \mathcal{D}_b(z,y)]$.

Then the pair $(\mathbb{W}, \mathcal{D}_b)$ is called a b-metric space." [26]

The concepts of metric space and b-metric space are identical when $b = 1$.

Example 2.2.2. Consider $\mathbb{W} = \mathbb{R}$ and a mapping $\mathcal{D}_b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{D}_b(g,h) = |g-h|^2,$$

is a b-metric on $\mathbb{R}$ with $b = 2$.

The concept of convergence, Cauchy sequence and completeness in b-metric are described as follows:

Definition 2.2.3. "Let $(\mathbb{W}, \mathcal{D}_b)$ be a b-metric space. A sequence $\{x_n\}$ in $\mathbb{W}$ is called convergent if and only if there exist $x \in \mathbb{W}$ such that

$$\mathcal{D}_b(x_n, x) \to 0 \quad \text{as} \quad n \to \infty,$$

we write

$$\lim_{n\to\infty} x_n = x." \text{ [27]}$$

Definition 2.2.4. "Let $(\mathbb{W}, \mathcal{D}_b)$ be a b-metric space. A sequence $\{x_n\}$ in $\mathbb{W}$ is called Cauchy if and only if

$$\mathcal{D}_b(x_n, x_m) \to 0 \quad \text{as} \quad m, n \to \infty." \text{ [27]}$$

Definition 2.2.5. "The b-metric space $(\mathbb{W}, \mathcal{D}_b)$ is said to be complete if every Cauchy sequence in $\mathbb{W}$ is convergent." [27]

In general b-metric space is not a continuous function.

Example 2.2.6. Consider $\mathbb{W} = \mathbb{N} \cup \{\infty\}$. A function $\mathcal{D}_b : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$

$$\mathcal{D}_b(g_1, g_2) = \begin{cases} 0 & \text{if } g_1 = g_2, \\ |\frac{1}{g_1} - \frac{1}{g_2}| & \text{if one of } g_1, g_2 \text{ is even and other is even or } \infty, \\ 5 & \text{if one of } g_1, g_2 \text{ is odd and other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

for all $g_1,\ g_2,\ g_3 \in \mathbb{W}$, we have

$$\mathcal{D}_b(g_1, g_3) \leq \tfrac{5}{2}(\mathcal{D}_b(g_1, g_2) + \mathcal{D}_b(g_2, g_3)),$$

so $(\mathbb{W}, \mathcal{D}_b)$ is a b-metric space with $b = \frac{5}{2}$.
Assume a sequence $\{a_n\} = 2n$ for each $n \in \mathbb{N}$, then

$$\mathcal{D}_b(2n, \infty) = \tfrac{1}{2n} \to 0 \text{ as } n \to \infty,$$

further

$$\lim_{n\to\infty} \mathcal{D}_b(2n, \infty) = 0,$$

but

$$\begin{aligned} \lim_{n\to\infty} \mathcal{D}_b(a_n, 1) = 2 &\not\to 5 \\ &= \mathcal{D}_b(\infty, 1). \end{aligned}$$

$\Rightarrow \mathcal{D}_b$ is discontinuous.

2.3 Fixed Point and Contractions

A wide range of problems from diverse fields of mathematics such as differential equations, discrete, and continuous systems of dynamics, can be demonstrated as a fixed point problem. The definition of fixed point and various kinds of contractions will be discussed in this section.

Definition 2.3.1. "Let $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ be mapping on a set $\mathbb{W}$. A point $x \in \mathbb{W}$ is said to be a fixed point of $\mathbb{T}$ if

$$\mathbb{T}x = x$$

that is, a point is mapped onto itself." [28]

Geometrically, if $y = \mathbb{T}x$ is a real-valued function, a fixed point of $\mathbb{T}$ is defined as, the point where the line $y = x$ intersects the graph of $\mathbb{T}$. As a result, a function may have a fixed point or not. Furthermore, the fixed point could be unique or not.

The graph mention below represents a function having three fixed point.

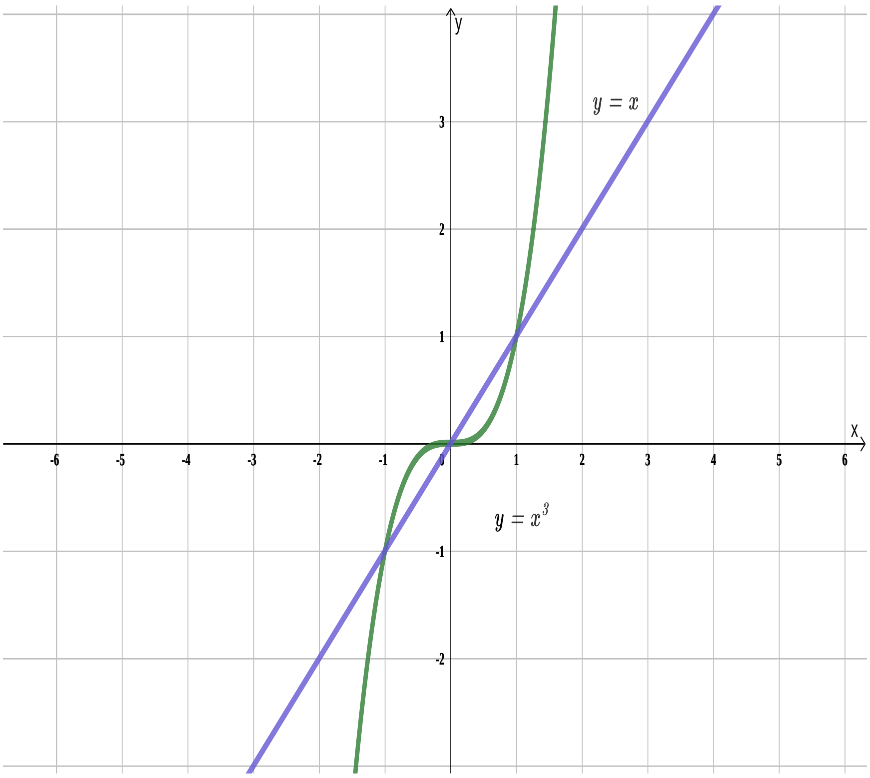

FIGURE 2.1: Three fixed point

Example 2.3.2. Assume $\mathbb{W} = \mathbb{R}$ along with metric $\mathcal{D}(g, h) = |g - h|$. Suppose a mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ is defined as

$$\mathbb{T}(g) = 2g + 1 \quad \text{for all} \quad g \in \mathbb{W}$$

then, $\mathbb{T}$ has a unique fixed point $g = -1$.

Example 2.3.3. Consider $\mathbb{W} = \mathbb{R}$ along with metric $\mathcal{D}(g, h) = |g - h|$. Suppose mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ defined as

$$\mathbb{T}(g) = g + 3 \quad \text{for all} \quad g \in \mathbb{W}$$

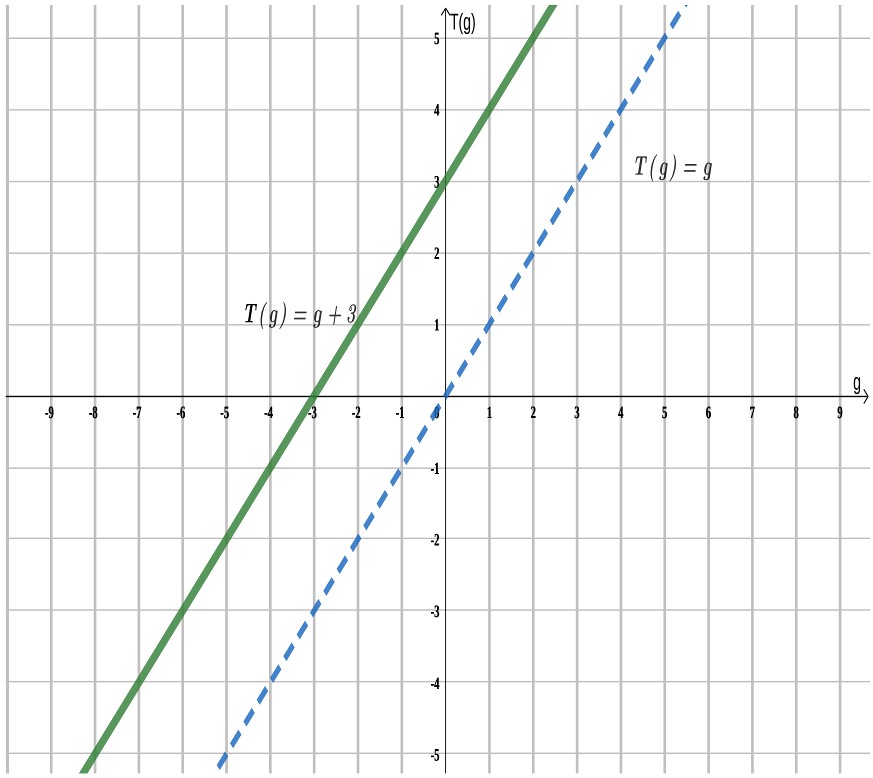

FIGURE 2.2: No fixed point

then, there is no fixed point of $\mathbb{T}$.

Definition 2.3.4. "Let $\mathbb{W} = (\mathbb{W}, \mathcal{D})$ be a metric space. A mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ is called a contraction on $\mathbb{W}$ if there is a positive real number $\alpha < 1$ such that for all $x, y \in \mathbb{W}$

$$\mathcal{D}(\mathbb{T}x, \mathbb{T}y) \leq \alpha \mathcal{D}(x, y)." \text{ [25]}$$

Example 2.3.5. Consider $\mathbb{W} = [0, 1]$ with metric $\mathcal{D}(g, h) = |g - h|$. A mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ defined as

$$\mathbb{T}(g) = \tfrac{3}{4}g.$$

is a contraction mapping, here $\alpha = \frac{3}{4}$.

The following fixed point result, known as the Banach contraction theorem, was established by Banach [25] in 1922.

Theorem 2.3.6. "Consider a metric space $\mathbb{W} = (\mathbb{W}, \mathcal{D})$, where $\mathbb{W} \neq \phi$. Suppose that $\mathbb{W}$ is complete and let $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ be a contraction on $\mathbb{W}$. Then $\mathbb{T}$ has precisely one fixed point." [25]

Example 2.3.7. Assume $\mathbb{W} = \mathbb{R}$ endowed with usual metric $\mathcal{D}(g, h) = |g - h|$. Define a mapping $\mathbb{T} : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ by

$$\mathbb{T}(g) = 1 + \tfrac{g}{3},$$

here $g = \frac{3}{2}$ is a unique fixed point of $\mathbb{T}$.

Samet et al. [29] defined α-ψ-contractive type mappings and prove different fixed point theorems for such mappings in complete metric spaces. He introduces the family Ψ of nondecreasing functions defined as follows
"For every function $\psi : [0, \infty) \to [0, \infty)$ the following holds:

(ψ_1) ψ is nondecreasing,

(ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ and $\psi(t) < t$ for any $t > 0$."

Definition 2.3.8. "Let $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ and $\alpha : \mathbb{W} \times \mathbb{W} \to [0, +\infty)$. $\mathbb{T}$ is α- admissible if for all $x, y \in \mathbb{W}$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(\mathbb{T}x, \mathbb{T}y) \geq 1." \text{ [29]}$$

Example 2.3.9. Consider $\mathbb{W} = (0, \infty)$. Define a mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ by

$$\mathbb{T}(g) = \ln g \qquad \forall\ g \in \mathbb{W}$$

and define $\alpha : \mathbb{W} \times \mathbb{W} \to [0, +\infty)$ by

$$\alpha(g,h) = \begin{cases} 2 & \text{if } g \geq h, \\ 0 & \text{if } g < h, \end{cases}$$

then, $\mathbb{T}$ is α-admissible.

Example 2.3.10. Consider $\mathbb{W} = (0, \infty)$. Define a mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ by

$$\mathbb{T}(g) = \sqrt{g} \qquad \forall \ g \in \mathbb{W}$$

and define $\alpha : \mathbb{W} \times \mathbb{W} \to [0, +\infty)$ by

$$\alpha(g,h) = \begin{cases} e^{g-h} & \text{if } g \geq h, \\ 0 & \text{if } g < h, \end{cases}$$

then, $\mathbb{T}$ is α-admissible.

Definition 2.3.11. "Let $(\mathbb{W}, \mathcal{D})$ be a metric space and $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ be a given mapping. we say that $\mathbb{T}$ is an α-ψ-contractive mapping if there exist two functions $\alpha : \mathbb{W} \times \mathbb{W} \to [0, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)\mathcal{D}(\mathbb{T}x, \mathbb{T}y) \leq \psi(\mathcal{D}(x,y)),$$

for all $x, y \in \mathbb{W}$." [29]

Jleli et al. [30] established the contractive condition in 2014 by taking the following function.
"$\Theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions,

(Θ_1) Θ is nondecreasing,

(Θ_2) for each sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1$ if and only if $\lim_{n\to\infty} (\alpha_n) = 0$,

(Θ_3) there exist $0 < m < 1$ and $l \in (0, \infty)$ such that $\lim_{\alpha\to 0^+} \frac{\Theta(\alpha)-1}{\alpha^m} = l$,"

Definition 2.3.12. "Let $(\mathbb{W}, \mathcal{D})$ be a metric space. A mapping $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ is said to be Θ-contraction such that

$$\Theta(\mathcal{D}(\mathbb{T}x, \mathbb{T}y)) \leq [\Theta(\mathcal{D}(x, y)]^m. \tag{2.1}$$

where $m \in (0, 1)$ and $x, y \in \mathbb{W}$." [30]

Jleli et al. [30] used the above contractive condition and proved the following fixed point result in complete metric space.

Theorem 2.3.13. "*Suppose that $\mathbb{T} : \mathbb{W} \to \mathbb{W}$ is a Θ-contraction, where $(\mathbb{W}, \mathcal{D})$ be a complete metric space. Then, $\mathbb{T}$ possesses a unique $x \in \mathbb{W}$ such that $\mathbb{T}x = x$.*" [24]

Instead of the condition (Θ_3), Ahmad et al. [31] used the following weaker condition and prove some fixed point results in complete metric space.

(Θ_4) Θ is continuous on $(0, \infty)$.

Throughout this thesis by Ω we mean the set of all functions Θ satisfying Θ_1, Θ_2 and Θ_4.

Chapter 3

Best Proximity Point Results for Generalized Θ-Contractions

3.1 Introduction

In this chapter, we discuss some fundamental definitions and few best proximity point results for generalized Θ-contractions (BBPR for GΘC) presented by Ma et al. [24]

3.2 Best Proximity Point in Metric Space

Many problems can be represented as an equation of the type $\mathbb{T}g = g$, where $\mathbb{T}$ denotes self-mapping in relevant domains. Consider a mapping $\mathbb{T} = \mathbb{U} \to \mathbb{V}$, define by

$$\mathbb{T}g = g,$$

here $\mathbb{T}$ is not a self mapping, the preceding equation $\mathbb{T}g = g$ has not always a fixed point. In this case, it is important to manage the estimated solution g so that the error $\mathcal{D}(g, \mathbb{T}g)$ is minimal. This study initiated the concept of BPP [32].

3.3 Some Basic Definitions

In the following section, we discuss some primary definitions and findings to be utilized in the subsequent chapter. We start by using the important notations. Let $(\mathbb{W}, \mathcal{D})$ be metric space, $\mathbb{U}$ and $\mathbb{V}$ two nonempty subsets of $\mathbb{W}$. Define

$$\begin{aligned}\mathcal{D}(\mathbb{U}, \mathbb{V}) &= \inf\{\mathcal{D}(a, b) : a \in \mathbb{U}, b \in \mathbb{V}\},\\ \mathbb{U}_0 &= \{a \in \mathbb{U} : \mathcal{D}(a, b) = \mathcal{D}(\mathbb{U}, \mathbb{V}) \text{ for some } b \in \mathbb{V}\},\\ \mathbb{V}_0 &= \{b \in \mathbb{V} : \mathcal{D}(a, b) = \mathcal{D}(\mathbb{U}, \mathbb{V}) \text{ for some } a \in \mathbb{U}\}.\end{aligned}$$

Hussain et al. [33] has introduced the following BPP notion.

Definition 3.3.1. "Let $(\mathbb{W}, \mathcal{D})$ be a metric space, $\mathbb{U}$ and $\mathbb{V}$ be two non-empty subsets of $\mathbb{W}$. An element $a \in \mathbb{U}$ is said to be best proximity point of mapping $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ if $\mathcal{D}(a, \mathbb{T}a) = \mathcal{D}(\mathbb{U}, \mathbb{V})$."

Zhang et al. [34] introduced the notion of weak P-property.

Definition 3.3.2. "Let $(\mathbb{U}, \mathbb{V})$ be a pair of nonempty subsets of a metric space $(\mathbb{W}, \mathcal{D})$ with $\mathbb{U}_0 \neq \phi$. Then the pair $(\mathbb{U}, \mathbb{V})$ is said to have the weak P-property (WPP) if and only if for any $a_1, a_2 \in \mathbb{U}_0$ and $b_1, b_2 \in \mathbb{V}_0$,

$$\begin{cases}\mathcal{D}(a_1, b_1) = \mathcal{D}(\mathbb{U}, \mathbb{V}),\\ \mathcal{D}(a_2, b_2) = \mathcal{D}(\mathbb{U}, \mathbb{V})\end{cases} \Rightarrow \mathcal{D}(a_1, a_2) \leq \mathcal{D}(b_1, b_2)."$$

In order to elaborate the above definition we have the following example.

Example 3.3.3. Consider $\mathbb{W} = \{(0, 1), (1, 0), (0, 3), (3, 0)\}$ along with a usual metric $\mathcal{D}$ on $\mathbb{R}^2$. Let

$$\begin{aligned}\mathbb{U} &= \{(0, 1), (1, 0)\},\\ \mathbb{V} &= \{(0, 3), (3, 0)\},\end{aligned}$$

then

$$\mathcal{D}(\mathbb{U}, \mathbb{V}) = 2$$

Now $\mathbb{U} = \mathbb{U}_0$ and $\mathbb{V} = \mathbb{V}_0$

$$\mathcal{D}((0,1),(0,3)) = 2 = \mathcal{D}(\mathbb{U},\mathbb{V}),$$
$$\mathcal{D}((1,0),(3,0)) = 2 = \mathcal{D}(\mathbb{U},\mathbb{V}),$$

we have

$$\mathcal{D}((0,1),(1,0)) < \mathcal{D}((0,3),(3,0)).$$

Also $\mathbb{U}_0 \neq \phi$. Thus, the pair $(\mathbb{U},\mathbb{V})$ satisfies weak P-property.

Definition 3.3.4. "Let $(\mathbb{W},\mathcal{D})$ be a metric space and $\mathbb{U},\mathbb{V}$ two subsets of $\mathbb{W}$, a non-self mapping $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ is called α-proximal admissible if

$$\begin{cases} \alpha(x_1,x_2) \geq 1, \\ \mathcal{D}(a_1,\mathbb{T}x_1) = \mathcal{D}(\mathbb{U},\mathbb{V}) \\ \mathcal{D}(a_2,\mathbb{T}x_2) = \mathcal{D}(\mathbb{U},\mathbb{V}) \end{cases} \Rightarrow \alpha(a_1,a_2) \geq 1.$$

$\forall\ x_1,x_2,a_1,a_2 \in \mathbb{U}$,where $\alpha : \mathbb{U}\times\mathbb{U} \to [0,\infty)$." [30]

Example 3.3.5. Let $\mathbb{W} = \mathbb{R}\times\mathbb{R}$. Define metric $\mathcal{D}$ as

$$\mathcal{D}((g_1,g_2),(h_1,h_2)) = |g_1 - h_1| + |g_2 - h_2|.$$

then $(\mathbb{W},\mathcal{D})$ is a metric space. Let x be any fixed positive real number, $\mathbb{U} = \{(x,g) : g \in \mathbb{R}\}$ and $\mathbb{V} = \{(0,g) : g \in \mathbb{R}\}$ then $\mathcal{D}(\mathbb{U},\mathbb{V}) = x$.
Define $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ by

$$\mathbb{T}(x,g) = \begin{cases} (0,\frac{g}{4}), & \text{if } g \geq 0, \\ (0,4g) & \text{if } g \leq 0, \end{cases}$$

and $\alpha : \mathbb{U}\times\mathbb{U} \to [0,\infty)$ by

$$\alpha((x,g),(x,h)) = \begin{cases} 2, & \text{if } g,h \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now let

$$\alpha((x, g_1), (x, g_2)) = 2. \tag{3.1}$$

$$\begin{aligned} \mathcal{D}((x, g_3), \mathbb{T}(x, g_1)) = x = \mathcal{D}(\mathbb{U}, \mathbb{V}), \\ \mathcal{D}((x, g_4), \mathbb{T}(x, g_3)) = x = \mathcal{D}(\mathbb{U}, \mathbb{V}), \end{aligned} \tag{3.2}$$

it follows from (3.1) that $g_1, g_2 \geq 0$. Further, from (3.2) $g_3 = \frac{g_1}{4}$ and $g_4 = \frac{g_2}{4}$, which implies that $g_3, g_4 \geq 0$. Hence, $\alpha((x, g_3), (x, g_4)) = 2$. Therefore $\mathbb{T}$ is an α-proximal admissible map.

3.4 Ćirić Type Contraction

Ma et al. [24] has introduced the following Ćirić type contraction to prove the BPPR in complete metric space.

Definition 3.4.1. "Let $\mathbb{U}$ and $\mathbb{V}$ be two subsets of a metric space $(\mathbb{W}, \mathcal{D})$ and $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$ be a function. A mapping $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ is said to be Ćirić type α-ψ-Θ-contraction if for $\psi \in \Psi, \Theta \in \Omega$, there exists $m \in (0, 1)$ and for $a, b \in \mathbb{U}$ with $\alpha(a, b) \geq 1$ and $\mathcal{D}(\mathbb{T}a, \mathbb{T}b) > 0$, we have

$$\alpha(a, b)\Theta[(\mathcal{D}(\mathbb{T}a, \mathbb{T}b))] \leq [\psi(\Theta(\mathbb{M}(a, b)))]^m, \tag{3.3}$$

where

$$\begin{aligned} \mathbb{M}(a, b) = \max\{\mathcal{D}(a, b), \frac{\mathcal{D}(a, \mathbb{T}a) + \mathcal{D}(b, \mathbb{T}b)}{2} - \mathcal{D}(\mathbb{U}, \mathbb{V}), \\ \frac{\mathcal{D}(a, \mathbb{T}b) + \mathcal{D}(b, \mathbb{T}a)}{2} - \mathcal{D}(\mathbb{U}, \mathbb{V})\}." \end{aligned}$$

Theorem 3.4.2. *Consider a complete metric space $(\mathbb{W}, \mathcal{D})$. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying the following conditions*

(i) *$\mathbb{T}$ is α-proximal admissible,*

(ii) *$\mathbb{T}$ is continuous,*

(iii) *$\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U}, \mathbb{V})$ satisfies the WPP,*

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}(a_1, \mathbb{T}a_0) = \mathcal{D}(\mathbb{U}, \mathbb{V})$ such that $\alpha(a_0, a_1) \geq 1$,

then, $\exists$ a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x, \mathbb{T}x) = \mathcal{D}(\mathbb{U}, \mathbb{V}).$$

Proof. Let $a_0 \in \mathbb{U}_0$, since $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$, then by assumption *(iv)* there exist an element a_1 in $\mathbb{U}_0$

$$\mathcal{D}(a_1, \mathbb{T}a_0) = \mathcal{D}(\mathbb{U}, \mathbb{V}),$$

such that $\alpha(a_0, a_1) \geq 1$.

Since $a_1 \in \mathbb{U}_0$ and $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$, there exist $a_2 \in \mathbb{U}_0$ such that

$$\mathcal{D}(a_2, \mathbb{T}a_1) = \mathcal{D}(\mathbb{U}, \mathbb{V}).$$

Since $\mathbb{T}$ is α-proximal admissible, which implies $\alpha(a_1, a_2) \geq 1$.

Similarly, since $a_2 \in \mathbb{U}_0$ and $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$, there exist $a_3 \in \mathbb{U}_0$, such that

$$\mathcal{D}(a_3, \mathbb{T}a_2) = \mathcal{D}(\mathbb{U}, \mathbb{V}),$$

and by α-proximal admissibility of $\mathbb{T}$, $\alpha(a_2, a_3) \geq 1$.

Continue this process to get $a_{n+1}, a_n \in \mathbb{U}_0$ which satisfy for all $n \in \mathbb{N}$

$$\mathcal{D}(a_{n+1}, \mathbb{T}a_n) = \mathcal{D}(\mathbb{U}, \mathbb{V}) \ \text{ and } \ \alpha(a_n, a_{n+1}) \geq 1. \tag{3.4}$$

Now suppose that $a_{n_0} = a_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then we have

$$\mathcal{D}(a_{n_0}, \mathbb{T}a_{n_0}) = \mathcal{D}(a_{n_0+1}, \mathbb{T}a_{n_0}),$$

from (3.4), we obtain

$$\mathcal{D}(a_{n_0}, \mathbb{T}a_{n_0}) = \mathcal{D}(\mathbb{U}, \mathbb{V}).$$

Hence, a_{n_0} is best proximity point of $\mathbb{T}$.

Therefore, we assume that $a_n \neq a_{n+1}$, that is $\mathcal{D}(a_n, a_{n+1}) > 0 \quad \forall\ n \in \mathbb{N} \cup \{0\}$.

$$\Rightarrow 1 < \Theta[\mathcal{D}(a_{n+1}, a_n)].$$

As Θ is nondecreasing and from weak P-property of $(\mathbb{U},\mathbb{V})$,

$$1<\Theta[\mathcal{D}(a_{n+1},a_n)]\leq\Theta[\mathcal{D}(\mathbb{T}a_n,\mathbb{T}a_{n-1})],$$

from (3.4), we have

$$1<\Theta[\mathcal{D}(a_{n+1},a_n)]\leq\alpha(a_n,a_{n-1})\Theta[\mathcal{D}(\mathbb{T}a_n,\mathbb{T}a_{n-1})],$$

and from (3.3), we have

$$1<\Theta[\mathcal{D}(a_{n+1},a_n)]\leq[\psi(\Theta(\mathbb{M}(a_n,a_{n-1})))]^m, \tag{3.5}$$

where

$$\begin{aligned}\mathbb{M}(a_n,a_{n-1})=\max\Big\{&\mathcal{D}(a_n,a_{n-1}),\frac{\mathcal{D}(a_n,\mathbb{T}a_n)+\mathcal{D}(a_{n-1},\mathbb{T}a_{n-1})}{2}\\&-\mathcal{D}(\mathbb{U},\mathbb{V}),\frac{\mathcal{D}(a_n,\mathbb{T}a_{n-1})+\mathcal{D}(a_{n-1},\mathbb{T}a_n)}{2}-\mathcal{D}(\mathbb{U},\mathbb{V})\Big\},\end{aligned}$$

by triangular inequality

$$\begin{aligned}\mathbb{M}(a_n,a_{n-1})\leq\max\Big\{&\mathcal{D}(a_n,a_{n-1}),\\&\frac{\mathcal{D}(a_n,a_{n+1})+\mathcal{D}(a_{n+1},\mathbb{T}a_n)+\mathcal{D}(a_{n-1},a_n)+\mathcal{D}(a_n,\mathbb{T}a_{n-1})}{2}-\mathcal{D}(\mathbb{U},\mathbb{V}),\\&\frac{\mathcal{D}(a_n,\mathbb{T}a_{n-1})+\mathcal{D}(a_{n-1},a_{n+1})+\mathcal{D}(a_{n+1},\mathbb{T}a_n)}{2}-\mathcal{D}(\mathbb{U},\mathbb{V})\Big\},\end{aligned}$$

from (3.4), we obtain

$$\begin{aligned}\mathbb{M}(a_n,a_{n-1})\leq\max\Big\{&\mathcal{D}(a_n,a_{n-1}),\\&\frac{\mathcal{D}(a_n,a_{n+1})+\mathcal{D}(\mathbb{U},\mathbb{V})+\mathcal{D}(a_{n-1},a_n)+\mathcal{D}(\mathbb{U},\mathbb{V})}{2}-\mathcal{D}(\mathbb{U},\mathbb{V}),\\&\frac{\mathcal{D}(\mathbb{U},\mathbb{V})+\mathcal{D}(a_{n-1},a_{n+1})+\mathcal{D}(\mathbb{U},\mathbb{V})}{2}-\mathcal{D}(\mathbb{U},\mathbb{V})\Big\},\\=\max\Big\{&\mathcal{D}(a_n,a_{n-1}),\\&\frac{\mathcal{D}(a_n,a_{n+1})+\mathcal{D}(\mathbb{U},\mathbb{V})+\mathcal{D}(a_{n-1},a_n)+\mathcal{D}(\mathbb{U},\mathbb{V})-2\mathcal{D}(\mathbb{U},\mathbb{V})}{2},\\&\frac{\mathcal{D}(\mathbb{U},\mathbb{V})+\mathcal{D}(a_{n-1},a_{n+1})+\mathcal{D}(\mathbb{U},\mathbb{V})-2\mathcal{D}(\mathbb{U},\mathbb{V})}{2}\Big\},\end{aligned}$$

$$\Rightarrow \mathbb{M}(a_n, a_{n-1}) \leq \max\left\{\mathcal{D}(a_n, a_{n-1}), \frac{\mathcal{D}(a_n, a_{n+1}) + \mathcal{D}(a_{n-1}, a_n)}{2}, \frac{\mathcal{D}(a_{n-1}, a_{n+1})}{2}\right\},$$

again by triangular inequality

$$\mathbb{M}(a_n, a_{n-1}) \leq \max\left\{\mathcal{D}(a_n, a_{n-1}), \frac{\mathcal{D}(a_n, a_{n+1}) + \mathcal{D}(a_{n-1}, a_n)}{2}, \frac{\mathcal{D}(a_n, a_{n+1}) + \mathcal{D}(a_{n-1}, a_n)}{2}\right\}$$
$$= \max\left\{\mathcal{D}(a_n, a_{n-1}), \frac{\mathcal{D}(a_n, a_{n+1}) + \mathcal{D}(a_{n-1}, a_n)}{2}\right\},$$

$$\Rightarrow M(a_n, a_{n-1}) \leq \max\left\{\mathcal{D}(a_n, a_{n-1}), \mathcal{D}(a_n, a_{n+1})\right\},$$

using above inequality in (3.5) gives

$$1 < \Theta[\mathcal{D}(a_n, a_{n+1})] \leq [\psi(\Theta(\max\{\mathcal{D}(a_n, a_{n-1}), \mathcal{D}(a_n, a_{n+1})\}))]^m, \tag{3.6}$$

if

$$\max\{\mathcal{D}(a_n, a_{n-1}), \mathcal{D}(a_n, a_{n+1})\} = \mathcal{D}(a_n, a_{n+1}),$$

then, inequality (3.6) yield

$$1 < \Theta[\mathcal{D}(a_n, a_{n+1})] \leq [\psi(\Theta(\mathcal{D}(a_n, a_{n+1}))]^m,$$

since $\psi(t) < t$ and $m \in (0, 1)$ so the above inequality becomes

$$1 < \Theta[\mathcal{D}(a_n, a_{n+1})] < (\Theta(\mathcal{D}(a_n, a_{n+1})).$$

which is a contradiction therefore

$$\max\{\mathcal{D}(a_n, a_{n-1}), \mathcal{D}(a_n, a_{n+1})\} = \mathcal{D}(a_n, a_{n-1}),$$

(3.6) yields

$$\begin{aligned} 1 < \Theta[\mathcal{D}(a_n, a_{n+1})] \leq & [\psi(\Theta(\mathcal{D}(a_{n-1}, a_n))]^m \\ \leq & [\psi(\Theta(\mathcal{D}(a_{n-2}, a_{n-1}))]^{m^2} \\ \leq & [\psi(\Theta(\mathcal{D}(a_{n-3}, a_{n-2}))]^{m^3} \\ & \vdots \\ \leq & [\psi(\Theta(\mathcal{D}(a_0, a_1))]^{m^n}. \end{aligned}$$

letting $n \to \infty$

$$\lim_{n\to\infty} 1 < \lim_{n\to\infty} [\Theta(\mathcal{D}(a_n, a_{n+1}))] \leq \lim_{n\to\infty} [\psi(\Theta(\mathcal{D}(a_0, a_1))]^{m^n},$$

where $m \in (0, 1)$, which implies

$$1 < \lim_{n\to\infty} [\Theta(\mathcal{D}(a_n, a_{n+1}))] \leq 1,$$

therefore we get

$$\Theta[\mathcal{D}(a_n, a_{n+1})] \to 1,$$

and by using (Θ_3), we obtain

$$\lim_{n\to\infty} \mathcal{D}(a_n, a_{n+1}) = 0. \tag{3.7}$$

Now we prove that $\{a_n\}$ is a Cauchy sequence in $\mathbb{U}$ we assume $\{a_n\}$ is not a Cauchy sequence in $\mathbb{U}$, then $\exists\ \epsilon > 0$ for all $j,\ k \in \mathbb{N}$ such that $j > k > n$, we have

$$\mathcal{D}(a_j, a_k) \geq \epsilon,$$

then,

$$\mathcal{D}(a_{j-1}, a_k) < \epsilon. \tag{3.8}$$

Thus, by the use of triangular inequality and (3.8), we get

$$\epsilon \leq \mathcal{D}(a_j, a_k) \leq \mathcal{D}(a_j, a_{j-1}) + \mathcal{D}(a_{j-1}, a_k) < \mathcal{D}(a_j, a_{j-1}) + \epsilon,$$

applying limit

$$\epsilon \leq \lim_{j,k\to\infty} \mathcal{D}(a_j, a_k) < \lim_{j\to\infty} \mathcal{D}(a_j, a_{j-1}) + \epsilon.$$

Using (3.7)

$$\epsilon \leq \lim_{j,k\to\infty} \mathcal{D}(a_j, a_k) < \epsilon.$$

$$\Rightarrow \lim_{j,k\to\infty} \mathcal{D}(a_j, a_k) = \epsilon, \tag{3.9}$$

by using triangular inequality, we get

$$\mathcal{D}(a_j, a_k) \leq \mathcal{D}(a_j, a_{j+1}) + \mathcal{D}(a_{j+1}, a_{k+1}) + \mathcal{D}(a_{k+1}, a_k),$$

letting limit as $j, k \to \infty$ and from (3.7)

$$\lim_{j,k\to\infty} \mathcal{D}(a_j, a_k) \leq \lim_{j,k\to\infty} \mathcal{D}(a_{j+1}, a_{k+1}) \tag{3.10}$$

and again by triangular inequality

$$\mathcal{D}(a_{j+1}, a_{k+1}) \leq \mathcal{D}(a_{j+1}, a_j) + \mathcal{D}(a_j, a_k) + \mathcal{D}(a_k, a_{k+1}).$$

Applying limit as $j, k \to \infty$ and from (3.7) and (3.9),

$$\begin{aligned}\lim_{j,k\to\infty} \mathcal{D}(a_{j+1}, a_{k+1}) &\leq \lim_{j,k\to\infty} \mathcal{D}(a_{j+1}, a_j) + \lim_{j,k\to\infty} \mathcal{D}(a_j, a_k) + \lim_{j,k\to\infty} \mathcal{D}(a_k, a_{k+1}) \\ &= 0 + \epsilon + 0.\end{aligned}$$

therefore we get

$$\lim_{j,k\to\infty} \mathcal{D}(a_{j+1}, a_{k+1}) = \epsilon.$$

Thus, (3.9) holds. Then by weak P- property of $(\mathbb{U}, \mathbb{V})$

$$1 \leq \Theta(\mathcal{D}(a_{j+1}, a_{k+1})) \leq \Theta(\mathcal{D}(\mathbb{T}a_j, \mathbb{T}a_k)),$$

by assumption, $\alpha(a_j, a_k) \geq 1$, we have that

$$1 \leq \Theta(\mathcal{D}(a_{j+1}, a_{k+1})) \leq \alpha(a_j, a_k)\Theta(\mathcal{D}(\mathbb{T}a_j, \mathbb{T}a_k)),$$

since $\mathbb{T}$ is Θ-contraction which implies

$$\begin{aligned}1 \leq \Theta(\mathcal{D}(a_{j+1}, a_{k+1})) &\leq [\psi(\Theta(\mathbb{M}(a_j, a_k)))]^m \\ &< \Theta(\mathbb{M}(a_j, a_k)).\end{aligned}$$

By proceeding as (3.5) till (3.7), taking limit as $j, k \to \infty$ in above inequality and using (Θ_4), we have that

$$\lim_{j,k\to\infty} \mathcal{D}(a_{j+1}, a_{k+1}) = 0,$$

which implies

$$\lim_{j,k\to\infty} \mathcal{D}(a_j, a_k) = 0 < \epsilon \tag{3.11}$$

Hence, this implies that $\{a_n\}$ is a Cauchy sequence in $\mathbb{U}$. Since $\mathbb{W}$ is complete and $\mathbb{U}$ is closed, then $\exists\, x \in \mathbb{U}$ such that $a_n \to x$ and continuity of $\mathbb{T}$ implies $\mathbb{T}a_n \to \mathbb{T}x$. So, from equation (3.4) we get

$$\begin{aligned}\mathcal{D}(\mathbb{U}, \mathbb{V}) &= \lim_{n\to\infty} \mathcal{D}(a_{n+1}, \mathbb{T}a_n), \\ \Rightarrow \mathcal{D}(\mathbb{U}, \mathbb{V}) &= \mathcal{D}(x, \mathbb{T}x).\end{aligned}$$

This complete the proof. □

Following examples illustrate the above result.

Example 3.4.3. Assume $\mathbb{W} = \mathbb{R}^2$ with metric $\mathcal{D}$ defined as

$$\mathcal{D}((g_1, g_2), (h_1, h_2)) = |g_1 - h_1| + |g_2 - h_2|.$$

Suppose

$$\begin{aligned}\mathbb{U} &= \{(-13, -10), (-4, -4), (-7, -8), (20, 0), (25, 30)\} \\ \mathbb{V} &= \{(-13, -6), (-11, -8), (-9, -10), (-4, 0), (0, -4)\},\end{aligned}$$

be a two non-empty subsets of $\mathbb{W}$. Now,

$$\begin{aligned}
\mathcal{D}(\mathbb{U},\mathbb{V}) =& \inf\{\mathcal{D}(g,h) : g \in \mathbb{U}, h \in \mathbb{V}\},\\
=& \inf\{\mathcal{D}((-13,-10),(-13,-6)), \mathcal{D}((-13,-10),(-11,-8)),\\
& \mathcal{D}((-13,-10),(-9,-10)), \mathcal{D}((-13,-10),(-4,0)), \mathcal{D}(((-13,-10),\\
& (0,-4)), \mathcal{D}((-4,-4),(-13,-6)), \mathcal{D}((-4,-4),(-11,-8)),\\
& \mathcal{D}((-4,-4),(-9,-10)), \mathcal{D}((-4,-4),(-4,0)), \mathcal{D}((-4,-4),(0,-4)),\\
& \mathcal{D}((-7,-8),(-13,-6)), \mathcal{D}((-7,-8),(-11,-8)), \mathcal{D}((-7,-8),(-9,-10)),\\
& \mathcal{D}((-7,-8),(-4,0)), \mathcal{D}((-7,-8),(0,-4)), \mathcal{D}((20,0),(-13,-6)),\\
& \mathcal{D}((20,0),(-11,-8)), \mathcal{D}((20,0),(-9,-10)), \mathcal{D}((20,0),(-4,0)),\\
& \mathcal{D}((20,0),(0,-4)), \mathcal{D}_b((25,30),(-13,-6)), \mathcal{D}((25,30),(-11,-8)),\\
& \mathcal{D}((25,30),(-9,-10)), \mathcal{D}((25,30),(-4,0)), \mathcal{D}((25,30),(0,-4))\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}_b(\mathbb{U},\mathbb{V}) =& \inf\{|0|+|4|, |2|+|2|, |4|+|0|, |9|+|10|, |13|+|6|,\\
& |9|+|2|, |7|+|4|, |5|+|6|, |0|+|4|, |4|+|0|,\\
& |6|+|2|, |4|+|0|, |2|+|2|, |3|+|8|, |7|+|4|,\\
& |33|+|6|, |31|+|8|, |29|+|10|, |24|+|0|, |20|+|4|,\\
& |38|+|36|, |36|+|38|, |34|+|40|, |29|+|30|, |25|+|34|\}\\
=& \inf\{4,4,4,19,19,11,11,11,4,4,8,4,4,11,11,39,39,39,\\
& 24,24,74,74,74,54,54\}.\\
=& 4
\end{aligned}$$

and

$$\mathbb{U}_0 = \{(-4,-4),(-7,-8)\}.$$
$$\mathbb{V}_0 = \{(-4,0),(0,-4),(-9,-10),(-11,-8)\}.$$

Define $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ by

$$\begin{aligned}
\mathbb{T}(-4,-4) &= (-9,-10).\\
\mathbb{T}(-7,-8) &= (-11,-8).\\
\mathbb{T}(-13,-10) &= (-13,-6).\\
\mathbb{T}(20,0) &= (-4,0).\\
\mathbb{T}(25,30) &= (0,-4),
\end{aligned}$$

and $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$ by $\alpha((g,h),(r,s)) = \frac{11}{10}$.

As $\mathbb{T}(-4,-4) = (-9,-10) \in \mathbb{V}_0$ and $\mathbb{T}(-7,-8) = (-11,-8) \in \mathbb{V}_0$. So it is clear from the mapping $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$. Suppose $(-4,-4),(-7,-8) \in \mathbb{U}_0$ and $(-4,0),(-9,-10) \in \mathbb{V}_0$, such that

$$\begin{cases} \mathcal{D}((-4,-4),(-4,0)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 4, \\ \mathcal{D}((-7,-8),(-9,-10)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 4, \end{cases}$$

$\Rightarrow \mathcal{D}((-4,-4),(-7,-8)) < \mathcal{D}((-4,0),(-9,-10))$.

Similarly, for all $(g_1,h_1),(g_2,h_2) \in \mathbb{U}$ and $(r_1,s_1),(r_2,s_2) \in \mathbb{V}$

$$\begin{cases} \mathcal{D}((g_1,h_1),(r_1,s_1)) = \mathcal{D}(\mathbb{U},\mathbb{V}), \\ \mathcal{D}((g_2,h_2),(r_2,s_2)) = \mathcal{D}(\mathbb{U},\mathbb{V}), \end{cases}$$

$\Rightarrow \mathcal{D}((g_1,h_1),(g_2,h_2)) < \mathcal{D}((r_1,s_1),(r_2,s_2))$

which satisfy $(\mathbb{U},\mathbb{V})$ has WPP. Now, to prove α-proximal admissible we proceed as follows

$$\begin{cases} \alpha((-7,-8),(20,0)) \geq 1, \\ \mathcal{D}((-9,-10),(-11,-8)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 4, \\ \mathcal{D}((-4,-4),(-4,0)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 4, \end{cases}$$

$\Rightarrow \alpha((-9,-10),(-4,-4)) = \frac{11}{10} > 1$.

Hence, $\alpha((g,h),(r,s)) \geq 1$ for all $g,h,r,s \in \mathbb{U}$. Which means that $\mathbb{T}$ is α-proximal admissible. Now, to demonstrate α-ψ-Θ-contraction. Define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{999}{1000}t, \tag{3.12}$$

and $\Theta : (0,\infty) \to (1,\infty)$ by

$$\Theta(t) = t + 1. \tag{3.13}$$

Let for $((-4,-4),(20,0)) \in \mathbb{U}$ we have to prove the following inequality.

$$\alpha((-4,-4),(20,0))\Theta[(\mathcal{D}(\mathbb{T}(-4,-4),\mathbb{T}(20,0))] \leq [\psi(\Theta(\mathbb{M}(-4,-4),(20,0))))]^m \tag{3.14}$$

Now, considering left hand side of the above inequality.

$$\begin{aligned}
&\alpha((-4,-4),(20,0))\Theta[(\mathcal{D}(\mathbb{T}(-4,-4),\mathbb{T}(20,0))]\\
&=\alpha((-4,-4),(20,0))\Theta[(\mathcal{D}(-9,-10),\mathcal{D}(-4,0))]\\
&=\alpha((-4,-4),(20,0))\Theta[|5|+|10|]\\
&=\alpha((-4,-4),(20,0))\Theta(15),
\end{aligned}$$

using (3.13), we get

$$\begin{aligned}
\alpha((-4,-4),(20,0))\Theta[(\mathcal{D}(\mathbb{T}(-4,-4),\mathbb{T}(20,0))] &= \alpha((-4,-4),(20,0))(15+1),\\
&=\frac{11}{10}(16)\\
&=\frac{88}{5},
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
\mathbb{M}((-4,-4),(20,0)) = \max\Big\{&\mathcal{D}((-4,-4),(20,0)),\\
&\frac{\mathcal{D}((-4,-4),\mathbb{T}(-4,-4))+\mathcal{D}((20,0),\mathbb{T}(20,0))}{2}-\mathcal{D}(\mathbb{U},\mathbb{V}),\\
&\frac{\mathcal{D}((-4,-4),\mathbb{T}(20,0))+\mathcal{D}((20,0),\mathbb{T}(-4,-4))}{2}-\mathcal{D}(\mathbb{U},\mathbb{V})\Big\}.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \mathbb{M}((-4,-4),(20,0)) &= \max\Big\{28, \frac{\mathcal{D}((-4,-4),(-9,-10))+\mathcal{D}((20,0),(-4,0))}{2}-4,\\
&\qquad \frac{\mathcal{D}((-4,-4),(-4,0))+\mathcal{D}((20,0),(-9,-10))}{2}-4\Big\}.\\
&=\max\Big\{28, \frac{35}{2}-4, \frac{43}{2}-4\Big\}.\\
&=\max\Big\{28, \frac{27}{2}, \frac{35}{2}\Big\}.\\
&=28
\end{aligned}$$

using (3.12) and (3.13), we have

$$\begin{aligned}
\big[\psi(\Theta(M((-4,-4),(20,0))))\big]^m &= \big[\psi(\Theta(28))\big]^m\\
&=\big[\psi(29)\big]^m\\
&=\left[\frac{999}{1000}(29)\right]^m.
\end{aligned} \tag{3.16}$$

From (3.15), (3.16) and for $m = 0.83$, we have

$$\frac{88}{5} < \left[\frac{999}{1000}(29)\right]^m,$$

which means that $\mathbb{T}$ is Ćirić type α-ψ-Θ contraction. Similarly inequality (3.14) holds for remaining cases.

To demonstrate that $\mathbb{T}$ has unique best proximity point, we proceed as follows

$$\begin{aligned}\mathcal{D}(x, \mathbb{T}x) =& \mathcal{D}((-7,-8), \mathbb{T}(-7,-8))\\ =& \mathcal{D}((-7,-8),(-11,-8))\\ =& 4\\ =& \mathcal{D}(\mathbb{U}, \mathbb{V}).\end{aligned}$$

Therefore, all axioms are true. Hence, $\mathbb{T}$ has a BPP (-7, -8).

Example 3.4.4. Consider $\mathbb{W} = \mathbb{R}^2$ with metric $\mathcal{D}$ defined as

$$\mathcal{D}((g_1, g_2),(h_1, h_2)) = |g_1 - h_1| + |g_2 - h_2|.$$

Suppose

$$\begin{aligned}\mathbb{U} =& \{1\} \times [0, \infty)\\ \mathbb{V} =& \{0\} \times [0, \infty),\end{aligned}$$

then

$$\mathcal{D}(\mathbb{U}, \mathbb{V}) = \mathcal{D}((1,0),(0,0)) = 1,$$

and $\mathbb{U}_0 = \mathbb{U}, \mathbb{V}_0 = \mathbb{V}$. Define $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ by

$$\mathbb{T}(1, g) = \begin{cases} (0, \frac{g}{3}) & \text{if } g \in [0,1], \\ (0, g - \frac{2}{3}) & \text{if } g > 1. \end{cases}$$

and $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$ by

$$\alpha((g,h),(r,s)) = \begin{cases} 1, & \text{if}(g,h),(r,s) \in [0,1] \times [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

As $\mathbb{T}(1,g) = (0,g) \in \mathbb{V}_0$ so $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$. Suppose $(1,g_1),(1,g_2) \in \mathbb{U}_0$ and $(0,r_1),(0,r_2) \in \mathbb{V}_0$, such that

$$\begin{cases} \mathcal{D}((1,g_1),(0,r_1)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 1, \\ \mathcal{D}((1,g_2),(0,r_2)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 1, \end{cases} \Rightarrow \mathcal{D}((1,g_1),(1,g_2)) = \mathcal{D}((0,r_1),(0,r_2))$$

Necessarily, $(g_1 = r_1 \in [0,1])$ and $(g_2 = r_2 \in [0,1])$. Which means that $(\mathbb{U},\mathbb{V})$ has WPP.

To demonstrate $\mathbb{T}$ is α-proximal admissible we suppose,

$$\begin{cases} \alpha((1,g_1),(1,g_2)) \geq 1, \\ \mathcal{D}((1,r_1),\mathbb{T}(1,g_1)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 1, \\ \mathcal{D}((1,r_2),\mathbb{T}(1,g_2)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 1, \end{cases}$$

then

$$\begin{cases} (1,g_1),(1,g_2) \in [0,1], \\ \mathcal{D}((1,r_1),\mathbb{T}(1,g_1)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 1, \\ \mathcal{D}((1,r_2),\mathbb{T}(1,g_2)) = \mathcal{D}(\mathbb{U},\mathbb{V}) = 1. \end{cases}$$

Thus, $(g_1,g_2) \in [0,1] \times [0,1]$. Further, $r_1 = \frac{g_1}{3}$ and $r_2 = \frac{g_2}{3}$, which implies that $\alpha(\,(1,\,r_1),\,(1,\,r_2)) \geq 1$. Therefore $\mathbb{T}$ is an α-proximal admissible mapping.

Now, to prove that $\mathbb{T}$ is α-ψ-Θ-contraction. Define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{999}{1000}t, \tag{3.17}$$

and $\Theta : (0,\infty) \to (1,\infty)$ by

$$\Theta(t) = t + 1. \tag{3.18}$$

Let for $((1,g),(1,h)) \in \mathbb{U}$, we have to prove the following α-ψ-Θ-contraction

$$\alpha((1,g),(1,h))\Theta[(\mathcal{D}(\mathbb{T}(1,g),\mathbb{T}(1,h))] \leq [\psi(\Theta(\mathbb{M}(1,g),(1,h))))]^m \tag{3.19}$$

where $m \in (0,1)$. Considering the left side of inequality (3.19) which gives

$$\begin{aligned}
&\alpha((1,g),(1,h))\Theta[(\mathcal{D}(\mathbb{T}(1,g),\mathbb{T}(1,h))]\\
&= \alpha((1,g),(1,h))\Theta\left[\left(\mathcal{D}((0,\frac{g}{3}),(0,\frac{h}{3})\right)\right]\\
&= \Theta\left[\left|\frac{g}{3}-\frac{h}{3}\right|\right]\\
&= \frac{|g-h|}{3}+1.
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{M}((1,g),(1,h)) = \max\Bigg\{&\mathcal{D}((1,g),(1,h)),\\
&\frac{\mathcal{D}((1,g),\mathbb{T}(1,g))+\mathcal{D}((1,h),\mathbb{T}(1,h))}{2}-\mathcal{D}(\mathbb{U},\mathbb{V}),\\
&\frac{\mathcal{D}((1,g),\mathbb{T}(1,h))+\mathcal{D}((1,h),\mathbb{T}(1,g))}{2}-\mathcal{D}(\mathbb{U},\mathbb{V})\Bigg\},
\end{aligned}$$

using $\mathbb{T}(1,g) = (0,\frac{g}{3})$ and $\mathbb{T}(1,h) = (0,\frac{h}{3})$

$$\begin{aligned}
\mathbb{M}((1,g),(1,h)) = \max\Bigg\{&\mathcal{D}((1,g),(1,h)),\\
&\frac{\mathcal{D}((1,g),(0,\frac{g}{3}))+\mathcal{D}((1,h),(0,\frac{h}{3}))}{2}-1,\\
&\frac{\mathcal{D}((1,g),(0,\frac{h}{3}))+\mathcal{D}((1,h),(0,\frac{g}{3}))}{2}-1)\Bigg\},
\end{aligned}$$

simplifying the above equation, we get

$$\begin{aligned}
\mathbb{M}((1,g),(1,h)) &= \max\Bigg\{|g-h|,\frac{|1|+|g-\frac{g}{3}|+|1|+|h-\frac{h}{3}|}{2}-1,\\
&\qquad\frac{|1|+|g-\frac{h}{3}|+|1|+|h-\frac{g}{3}|}{2}-1\Bigg\},\\
&= \max\Bigg\{|g-h|,\frac{1+|g-\frac{g}{3}|+1+|h-\frac{h}{3}|-2}{2},\\
&\qquad\frac{1+|g-\frac{h}{3}|+1+|h-\frac{g}{3}|-2}{2}\Bigg\}.
\end{aligned}$$

$$\Rightarrow \mathbb{M}((1,g),(1,h)) = \max\left\{|g-h|, \frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}, \frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}\right\}.$$

If $\max\left\{|g-h|, \frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}, \frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}\right\} = |g-h|$, then inequality (3.19) becomes

$$\begin{aligned}\frac{|g-h|}{3}+1 &\leq [\psi(\Theta(|g-h|)))]^m \\ &= [\psi(|g-h|+1))]^m \\ &< \psi(|g-h|+1), \\ &= \frac{999}{1000}(|g-h|+1),\end{aligned}$$

which implies

$$\frac{|g-h|}{3}+1 \leq \frac{999}{1000}(|g-h|+1),$$

in this case (3.19) holds. Now, if

$$\max\left\{|g-h|, \frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}, \frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}\right\} = \frac{|\frac{g}{3}|+|\frac{h}{3}|}{2}.$$

Then inequality (3.19) becomes

$$\begin{aligned}\frac{|g-h|}{3}+1 &\leq \left[\psi\left(\Theta\left(\frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}\right)\right)\right]^m \\ &= \left[\psi\left(\frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}+1\right)\right]^m \\ &< \psi\left(\frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}+1\right),\end{aligned}$$

$$\Rightarrow \frac{|g-h|}{3}+1 < \frac{999}{1000}\left[\frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}+1\right],$$

so inequality (3.19) holds. Also if

$$\max\left\{|g-h|, \frac{|\frac{2g}{3}|+|\frac{2h}{3}|}{2}, \frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}\right\} = \frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2},$$

then inequality (3.19) becomes

$$
\begin{aligned}
\frac{|g-h|}{3}+1 &\leq \left[\psi\left(\Theta\left(\frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}\right)\right)\right]^m \\
&= \left[\psi\left(\frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}+1\right)\right]^m \\
&< \psi\left(\frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}+1\right),
\end{aligned}
$$

$$
\Rightarrow \frac{|g-h|}{3}+1 < \frac{999}{1000}\left[\frac{|g-\frac{h}{3}|+|h-\frac{g}{3}|}{2}+1\right],
$$

in case inequality (3.19) also holds. Therefore, $\mathbb{T}$ is Ćirić type α-ψ-Θ contraction. Similarly, inequality holds for remaining cases. Now, to prove $(1,0)$ is best proximity point we proceed as follow.

$$
\begin{aligned}
\mathcal{D}(x,\mathbb{T}x) =& \mathcal{D}((1,0),\mathbb{T}(1,0)) \\
&= \mathcal{D}((1,0),(0,0)) \\
&= 1
\end{aligned}
$$

this implies

$$
\mathcal{D}(x,\mathbb{T}x) = \mathcal{D}(\mathbb{U},\mathbb{V}).
$$

Hence, $\mathbb{T}$ has a BPP $(1,0)$

Ma et al. [24] introduced the following condition $\mathcal{U}$ for the uniqueness of BPP.

$(\mathcal{U})\ :\ \forall\ x,y \in BPP(\mathbb{T}),\ \alpha(x,y) \geq 1,$

where $BBP(\mathbb{T})$ denote the set of best proximity points of $\mathbb{T}$.

Theorem 3.4.5. To obtain a unique $x \in \mathbb{U}$ which satisfy

$$
\mathcal{D}(x,\mathbb{T}x) = \mathcal{D}(\mathbb{U},\mathbb{V}),
$$

we are going to add the condition $(\mathcal{U})$ to the hypotheses of Theorem *(3.4.2)*

Proof. Suppose that x and y are two BPP of $\mathbb{T}$ and $x \neq y$ that is, $\mathcal{D}(x, \mathbb{T}x) = \mathcal{D}(\mathbb{U}, \mathbb{V}) = \mathcal{D}(y, \mathbb{T}y)$. Then by condition $(\mathcal{U})$

$$\alpha(x, y) \geq 1. \tag{3.20}$$

Since the pair $(\mathbb{U}, \mathbb{V})$ has the WPP and from (3.3), we get

$$\begin{aligned}
\Theta(\mathcal{D}(x, y)) \leq \Theta(\mathcal{D}(\mathbb{T}x, \mathbb{T}y) &\leq \alpha(x, y)\Theta(\mathcal{D}(\mathbb{T}x, \mathbb{T}y)) \\
&\leq [\psi(\Theta(\mathbb{M}(x, y)))]^m \\
&= [\psi(\Theta(\mathcal{D}(x, y)))]^m \\
&< \Theta(\mathcal{D}(x, y)),
\end{aligned}$$

which is contradiction, so $x = y$.

Corollary 3.4.6. Consider a complete metric space $(\mathbb{W}, \mathcal{D})$. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\alpha(a, b)\Theta[\mathcal{D}(\mathbb{T}a, \mathbb{T}b)] \leq [\psi(\Theta(\mathcal{D}(a, b)))]^m$;

(ii) $\mathbb{T}$ is α-proximal admissible*;*

(iii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U}, \mathbb{V})$ satisfies the WPP*;*

(iv) $\mathbb{T}$ is continuous*;*

(v) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}(a_1, \mathbb{T}a_0) = \mathcal{D}(\mathbb{U}, \mathbb{V})$ such that $\alpha(a_0, a_1) \geq 1$*;*

then there exist $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x, \mathbb{T}x) = \mathcal{D}(\mathbb{U}, \mathbb{V}).$$

Proof. By proceeding as Theorem (3.4.2) the pair $(\mathbb{U}, \mathbb{V})$ has the WPP and from (3.5), we get

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}(a,b)) &\leq \Theta(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b) \\ &\leq \alpha(a,b)\Theta(\mathcal{D}(\mathbb{T}a, \mathbb{T}b)) \\ &\leq [\psi(\Theta(\mathbb{M}(a,b)))]^m. \end{aligned}$$

If $\mathbb{M}(a,b) = \mathcal{D}(a,b)$

$$1 \leq \Theta(\mathcal{D}(a,b)) \leq [\psi(\Theta(\mathcal{D}(a,b)))]^m.$$

Now, proceeding as Equation (3.6) till (3.11) there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x, \mathbb{T}x) = \mathcal{D}(\mathbb{U}, \mathbb{V}).$$

Corollary 3.4.7. *Consider a complete metric space $(\mathbb{W}, \mathcal{D})$. Suppose $\mathbb{U}, \mathbb{V} \neq \phi$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying*

(i) $\Theta[\mathcal{D}(\mathbb{T}a, \mathbb{T}b)] \leq [\psi(\Theta(\mathcal{D}(a,b)))]^m$,

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ *and the pair* $(\mathbb{U}, \mathbb{V})$ *satisfies the WPP,*

(iii) $\mathbb{T}$ *is continuous,*

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ *with* $\mathcal{D}(a_1, \mathbb{T}a_0) = \mathcal{D}(\mathbb{U}, \mathbb{V})$ *such that* $\alpha(a_0, a_1) \geq 1$,

then there exist a point $x \in \mathbb{U}$ which satisfy $\mathcal{D}(x, \mathbb{T}x) = \mathcal{D}(\mathbb{U}, \mathbb{V})$.

Proof. By proceeding as Theorem (3.4.2) the pair $(\mathbb{U}, \mathbb{V})$ has the WPP and from (3.5), we get

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}(a,b)) &\leq \Theta(\mathcal{D}(\mathbb{T}a, \mathbb{T}b) \\ &\leq \alpha(a,b)\Theta(\mathcal{D}(\mathbb{T}a, \mathbb{T}b)). \end{aligned}$$

If $\alpha(a,b) = 1 \;\forall\; a,b \in \mathbb{U}$, we obtain

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}(a,b)) &\leq \Theta(\mathcal{D}(\mathbb{T}a,\mathbb{T}b)). \\ &\leq [\psi(\Theta(\mathbb{M}(a,b)))]^m. \end{aligned}$$

Now, proceeding as Equation (3.5) till (3.11) there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x,\mathbb{T}x) = \mathcal{D}(\mathbb{U},\mathbb{V}).$$

Corollary 3.4.8. Consider a complete metric space $(\mathbb{W},\mathcal{D})$. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U},\mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\Theta[\mathcal{D}(\mathbb{T}a,\mathbb{T}b)] \leq [\psi(\Theta(\mathcal{D}(a,b)))]^m$,

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U},\mathbb{V})$ satisfies the WPP,

(iii) $\mathbb{T}$ is continuous,

(iv) $\exists\; a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}(a_1,\mathbb{T}a_0) = \mathcal{D}(\mathbb{U},\mathbb{V})$ such that $\alpha(a_0,a_1) \geq 1$,

then there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x,\mathbb{T}x) = \mathcal{D}(\mathbb{U},\mathbb{V}).$$

Proof. By proceeding as Theorem (3.4.2) the pair $(\mathbb{U},\mathbb{V})$ has the WPP and from (3.5), we get

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}(a,b)) &\leq \Theta(\mathcal{D}(\mathbb{T}a,\mathbb{T}b) \\ &\leq \alpha(a,b)\Theta(\mathcal{D}(\mathbb{T}a,\mathbb{T}b)). \end{aligned}$$

By taking $\mathbb{M}(a,b) = \mathcal{D}(a,b)$ in Corollary (3.4.7), we obtain.

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}(a,b)) &\leq \Theta(\mathcal{D}(\mathbb{T}a,\mathbb{T}b)). \\ &\leq [\psi(\Theta(\mathbb{M}(a,b)))]^m. \\ &\leq [\psi(\Theta(\mathcal{D}(a,b)))]^m. \end{aligned}$$

Now, proceeding as Equation (3.6) till (3.11) there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x,\mathbb{T}x) = \mathcal{D}(\mathbb{U},\mathbb{V}).$$

Corollary 3.4.9. Consider a complete metric space $(\mathbb{W},\mathcal{D})$. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U},\mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T}:\mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\mathcal{D}(\mathbb{T}a,\mathbb{T}b) \leq m\mathcal{D}(a,b)$,

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U},\mathbb{V})$ satisfies the WPP,

(iii) $\mathbb{T}$ is continuous,

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}(a_1,\mathbb{T}a_0) = \mathcal{D}(\mathbb{U},\mathbb{V})$ such that $\alpha(a_0,a_1) \geq 1$,

then there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x,\mathbb{T}x) = \mathcal{D}(\mathbb{U},\mathbb{V}).$$

By taking $\psi(x) = mx$ for $m \in (0,1)$ and $\Theta(x) = e^x$ in Corollary (3.4.8), we obtained the main result of Jleli et al. [22].

Corollary 3.4.10. Consider a complete metric space $(\mathbb{W},\mathcal{D})$. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U},\mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T}:\mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\mathcal{D}(\mathbb{T}a,\mathbb{T}b) \leq m\mathcal{D}(a,b)$,

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U}, \mathbb{V})$ satisfies the WPP,

(iii) $\mathbb{T}$ is continuous,

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}(a_1, \mathbb{T}a_0) = \mathcal{D}(\mathbb{U}, \mathbb{V})$ such that $\alpha(a_0, a_1) \geq 1$,

then there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}(x, \mathbb{T}x) = \mathcal{D}(\mathbb{U}, \mathbb{V}).$$

By taking $\psi(x) = mx$ for $m \in (0, 1)$ and $\Theta(x) = e^x$ in Corollary (3.4.8), we obtained the main result of Suzuki [35].

Chapter 4

Best Proximity Point Results for Generalized Θ-Contractions in b-Metric Spaces

This chapter is about the extension of best proximity point results for Θ-contractions in b-metric spaces (BPPR for GΘC in b-Metric spaces).

4.1 Best Proximity Point in b-Metric Spaces

Consider a b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$. Suppose $\mathbb{U}$, $\mathbb{V} \subseteq \mathbb{W}$, where $\mathbb{U}$, $\mathbb{V} \neq \phi$. Define

$$\mathcal{D}_b(\mathbb{U}, \mathbb{V}) = \inf\{\mathcal{D}_b(a, b) : a \in \mathbb{U},\ b \in \mathbb{V}\},$$

$$\mathbb{U}_0 = \{a \in \mathbb{U} : \mathcal{D}_b(a, b) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \text{ for some } b \in \mathbb{V}\},$$

$$\mathbb{V}_0 = \{b \in \mathbb{V} : \mathcal{D}_b(a, b) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \text{ for some } a \in \mathbb{U}\}.$$

Definition 4.1.1. Consider $(\mathbb{W}, \mathcal{D}_b)$ be a b-metric space with coefficient $s \geq 1$ and $\mathbb{U}$, $\mathbb{V} \subseteq \mathbb{W}$. An element $a \in \mathbb{U}$ is said to be BPP of mapping $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ if $\mathcal{D}_b(a, \mathbb{T}a) = \mathcal{D}_b(\mathbb{U}, \mathbb{V})$.

Definition 4.1.2. Consider a pair $(\mathbb{U}, \mathbb{V})$ be nonempty subsets of b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$ and $\mathbb{U}_0 \neq \phi$. Then the pair $(\mathbb{U}, \mathbb{V})$ is said to have a WPP if and only if for any $a_1, a_2 \in \mathbb{U}_0$ and $b_1, b_2 \in \mathbb{V}_0$.

$$\begin{cases} \mathcal{D}_b(a_1, b_1) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \\ \mathcal{D}_b(a_2, b_2) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \end{cases} \Rightarrow \mathcal{D}_b(a_1, a_2) \leq \mathcal{D}_b(b_1, b_2).$$

Definition 4.1.3. Consider a b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$ and $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$, a non-self mapping $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ is called α-proximal admissible if

$$\begin{cases} \alpha(a_1, a_2) \geq 1, \\ \mathcal{D}_b(x_1, \mathbb{T}a_1) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \\ \mathcal{D}_b(x_2, \mathbb{T}a_2) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \end{cases} \Rightarrow \alpha(x_1, x_2) \geq 1.$$

$\forall\ a_1, a_2, x_1, x_2 \in \mathbb{U}$, where $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$.

Now we are going to determine BPPR for Ćirić type contraction in b-metric spaces.

4.2 Ćirić Type Contraction in b-Metric Space

Definition 4.2.1. Consider a b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ and $\alpha : \mathbb{U} \times \mathbb{U} \to [0, \infty)$ be a function. A mapping $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ is said to be Ćirić type α-ψ-Θ-contraction if for $\psi \in \Psi, \Theta \in \Omega$, $\exists\ m \in (0, 1)$ and for $a, b \in \mathbb{U}$ with $\alpha(a, b) \geq 1$ and $\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b) > 0$, we have

$$\alpha(a, b)\Theta[(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b))] \leq [\psi(\Theta(\mathbb{M}(a, b)))]^m, \tag{4.1}$$

where

$$\begin{aligned} \mathbb{M}(a, b) = &\max\{\mathcal{D}_b(a, b), \frac{\mathcal{D}_b(a, \mathbb{T}a) + \mathcal{D}_b(b, \mathbb{T}b)}{2} - s\mathcal{D}_b(\mathbb{U}, \mathbb{V}), \\ &\frac{s\mathcal{D}_b(a, \mathbb{T}b) + \mathcal{D}_b(b, \mathbb{T}a)}{2} - s\mathcal{D}_b(\mathbb{U}, \mathbb{V})\}. \end{aligned}$$

The following theorem is the extension of theorem (3.4.2) in the setting of b-metric space.

Theorem 4.2.2. Consider a complete b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$, where b-metric is continuous. Suppose that $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\mathbb{T}$ is α-proximal admissible;

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U}, \mathbb{V})$ satisfies the WPP,

(iii) $\mathbb{T}$ is continuous,

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}_b(a_1, \mathbb{T}a_0) = \mathcal{D}_b(\mathbb{U}, \mathbb{V})$ which satisfy $\alpha(a_0, a_1) \geq 1$,

then, $\exists$ a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}_b(x, \mathbb{T}x) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}).$$

Proof. Let $a_0 \in \mathbb{U}_0$, since $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$, then by assumption (iv) there exist an element a_1 in $\mathbb{U}_0$ such that

$$\mathcal{D}_b(a_1, \mathbb{T}a_0) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}),$$

such that $\alpha(a_0, a_1) \geq 1$. Since $a_1 \in \mathbb{U}_0$ and $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$, there exist $a_2 \in \mathbb{U}_0$ such that

$$\mathcal{D}_b(a_2, \mathbb{T}a_1) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}),$$

by α-proximal admissibility of $\mathbb{T}$, $\alpha(a_1, a_2) \geq 1$. Similarly, since $a_2 \in \mathbb{U}_0$ and $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$, there exist $a_3 \in \mathbb{U}_0$, such that

$$\mathcal{D}_b(a_3, \mathbb{T}a_2) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}),$$

and by α-proximal admissibility of $\mathbb{T}$, $\alpha(a_2, a_3) \geq 1$.

Continuing this process, we get for all $n \in \mathbb{N}$

$$\mathcal{D}_b(a_{n+1}, \mathbb{T}a_n) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) \text{ and } \alpha(a_n, a_{n+1}) \geq 1. \tag{4.2}$$

Now suppose that $a_{n_0} = a_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then we have

$$\mathcal{D}_b(a_{n_0}, \mathbb{T}a_{n_0}) = \mathcal{D}_b(a_{n_0+1}, \mathbb{T}a_{n_0}),$$

from equation (4.2) we obtain

$$\mathcal{D}_b(a_{n_0}, \mathbb{T}a_{n_0}) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}).$$

Hence, a_{n_0} is BPP of $\mathbb{T}$. Therefore, we suppose that $a_n \neq a_{n+1}$, that is $\mathcal{D}_b(a_n, a_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. This implies

$$1 < \Theta[\mathcal{D}_b(a_{n+1}, a_n)].$$

As Θ is nondecreasing and from weak P-property of $(\mathbb{U}, \mathbb{V})$,

$$1 < \Theta[\mathcal{D}_b(a_{n+1}, a_n)] \leq \Theta[\mathcal{D}_b(\mathbb{T}a_n, \mathbb{T}a_{n-1})],$$

from equation (4.2)and (4.1), we obtain

$$\begin{aligned} 1 < \Theta[\mathcal{D}_b(a_{n+1}, a_n)] &\leq \alpha(a_n, a_{n-1})\Theta[\mathcal{D}_b(\mathbb{T}a_n, \mathbb{T}a_{n-1})], \\ &\leq [\psi(\Theta(\mathbb{M}(a_n, a_{n-1})))]^m, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \mathbb{M}(a_n, a_{n-1}) = \max\Big\{ &\mathcal{D}_b(a_n, a_{n-1}), \frac{\mathcal{D}_b(a_n, \mathbb{T}a_n) + \mathcal{D}_b(a_{n-1}, \mathbb{T}a_{n-1})}{2} \\ &- s\mathcal{D}_b(\mathbb{U}, \mathbb{V}), \frac{s\mathcal{D}_b(a_n, \mathbb{T}a_{n-1}) + \mathcal{D}_b(a_{n-1}, \mathbb{T}a_n)}{2} - s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) \Big\}, \end{aligned}$$

by triangular inequality

$$\begin{aligned}
\mathbb{M}(a_n, a_{n-1}) \leq \max\Big\{ & \mathcal{D}_b(a_n, a_{n-1}), \\
& \frac{s[\mathcal{D}_b(a_n, a_{n+1}) + \mathcal{D}_b(a_{n+1}, \mathbb{T}a_n)] + s[\mathcal{D}_b(a_{n-1}, a_n) + \mathcal{D}_b(a_n, \mathbb{T}a_{n-1})]}{2} \\
& - s\mathcal{D}_b(\mathbb{U}, \mathbb{V}), \frac{s\mathcal{D}_b(a_n, \mathbb{T}a_{n-1}) + s[\mathcal{D}_b(a_{n-1}, a_{n+1}) + \mathcal{D}_b(a_{n+1}, \mathbb{T}a_n)]}{2} \\
& - s\mathcal{D}_b(\mathbb{U}, \mathbb{V})\Big\}, \\
= \max\Big\{ & \mathcal{D}_b(a_n, a_{n-1}), \\
& \frac{s\mathcal{D}_b(a_n, a_{n+1}) + s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) + s\mathcal{D}_b(a_{n-1}, a_n) + s\mathcal{D}_b(\mathbb{U}, \mathbb{V})}{2} - s\mathcal{D}_b(\mathbb{U}, \mathbb{V}), \\
& \frac{s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) + s\mathcal{D}_b(a_{n-1}, a_{n+1}) + s\mathcal{D}_b(\mathbb{U}, \mathbb{V})}{2} - s\mathcal{D}_b(\mathbb{U}, \mathbb{V})\Big\}, \\
= \max\Big\{ & \mathcal{D}_b(a_n, a_{n-1}), \\
& \frac{s\mathcal{D}_b(a_n, a_{n+1}) + s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) + s\mathcal{D}_b(a_{n-1}, a_n) + s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) - 2s\mathcal{D}_b(\mathbb{U}, \mathbb{V})}{2}, \\
& \frac{s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) + s\mathcal{D}_b(a_{n-1}, a_{n+1}) + s\mathcal{D}_b(\mathbb{U}, \mathbb{V}) - 2s\mathcal{D}_b(\mathbb{U}, \mathbb{V})}{2}\Big\},
\end{aligned}$$

which implies

$$\begin{aligned}
\mathbb{M}(a_n, a_{n-1}) \leq \max\Big\{ & \mathcal{D}_b(a_n, a_{n-1}), \frac{s[\mathcal{D}_b(a_n, a_{n+1}) + \mathcal{D}_b(a_{n-1}, a_n)]}{2}, \\
& \frac{s\mathcal{D}_b(a_{n-1}, a_{n+1})}{2}\Big\}.
\end{aligned}$$

Again by using triangular inequality

$$\begin{aligned}
\mathbb{M}(a_n, a_{n-1}) \leq \max\Big\{ & \mathcal{D}_b(a_n, a_{n-1}), \frac{s[\mathcal{D}_b(a_n, a_{n+1}) + \mathcal{D}_b(a_{n-1}, a_n)]}{2}, \\
& \frac{s^2[\mathcal{D}_b(a_n, a_{n+1}) + \mathcal{D}_b(a_{n-1}, a_n)]}{2}\Big\},
\end{aligned}$$

since $s \geq 1$ so we have

$$\begin{aligned}
\mathbb{M}(a_n, a_{n-1}) &\leq \max\Big\{\mathcal{D}_b(a_n, a_{n-1}), \frac{s^2[\mathcal{D}_b(a_n, a_{n+1}) + \mathcal{D}_b(a_{n-1}, a_n)]}{2}\Big\}, \\
&\leq \max\Big\{\mathcal{D}_b(a_n, a_{n-1}), s^2\mathcal{D}_b(a_n, a_{n+1})\Big\},
\end{aligned}$$

using above inequality in (4.3) we obtain

$$1 < \Theta[\mathcal{D}_b(a_n, a_{n+1})] \leq [\psi(\Theta(\max\{\mathcal{D}_b(a_n, a_{n-1}), s^2\mathcal{D}_b(a_n, a_{n+1})\}))]^m. \tag{4.4}$$

If

$$\max\{\mathcal{D}_b(a_n, a_{n-1}), \mathcal{D}_b(a_n, a_{n+1})\} = s^2\mathcal{D}_b(a_n, a_{n+1}).$$

Then, inequality (4.4) becomes

$$1 < \Theta[\mathcal{D}_b(a_n, a_{n+1})] \leq [\psi(\Theta(s^2\mathcal{D}_b(a_n, a_{n+1}))]^m,$$

by induction, we get

$$\begin{aligned} 1 < \Theta[\mathcal{D}(a_n, a_{n+1})] \leq & [\psi(\Theta(s^2\mathcal{D}_b(a_n, a_{n+1}))]^m \\ \leq & [\psi(\Theta(s^2\mathcal{D}_b(a_{n-1}, a_n))]^{m^2} \\ \leq & [\psi(\Theta(s^2\mathcal{D}_b(a_{n-2}, a_{n-1}))]^{m^3} \\ & \vdots \\ \leq & [\psi(\Theta(s^2\mathcal{D}_b(a_0, a_1))]^{m^n}. \end{aligned}$$

Now by applying limit $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} 1 < \lim_{n\to\infty}[\Theta(\mathcal{D}(a_n, a_{n+1}))] \leq \lim_{n\to\infty}[\psi(\Theta(s^2\mathcal{D}_b(a_0, a_1))]^{m^n},$$

where $m \in (0, 1)$, which implies

$$1 < \lim_{n\to\infty}[\Theta(\mathcal{D}(a_n, a_{n+1}))] \leq 1,$$

therefore we get

$$\Theta[\mathcal{D}_b(a_n, a_{n+1})] \to 1,$$

and by using (Θ_3), we obtain

$$\lim_{n\to\infty} \mathcal{D}_b(a_n, a_{n+1}) = 0,$$

similarly if

$$\max\{\mathcal{D}_b(a_n, a_{n-1}), s^2\mathcal{D}_b(a_n, a_{n+1})\} = \mathcal{D}_b(a_n, a_{n-1}),$$

then, inequality (4.4) becomes

$$1 < \Theta[\mathcal{D}_b(a_n, a_{n+1})] \leq [\psi(\Theta(\mathcal{D}_b(a_n, a_{n-1}))]^m,$$

by induction, we get

$$\begin{aligned} 1 < \Theta[\mathcal{D}_b(a_n, a_{n+1})] &\leq [\psi(\Theta(\mathcal{D}_b(a_n, a_{n-1}))]^m \\ &\leq [\psi(\Theta(\mathcal{D}_b(a_{n-1}, a_n))]^{m^2} \\ &\leq [\psi(\Theta(\mathcal{D}_b(a_{n-2}, a_{n-1}))]^{m^3} \\ &\vdots \\ &\leq [\psi(\Theta(\mathcal{D}_b(a_0, a_1))]^{m^n}, \end{aligned}$$

letting $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} 1 < \lim_{n\to\infty} [\Theta(\mathcal{D}_b(a_n, a_{n+1}))] \leq \lim_{n\to\infty} [\psi(\Theta(\mathcal{D}_b(a_0, a_1))]^{m^n},$$

where $m \in (0, 1)$, which implies

$$1 < \lim_{n\to\infty} [\Theta(\mathcal{D}_b(a_n, a_{n+1}))] \leq 1,$$

therefore we get

$$\Theta[\mathcal{D}_b(a_n, a_{n+1})] \to 1,$$

and by using (Θ_4), we obtain

$$\lim_{n\to\infty} \mathcal{D}_b(a_n, a_{n+1}) = 0. \tag{4.5}$$

Now we prove that $\{a_n\}$ is a cauchy sequence in $\mathbb{U}$. Assume contrary $\{a_n\}$ is not a Cauchy sequence in $\mathbb{U}$, then $\exists\ s\epsilon > 0$ for all $j,\ k \in \mathbb{N}$ such that $j > k > n$ for all $j, k \in \mathbb{N}$

$$\mathcal{D}_b(a_j, a_k) \geq s\epsilon,$$

then,

$$\mathcal{D}_b(a_{j-1}, a_k) < s\epsilon, \tag{4.6}$$

thus, by triangular inequality and (4.6), we get

$$s\epsilon \leq \mathcal{D}_b(a_j, a_k) \leq s[\mathcal{D}_b(a_j, a_{j-1}) + \mathcal{D}_b(a_{j-1}, a_k)] < s\mathcal{D}_b(a_j, a_{j-1}) + s\epsilon,$$

applying limit

$$s\epsilon \leq \lim_{j,k\to\infty} \mathcal{D}_b(a_j, a_k) < \lim_{j\to\infty} s\mathcal{D}_b(a_j, a_{j-1}) + s\epsilon.$$

using (4.5)

$$s\epsilon \leq \lim_{j,k\to\infty} \mathcal{D}_b(a_j, a_k) < s\epsilon.$$

$$\lim_{j,k\to\infty} \mathcal{D}_b(a_j, a_k) = s\epsilon, \tag{4.7}$$

then by again triangular inequality, we get

$$\mathcal{D}_b(a_j, a_k) \leq s[\mathcal{D}_b(a_j, a_{j+1}) + \mathcal{D}_b(a_{j+1}, a_{k+1}) + \mathcal{D}_b(a_{k+1}, a_k)],$$

letting limit as $j, k \to \infty$ and from (4.5)

$$\lim_{j,k\to\infty} \mathcal{D}_b(a_j, a_k) \leq s \lim_{j,k\to\infty} \mathcal{D}_b(a_{j+1}, a_{k+1}) \tag{4.8}$$

and also by triangular inequality

$$\mathcal{D}_b(a_{j+1}, a_{k+1}) \leq s[\mathcal{D}_b(a_{j+1}, a_j) + \mathcal{D}_b(a_j, a_k) + \mathcal{D}_b(a_k, a_{k+1})].$$

Applying limit as $j, k \to \infty$ and from (4.5) and (4.7),

$$\begin{aligned}\lim_{j,k\to\infty} \mathcal{D}_b(a_{j+1}, a_{k+1}) &\leq \lim_{j,k\to\infty} s\mathcal{D}_b(a_{j+1}, a_j) + \lim_{j,k\to\infty} s\mathcal{D}_b(a_j, a_k) + \lim_{j,k\to\infty} s\mathcal{D}_b(a_k, a_{k+1})\\ &= 0 + s^2\epsilon + 0,\end{aligned}$$

therefore we get

$$\lim_{j,k\to\infty} \mathcal{D}_b(a_{j+1}, a_{k+1}) = s^2\epsilon. \tag{4.9}$$

Thus, Equation (4.8) holds. Then by assumption ,$\alpha(a_j, a_k) \geq 1$ and weak P-property of $(\mathbb{U}, \mathbb{V})$, we have that

$$\begin{aligned}1 \leq \Theta(\mathcal{D}_b(a_{j+1}, a_{k+1})) &\leq \Theta(\mathcal{D}_b(\mathbb{T}a_j, \mathbb{T}a_k))\\ &\leq \alpha(a_j, a_k)\Theta(\mathcal{D}_b(\mathbb{T}a_j, \mathbb{T}a_k))\\ &\leq [\psi(\Theta(\mathbb{M}(a_j, a_k)))]^m\\ &< \Theta(\mathbb{M}(a_j, a_k)).\end{aligned}$$

By proceeding as (4.3) till (4.5), letting limit as $j, k \to \infty$ in above inequality and using (Θ_4), we have that

$$\lim_{j,k\to\infty} \mathcal{D}_b(a_{j+1}, a_{k+1}) = 0, \tag{4.10}$$

using (4.10) in (4.8)

$$\lim_{j,k\to\infty} \mathcal{D}_b(a_j, a_k) \leq s(0)$$

which implies

$$\lim_{j,k\to\infty} \mathcal{D}_b(a_j, a_k) = 0 < s\epsilon,$$

which is contradiction to equation (4.7). Hence, this implies that $\{a_n\}$ is Cauchy sequence in $\mathbb{U}$. Since $\mathbb{W}$ is complete, then $\exists\, x \in \mathbb{U}$ such that $a_n \to x$ and continuity of $\mathbb{T}$ implies $\mathbb{T}a_n \to \mathbb{T}x$.
So, from (4.2) we

$$\mathcal{D}_b(\mathbb{U},\mathbb{V}) = \lim_{n\to\infty} \mathcal{D}_b(a_{n+1}, \mathbb{T}a_n).$$
$$\Rightarrow \mathcal{D}_b(\mathbb{U},\mathbb{V}) = \mathcal{D}_b(x, \mathbb{T}x).$$

This complete the proof. □

Example 4.2.3. Suppose $\mathbb{W} = \mathbb{R}^2$ define as

$$\mathcal{D}_b((g_1, g_2),(h_1, h_2)) = |g_1 - h_1|^2 + |g_2 - h_2|^2$$

then $(\mathbb{W}, \mathcal{D}_b)$ is a b-metric space having coefficient $s = 2$.
Suppose

$$\mathbb{U} = \{(-6,-2),(-7,-8),(-11,-12),(20,0),(25,30)\}$$
$$\mathbb{V} = \{(-13,-6),(-11,-8),(-9,-10),(-4,0),(0,-4)\}.$$

be a two non-empty subsets of $\mathbb{W}$. Now,

$\mathcal{D}_b(\mathbb{U},\mathbb{V}) = \inf\{\mathcal{D}_b(g,h) : g \in \mathbb{U}, h \in \mathbb{V}\},$

$$\begin{aligned}
&= \inf\{\mathcal{D}_b((-6,-2),(-13,-6)), \mathcal{D}_b((-6,-2),(-11,-8)),\\
&\mathcal{D}_b((-6,-2),(-9,-10)), \mathcal{D}_b((-6,-2),(-4,0)), \mathcal{D}_b((-6,-2),(0,-4)),\\
&\mathcal{D}_b((-7,-8),(-13,-6)), \mathcal{D}_b((-7,-8),(-11,-8)), \mathcal{D}_b((-7,-8),\\
&(-9,-10)), \mathcal{D}_b((-7,-8),(-4,0)), \mathcal{D}_b((-7,-8),(0,-4)),\\
&\mathcal{D}_b((-11,-12),(-13,-6)), \mathcal{D}_b((-11,-12),(-11,-8)), \mathcal{D}_b((-11,-12),\\
&(-9,-10)), \mathcal{D}_b((-11,-12),(-4,0)), \mathcal{D}_b((-11,-12),(0,-4)),\\
&\mathcal{D}_b((20,0),(-13,-6)), \mathcal{D}_b((20,0),(-11,-8)), \mathcal{D}_b((20,0),(-9,-10)),\\
&\mathcal{D}_b((20,0),(-4,0)), \mathcal{D}_b((20,0),(0,-4)), \mathcal{D}_b((25,30),(-13,-6))\\
&\mathcal{D}_b((25,30),(-11,-8)), \mathcal{D}_b((25,30),(-9,-10)), \mathcal{D}_b((25,30),(-4,0)),\\
&\mathcal{D}_b((25,30),(0,-4))\}
\end{aligned}$$

$$\begin{aligned}\mathcal{D}_b(\mathbb{U},\mathbb{V}) =& \inf\{|7|^2+|4|^2, |5|^2+|6|^2, |3|^2+|8|^2, |2|^2+|2|^2, |6|^2+|2|^2,\\ & |6|^2+|2|^2, |4|^2+|0|^2, |2|^2+|2|^2, |3|^2+|8|^2, |7|^2+|4|^2,\\ & |2|^2+|6|^2, |0|^2+|4|^2, |2|^2+|2|^2, |7|^2+|12|^2, |11|^2+|8|^2,\\ & |33|^2+|6|^2, |31|^2+|8|^2, |29|^2+|10|^2, |24|^2+|0|^2, |20|^2+|4|^2,\\ & |38|^2+|36|^2, |36|^2+|38|^2, |34|^2+|40|^2, |29|^2+|30|^2, |25|^2+|34|^2\}\\ =& \inf\{65, 61, 73, 8, 40, 40, 16, 8, 73, 65, 40, 16, 8, 193, 185, 1125, 1025, 941,\\ & 576, 416, 2740, 2740, 2756, 1741, 1781\}.\\ =& 8\end{aligned}$$

and

$$\begin{aligned}\mathbb{U}_0 &= \{(-6,-2), (-7,-8)\},\\ \mathbb{V}_0 &= \{(-4,0), (0,-4), (-9,-10), (-11,-8)\}.\end{aligned}$$

Define $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ by

$$\begin{aligned}\mathbb{T}(-6,-2) &= (-11,-8),\\ \mathbb{T}(-7,-8) &= (-9,-10),\\ \mathbb{T}(-11,-12) &= (-13,-6),\\ \mathbb{T}(20,0) &= (-4,0),\\ \mathbb{T}(25,30) &= (0,-4),\end{aligned}$$

and $\alpha : \mathbb{U} \times \mathbb{U} \to [0,\infty)$ by $\alpha((g,h),(r,s)) = \frac{11}{10}$.

As $\mathbb{T}(-6,-2) = (-11,-8) \in \mathbb{V}_0$, $\mathbb{T}(-7,-8) = (-9,-10) \in \mathbb{V}_0$. Certainly $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$. Now, let $(-6,-2), (-7,-8) \in \mathbb{U}_0$ and $(-4,0), (-9,-10) \in \mathbb{V}_0$, such that

$$\begin{cases}\mathcal{D}_b((-6,-2),(-4,0)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 8,\\ \mathcal{D}_b((-7,-8),(-9,-10)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 8,\end{cases}$$

$\Rightarrow \mathcal{D}_b((-6,-2),(-7,-8)) < \mathcal{D}_b((-4,0),(-9,-10))$.

Similarly, for all $(g_1,h_1),(g_2,h_2) \in \mathbb{U}_0$ and $(r_1,s_1),(r_2,s_2) \in \mathbb{V}_0$

$$\begin{cases} \mathcal{D}_b((g_1,h_1),(r_1,s_1)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}), \\ \mathcal{D}_b((g_2,h_2),(r_2,s_2)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}), \end{cases}$$

$\Rightarrow \mathcal{D}_b((g_1,h_1),(g_2,h_2)) < \mathcal{D}_b((r_1,s_1),(r_2,s_2))$

Thus, the pair $(\mathbb{U},\mathbb{V})$ has WPP. Now, to prove α-proximal admissibility of $\mathbb{T}$, we proceed as follows

$$\begin{cases} \alpha((-7,-8),(20,0)) \geq 1, \\ \mathcal{D}_b((-11,-12),(-9,-10)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 8, \\ \mathcal{D}_b((-6,-2),(-4,0)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 8, \end{cases}$$

$\Rightarrow \alpha((-9,-10),(-6,-2) = \frac{11}{10} > 1.$

Hence, $\alpha((g,h),(r,s)) \geq 1$ for all $g,h,r,s \in \mathbb{U}$. Which means that $\mathbb{T}$ is α-proximal admissible. Now, to demonstrate α-ψ-Θ-contraction. Define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{999}{1000}t. \tag{4.11}$$

And $\Theta : (0,\infty) \to (1,\infty)$ by

$$\Theta(t) = t+1. \tag{4.12}$$

Suppose for $((-6,-2),(20,0))$ we will check the following inequality.

$$\alpha((-6,-2),(20,0))\Theta[(\mathcal{D}(\mathbb{T}(-6,-2),\mathbb{T}(20,0))] \leq [\psi(\Theta(\mathbb{M}(-6,-2),(20,0)))]^m$$

Now, using (4.12) in left hand side of above inequality.

$$\begin{aligned} &\alpha((-6,-2),(20,0))\Theta[(\mathcal{D}(\mathbb{T}(-6,-2),\mathbb{T}(20,0))] \\ &= \alpha((-6,-2),(20,0))\Theta[(\mathcal{D}(-11,-8),\mathcal{D}(-4,0))] \\ &= \alpha((-6,-2),(20,0))\Theta(113) \\ &= \alpha((-6,-2),(20,0))(113+1) \\ &= \frac{11}{10}(114) \\ &= \frac{627}{5}, \end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\mathbb{M}((-6,-2),(20,0)) = \max\Big\{&\mathcal{D}_b((-6,-2),(20,0)),\\ &\frac{\mathcal{D}_b((-6,-2),\mathbb{T}(-6,-2))+\mathcal{D}_b((20,0),\mathbb{T}(20,0))}{2}\\ &-2\mathcal{D}_b(\mathbb{U},\mathbb{V}),\frac{2\mathcal{D}_b((-6,-2),\mathbb{T}(20,0))+\mathcal{D}_b((20,0),\mathbb{T}(-6,-2))}{2}\\ &-2\mathcal{D}_b(\mathbb{U},\mathbb{V})\Big\},\\ = \max\Big\{&680,\frac{\mathcal{D}_b((-6,-2),(-11,-8))+\mathcal{D}_b((20,0),(-4,0))}{2}\\ &-2(8),\frac{2\mathcal{D}_b((-6,-2),(-4,0))+\mathcal{D}_b((20,0),(-11,-8))}{2}\\ &-2(8)\Big\},\end{aligned}$$

after simplification, we get

$$\begin{aligned}\mathbb{M}((-6,-2),(20,0)) &= \max\left\{680,\frac{637}{2}-16,\frac{1041}{2}-16\right\}.\\ &= \max\left\{680,\frac{605}{2},\frac{1009}{2}\right\}.\\ &= 680\end{aligned}$$

using (4.11) and (4.12), we have

$$\begin{aligned}\left[\psi(\Theta(\mathbb{M}((-6,-2),(20,0))))\right]^m &= \left[\psi(\Theta(680))\right]^m\\ &= \left[\psi(681)\right]^m \\ &= \left[\frac{999}{1000}(681)\right]^m.\end{aligned} \tag{4.14}$$

Hence,from (4.13), (4.14) and for $m = 0.83$, we have

$$\frac{627}{5} < \left[\frac{999}{1000}(681)\right]^m.$$

Which means that $\mathbb{T}$ is Ćirić type α-ψ-Θ-contraction. Similarly, inequality holds for remaining cases. To demonstrate that $\mathbb{T}$ has unique best proximity point we proceed as follows.

$$\begin{aligned}\mathcal{D}_b(x,\mathbb{T}x) =&\mathcal{D}_b((-7,-8),\mathbb{T}(-7,-8))\\ &= \mathcal{D}_b((-7,-8),(-9,-10))\\ &= 8\\ &= \mathcal{D}_b(\mathbb{U},\mathbb{V}).\end{aligned}$$

Therefore all axioms are true. Hence, $(-7,-8)$ has a BPP of $\mathbb{T}$.

Example 4.2.4. Consider $\mathbb{W} = \mathbb{R}^2$ define $\mathcal{D}_b$ as

$$\mathcal{D}_b((g_1,g_2),(h_1,h_2)) = |g_1-h_1|^2 + |g_2-h_2|^2.$$

then $(\mathbb{W},\mathcal{D}_b)$ is a b-metric having coefficient $s=2$. Suppose

$$\begin{aligned}\mathbb{U} =&\{1\}\times[0,\infty)\\ \mathbb{V} =&\{0\}\times[0,\infty),\end{aligned}$$

then

$$\mathcal{D}_b(\mathbb{U},\mathbb{V}) = \mathcal{D}_b((1,0),(0,0)) = 1,$$

and $\mathbb{U}_0 = \mathbb{U}, \mathbb{V}_0 = \mathbb{V}$.

Define $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ by

$$\mathbb{T}(1,g) = \begin{cases} (0,\frac{g}{3}) & \text{if } g\in[0,1],\\ (0,g-\frac{2}{3}) & \text{if } g>1.\end{cases}$$

and $\alpha : \mathbb{U}\times\mathbb{U} \to [0,\infty)$ by

$$\alpha((g,h),(r,s)) = \begin{cases} 1, & \text{if}(g,h),(r,s)\in[0,1]\times[0,1],\\ 0, & \text{otherwise.}\end{cases}$$

As $\mathbb{T}(1,g) = (0,g) \in \mathbb{V}_0$ so $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$. Suppose $(1,g_1),(1,g_2) \in \mathbb{U}_0$ and $(0,r_1),(0,r_2) \in \mathbb{V}_0$, such that

$$\begin{cases} \mathcal{D}_b((1,g_1),(0,r_1)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 1, \\ \mathcal{D}_b((1,g_2),(0,r_2)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 1, \end{cases} \Rightarrow \mathcal{D}_b((1,g_1),(1,g_2)) = \mathcal{D}_b((0,r_1),(0,r_2))$$

Necessarily, $(g_1 = r_1 \in [0,1])$ and $(g_2 = r_2 \in [0,1])$. Which means that $(\mathbb{U},\mathbb{V})$ has WPP.

To demonstrate $\mathbb{T}$ is α-proximal admissible we suppose,

$$\begin{cases} \alpha((1,g_1),(1,g_2)) \geq 1, \\ \mathcal{D}_b((1,r_1),\mathbb{T}(1,g_1)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 1, \\ \mathcal{D}_b((1,r_2),\mathbb{T}(1,g_2)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 1, \end{cases}$$

then

$$\begin{cases} (1,g_1),(1,g_2) \in [0,1], \\ \mathcal{D}_b((1,r_1),\mathbb{T}(1,g_1)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 1, \\ \mathcal{D}_b((1,r_2),\mathbb{T}(1,g_2)) = \mathcal{D}_b(\mathbb{U},\mathbb{V}) = 1. \end{cases}$$

Thus, $(g_1,g_2) \in [0,1] \times [0,1]$. Further, $r_1 = \frac{g_1}{3}$ and $r_2 = \frac{g_2}{3}$, which implies that $\alpha(\,(1,\,r_1),\,(1,\,r_2)) \geq 1$. Therefore $\mathbb{T}$ is an α-proximal admissible mapping.

Now, to prove that $\mathbb{T}$ is α-ψ-Θ-contraction. Define $\psi : [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \frac{999}{1000}t, \tag{4.15}$$

and $\Theta : (0,\infty) \to (1,\infty)$ by

$$\Theta(t) = t + 1. \tag{4.16}$$

Let for $((1,g),(1,h)) \in \mathbb{U}$, we have to prove the following α-ψ-Θ-contraction

$$\alpha((1,g),(1,h))\Theta[(\mathcal{D}(\mathbb{T}(1,g),\mathbb{T}(1,h))] \leq [\psi(\Theta(\mathbb{M}(1,g),(1,h))))]^m \tag{4.17}$$

where $m \in (0,1)$. Using left side of inequality (4.17) which gives

$$\begin{aligned}
&\alpha((1,g),(1,h))\Theta[(\mathcal{D}(\mathbb{T}(1,g),\mathbb{T}(1,h))]\\
&= \alpha((1,g),(1,h))\Theta\left[(\mathcal{D}_b((0,\frac{g}{3}),(0,\frac{h}{3}))\right]\\
&= \Theta\left[|\frac{g}{3}-\frac{h}{3}|^2\right]\\
&= \frac{|g-h|^2}{9}+1.
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{M}((1,g),(1,h)) = \max\Big\{ &\mathcal{D}_b((1,g),(1,h)),\\
&\frac{\mathcal{D}_b((1,g),\mathbb{T}(1,g))+\mathcal{D}_b((1,h),\mathbb{T}(1,h))}{2} - b\mathcal{D}_b(\mathbb{U},\mathbb{V}),\\
&\frac{b\mathcal{D}_b((1,g),\mathbb{T}(1,h))+\mathcal{D}((1,h),\mathbb{T}(1,g))}{2} - b\mathcal{D}_b(\mathbb{U},\mathbb{V})\Big\},
\end{aligned}$$

using $\mathbb{T}(1,g) = (0,\frac{g}{3})$ and $\mathbb{T}(1,h) = (0,\frac{h}{3})$

$$\begin{aligned}
\mathbb{M}((1,g),(1,h)) = \max\Big\{ &\mathcal{D}_b((1,g),(1,h)),\\
&\frac{\mathcal{D}_b((1,g),(0,\frac{g}{3}))+\mathcal{D}_b((1,h),(0,\frac{h}{3}))}{2} - 2(1),\\
&\frac{2\mathcal{D}_b((1,g),(0,\frac{h}{3}))+\mathcal{D}_b((1,h),(0,\frac{g}{3}))}{2} - 2(1)\Big\},
\end{aligned}$$

simplifying the above equation, we get

$$\begin{aligned}
\mathbb{M}((1,g),(1,h)) &= \max\Big\{ |g-h|^2, \frac{|1|^2+|g-\frac{g}{3}|^2+|1|^2+|h-\frac{h}{3}|^2}{2} - 2,\\
&\frac{2|1|^2+3|g-\frac{h}{3}|^2+|1|^2+|h-\frac{g}{3}|^2}{2} - 2\Big\},\\
&= \max\Big\{ |g-h|^2, \frac{1+|g-\frac{g}{3}|^2+1+|h-\frac{h}{3}|^2-4}{2},\\
&\frac{2+2|g-\frac{h}{3}|^2+1+|h-\frac{g}{3}|^2-4}{2}\Big\}.\\
&= \max\Big\{ |g-h|^2, \frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}, \frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}\Big\}.
\end{aligned}$$

If $\max\left\{|g-h|^2, \frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}, \frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}\right\} = |g-h|^2$, then inequality (4.17) becomes

$$\begin{aligned}\frac{|g-h|^2}{9}+1 &\leq [\psi(\Theta(|g-h|^2)))]^m\\ &= [\psi(|g-h|^2+1))]^m\\ &< \psi(|g-h|^2+1),\\ &= \frac{999}{1000}(|g-h|^2+1),\end{aligned}$$

which implies

$$\frac{|g-h|^2}{9}+1 \leq \frac{999}{1000}(|g-h|^2+1),$$

so, the inequality (4.17) holds. Now, if

$$\max\left\{|g-h|^2, \frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}, \frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}\right\} = \frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2},$$

then inequality (4.17) becomes

$$\begin{aligned}\frac{|g-h|^2}{9}+1 &\leq \left[\psi\left(\Theta\left(\frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}\right)\right)\right]^m\\ &= \left[\psi\left(\frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}+1\right)\right]^m\\ &< \psi\left(\frac{|\frac{2g}{3}|+|\frac{2h}{3}|^2-2}{2}+1\right),\end{aligned}$$

$$\Rightarrow \frac{|g-h|^2}{9}+1 < \frac{999}{1000}\left[\frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}+1\right].$$

in this case inequality (4.17) holds. Also if

$$\max\left\{|g-h|^2, \frac{|\frac{2g}{3}|^2+|\frac{2h}{3}|^2-2}{2}, \frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}\right\} = \frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2},$$

then inequality (4.17) becomes

$$\begin{aligned}\frac{|g-h|^2}{9}+1 &\leq \left[\psi\left(\Theta\left(\frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}\right)\right)\right]^m\\ &= \left[\psi\left(\frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}+1\right)\right]^m\\ &< \psi\left(\frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}+1\right),\end{aligned}$$

$$\Rightarrow \frac{|g-h|^2}{9}+1 < \frac{999}{1000}\left[\frac{2|g-\frac{h}{3}|^2+|h-\frac{g}{3}|^2-1}{2}+1\right].$$

Inequality (4.17) also holds in this case. Therefore, $\mathbb{T}$ is Ćirić type α-ψ-Θ contraction. Now, to prove best proximity point we proceed as follow.

$$\begin{aligned}\mathcal{D}_b(x,\mathbb{T}x) =& \mathcal{D}_b((1,0),\mathbb{T}(1,0))\\ &= \mathcal{D}_b((1,0),(0,0))\\ &= 1\end{aligned}$$

this implies

$$\mathcal{D}_b(x,\mathbb{T}x) = \mathcal{D}_b(\mathbb{U},\mathbb{V})$$

Hence, $(1,0)$ is BPP of $\mathbb{T}$.

we use the following condition $\mathcal{U}$ for the uniqueness of BPP.

($\mathcal{U}$) $\forall\ a,b \in BPP(\mathbb{T}),\ \alpha(a,b) \geq 1,$

where $BBP(\mathbb{T})$ denote the set of best proximity points of $\mathbb{T}$.

Theorem 4.2.5. To obtain a unique $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}_b(x,\mathbb{T}x) = \mathcal{D}_b(\mathbb{U},\mathbb{V}).$$

We are going to add condition *($\mathcal{U}$)* to the hypotheses of Theorem *(4.2.2)*

Proof. Suppose that x and y are two BPP of $\mathbb{T}$ and $x \neq y$ that is, $\mathcal{D}_b(x, \mathbb{T}x) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}) = \mathcal{D}_b(y, \mathbb{T}y)$. Then by condition $(\mathcal{U})$

$$\alpha(x, y) \geq 1. \tag{4.18}$$

Since the pair $(\mathbb{U}, \mathbb{V})$ has the WPP and from inequality (4.3), we get

$$\begin{aligned}
\Theta(\mathcal{D}_b(x, y)) \leq \Theta(\mathcal{D}_b(\mathbb{T}x, \mathbb{T}y) &\leq \alpha(x, y)\Theta(\mathcal{D}_b(\mathbb{T}x, \mathbb{T}y)) \\
&\leq [\psi(\Theta(\mathbb{M}(x, y)))]^m \\
&= [\psi(\Theta(\mathcal{D}_b(x, y)))]^m \\
&< \Theta(\mathcal{D}_b(x, y)),
\end{aligned}$$

which is contradiction, so $x = y$.

Corollary 4.2.6. Consider a complete b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$, where b-metric is continuous. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\alpha(a, b)\Theta[\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b] \leq [\psi(\Theta(\mathcal{D}_b(a, b)))]^m$;

(ii) $\mathbb{T}$ is α-proximal admissible;

(iii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U}, \mathbb{V})$ satisfies the WPP;

(iv) $\mathbb{T}$ is continuous;

(v) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}_b(a_1, \mathbb{T}a_0) = \mathcal{D}_b(\mathbb{U}, \mathbb{V})$ such that $\alpha(a_0, a_1) \geq 1$;

then there exist $x \in \mathbb{U}$ which satisfy $\mathcal{D}_b(x, \mathbb{T}x) = \mathcal{D}_b(\mathbb{U}, \mathbb{V})$.

Proof. By proceeding as Theorem (4.2.2) the pair $(\mathbb{U}, \mathbb{V})$ has the WPP and from (4.3), we get

$$\begin{aligned}
1 \leq \Theta(\mathcal{D}_b(a, b)) \leq \Theta(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b) \\
&\leq \alpha(a, b)\Theta(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b)) \\
&\leq [\psi(\Theta(\mathbb{M}(a, b)))]^m
\end{aligned}$$

If $\mathbb{M}(a,b) = \mathcal{D}_b(a,b)$

$$1 \leq \Theta(\mathcal{D}_b(a,b)) \leq [\psi(\Theta(\mathcal{D}_b(a,b)))]^m.$$

Now, proceeding as Equation (4.4) till (4.10) there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}_b(x,\mathbb{T}x) = \mathcal{D}_b(\mathbb{U},\mathbb{V}).$$

Corollary 4.2.7. Consider a complete b-metric space $(\mathbb{W},\mathcal{D}_b)$ with coefficient $s \geq 1$, where b-metric is continuous. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U},\mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying

(i) $\Theta[\mathcal{D}_b(\mathbb{T}a,\mathbb{T}b)] \leq [\psi(\Theta(\mathcal{D}_b(a,b)))]^m$,

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U},\mathbb{V})$ satisfies the WPP,

(iii) $\mathbb{T}$ is continuous,

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}_b(a_1,\mathbb{T}a_0) = \mathcal{D}_b(\mathbb{U},\mathbb{V})$ such that $\alpha(a_0,a_1) \geq 1$,

then there exist a point $x \in \mathbb{U}$ which satisfy $\mathcal{D}_b(x,\mathbb{T}x) = \mathcal{D}_b(\mathbb{U},\mathbb{V})$.

Proof. By proceeding as Theorem (4.2.2) the pair $(\mathbb{U},\mathbb{V})$ has the WPP and from (4.3), we get

$$\begin{aligned}1 \leq \Theta(\mathcal{D}_b(a,b)) &\leq \Theta(\mathcal{D}_b(\mathbb{T}a,\mathbb{T}b)\\ &\leq \alpha(a,b)\Theta(\mathcal{D}_b(\mathbb{T}a,\mathbb{T}b)).\end{aligned}$$

If $\alpha(a,b) = 1\ \forall\ a,b \in \mathbb{U}$, we obtain

$$\begin{aligned}1 \leq \Theta(\mathcal{D}_b(a,b)) &\leq \Theta(\mathcal{D}_b(\mathbb{T}a,\mathbb{T}b)).\\ &\leq [\psi(\Theta(\mathbb{M}(a,b)))]^m.\end{aligned}$$

Now, proceeding as Equation (4.3) till (4.10) there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}_b(x, \mathbb{T}x) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}).$$

Corollary 4.2.8. *Consider a complete b-metric space $(\mathbb{W}, \mathcal{D}_b)$ with coefficient $s \geq 1$, where b-metric is continuous. Suppose $\mathbb{U}$, $\mathbb{V} \neq \phi$, where $\mathbb{U}, \mathbb{V} \subseteq \mathbb{W}$ with $\mathbb{U}_0$ is nonempty. Let $\mathbb{T} : \mathbb{U} \to \mathbb{V}$ be a Ćirić type α-ψ-Θ-contraction satisfying*

(i) $\Theta[\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b)] \leq [\psi(\Theta(\mathcal{D}_b(a, b)))]^m$,

(ii) $\mathbb{T}(\mathbb{U}_0) \subseteq \mathbb{V}_0$ and the pair $(\mathbb{U}, \mathbb{V})$ satisfies the WPP,

(iii) $\mathbb{T}$ is continuous,

(iv) $\exists\ a_0, a_1 \in \mathbb{U}_0$ with $\mathcal{D}_b(a_1, \mathbb{T}a_0) = \mathcal{D}_b(\mathbb{U}, \mathbb{V})$ such that $\alpha(a_0, a_1) \geq 1$,

then there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}_b(x, \mathbb{T}x) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}).$$

Proof. By proceeding as Theorem (4.2.2) the pair $(\mathbb{U}, \mathbb{V})$ has the WPP and from (4.3), we get

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}_b(a, b)) &\leq \Theta(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b) \\ &\leq \alpha(a, b)\Theta(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b)). \end{aligned}$$

By taking $\mathbb{M}(a, b) = \mathcal{D}_b(a, b)$ in Corollary (4.2.7), we obtain

$$\begin{aligned} 1 \leq \Theta(\mathcal{D}_b(a, b)) &\leq \Theta(\mathcal{D}_b(\mathbb{T}a, \mathbb{T}b)). \\ &\leq [\psi(\Theta(\mathbb{M}(a, b)))]^m. \\ &\leq [\psi(\Theta(\mathcal{D}_b(a, b)))]^m. \end{aligned}$$

Now, proceeding as Equation (4.4) till (4.10) there exist a point $x \in \mathbb{U}$ which satisfy

$$\mathcal{D}_b(x, \mathbb{T}x) = \mathcal{D}_b(\mathbb{U}, \mathbb{V}).$$

Remark 4.2.9. By taking $\psi(x) = mx$ for $m \in (0, 1)$ and $\Theta(x) = e^x$ in Corollary (3.4.8), we obtained the main result of Jleli et al. [22] in the setting of b-metric space.

Remark 4.2.10. By taking $\psi(x) = mx$ for $m \in (0, 1)$ and $\Theta(x) = e^x$ in Corollary (3.4.8), we obtained the main result of Suzuki [35] in the setting of b-metric space.

Chapter 5

Conclusion

In this thesis the work of Ma et al. [24] on “Best Proximity Point Results for Generalized Θ-Contraction” is examined and elaborate to represent the complete analysis of this article.

This research aimed mainly to extend the above results in the setting of b-metric spaces. For this purpose, the notion of best proximity point and Θ-contractions in b-metric spaces is established. Moreover for Θ-contractions, fixed point theorems are established in the setting of b-metric space. Our results might be beneficial in determining specific best proximity points in perception of b-metric spaces.

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