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Central Configuration in a Symmetric Five Body Problem

by

Muhammad Zahid

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I dedicate this sincere effort to my beloved parents and my elegant teachers whose devotions and contributions to my life are really worthless and whose deep consideration on part of my academic career,made me consolidated and inspired me as I am upto this grade now.

CERTIFICATE OF APPROVAL

Central Configuration in a Symmetric Five Body Problem

by

Muhammad Zahid

(MMT181006)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Ghulam Shabbir	GIKI, KPK
(b)	Internal Examiner	Dr. Shafqat Hussain	CUST, Islamabad
(c)	Supervisor	Dr. Abdul Rehman Kashif	CUST, Islamabad

Dr. Abdul Rehman Kashif
Thesis Supervisor
October, 2020

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
October, 2020

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
October, 2020

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(Muhammad Zahid)

Registration No: MMT181006

Abstract

In this thesis we model a symmetric five-body problem with position coordinates for the five bodies as $(-x, 0)$, $(y, 0)$, $(x, 0)$, $(-y, 0)$ and $(0, 0)$. $(0, 0)$ is the centre of mass of the system. Regions of central configurations, where it is possible to choose positive masses, are derived using both analytical and numerical tools. We also identify regions in the phase space where no central configurations are possible. A certain relationship exists between the mass placed at the center of mass of the systems i.e., $(0, 0)$ and the remaining four masses. This relationship investigated both numerically and analytically. Similarly restrictions on the geometry and restrictions on the inter-body distances are investigated.

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Abbreviations

2BP	Two-Body Problem
3BP	Three-Body Problem
4BP	Four-Body Problem
5BP	Five-Body Problem
CC	Central Configuration
D	Denominator
$\mathbf{M_s}$	Mass of the Sun
$\mathbf{M_e}$	Mass of the Earth
N	Numerator
NBP	N-Body Problem
R	Region
SI	System International

Physical Constants

Name	Symbol	Unit
Universal Gravitational Constant	G	$m^3kg^{-1}s^{-2}$
Gravitational Force	$\mathbf{F}$	$Newton$
Distance	r	$Meter$
Linear Momentum	$\mathbf{P}$	$kgms^{-1}$
Angular Momentum	$\mathbf{L}$	kgm^2s^{-1}
Point Mass	m_i	kg
Real Number	$\mathbb{R}$	
Theta	θ	
phi	π	
Union	$\cup$	
Intersection	$\cap$	
Belong	$\in$	
Does Not Belong	$\notin$	
Such That	$\ni$	
For All	$\forall$	
Less	$<$	
Greater	$>$	
Less or Equal	$\leq$	
Greater or Equal	$\geq$	
Infinity	∞	

Chapter 1

Introduction

The n-body problem is the problem in celestial mechanics, which predict the individual motion of a number of celestial bodies colliding with each other gravitationally. The goal of solving this kind of issue was inspired by the desire to infer the motion of the sun , moon , planets, and other measurable stars etc. The n-body problem [1] was become a challenging problem for mathematician and astronomers in the 17^{th} century.

The statement of the problem can be defined as "*what would happen to the orbit if n-number of celestial bodies collide under the gravitational forces.*" Newton used his universal law of gravitational to solved two body problem. There is no general solution of the problem of $n \geq 3$. Then we have to apply some restriction to solved $n \geq 3$ body problem. Mathematician and astronomers are continuously working on n-body for last 4 centuries. Kepler's explained the elliptical trajectories of the planets around the Sun under his law of planetary motion in the end of 17^{th} century, which he wrote in **Philosophiae Naturails Principia Mathematica** [2]. This has proven to be the most important discovery in the history of science. The law for two bodies when they are moving under the mutual gravitational force can be defined as:

$$\mathbf{F} = G\frac{m_1 m_2}{r^3}\mathbf{r},$$

m_1, m_2 represent two masses, r is the distance between their center and G is the universal gravitational constant whose value is $6.67 \times 10^{-11} Nm^3kg^{-1}s^{-2}$. Since the gravitational force is directly proportional to the product of their masses, therefore the gravitational force increases by increasing the numerical value of each mass. The gravitational force of attraction become dubbled, if we dubbled the mass of one of them. So the force of gravity between them is tripled, if the mass of one of the bodies is tripled and so on. Since gravity is inversely proportional to the square of the distance between the objects. The gravitational force therefore decreases by increasing the distance between the bodies. If the distance between them is doubled, the force of gravitational attraction decreases by a factor of 4. Newton's turned his consideration to the system more complex as comparison to the Sun-Earth-Moon system after the testification of the flow given by Kepler's. Sun-Moon-Earth system are one of the most important discussion of Kepler's. However Newton's faced a lot problem and difficulties in this regard. He was not able through out his life to gain any breakthrough in the three-body problem after a lot of effort.

Alexis Clairaut successively perform an approximation for the three-body problem in 1747 after 20th years the death of Newton's. After some change, his effort considered for the periapsis of the Moon, which was the main purpose of Newton's. In 1752 he won the Petersburg academy prize. The value of his approximation was properly determined its motion in 1759, when Halley's comet passed by Earth. The margin of error which is forcecasting in his equation take off himself within a month. Leonhard Eular also discuss the three-body problem. The end of the classical period of work was the most significant work of Henri Poincare on three body-problem. King Oscar II of Swedon, solved the n-body problem (a more general form of the problem with n rather than three masses) for settled an award in 19^{th} century. The statement of the problem are as under: [3] "*Given a system of arbitrary many points masses that attract each other according to Newton's law, under the assumption that no two objects over collide, try to find a representation of the coordinate of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly*".

Many well known astronomers and mathematician including Euler, Lagrange and Carl Gustav Jacob Jacobi working on it in the 19*th* century. Qiudong Wong presented the comprehensive solution of the n-body problem [4], untill in 1991 the general solution remained unsolved. While his work [5] "*The global solution of n-body Problem"* met the requirements of King Oscars problem, Wang himself would symbolize the result as a tricky, simple and useless result while praising the publication that Rom-Kedar and Zaslavsky did complete [6].

1.1 Central Configuration

A CC is a particular setting of points masses interacting by Newtons law of gravitation with the help of the following property *the gravitational acceleration vector generated on each mass by all the others should point towards the centre of mass and be proportional to the distance to the centre of mass.* Central configuration plays a very significant role in the study of the Newtonian n-body problem, because they obtained a very clear and obvious solution of the equation of motion. They determined the topology of the integrals manifolds, and govern the behavior of solution near collision. Four-body and five body problem are very important, because generally two-third of our galaxys are constituted in multi-stellar system. CC plays a very important role for understanding the n-body problem [7, 8]. In finding explicit homographic solution of the equation of motion and periodic solutions [9] central configuration are very useful. Also CC helpful to determined that type of solution, which is nearby the collisions and the energy level sets that keeps the central configuration to find topology of the integrals manifolds. After the linear stability analysis of the rhomboidal 4-body problem Bakker and Simmons [10] clearly state that collision can be formalized at the origin. The same kind of problem analyzed earlier Perez-Chavela and Lacomba [11] for regularization of binary collisions. They studied its escape and capture orbits in [12]. Yan [13] studied the existence and linear stability of periodic orbits of the same model for equal masses. For finding the region of stability for the rhomboidal 4-body problem used the Delagado-Fernandez and Perez- chavela [14]. Corbera and Llibre [15] and Shoaib, Faye, and Sivasankaran [16] also work on 4-body problem. Many periodic orbit which calculated by Marchesian and Vidal [17] for a rhomboidal 5-body problem, when we put 2- couples masses at the vertices of the rhombus and the 5^{th} mass taken stationary at the center of system mass. For this model Shoaib and Kashif in [18] derived CC regions. Different form of the restricted rhomboidal five-body problem which has four positive masses on the vertices of a rhombus and the fifth infinitesimal mass placed in the planet of the four masses are studied by Roy [19] MacMillan and Bartky [20].

1.2 Thesis Contribution

In this thesis we have reviewed [21] symmetric five body problem, where four of the masses placed symmetrically at the vertices and the remaining fifth mass placed stationary at the center of mass of the system as shown in Figure 4.1. Figure 4.2 clearly shows that all the masses lies on a straight line through out their revolution due to centripetal force which provide gravitational force. Central configuration region were obtained for 5BP. The region where central configuration exist explained graphically in different cases.

1.3 Dissertation Outlines

This dissertation is divided into five chapters.

In Chapter 1. The objective of this research and the introduction of the problem are discussed briefly in this chapter. Also we discuss the two-body problem and n-body problem briefly in the last of this chapter.

In Chapter 2. We discuss some laws of motion and some basic definition related to my thesis.

In Chapter 3. We discuss restricted 3-body problem including Jacobin Integral and Lagrange Points.

In Chapter 4. The thesis [22] is reviewed comprehensively.

In Chapter 5. We conclude our thesis.

Chapter 2

introductory

This chapter include some basic definitions, fundamental concepts and governing laws which are essential to understand the work presented in next chapters.

2.1 Important Definitions

2.1.1 Motion [23]

"Motion is the action used to change the location or position of an object with respect to the surroundings over time."

2.1.2 Mechanics [23]

"Mechanics is a branch of physics concerned with motion or change in position of physical objects. It is sometimes further subdivided into:

1. **Kinematics**, which is concerned with the geometry of the motion,
2. **Dynamics**, which is concerned with the physical causes of the motion,
3. **Statics**, which is concerned with conditions under which no motion is apparent."

2.1.3 Scalar [23]

"Various quantities of physics, such as length, mass and time, requires for their specification a single real number (apart from units of measurement which are decided upon in advance). Such quantities are called **Scalars** and the real number is called the magnitude of the quantity."

2.1.4 Vector [23]

"Other quantities of physics, such as displacement, velocity, momentum, force etc require for their specification a direction as well as magnitude. Such quantities are called **Vectors**."

2.1.5 Field [23]

"A field is a physical quantity associated with every point of spacetime. The physical quantity may be either in vector form, scalar form or tensor form."

2.1.6 Scalar Field [23]

"If at every point in a region, a scalar function has a defined value, the region is called a scalar field. i.e.,

$$f : \mathbb{R}^3 \to \mathbb{R},$$

e.g. temperature and pressure fields around the earth."

2.1.7 Vector Field [23]

"If at every point in a region, a vector function has a defined value, the region is called a vector field.

$$V : \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

e.g. tangent vector around a smooth curve."

2.1.8 Conservative Vector Field [23]

"A vector field $\mathbf{V}$ is conservative if and only if there exists a continuously differentiable scalar field f such that $\mathbf{V} = -\nabla f$ or equivalently if and only if

$$\nabla \times \mathbf{V} = \texttt{Curl}\mathbf{V} = \mathbf{0}."$$

2.1.9 Uniform Force Field [23]

"A force field which has constant magnitude and direction is called a uniform or constant force field. If the direction of the field is taken as negative z direction and magnitude is constant $F_0 > 0$, then the force field is given by

$$\mathbf{F} = -F_0\hat{\mathbf{k}}."$$

2.1.10 Central Force [23]

"Suppose that a force acting on a particle of mass m such that
(a) it is always directed from m toward or away from a fixed point O,
(b) its magnitude depends only on the distance r from O.
then we call the force a central force or central force field with O as the center of force. In symbols $\mathbf{F}$ is a central force if and only if

$$\mathbf{F} = f(r)\mathbf{r}_1 = f(r)\tfrac{\mathbf{r}}{r},$$

where $\mathbf{r}_1 = \frac{\mathbf{r}}{r}$ is a unit vector in the direction of $\mathbf{r}$. The central force is one of attraction towards O or repulsion from O according as *f(r)* < 0 or *f(r)* > 0 respectively."

2.1.11 Degree of Freedom [23]

"The number of coordinates required to specify the position of a system of one or more particles is called number of degree of freedom of the system."

2.1.12 Center of Mass [24]

"Let $r_1, r_2, ..., r_n$ be the position vector of a system of n particles of masses $m_1, m_2, ...m_n$ respectively. The center of mass or centroid of the system of particles is defined as that point having position vector

$$\hat{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... + m_n\mathbf{r}_n}{m_1 + m_2 + ... + m_n} = \frac{1}{\mathbf{M}}\sum_{\nu=1}^{n} m_\nu \mathbf{r}_\nu,$$

where

$$\mathbf{M} = \sum_{\nu=1}^{n} m_\nu,$$

is the total mass of the system."

2.1.13 Center of Gravity [24]

"If a system of particles is in a uniform gravitational field, the center of mass is sometimes called the center of gravity."

2.1.14 Torque [24]

"Torque is the measure of the force that can cause an object to rotate about an axis. If a particle with a position vector $\mathbf{r}$ moves in a force field $\mathbf{F}$, we define $\boldsymbol{\tau}$ as torque or moment of the force as:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

The magnitude of $\boldsymbol{\tau}$ is,

$$\tau = rF\sin\theta.$$

The magnitude of torque is a measure of the turning effect produced on the particle by the force."

2.1.15 Momentum [24]

"The linear momentum $\mathbf{p}$ of an object with mass m and velocity $\mathbf{v}$ is defined as:

$$\mathbf{p} = m\mathbf{v}.$$

Under certain circumstances the linear momentum of a system is conserved. The linear momentum of a particle is related to the net force acting on that object:

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt}.$$

The rate of change of linear momentum of a particle is equal to the net force acting on the object, and is pointed in the direction of the force. If the net force acting on an object is zero, its linear momentum is constant (conservation of linear momentum). The total linear momentum $\mathbf{p}$ of a system of particles is defined as the vector sum of the individual linear momentum.

$$\mathbf{p} = \sum_{1}^{n} \mathbf{p}_i."$$

2.1.16 Point-like Particle [24]

"A point-like particle is an idealization of particles mostly used in different fields of physics. Its defining features is the lacks of spatial extension:being zero-dimensional, it does not take up space. A point-like particle is an appropriate representation of an object whose structure, size and shape is irrelevant in a given context. e.g., from far away, a finite-size mass (object) will look like a point-like particle."

2.1.17 Angular Momentum [24]

"Angular momentum for a point-like particle of mass m with linear momentum $\mathbf{p}$ about a point O, defined by the equation

$$\mathbf{L} = \mathbf{r} \times \mathbf{p},$$

where $\mathbf{r}$ is the vector from the point O to the particle. The torque about the point O acting on the particle is equal to the rate of change of the angular momentum about the point O of the particle i.e.,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}.$$

The angular momentum is conserved if torque of an object is equal to zero."

2.1.18 Lorentz Transformation [24]

"Lorentz transformation is the relationship between two different coordinate frames that move at a constant velocity and are relative to each other. The name of the transformation comes from a Dutch physicist Hendrik Lorentz. There are two frames of reference, which are:

2.1.18.1 Inertial Frame of Reference

A frame of reference that remains at rest or moves with constant velocity with respect to other frames of reference is called inertial frame of reference. Actually, an unaccelerated frame of reference is an inertial frame of reference that is, such a body is at rest or moving at a constant velocity. In this frame of reference a body does not acted upon by external forces. Newton's laws of motion are valid in all inertial frames of reference. Inertial reference frames are frames in which the Principle of Inertia is true. All inertial frames of reference are equivalent. A train moving with constant velocity is a best example of inertial frame of reference.

2.1.18.2 Non-Inertial Frame of Reference

A non-inertial reference frame is a frame of reference that is undergoing acceleration with respect to an inertial frame. While the laws of motion are the same in all inertial frames, in non-inertial frames, they vary from frame to frame depending on the acceleration."

2.1.19 Lagrange Points [25]

"A point in space where a small body with negligible mass under the gravitational influence of two large bodies will remain at rest relative to the larger ones. These points are locations in an orbital arrangement of two large bodies where a third smaller body, affected solely by gravity, is capable of maintaining a stable position relative to the two larger bodies. A lagrange point is also known as a equilibrium point and Liberation point named after a French mathematician and atronomer Joseph-Louis Lagrange. He was first to find these equlibrium points for the earth, sun, and moon system. He found five points out of these three are collinear."

2.1.20 Basin of Attraction [26]

"Newton method is used to find the roots of equations but Arthur Cayley found that if the roots of a function are already know then Newton's method can guide to another problem that is which initial guesses iterate to which roots and the region of these initial guesses is called basins of attraction of the roots."

2.1.21 Holonomic and non holonomic Constraints [27]

"In classical mechanics, a constraint on a system is a parameter that the system must obey. The limitation on the motion are often called constraints. If the constraints condition can be expressed as an equation,

$$\phi(\mathbf{r_1}, \mathbf{r_2}, ...\mathbf{r_n}, \mathbf{t}) = 0,$$

connecting the position vector of the particles and the time, then the constraints are called holonomic, otherwise non-holonomic."

2.1.22 Galilean Transformation [28]

"In physics, a Galilean transformation is used to transform between the coordinates of two reference frames which differ only by constant relative motion within the constructs of Newtonian physics. Without the translations in space and time the group is the homogeneous Galilean group. Galilean transformations, also called Newtonian transformations, set of equations in classical physics that relate the space and time coordinates of two systems moving at a constant velocity relative to each other."

2.1.23 Celestial Mechanics

"Celestial mechanics is the branch of astronomy that deals with the motions of objects in outer space. Historically, celestial mechanics applies principles of physics (classical mechanics) to astronomical objects, such as stars and planets, to produce ephemeris data. Actually celestial mechanics is the science devoted to the study of the motion of the celestial bodies on the basis of the laws of gravitation. It was founded by Newton and it is the oldest of the chapters of Physical Astronomy."

2.2 Kepler's Laws of Planetary Motion [29]

"Kepler's three laws of planetary motion can be described as follows:

1. Keplers first law states that every planet moves along an ellipse, with the Sun located at a focus of the ellipse. An ellipse is defined as the set of all points such that the sum of the distance from each point to two foci is a constant.

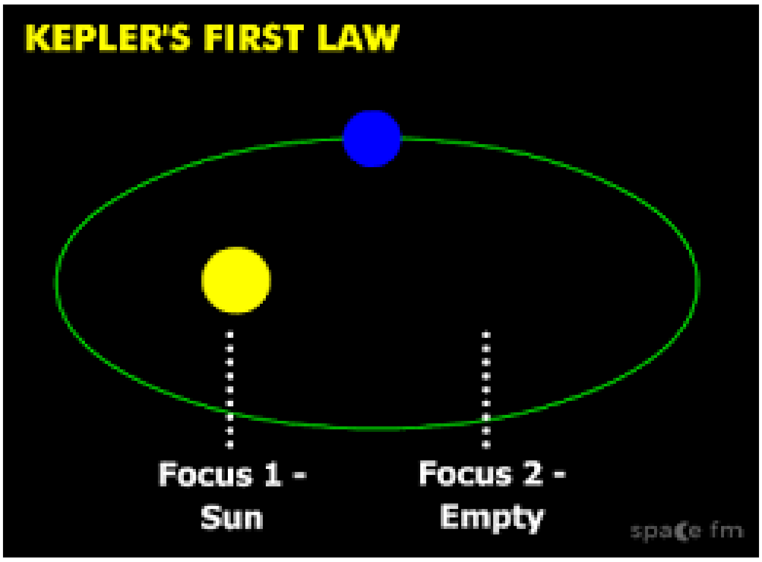

FIGURE 2.1: The orbit of a planet around the sun is an ellipse, with the sun at one focus.

Clearly shown from the figure that, the sun moving in elliptical orbit at single point (focus).

2. Keplers second law states that a planet moves in its ellipse so that the line between it and the Sun placed at a focus sweeps out equal areas in equal times.

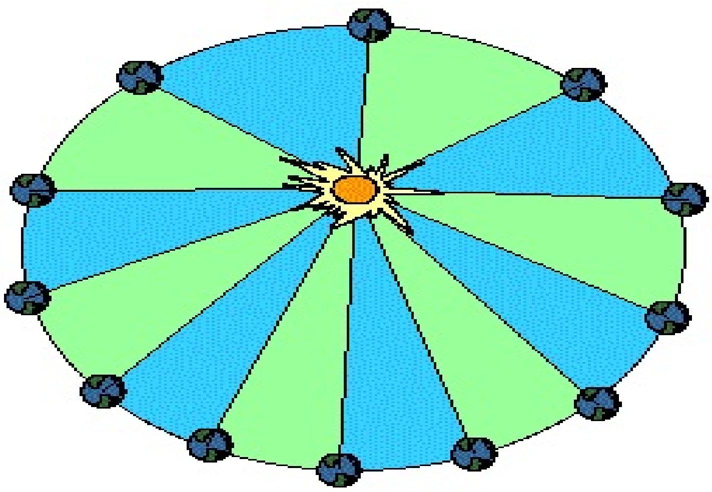

FIGURE 2.2: The distance from the sun to any planets having same area.

3. The cube of the semi major axis of the planetary orbits are proportional to the square of the planets periods of revolution. Mathematically, Kepler's

third law can be written as:

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3,$$

where T is the time period, r is the semi major axis, M_s is the mass of sun and G is the universal gravitational constant."

2.3 Newton's Laws of Motion [30]

"The following three laws of motion given by Newton are considered the axioms of mechanics:

1. **First Law of Motion**

   Every particle persists in a state of rest or of uniform motion in a straight line unless acted upon by a force.

2. **Second Law of Motion**

   If $\mathbf{F}$ is the external force acting on a particle of mass m which as a reaction is moving with velocity $\mathbf{v}$, then

   $$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{P}}{dt}.$$

   If m is independent of time this becomes

   $$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

   where $\mathbf{a}$ is the acceleration of the particle.

3. **Third Law of Motion**

   For every action, there is an equal and opposite reaction. Mathematically can be defined as,

   $$\mathbf{F_1} = -\mathbf{F_2}."$$

2.4 Newton's Universal Law of Gravitation [31]

"Every particle of matter in the universe attracts every other particle of matter with a force which is directly proportional to the product of the masses and inversely proportional to the square of the distance between them. Hence, for any two particles separated by a distance r, the magnitude of the gravitational force $\mathbf{F}$ is:

$$\mathbf{F} = G\frac{m_1 m_2}{r^2}\mathbf{r}$$

where m_1, m_2 represent two masses, r is the distance between them and G is the universal gravitational constant. Its numerical value in SI units is
$6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2}$."

2.4.1 Two Body Problem [32]

"In classical mechanics, the 2BP is to predict the motion of two massive objects which are abstractly viewed as point particles. 2BP, first studied and resolved by Newton, states: Suppose that t is given at some time the positions and velocities of two heavy bodies moving under their mutual gravitational force, then what should be their location and velocity t at any other time, if the masses are known." Earth circling around a sun, two stars circling around each other, orbiting a satellite, for instance. Because of the facts below, the 2BP problem is most important.

1. In celestial mechanics, it is the only gravity problem, apart from very limited solutions to the 3BP for which I have a detailed and a suitable solution.

2. A large scope of realistic elliptic motion issues can be viewed as approximately 2BP.

3. In order to provide approximate orbital parameters and forecasts, the two-body solution may be used or deliver as a initial point for producing analytical solutions that are accurate for higher precision orders.

2.5 The Solution to the Two-Body Problem [33]

"Newton's universal gravitational law is the governing law for the two bodies:

$$\mathbf{F} = G\frac{m_1 m_2}{d^3}\mathbf{d}, \tag{2.1}$$

For two masses, m_1 and m_2 are separated by a $\mathbf{d}$ distance, and the universal gravitational constant is G."

The purpose here is to decide if the initial locations and velocities are known, the direction of the particles for some time t. The force of attraction $\mathbf{F}_{12}$ in Figure 2.1 is directed towards m_1 along d, while the force $\mathbf{F}_{21}$ on m_2 is directed in the opposite direction. According to Newton's third law of motion,

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{2.2}$$

From Figure 2.1,

$$\mathbf{F}_{12} = G\frac{m_1 m_2}{d^3}\mathbf{d}. \tag{2.3}$$

Particles under their gravity equations are given by (2.1) and (2.2) respectively, contributing Newton's 2nd law and by equations (2.1) and (2.2).

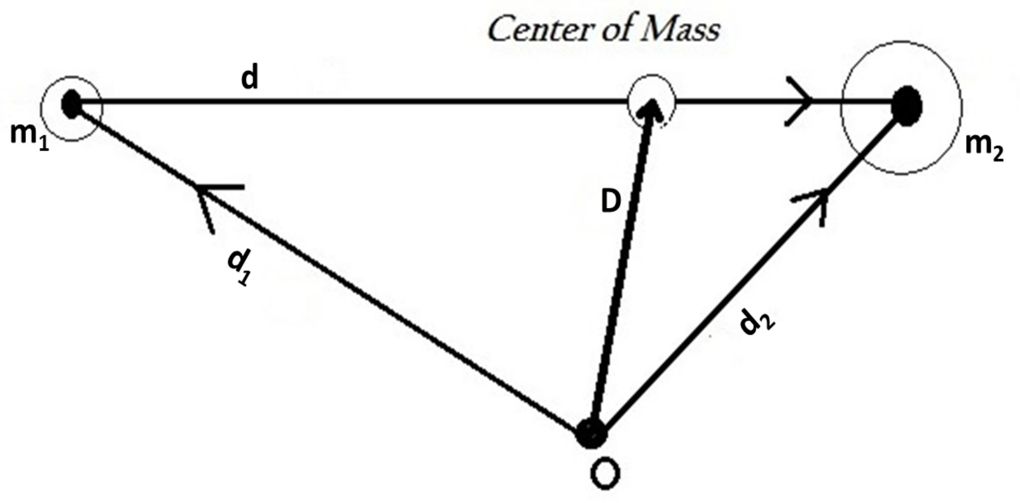

FIGURE 2.3: Center of mass.

$$m_1\ddot{\mathbf{d}}_1 = G\frac{m_1m_2}{d^3}\mathbf{d}, \tag{2.4}$$

$$m_2\ddot{\mathbf{d}}_2 = -G\frac{m_1m_2}{d^3}\mathbf{d}, \tag{2.5}$$

Where the location vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are from the position point O, as shown in Figure 2.1. When the equations (2.4) and (2.5) are applied, we obtained:

$$m_1\ddot{\mathbf{d}}_1 + m_2\ddot{\mathbf{d}}_2 = \mathbf{0}. \tag{2.6}$$

The integration of the equations above yields:

$$m_1\dot{\mathbf{d}}_1 + m_2\dot{\mathbf{d}}_2 = \mathbf{k}_1, \tag{2.7}$$

The total linear momentum of the system is a constant, i.e., $m_1\mathbf{v}_{m1} + m_2\mathbf{v}_{m_2} = \mathbf{k}_1$. Again integrating equation (2.7) implies that:

$$m_1\mathbf{d}_1 + m_2\mathbf{d}_2 = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.8}$$

Where $\mathbf{k}_1$ and $\mathbf{k}_2$ represent the constant of integration. Using 2BP's description of the centre of mass, $\mathbf{D}$ is defined as:

$$(m_1 + m_2)\mathbf{D} = m_1\mathbf{d}_1 + m_2\mathbf{d}_2,$$

$$m_t\mathbf{D} = m_1\mathbf{d}_1 + m_2\mathbf{d}_2, \tag{2.9}$$

where $m_t = m_1 + m_2$. We get the derivative of the (2.9) equation and compare it with the (2.7) equation.

$$m_t\dot{\mathbf{D}} = \mathbf{k}_1,$$
$$\dot{\mathbf{D}} = \frac{\mathbf{k}_1}{m_t} = constant$$

i.e. linear momentum of the system is constant i.e., $m_1\mathbf{v}_{m1} + m_2\mathbf{v}_{m_2} = \mathbf{k}_1$. Again integrating equation (2.7) implies that:

$$m_1\mathbf{d}_1 + m_2\mathbf{d}_2 = \mathbf{k}_1 t + \mathbf{k}_2, \tag{2.10}$$

Where $\mathbf{k}_1$and $\mathbf{k}_2$ represent the constant of integration. Using 2BP's description of the centre of mass, $\mathbf{D}$ is defined as $\mathbf{D}$:

$$\begin{aligned}(m_1 + m_2)\mathbf{D} &= m_1\mathbf{d}_1 + m_2\mathbf{d}_2,\\ m_t\mathbf{D} &= m_1\mathbf{d}_1 + m_2\mathbf{d}_2, \qquad (2.11)\end{aligned}$$

where $m_t = m_1 + m_2$. We get the derivative of the (2.9) equation and compare it with the (2.7) equation.

$$m_t\dot{\mathbf{D}} = \mathbf{k}_1, \tag{2.12}$$

$$\dot{\mathbf{D}} = \frac{\mathbf{k}_1}{m_t} = constant \tag{2.13}$$

show that $\dot{\mathbf{D}} = \mathbf{v}_c$ is constant.
Subtracting (2.6) from (2.5) from the equations gives:

$$\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 = \frac{Gm_2}{d^3}d + \frac{Gm_1}{d^3}\mathbf{d}, \tag{2.14}$$

$$\ddot{\mathbf{d}}_1 - \ddot{\mathbf{d}}_2 = G(m_1 + m_2)\frac{\mathbf{d}}{d^3}$$

$$\Rightarrow \ddot{\mathbf{d}} = \beta\frac{\mathbf{d}}{d^3},$$

$$\Rightarrow \ddot{\mathbf{d}} - \beta\frac{\mathbf{d}}{d^3} = \mathbf{0}, \tag{2.15}$$

Where a reduced mass is known as $\beta = G(m_1 + m_2)$ and $\mathbf{d}_1 - \mathbf{d}_2 = -\mathbf{d}$, is shown in figure 2.1.

Multiplying **r** with the (2.12) equation Such that,

$$\begin{aligned} &\mathbf{d}\times\beta\ddot{\mathbf{d}}+\frac{\beta^2}{d^3}\mathbf{d}\times\mathbf{d}=\mathbf{0},\\ &\mathbf{d}\times\ddot{\mathbf{d}}=\mathbf{0}, \end{aligned} \tag{2.16}$$

integrating above equation yields:

$$\mathbf{d}\times\dot{\mathbf{d}}=\mathbf{H}, \tag{2.17}$$

Where **H** is integration constant. We should write the equation (2.12),

$$\begin{aligned} &\Rightarrow \mathbf{d}\times\beta\ddot{\mathbf{d}}=\mathbf{0},\\ &\Rightarrow \mathbf{d}\times\mathbf{F}=\mathbf{0}, \end{aligned} \tag{2.18}$$

where,

$$\mathbf{F}=\beta\ddot{\mathbf{d}}.$$

The description of torque and angular momentum is taken from Chapter 2:

$$T=\mathbf{d}\times\mathbf{F}, \tag{2.19}$$

Comparing (2.14) and (2.15) equation, we obtained:

$$T=\mathbf{d}\times\mathbf{F}=\mathbf{0}, \tag{2.20}$$

$$\frac{d\mathbf{L}}{dt}=\mathbf{0}$$

$$\mathbf{L}=\text{conserved.}$$

This means that the angular momentum is conserved.
From chapter 2 we know the definition that angular momentum is constant or conserved, if external torque of an object is equal to zero, and we know that external torque of an object is equal to zero when, any one of them **d** or **F** are equal to zero. Angular momentum is a vector quantity, requiring the specification

of both a magnitude and a direction for its complete description. Example, the conservation of angular momentum explains the angular acceleration of an ice skater as she brings her arms and legs close to the vertical axis of rotation.

2.6 Velocity and Acceleration Components of Radial and Transverse

The velocity components along and perpendicular to the radius vector joining m_1 to m_2 are $\dot{d}$ and $d\dot{\theta}$ if the polar co-ordinates d and θ are chosen in this region as shown in fig (2.2), then,

$$\dot{\mathbf{d}} = \dot{d}\mathbf{i} + d\dot{\theta}\mathbf{j}, \tag{2.21}$$

Where the unit vectors $\hat{i}$ and $\hat{j}$ are located along and perpendicular to the vector radius. Thus, by means of equ (2.13) and (2.16),

$$\mathbf{d} \times (\dot{d}\mathbf{i} + d\dot{\theta}\mathbf{j}) = d^2\dot{\theta}\mathbf{k} = L\mathbf{k}, \tag{2.22}$$

Where the unit vector $\mathbf{k}$ is Perpendicular to the orbital plane. Which can be written as,

$$d^2\dot{\theta} = L, \tag{2.23}$$

Since L is a constant shown to be dubble the radius vector definition rate of the field. This is a mathematical version of the second law of Kepler. Now, if we use the scalar product $\dot{\mathbf{d}}$ with the equation (2.11), we obtain equation (2.11) are as under.

$$\dot{\mathbf{d}}.\frac{d^2\mathbf{d}}{dt^2} + \beta\frac{\dot{\mathbf{d}}.\mathbf{d}}{d^3} = 0,$$

after integrated we have get,

$$\frac{1}{2}\dot{\mathbf{d}}.\dot{\mathbf{d}} - \frac{m_1 u}{d} = H,$$

$$\frac{1}{2}v^2 - \frac{\beta}{d} = H, \tag{2.24}$$

where H is a constant of integration. This is the sort of energy system preservation. The H quantity does not include absolute energy, $\frac{1}{2}\beta^2$ is associated with KE, and $\frac{-\beta}{d}$ is associated with PE of the system's, i.e., the system's total energy is constant. Remember the elements of the acceleration vector from celestial mechanics the radius vector is perpendicular to and along.

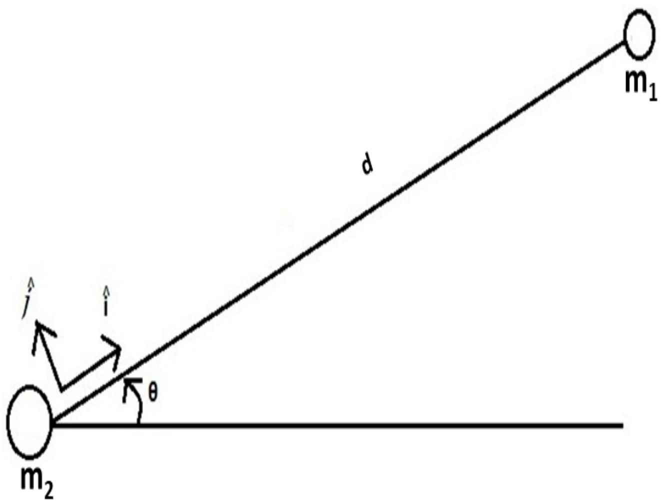

FIGURE 2.4: Velocity and acceleration components.

$$\mathbf{a} = (\ddot{d} - d\dot{\theta}^2)\hat{\imath} + \frac{1}{d}\frac{d}{dt}(d^2\dot{\theta})\hat{\jmath},$$

The above equation is used in (2.11), we obtained,

$$\ddot{d} - d\dot{\theta}^2 = -\frac{\beta}{d^2}, \tag{2.25}$$

$$\frac{1}{d}\frac{d}{dt}(d^2\dot{\theta}) = 0. \tag{2.26}$$

We get the following angular momentum integral after further integrating equation (2.21):

$$d^2\dot{\theta} = L, \tag{2.27}$$

under such type of substitution,

$$u = \frac{1}{d}, \tag{2.28}$$

The exclusion of the time between the equations (2.20) and (2.22) means that:

$$\frac{d^2u}{d\theta^2} + u = \frac{\beta}{L^2}. \tag{2.29}$$

The familiar shape of the equation above is:

$$u = \frac{\beta}{L^2} + B\cos(\theta - \theta_0), \tag{2.30}$$

Where B and θ_0 represent integration constants.

In the above equation, substitute $u = \frac{1}{d}$, equation:

$$\frac{1}{d} = \frac{\beta}{L^2} + B\cos(\theta - \theta_0)$$

$$d = \frac{\frac{L^2}{\beta}}{1 + \frac{L^2 B_1}{\beta}\cos(\theta - \theta_0)}.$$

The polar shape of the conical equation can be written as:

$$d = \frac{q}{1 + e\cos(\theta - \theta_0)},$$

where

$$q = \frac{L^2}{\beta},$$
$$e = \frac{BL^2}{\beta}. \tag{2.31}$$

The orbit of one celestial body around another is defined by eccentricity e. Thus,

1. The orbit is elliptical if $0 < e < 1$,
2. Parabolic if $e = 1$,
3. Hyperbolic if $e > 1$.

Therefore, a conic is the solution to the 2BP, including the first law of Kepler as a special case. Mathematically can be defined as,

$$e = \frac{c}{a},$$

$c \Rightarrow$ is the distance from focus to the center and,
$a \Rightarrow$ represent the semi major axis,
$b \Rightarrow$ represent the minor axis,
where,

$$a^2 = b^2 + c^2,$$
$$c^2 = a^2 - b^2.$$

2.7 In the n-Body Problem Equations of Motion

The two body issue deals with much of the crucial work in astrodynamics, but we also need to model the natural world using alternate bodies. Producing 3BP formulas is the next logical step. The n-body problem is a further generalisation of three body problems. In general, it takes a fixed number of integration constants to solve general differential equations of movements in the n-body problem. Consider a basic gravity question in which over time we have constant acceleration, $a(t) = a_0$. We get the velocity, $v(t) = a_0t + v_0$, if we integrate this equation. Again integrating provides, $d(t) = d_0 + v_0t + \frac{1}{2}$ and a_0t^2. Once again integration provides $d(t) = d_0 + v_0t + \frac{1}{2}$ and a_0t^2 respectively. The initial conditions must be known in order to finalise the solution. This example is a straight-forward analytical approach using the initial values, or a function of integration time and constants, called movement integrals. Unfortunately, this isn't always the easy scenario. If initial conditions alone do not provide a solution,The order of differential equations can be reduced by integrals of motion, also referred to as the degrees of freedom of the dynamic system, can be lowered by integrals of motion. Ideally, we should reduce it to order zero if the number of integrals is equal to the order of differential equations. These integrals are constant behaviour of the original conditions, as well as constants of the direction and velocity of motion at any time, hence the term constants.

We need $6n$ integrals of motion for a complete solution to the n-body problem, a system of $3n$ second order differential equations. Linear momentum conservation provides six, energy one conservation, and total angular momentum conservation three, for a total of ten. There are no rules similar to the first two laws of Kepler to obtain additional constants, so we are left with a $6n-10$ for $n \geq 3$ order structure. These equations defy all attempts at closed-form solutions for n bodies ie., $n \geq 3$. H.Brun, in 1887, proved that there were no other algebraic integrals. We still have only ten known integrals, although Poincaré later generalised Brun 's work. They provide us with insight into the three motions of the body and n-body problems. Conservation of complete linear momentum ensures that there are no outside forces in the system.

First of all, here we set up the n motion equations of large bodies whose radius vectors are $\mathbf{d}_i$ from an unexpedited point O, whose radius vectors are d_{ij} given as:

$$\mathbf{d}_{ij} = \mathbf{d}_j - \mathbf{d}_i, \tag{2.32}$$

According to Newton's law of gravitation,

$$m_i\ddot{\mathbf{d}}_i = G\sum_{j=1 j\neq i}^{n} \frac{m_i m_j}{d_{ij}^3}\mathbf{d}_{ij}, \qquad (i = 1, ...n). \tag{2.33}$$

Here we notice that $\mathbf{d}_{ij}$ indicates that the vector is oriented toward m_i and m_j between m_i and m_j, thus

$$\mathbf{d}_{ij} = -\mathbf{d}_{ji} \tag{2.34}$$

For n-body problem, the set of equations (2.27) is the necessary motion equation, G being the universal gravitational constant.

Chapter 3

The Circular Restricted Three Body Problem

3.1 Introduction

The problem were solved by Lagrange in 1772. Poiner, Hill and others of the so-called circular confined 3-body problem have done a great deal of research to gain insight into the possible form of motion in the three-body problem, where two massive particles pass around their centre of mass in a circle and attract a third particle of infinitesimal mass. Knowing the orbits and masses of two large particles, the problem is to determine the potential motions of the third particles, provided the coordinates and velocities of the time at any arbitrarily fixed moment of time. Therefore, the general three body problem is reduced from $9, 2^{nd}$ order differential equation to $3, 2^{nd}$ order ones i.e., a decrease from 18 integration constants to 6 constants. If the problem is further reduced, there are only two 2^{nd} order equations for the test particle to travel continuously in the orbital plane of the two massive bodies. So that the problem is of order 4. This special arrangement is called the co-planer circular restricted 3-body problem [36]. It is possible to find an integral part of motion that is critical in obtaining information on the behaviour of small particles.

3.2 Jacobin Integral [23]

Let the mass unit be such that there is a certain $m1 = 1-\mu$, where $\mu \leq 1$. We also select the distance unit as their separation constant, the unit of time is so selected that unity is also the gravitational constant G. If the $\zeta - axis$ is perpendicular to the plane of rotation of the two large bodies $\zeta_1 = \zeta_2 = 0$, the mean angular of the two bodies is now n. Now we take a set of x, y and z axes with the same origin as before, but with the x and y axes rotating perpendicular to the plane of rotation as seen in the figure, with the angular velocity unit around the $z-$axis that corresponds with the $\zeta-$axis. The position of the $x-$axis can be selected in such a way that the two large bodies always lie on it at p_1 and p_2, with coordinates $(-x_1, 0, 0)$ and $(x_2, 0, 0)$ respectively, i.e.,

$$2x_2 = 1 - \mu, \quad x_1 = -\mu, \tag{3.1}$$

$$r_1^2 = (x_1 - x)^2 + y^2 + z^2, \tag{3.2}$$

$$r_2^2 = (x_2 - x)^2 + y^2 + z^, \tag{3.3}$$

Where the infinitesimal particle coordinates with regard to the moving axes are (x, y, z). Through the following relation they are linked to the old coordination,

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \xi = \eta$$

$$\xi = x\cos t - y\sin t, \qquad \eta = x\sin t + y\cos t, \qquad \xi = \eta \tag{3.4}$$

$$\ddot{\xi} = (\ddot{x} - 2\dot{y} - x)\cos t + (\ddot{y} - 2\dot{x} + y)\sin t, \tag{3.5}$$

$$\ddot{\eta} = (\ddot{x} - 2\dot{y} - x)\sin t + (\ddot{y} + 2\dot{x} - y)\cos t, \tag{3.6}$$

$$\ddot{\zeta} = \ddot{z} \tag{3.7}$$

$$\begin{bmatrix} \xi_1 \\ \eta_1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad \xi_1 = x_1 \cos t; \qquad \eta_1 = x_1 \sin t$$

similarly

$$\begin{bmatrix} \xi_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \end{bmatrix}, \qquad \xi_2 = x_2 \cos t; \qquad \eta_2 = x_2 \sin t \quad (3.8)$$

$$n^2 a^3 = G(m_1 + m_2) = 1.$$

It is shown that unity is also due to units selected for the angular velocity of the two finite mass particles.

If the co-ordinates of $(1-\mu)$ and $-\mu$ are (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2), apply to the non-rotating axes ξ, η, ζ, and ζ respectively, then the equation of motion of the tiny particle are as under:

$$m_i \ddot{\mathbf{r}}_i = G \sum_{j=1 j\neq i}^{3} \frac{m_i m_j}{r_{ij}^3} \mathbf{r}_{ij},$$

for $i = 3$,

$$m_3 \ddot{\mathbf{r}}_3 = G \sum_{j=1}^{3} \frac{m_3 m_j}{r_{3j}^3} \mathbf{r}_{3j}, \quad (3.9)$$

$$\ddot{\mathbf{r}}_3 = \begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta} \end{bmatrix} = \frac{m_1}{r_{31}^3} \mathbf{r}_{31} + \frac{m_2}{r_{32}^3} \mathbf{r}_{32},$$

$$\ddot{\mathbf{r}}_3 = \begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta} \end{bmatrix} = \frac{1-\mu}{r_1^3} \begin{bmatrix} \xi_1 - \xi \\ \eta_1 - \eta \\ \zeta_1 - \zeta \end{bmatrix} + \frac{\mu}{r_2^3} \begin{bmatrix} \xi_2 - \xi \\ \eta_2 - \eta \\ \zeta_2 - \zeta \end{bmatrix}. \quad (3.10)$$

Also Euclidean distance is given by,

$$\left.\begin{aligned} r_1^2 &= (\xi_1 - \xi)^2 + (\eta_1 - \eta)^2 + (\zeta_1 - \zeta)^2, \\ r_2^2 &= (\xi_2 - \xi)^2 + (\eta_2 - \eta)^2 + (\zeta_2 - \zeta)^2. \end{aligned}\right\} \quad (3.10)$$

Above equations are,

$$\left.\begin{aligned} \ddot{\xi} &= 1-\mu\frac{(\xi_1-\xi)}{r_1^3}+\mu\frac{(\xi_2-\xi)}{r_2^3}, \\ \ddot{\eta} &= 1-\mu\frac{(\eta_1-\eta)}{r_1^3}+\mu\frac{(\eta_2-\eta)}{r_2^3}, \\ \ddot{\zeta} &= 1-\mu\frac{(\zeta_1-\zeta)}{r_1^3}+\mu\frac{(\zeta_2-\zeta)}{r_2^3}, \end{aligned}\right\} \tag{3.10}$$

$$(\ddot{x}-2\dot{y}-x)\cos t+(\ddot{y}-2\dot{x}+y)\sin t=[\frac{1-\mu}{r_1^3}(x_1-x)+\frac{\mu}{r_2^3}(x_2-x)]\cos t$$
$$+[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}]y\sin t,$$

$$(\ddot{y}-2\dot{y}-x)\sin t+(\ddot{y}+2\dot{x}-y)\cos t=[\frac{1-\mu}{r_1^3}(x_1-x)+\frac{\mu}{r_2^3}(x_2-x)]\sin t$$
$$-[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}]y\cos t,$$
$$\ddot{z}=-[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}]z.$$

If we multiply the equation (3.13) by *cost* and (3.14) by *sint* and add, also multiply the equation (3.13) by $-\sin t$ and (3.14) by $\cos t$ and adding, we get the following result are as under:

$$\begin{aligned} \ddot{y}-2\dot{x}-y &= -[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}]y, \\ \ddot{x}-2\dot{y}-x &= -[\frac{1-\mu}{r_1^3}(x_1-x)+\frac{\mu}{r_2^3}(x_2-x)], \\ \ddot{z} &= -[\frac{1-\mu}{r_1^3}+\frac{\mu}{r_2^3}]. \end{aligned}$$

Such equations that do not specifically contain the independent variable t are the equations of motion of tiny particles w.r.t rotating coordinates, Let us describe the function U to y,

$$U=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{r_1}+\frac{\mu}{r_2}.$$

It is readily seen that the set equation (3.16) and (3.17) may be written as:

$$\ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x},$$
$$\ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y},$$
$$\ddot{z} = \frac{\partial U}{\partial z}.$$

If we multiply equation (3.20) by $\dot{x}$, (3.21) by $\dot{y}$ and (3.22) by $\dot{z}$ and add we get,

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial z}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z},$$

This is a perfect differential because U is a function of integrating x, y, and z alone, so we get,

$$\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = 2U - C.$$

Where C is a constant of integration. The equation L.H.S of equation (3.24) is the square of the small particle's velocity in the revolving frame. Denoted it by V^2, then equation (3.24) becomes as:

$$V^2 = 2U - C = x^2 + y^2 + z^2 = x^2 + y^2 + 2\frac{1-\mu}{r_1} + \frac{\mu}{r_2} + c.$$

The Jacobi Integral often called the relative energy. It is the only movement aspect that can be accomplished in a circular 3-body problem.

3.3 Lagrange Points [23]

Although equation from (3.20) to (3.22) have no closed from analytical solution, we can use them to determine the location of the equilibrium points. These are the locations in space where the mass (m_3) would have zero velocity and zero acceleration i.e., where m_3 would appear permanently at rest relative to m_1 and m_2 and therefore appear to initial observer (to move in circular orbit around m_1 and m_2). One place of equilibrium position (also called Lagrangian point or liberation point). A body presumably stay there. The equilibrium points are therefore defined by the conditions. Substituting these conditions,

$$\dot{x} = \ddot{x} = \dot{y} = \ddot{y} = \dot{z} = \ddot{z} = 0.$$

Subtracting equation (3.20) from (3.22) we get,

$$\begin{aligned}
&-x = \frac{1-\mu}{r_1^3}(x - x_1) - \frac{\mu}{r_2^3}(x - x_2), \\
&-y = -[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}]y, \\
&= -[\frac{1-\mu}{r_1^3} + \frac{\mu}{r_2^3}]z.
\end{aligned}$$

Since $\frac{1-\mu}{r_1^3}$ and $\frac{\mu}{r_2^3} > 0$, it must therefore we true that $z = 0$ i-e., the equilibrium point lies in the orbital plane ($xy-$plane where $z = 0$).

Case-1:

If $y \neq 0$ then points are on y-axis,

$$\begin{aligned}
&\frac{1-\mu}{r_1^3} = \frac{1-\mu}{r_2^3}, \\
&x = \frac{1-\mu}{r_1^3}(x - x_1) - \frac{\mu}{r_2^3}(x - x_2), \\
&r_1^3 = 1, \qquad r_2^3 = 1, \\
&r_1 = r_2 = 1, \\
&1 = r_1^2 = (x_1 - x)^2 + y^2 + z^2, \quad z^2 = 0 \\
&1 = r_1^2 = (x_2 - x)^2 + y^2, \\
&0 = (x_1 - x)^2 - (x_2 - x)^2, \\
&x = \frac{x_1 + x_2}{2},
\end{aligned}$$

using $x = \frac{x_1+x_2}{2}$ in equation (3.30) or (3.31),

we get, $y = \sqrt{3}/2 = 60^0$,

we have thus found two equilibrium points L_4 and L_5 which are equidistant from each other.

Case-2:

If $y = 0$, then points are on x-axis,

$$r_1^2 = (x_1 - x)^2; \qquad r_2^2 = (x_2 - x)^2,$$
$$r_1^2 = (x + \mu)\hat{i} \qquad r_2^2 = (x - 1\mu)\hat{i},$$
$$r_1^2 = |x + \mu| \qquad r_2^2 = |x - 1 + \mu|.$$

Using this in equation (3.26) .

$$y = f(x) = \frac{(x - 1 + \mu)}{|x - 1 + \mu|^3}\mu + \frac{(1 - \mu)(x + \mu)}{|x + \mu|^3} - x = 0.$$

The algebraic equation (3.35) is non-linear. The root of $f(x) = 0$ yields other equilibrium points besides L_4 and L_5, i.e, L_1, L_2 and L_3. To find the first requires specifying the mass parameter μ and the using the numerical technique to obtain the other roots $(L_1 - L_3)$ for that particular value. Finally we obtain five equilibrium points $L_1 - L_5$ for restricted 3-body problem.

Chapter 4

Central Configuration in Symmetric Five Body Problem

A central configuration is a special arrangement of point masses interacting by Newton's law of gravitation with the following property: the gravitational acceleration vector produced on each mass by all the others should point toward the center of mass and be proportional to the distance to the center of mass i.e.,

$$\ddot{\mathbf{q}}_{\mathbf{i}} = -\lambda(\mathbf{q}_{\mathbf{i}} - \mathbf{c}).$$

Central configurations play an important role in the study of the Newtonian n-body problem. For example, they lead to the only explicit solutions of the equations. The equation of motion for the n-body problems can be defined as:

$$m_i \frac{d^2\mathbf{q}_{\mathbf{i}}}{dt^2} = \sum_{\substack{j=0,\\ j\neq i}}^{n} \frac{m_i m_j(\mathbf{q}_{\mathbf{j}} - \mathbf{q}_{\mathbf{i}})}{\mid \mathbf{q}_{\mathbf{i}} - \mathbf{q}_{\mathbf{j}} \mid^{\mathbf{3}}} \qquad\qquad i = 1, 2, 3.....n$$

$$\mathbf{q}_{\mathbf{0}} = (0, 0), \mathbf{q}_{\mathbf{1}} = (-x, 0), \mathbf{q}_{\mathbf{2}} = (y, 0), \mathbf{q}_{\mathbf{3}} = (x, 0), \mathbf{q}_{\mathbf{4}} = (-y, 0),$$

where unit of gravitational constant taken to be unity i.e $G = 1$. Using equations (4.1) and (4.2) we obtain the following central configuration equations for five bodies of masses $(m_0, m_1, m_2, m_3, m_4)$ i.e.,

$$\sum_{\substack{j=0,\\ j\neq i}}^{4} \frac{m_j(\mathbf{q_j}-\mathbf{q_i})}{\mid \mathbf{q_i}-\mathbf{q_j}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_i}-\mathbf{c}) \qquad\qquad i=0,1,2,3,4.$$

Now we have find four equations of motion from equation (4.3) such that.

for $i = 1$,

$$\sum_{\substack{j=0,\\ j\neq k}}^{4} \frac{m_j(\mathbf{q_j}-\mathbf{q_1})}{\mid \mathbf{q_j}-\mathbf{q_1}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_1}-\mathbf{c}),$$

$$\frac{m_0(\mathbf{q_0}-\mathbf{q_1})}{\mid \mathbf{q_0}-\mathbf{q_1}\mid^{\mathbf{3}}} + \frac{m_2(\mathbf{q_2}-\mathbf{q_1})}{\mid \mathbf{q_2}-\mathbf{q_1}\mid^{\mathbf{3}}} + \frac{m_3(\mathbf{q_3}-\mathbf{q_1})}{\mid \mathbf{q_3}-\mathbf{q_1}\mid^{\mathbf{3}}} + \frac{m_4(\mathbf{q_4}-\mathbf{q_1})}{\mid \mathbf{q_4}-\mathbf{q_1}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_1}-\mathbf{c}),$$

$$\frac{m_0\mathbf{q_1}}{x^3} + \frac{m_2\mathbf{q_{12}}}{(\sqrt{x^2+y^2})^3} + \frac{m_3\mathbf{q_{13}}}{8x^3} + \frac{m_4\mathbf{q_{14}}}{(\sqrt{x^2+y^2})^3} = -\lambda(\mathbf{q_1}-\mathbf{c}).$$

for $i = 2$,

$$\sum_{\substack{j=0,\\ j\neq k}}^{4} \frac{m_j(\mathbf{q_j}-\mathbf{q_2})}{\mid \mathbf{q_j}-\mathbf{q_2}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_2}-\mathbf{c}),$$

$$\frac{m_0(\mathbf{q_0}-\mathbf{q_2})}{\mid \mathbf{q_0}-\mathbf{q_2}\mid^{\mathbf{3}}} + \frac{m_1(\mathbf{q_1}-\mathbf{q_2})}{\mid \mathbf{q_1}-\mathbf{q_2}\mid^{\mathbf{3}}} + \frac{m_3(\mathbf{q_3}-\mathbf{q_2})}{\mid \mathbf{q_3}-\mathbf{q_2}\mid^{\mathbf{3}}} + \frac{m_4(\mathbf{q_4}-\mathbf{q_2})}{\mid \mathbf{q_4}-\mathbf{q_2}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_2}-\mathbf{c}),$$

$$\frac{m_0\mathbf{q_2}}{y^3} + \frac{m_1\mathbf{q_{21}}}{(\sqrt{x^2+y^2})^3} + \frac{m_3\mathbf{q_{23}}}{(\sqrt{x^2+y^2})^3} + \frac{m_4\mathbf{q_{24}}}{8y^3} = -\lambda(\mathbf{q_2}-\mathbf{c}).$$

for $i = 3$,

$$\sum_{\substack{j=0,\\ j\neq k}}^{4} \frac{m_j(\mathbf{q_j}-\mathbf{q_2})}{\mid \mathbf{q_j}-\mathbf{q_2}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_3}-\mathbf{c}),$$

$$\frac{m_0(\mathbf{q_0}-\mathbf{q_3})}{\mid \mathbf{q_0}-\mathbf{q_3}\mid^{\mathbf{3}}} + \frac{m_1(\mathbf{q_1}-\mathbf{q_3})}{\mid \mathbf{q_1}-\mathbf{q_3}\mid^{\mathbf{3}}} + \frac{m_2(\mathbf{q_2}-\mathbf{q_3})}{\mid \mathbf{q_2}-\mathbf{q_3}\mid^{\mathbf{3}}} + \frac{m_4(\mathbf{q_4}-\mathbf{q_3})}{\mid \mathbf{q_4}-\mathbf{q_3}\mid^{\mathbf{3}}} = -\lambda(\mathbf{q_3}-\mathbf{c}),$$

$$\frac{m_0\mathbf{q_3}}{x^3} + \frac{m_1\mathbf{q_{31}}}{8x^3} + \frac{m_2\mathbf{q_{32}}}{(\sqrt{x^2+y^2})^3} + \frac{m_4\mathbf{q_{34}}}{(\sqrt{x^2+y^2})^3} = -\lambda(\mathbf{q_3}-\mathbf{c}).$$

for $i = 4$,

$$\sum_{\substack{j=0,\\ j\neq k}}^{4} \frac{m_j(\mathbf{q_j} - \mathbf{q_4})}{\mid \mathbf{q_j} - \mathbf{q_4} \mid^{\mathbf{3}}} = -\lambda(\mathbf{q_4} - \mathbf{c}),$$

$$\frac{m_0(\mathbf{q_0} - \mathbf{q_4})}{\mid \mathbf{q_0} - \mathbf{q_4} \mid^{\mathbf{3}}} + \frac{m_1(\mathbf{q_1} - \mathbf{q_4})}{\mid \mathbf{q_1} - \mathbf{q_4} \mid^{\mathbf{3}}} + \frac{m_2(\mathbf{q_2} - \mathbf{q_4})}{\mid \mathbf{q_2} - \mathbf{q_4} \mid^{\mathbf{3}}} + \frac{m_3(\mathbf{q_3} - \mathbf{q_4})}{\mid \mathbf{q_3} - \mathbf{q_4} \mid^{\mathbf{3}}} = -\lambda(\mathbf{q_4} - \mathbf{c}),$$

$$\frac{m_0\mathbf{q_4}}{y^3} + \frac{m_1\mathbf{q_{41}}}{(\sqrt{x^2+y^2})^3} + \frac{m_2\mathbf{q_{42}}}{8y^3} + \frac{m_3\mathbf{q_{43}}}{(\sqrt{x^2+y^2})^3} = -\lambda(\mathbf{q_4} - \mathbf{c}).$$

Equations (4.5), (4.7), (4.9) and (4.11) are general central configuration equations which are using further in different cases for different purpose. This is the main equations on the base of this, we can find central configuration region.

4.1 Geometry of the Problem

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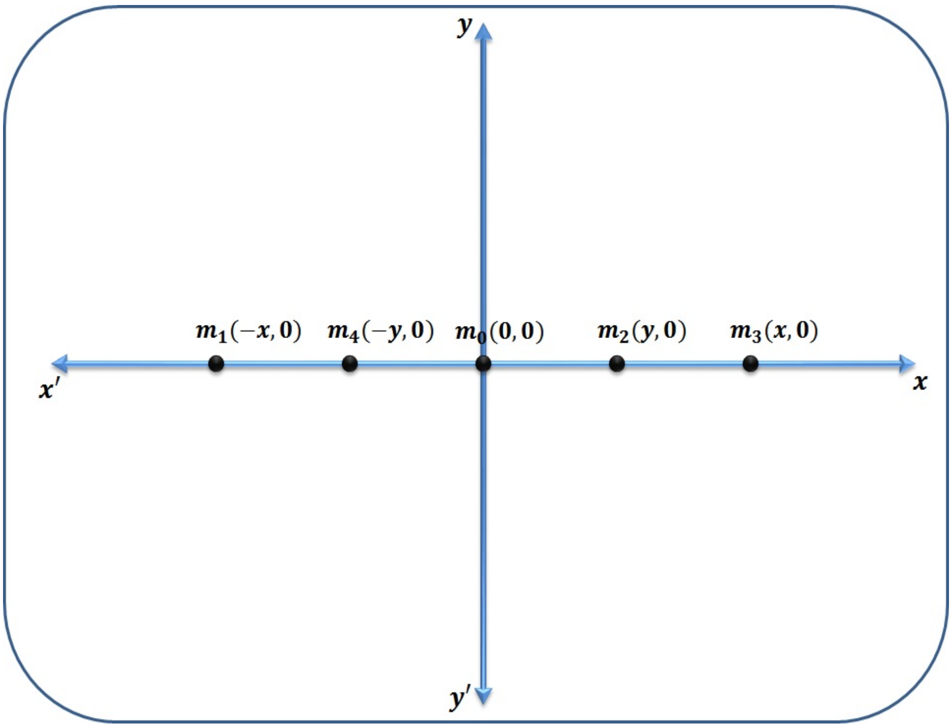

FIGURE 4.1: Five collinear masses (m_0, m_1, m_2, m_3 and m_4) along $x - axis$.

The geometry of the problem [37] showing that all the masses will remain collinear through out their motion. This idea was firstly given by the Swiss scientist Leonhard Eular. This is a special case in the five body problem where all the masses always lies on a straight line, which is clearly shown in the following Figures 4.1 and 4.2. Geometry of the problem remain the same means that all the masses lies only on x-axis. In this geometry there have no any mass on y-axis.

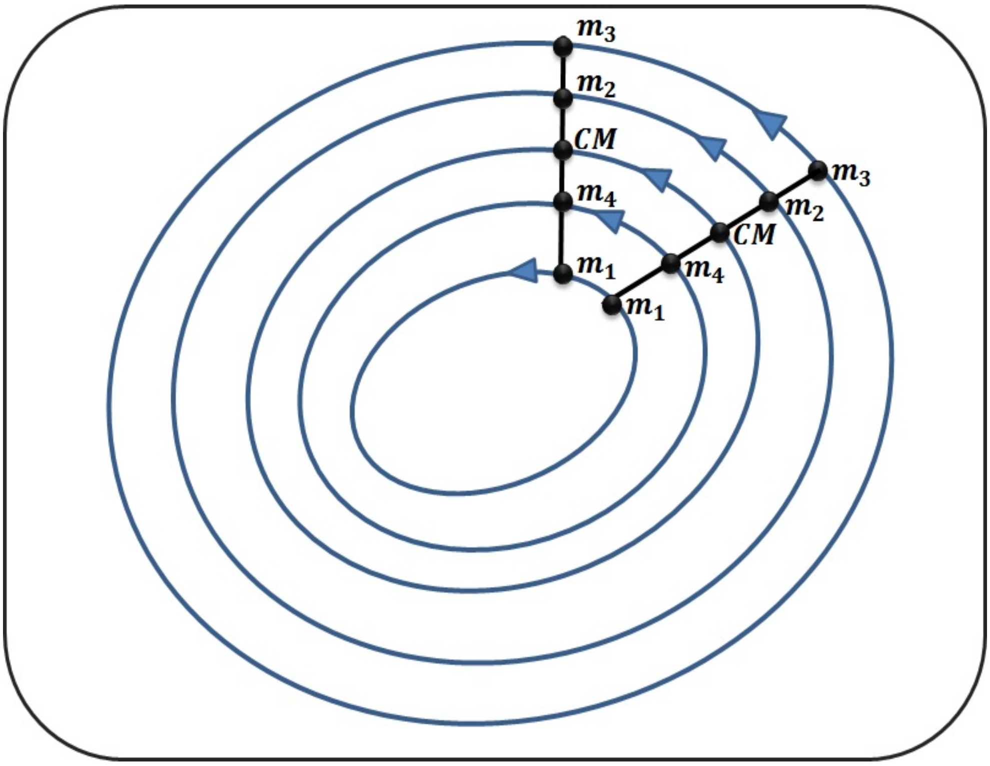

FIGURE 4.2: Masses lies in a straight line throughout their revolution.

This is some special geometry of our problem in which all the five masses rotating with their centre of mass. Actually centre of mass is the position where Newton's law of motions applies perfectly. Under by equation of central configuration all the five masses always on a straight line under their motion. Central configuration equation play very important rule in the geometry of our problem. This kind of motion is also known as rectilinear motion.

Theorem 4.1. Consider five bodies of masses $(m_0, m_1, m_2, m_3, m_4)$ located at $(0, 0)$, $(-x, 0)$, $(y, 0)$, $(x, 0)$ and $(-y, 0)$ consecutively. The mass m_0 is kept constant at the origin. Suppose that $m_1 = m_3 = 1, m_2 = m_4 = m$

1. Under this specific arrangement of polar coordinates in the collinear 5BP where $m_0(\theta) > 0$ and $r = 1$ will form CC when θ belongs to the union of this two open interval i.e., $\theta \in (-1.94, -1.04) \cup (0.74, 1.04)$. At least one of the masses would get negative for all other values of θ.

2. The CC area for r $\neq$ 1 is shown in Figures 4.3, 4.4 and 4.5.

4.2 Proof of Theorem 4.1

Equations of central configuration are,

$$\frac{m_0 q_1}{x^3} + \frac{m_2 q_{12}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{13}}{8x^3} + \frac{m_4 q_{14}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_1 - c),$$

$$\frac{m_0 q_2}{y^3} + \frac{m_1 q_{21}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{23}}{(\sqrt{x^2+y^2})^3} + \frac{m_4 q_{24}}{8y^3} = -\lambda(q_2 - c),$$

$$\frac{m_0 q_3}{x^3} + \frac{m_1 q_{31}}{8x^3} + \frac{m_2 q_{32}}{(\sqrt{x^2+y^2})^3} + \frac{m_4 q_{34}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_3 - c),$$

$$\frac{m_0 q_4}{y^3} + \frac{m_1 q_{41}}{(\sqrt{x^2+y^2})^3} + \frac{m_2 q_{42}}{8y^3} + \frac{m_3 q_{43}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_4 - c).$$

Using $m_1 = m_3 = 1$ and $m_2 = m_4 = m$ and $q_1 = (-x, 0), q_2 = (y, 0), q_3 = (x, 0)$, $q_4 = (-y, 0)$ in the imply equation (4.11),

$$\frac{m_0 q_1}{x^3} + \frac{m_2 q_{12}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{13}}{8x^3} + \frac{m_4 q_{14}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_1 - c),$$

$$\frac{m_0(-x,0)}{x^3} + \frac{m(x+y,0)}{(\sqrt{x^2+y^2})^3} + \frac{(2x,0)}{8x^3} + \frac{m(x+y,0)}{(\sqrt{x^2+y^2})^3} = -\lambda((-x,0) - c),$$

$$\frac{m_0}{x^2} + \frac{m(x+y)}{(\sqrt{x^2+y^2})^3} + \frac{1}{4x^2} + \frac{m(x-y)}{(\sqrt{x^2+y^2})^3} = -\lambda(x - c),$$

$$\frac{-m_0}{x^2}+\frac{2mx}{(\sqrt{x^2+y^2})^3}+\frac{1}{4x^2}=-\lambda(x-c).$$

Equation (4.12) become,

$$\frac{m_0q_2}{y^3}+\frac{m_1q_{21}}{(\sqrt{x^2+y^2})^3}+\frac{m_3q_{23}}{(\sqrt{x^2+y^2})^3}+\frac{m_4q_{24}}{8y^3}=-\lambda(q_2-c),$$

$$\frac{m_0q_2}{y^3}+\frac{m_1(q_1-q_2)}{(\sqrt{x^2+y^2})^3}+\frac{m_3(q_3-q_2)}{(\sqrt{x^2+y^2})^3}+\frac{m_4(q_4-q_2)}{8y^3}=-\lambda(q_2-c),$$

$$\frac{m_0(y,0)}{y^3}-\frac{(x+y,0)}{(\sqrt{x^2+y^2})^3}+\frac{(x-y,0)}{(\sqrt{x^2+y^2})^3}+\frac{m(-2y,0)}{8y^3}=-\lambda(y-c,0),$$

$$\frac{m_0}{y^2}-\frac{(x+y)}{(\sqrt{x^2+y^2})^3}+\frac{(x-y)}{(\sqrt{x^2+y^2})^3}-\frac{2my}{8y^3}=-\lambda(y-c),$$

$$\frac{m_0}{y^2}-\frac{2y}{(\sqrt{x^2+y^2})^3}-\frac{m}{4y^2}=-\lambda(y-c).$$

Equation (4.13) yield,

$$\frac{m_0q_3}{x^3}+\frac{m_1q_{31}}{8x^3}+\frac{m_2q_{32}}{(\sqrt{x^2+y^2})^3}+\frac{m_4q_{34}}{(\sqrt{x^2+y^2})^3}=-\lambda(q_3-c),$$

$$\frac{m_0q_3}{x^3}+\frac{m_1(q_1-q_3)}{8x^3}+\frac{m_2(q_2-q_3)}{(\sqrt{x^2+y^2})^3}+\frac{m_4(q_4-q_3)}{(\sqrt{x^2+y^2})^3}=-\lambda(q_3-c),$$

$$\frac{m_0(x,0)}{x^3}-\frac{(2x,0)}{8x^3}+\frac{(-x+y,0}{(\sqrt{x^2+y^2})^3}-\frac{m(x+y,0}{(\sqrt{x^2+y^2})^3}=-\lambda((x,0)-c),$$

$$\frac{m_0}{x^2}-\frac{1}{4x^2}+\frac{m(-x+y)}{(\sqrt{x^2+y^2})^3}-\frac{m(x+y)}{(\sqrt{x^2+y^2})^3}=-\lambda((x-c),$$

$$\frac{m_0}{x^2}-\frac{1}{4x^2}-\frac{2mx}{(\sqrt{x^2+y^2})^3}=-\lambda((x-c).$$

Finally equation (4.14) become,

$$\frac{m_0 q_4}{y^3} + \frac{m_1 q_{41}}{(\sqrt{x^2+y^2})^3} + \frac{m_2 q_{42}}{8y^3} + \frac{m_3 q_{43}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_4 - c),$$

$$\frac{m_0(-y,0)}{y^3} + \frac{1(q_1 - q_4)}{(\sqrt{x^2+y^2})^3} + \frac{m(q_2 - q_4)}{8y^3} + \frac{1(q_3 - q_4)}{(\sqrt{x^2+y^2})^3} = -\lambda(q_4 - c),$$

$$\frac{m_0(-y,0)}{y^3} + \frac{(-x+y,0)}{(\sqrt{x^2+y^2})^3} + \frac{(2y,0)}{8y^3} + \frac{(x+y,0)}{(\sqrt{x^2+y^2})^3} = \lambda(y+c,0),$$

$$\frac{-m_0}{y^2} + \frac{(-x+y)}{(\sqrt{x^2+y^2})^3} + \frac{2my}{8y^2} + \frac{(x+y)}{(\sqrt{x^2+y^2})^3} = -\lambda(y+c,0),$$

$$\frac{-m_0}{y^2} + \frac{2y}{(\sqrt{x^2+y^2})^3} + \frac{m}{4y^2} = \lambda(y+c).$$

Subtracting equation (4.15) from (4.17) get the following result,

$$\frac{2m_0}{x^2} - \frac{4mx}{(\sqrt{x^2+y^2})^3} - \frac{2}{4x^2} = -2\lambda x.$$

Dividing above equation (4.19) by 2 we have,

$$\frac{m_0}{x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} - \frac{1}{4x^2} = -\lambda x.$$

Also subtracting equation (4.16) from (4.18) we get,

$$\frac{-2m_0}{y^2} + \frac{2m}{4y^2} + \frac{4y}{(\sqrt{x^2+y^2})^3} = 2\lambda y.$$

Simplifying the above equation gives,

$$\frac{-m_0}{y^2} + \frac{m}{4y^2} + \frac{2y}{(\sqrt{x^2+y^2})^3} = \lambda y.$$

Using $\lambda = 1$ in equation (4.20) and (4.22) we get the followings,

$$\frac{m_0}{x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} - \frac{1}{4x^2} = -x,$$

$$\frac{-m_0}{y^2} + \frac{m}{4y^2} + \frac{2y}{(\sqrt{x^2+y^2})^3} = y.$$

Solving equations (4.23) and (4.24) simultaneously for $m(x,y)$ and $m_0(x,y)$ and eliminating m_0 from equation (4.23) yield,

$$\frac{m_0}{x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} - \frac{1}{4x^2} = -x,$$

$$m_0 = \frac{2mx^3}{(\sqrt{x^2+y^2})^3} + \frac{1}{4} - x^3.$$

Using the value of m_0 in (4.24) and solving for m,

$$\frac{m}{4y^2} - \frac{\left(\frac{2mx^3}{(\sqrt{x^2+y^2})^3} + \frac{1}{4} - x^3\right)}{y^2} + \frac{2y}{(\sqrt{x^2+y^2})^3} = y,$$

$$\frac{m}{4y^2} - \frac{2mx^3}{y^2(\sqrt{x^2+y^2})^3} - \frac{1}{4y^2} + \frac{x^3}{y^2} + \frac{2y}{(\sqrt{x^2+y^2})^3} = y,$$

$$m\left(\frac{1}{4y^2} - \frac{2x^3}{y^2(\sqrt{x^2+y^2})^3}\right) = \frac{1}{4y^2} - \frac{x^3}{y^2} - \frac{2y}{(\sqrt{x^2+y^2})^3} + y,$$

$$m\left((\sqrt{x^2+y^2})^3 - 8x^3\right) = -8y^3 + (\sqrt{x^2+y^2})^3(1 - 4x^3 + 4y^3),$$

$$m(x,y) = \frac{8y^3 - (\sqrt{x^2+y^2})^3(1 - 4x^3 + 4y^2)}{8x^3 - (\sqrt{x^2+y^2})^3}.$$

Using the value of m from equation (4.27) in (4.26) and simplifying the resulting expression for m_0,

$$m_0(x,y) = \frac{2mx^3}{(\sqrt{x^2+y^2})^3} + \frac{1}{4} - x^3,$$

$$m_0(x,y) = \frac{2x^3}{(\sqrt{x^2+y^2})^3}\left(\frac{8y^3 - (\sqrt{x^2+y^2})^3(1-4x^3+4y^2)}{8x^3-(\sqrt{x^2+y^2})^3}\right) + \frac{1}{4} - x^3,$$

$$m_0(x,y) = \frac{16x^3y^3 - 2x^3(\sqrt{x^2+y^2})^3(1-4x^3+4y^2)}{4(\sqrt{x^2+y^2})^3(8x^3-(\sqrt{x^2+y^2})^3)} + \frac{1-4x^3}{4},$$

we get,

$$m_0(x,y) = \frac{32x^3y^3(2-(\sqrt{x^2+y^2})^3) - (\sqrt{x^2+y^2})^3)(1-4x^3)}{4(\sqrt{x^2+y^2})^3(8x^3-(\sqrt{x^2+y^2})^3)}.$$

Equation (4.27) and (4.28) are impossible to solve easily for x and y. That's why we're turning it into a polar coordinate to redraft $m(x,y)$ and $m_0(x,y)$ as $m(r,\theta)$ taking $x = r\cos\theta$ and $y = r\sin\theta$. Equation (4.27) become. Using the following transformation,

$$m(x,y) = \frac{8y^3 - (\sqrt{x^2+y^2})^3(1-4x^3+4y^2)}{8x^3-(\sqrt{x^2+y^2})^3},$$

$$m(r,\theta) = \frac{8r^3\sin^3\theta) - (r^2\cos^2\theta + r^2\sin^2\theta)^{\frac{3}{2}}(1-4r^3\cos^3\theta + 4r^3\sin^3\theta)}{8(r\cos\theta)^3 - (r^2\cos^2\theta + r^2\sin^2\theta)^{\frac{3}{2}}},$$

$$m(r,\theta) = \frac{8r^3\sin^3\theta - r^3(1-4r^3\cos^3\theta + 4r^3\sin^3\theta)}{8r^3\cos^3\theta - r^3},$$

$$m(r,\theta) = \frac{8\sin^3\theta - 1 + 4r^3\cos^3\theta - 4r^3\sin^3\theta)}{8\cos^3\theta - 1},$$

$$m(r,\theta) = \frac{8\sin^3\theta - 1 + 4r^3\cos^3\theta - 4r^3\sin^3\theta)}{8(\frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta) - 1},$$

$$m(r,\theta) = \frac{8\sin^3\theta - 1 + 4r^3\cos^3\theta - 4r^3\sin^3\theta)}{6\cos\theta + 2\cos 3\theta - 1},$$

$$m(r,\theta)=\frac{1+4r^3\cos^3\theta-4(2+r^3)\sin^3\theta}{1-6\cos\theta-2\cos3\theta)},$$

This is the required expression for m in polar form, we can transform equation (4.28) using the following procedure,

$$m_0(x,y)=\frac{32x^3y^3(2-(\sqrt{x^2+y^2})^3)-(\sqrt{x^2+y^2})^3)(1-4x^3)}{4(\sqrt{x^2+y^2})^3(8x^3-(\sqrt{x^2+y^2})^3)},$$

$$m_0(r,\theta)=\frac{32(r\cos\theta)^3(r\sin\theta)^3)(2-(r^2\cos^2\theta+r^2\sin^2\theta)^{\frac{3}{2}})}{4(r^2\cos^2\theta+r^2\sin^2\theta)^{\frac{3}{2}}(8r^3\cos^3\theta-(r^2\cos^2\theta+r^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{(r^2\cos^2\theta+r^2\sin^2\theta)^3)(1-4r^3\cos^3\theta)}{4(r^2\cos^2\theta+r^2\sin^2\theta)^{\frac{3}{2}}(8r^3\cos^3\theta-(r^2\cos^2\theta+r^2\sin^2\theta)^{\frac{3}{2}}},$$

$$m_0(r,\theta)=\frac{32r^6\cos^3\theta\sin^3\theta(2-r^3)-r^6(1-4r^3\cos^3\theta)}{4r^3(8r^3\cos^3\theta-r^3)},$$

$$m_0(r,\theta)=\frac{32\cos^3\theta\sin^3\theta(2-r^3)-(1-4r^3\cos^3\theta)}{4r^3(8r^3\cos^3\theta-r^3)},$$

$$m_0(r,\theta)=\frac{32(\frac{3}{4}\cos\theta+\frac{1}{4}\cos3\theta)(\frac{3}{4}\sin\theta-\frac{1}{4}\sin3\theta)(2-r^3)-(1-4r^3\cos^3\theta)}{32(\frac{3}{4}\cos\theta+\frac{1}{4}\cos3\theta)-4},$$

$$m_0(r,\theta)=\frac{2(9\cos\theta\sin\theta-3\cos\theta\sin3\theta+3\sin\theta\cos3\theta-\sin3\theta\cos3\theta)(2-r^3)}{4(6\cos\theta+2\cos3\theta-1)}$$

$$+\frac{-1+4r^3(\frac{3}{4}\cos\theta+\frac{1}{4}\cos\theta)}{4(6\cos\theta+2\cos3\theta-1)},$$

$$m_0(r,\theta)=\frac{(18\cos\theta\sin\theta+3\cos3\theta\sin\theta-3\cos\theta\sin3\theta-\cos3\theta\sin3\theta)(2-r^3)}{4(6\cos\theta+2\cos3\theta-1)}$$

$$+\frac{-1+3r^3\cos\theta+r^3\cos3\theta}{4(6\cos\theta+2\cos3\theta-1)},$$

$$m_0(r,\theta) = \frac{9(2\sin\theta\cos\theta) - 6(\sin 3\theta\cos\theta - \cos 3\theta\sin\theta)}{4(6\cos\theta + 2\cos 3\theta - 1)}$$

$$+ \frac{-(2\sin 3\theta\cos 3\theta)(2 - r^3) - 1 + 3r^3\cos\theta + r^3\cos 3\theta}{4(6\cos\theta + 2\cos 3\theta - 1)},$$

$$m_0(r,\theta) = \frac{9\sin 2\theta - 6\sin(3\theta - \theta) - \sin 6\theta)(2 - r^3) - 1 + 3r^3\cos\theta + r^3\cos 3\theta}{4(6\cos\theta + 2\cos 3\theta - 1)},$$

$$m_0(r,\theta) = \frac{(3\sin 2\theta - \sin 6\theta)(2 - r^3) + 3r^3\cos\theta + r^3\cos 3\theta}{4(6\cos\theta + 2\cos 3\theta - 1)},$$

$$m_0(r,\theta) = \frac{6\sin 2\theta - 3r^3\sin 2\theta - 2\sin 6\theta + r^3\sin 6\theta + 3r^3\cos\theta + r^3\cos 3\theta - 1}{4(6\cos\theta + 2\cos 3\theta - 1)},$$

$$m_0(r,\theta) = \frac{(1 - 6\sin 2\theta + 2\sin 6\theta) - r^3(3\cos\theta - 3\sin 2\theta + \cos 3\theta + \sin 6\theta)}{4(1 - 6\cos\theta - 2\cos\theta)},$$

This is the required polar form of m_0.

A central configuration is generated in the 5-body problem if the position vector of each particle with regard to the center of mass is a typical scalar multiple of its own. Vector acceleration. Given a collinear configuration of five bodies, we consider the problem: under what circumstances it is possible to choose positive masses that make it central. We know that four positive masses can always be selected so that the four positions with the masses given form a central configuration. However, for an arbitrary configuration of five bodies, positive masses forming a central configuration can not always be identified. In this thesis, depending on the location x and the centre of mass $m0$,we define an expression of four masses, which gives a central configuration in the five-body collinear problem. In particular, we demonstrate that there is a compact area in which positive masses can not be centrally configured. In comparison, for any configuration in the complement of the compact area, positive masses can always be selected to make the configuration central.

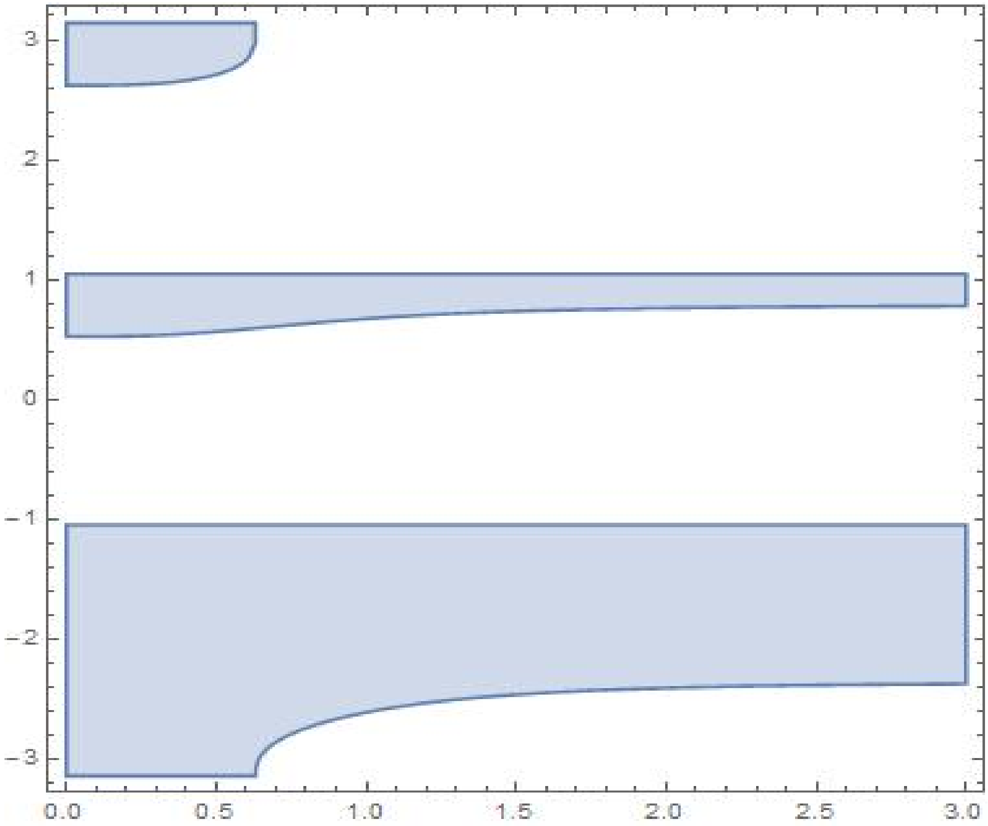

FIGURE 4.3: Central configuration region for $m(r, \theta) > 0$ where r and θ are in x and y-axis respectively.

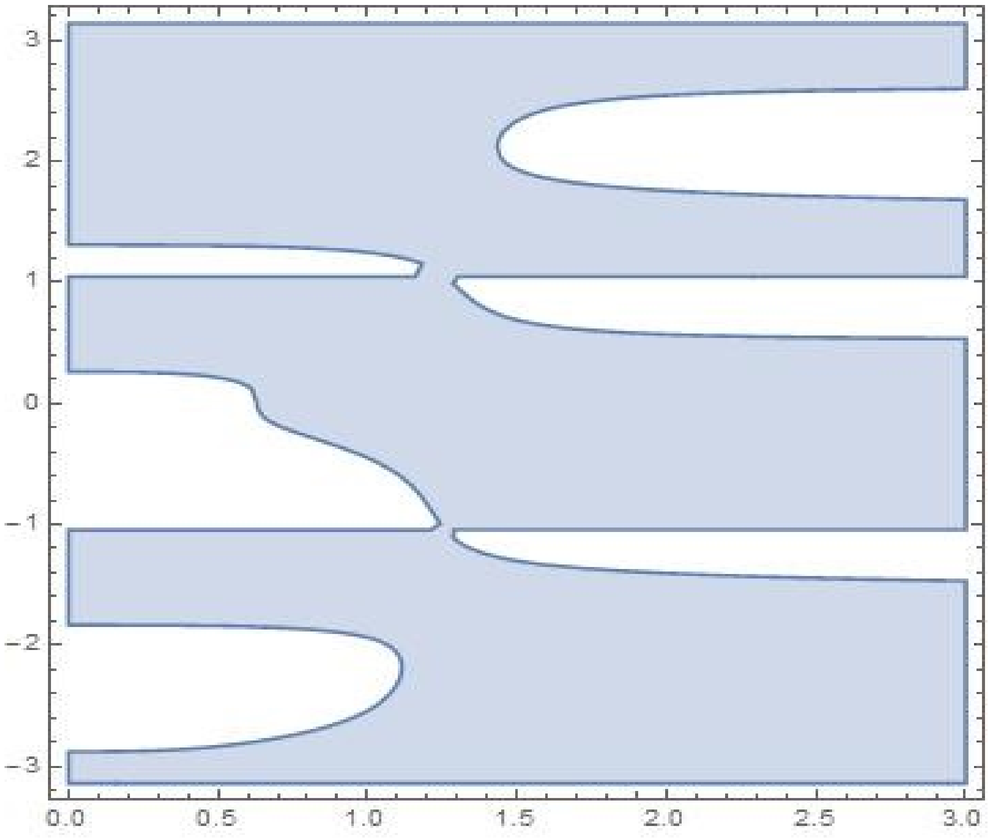

FIGURE 4.4: Similarly the shaded region (blue) show the central configuration region i.e., $m_0(r, \theta) > 0$.

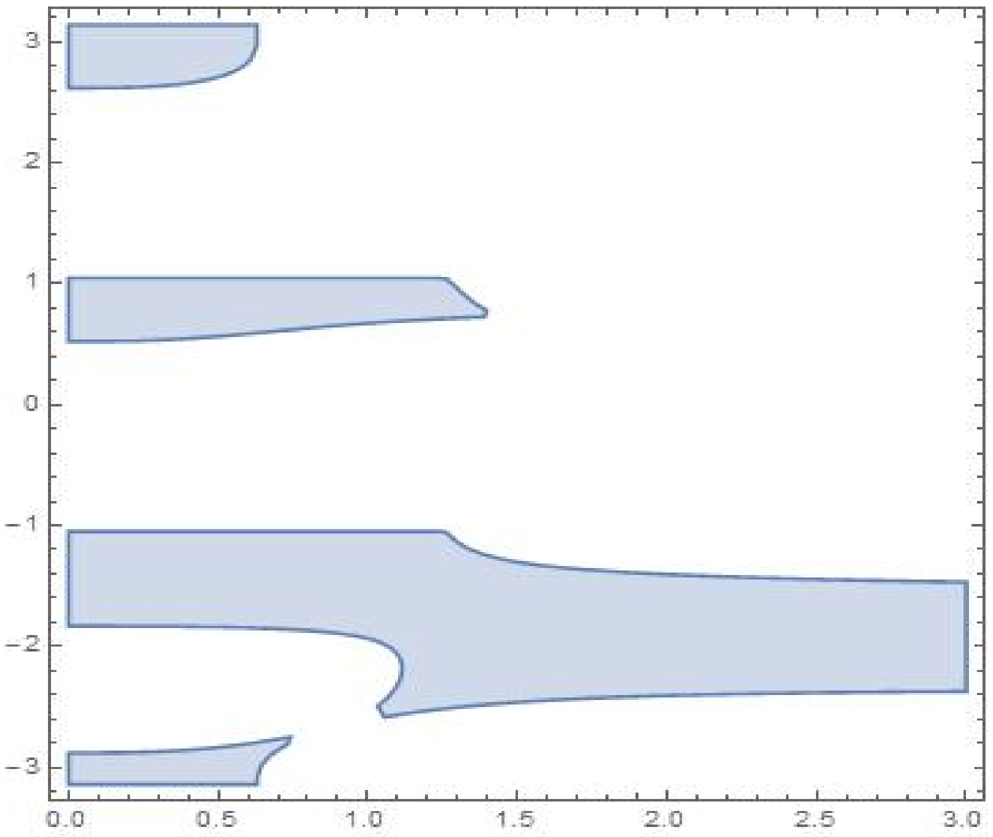

FIGURE 4.5: Central configuration region for both positive masses i.e., $m_0(r, \theta) > 0$ and $m(r, \theta) > 0$.

This is the required region where central configuration exist. That we have discussed in the earlier stage, that central configuration exist for positive masses. Taking $r = 1$, the denominator of m and m_0 goes to zero at $\theta = -\frac{\pi}{3}, \frac{\pi}{3}$. Similarly the denominator is negative when $\theta \in (-\frac{\pi}{3}, \frac{\pi}{3})$ and is positive elsewhere. At $r = 1$ the numerator of $m(\theta)$ disposed by $1 + \cos^3\theta - 12\sin^3\theta$. This has real zero when θ equal to -2.61 and 0.673. This numerator is positive when θ lies between the union of these two interval $(-2.61, -1.04)$ and $(0.673, 1.04)$, then clearly $m(\theta)$ goes to positive i.e., $m(\theta) > 0$. Hence when θ=$-2.541, -1.935, -0.449$ and 1.248 the numerator of m_0 at r equal to one is given by $-1 + 3cos\theta + cos3\theta + 3sin2\theta - sin6\theta$ has real zeros. Likewise the m_0 numerator is positive if θ belongs to $(\pi, -1.04) \cup (-1.935, -0.449) \cup (1.248, \pi)$. CC exist for this specific arrangement of collinear five body problem when, $\theta \in (-1.94, -1.04) \cup (0.74, 1.04)$ for $m(r, \theta) > 0$ and $m_0(r, \theta) > 0$ at $r = 1$. The CC region is given in the above Figures when $r \neq 1$.

Theorem 4.2. Using the symmetry of the masses i.e, $m_1 = m_3 = M$ and $m_2 = m_4 = m$ having vertices $m_1(-1,0), m_2(y,0), m_3(1,0), m_4(-y,0)$ and $m_0(0,0)$ placed on a straight line. The mass m_0 is kept constant at the origin. Then there have a region such that:

$$R_1 = (R_{1m} \cup R^*_{1m}) \cap (R_{1M} \cup R^*_{1M}).$$

The central configuration exist for positive masses,

$$R_{1m} = \{(y, m_0)|m_0 > \frac{y^3(8 - (\sqrt{1+y^2})^3)}{-8y^3 + (\sqrt{1+y^2})^3} \; and \; y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)\},$$

$$R^*_{1m} = \{(y, m_0)|m_0 > \frac{y^3(8 - (\sqrt{1+y^2})^3)}{-8y^3 + (\sqrt{1+y^2})^3} \; and \; y \in (2-\sqrt{3}, 2+\sqrt{3})\}.$$

Similarly $M(y, m_0)$ will also be positive in the following regions,

$$R_{1M} = \{(y, m_0)|m_0 > \frac{(8y^3 - (\sqrt{1+y^2})^3)}{-8 + (\sqrt{1+y^2})^3} \; and \; y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)\},$$

$$R^*_{1M} = \{(y, m_0)|m_0 > \frac{y^3(8 - (\sqrt{1+y^2})^3)}{-8y^3 + (\sqrt{1+y^2})^3} \; and \; y \in (2-\sqrt{3}, 2+\sqrt{3})\}.$$

CC does not exist in the complement of this region for $m, m_0 > 0$.

4.3 Proof of Theorem 4.2

Renaming the equations (4.11)- (4.14) as:

$$\frac{m_0 q_1}{x^3} + \frac{m_2 q_{12}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{13}}{8x^3} + \frac{m_4 q_{14}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_1 - c),$$

$$\frac{m_0 q_2}{y^3} + \frac{m_1 q_{21}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{23}}{(\sqrt{x^2+y^2})^3} + \frac{m_4 q_{24}}{8y^3} = -\lambda(q_2 - c),$$

$$\frac{m_0 q_3}{x^3} + \frac{m_1 q_{31}}{8x^3} + \frac{m_2 q_{32}}{(\sqrt{x^2+y^2})^3} + \frac{m_4 q_{34}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_3 - c),$$

$$\frac{m_0 q_4}{y^3} + \frac{m_1 q_{41}}{(\sqrt{x^2+y^2})^3} + \frac{m_2 q_{42}}{8y^3} + \frac{m_3 q_{43}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_4 - c).$$

Using the fact given in theorem 4.2 i.e., $m_1 = m_3 = M$, $m_2 = m_4 = m$ and $m_1(-1,0), m_2(y,0), m_3(1,0), m_4(-y,0)$ in the above equations. The equations (4.33)- (4.36) becomes as,

$$\frac{-m_0}{x^3} + \frac{2m}{(\sqrt{x^2+y^2})^3} + \frac{M}{4x^3} = \lambda + \lambda c,$$

$$\frac{m_0}{y^2} - \frac{2My}{(\sqrt{x^2+y^2})^3} - \frac{m}{4y^2} = -\lambda y + \lambda c,$$

$$\frac{m_0}{x^3} - \frac{2m}{(\sqrt{x^2+y^2})^3} - \frac{M}{4x^3} = -\lambda + \lambda c,$$

$$\frac{-m_0}{y^2} + \frac{2My}{(\sqrt{x^2+y^2})^3} + \frac{m}{4y^2} = \lambda y + \lambda c.$$

Subtract equation (4.37) from (4.39) to obtain the following equation,

$$\frac{m_0}{x^3} - \frac{2m}{(\sqrt{x^2+y^2})^3} - \frac{M}{4x^3} = -\lambda.$$

Also subtract equation (4.38) from (4.40) to get the following equation,

$$\frac{-m_0}{y^2} + \frac{2My}{(\sqrt{x^2+y^2})^3} + \frac{m}{4y^2} = \lambda y.$$

Now use $\lambda = x = 1$ in equations (4.41) and (4.42) we get the following equations,

$$-m_0 - \frac{2m}{(\sqrt{x^2+y^2})^3} - \frac{M}{4} = -1,$$

$$\frac{-m_0}{y^2} + \frac{2My}{(\sqrt{1+y^2})^3} + \frac{m}{4y^2} = y.$$

Solving equation (4.43) for M and simplifying to obtain the following result,

$$\frac{M}{4} = m_0 - \frac{2m}{(\sqrt{1+y^2})^3} + 1,$$

$$M = \frac{4m_0(\sqrt{1+y^2})^3 - 8m_0 + 4(\sqrt{1+y^2})^3}{(\sqrt{1+y^2})^3}.$$

Substituting the value of M from equation (4.46) in (4.44),

$$\frac{-m_0}{y^2} + \frac{8m_0 y}{(\sqrt{1+y^2})^3} - \frac{16my}{(\sqrt{1+y^2})^3} + \frac{8y}{(\sqrt{1+y^2})^3} + \frac{m}{4y^2} - y = 0,$$

$$m\left(\frac{1}{4y^2} - \frac{16y}{(1+y^2)^3}\right) = \frac{m_0(\sqrt{1+y^2})^3 - 8m_0y^3 - 8y^3 + y^3(\sqrt{1+y^2})^3}{y^2(\sqrt{1+y^2})^3},$$

$$m\left(\frac{(1+y^2)^3 - 64y^3}{4y^2(1+y^2)^3}\right) = \frac{m_0(\sqrt{1+y^2})^3) - 8y^3) + y^3(\sqrt{1+y^2})^3 - 8)}{y^2(\sqrt{1+y^2})^3},$$

$$m = \frac{m_0(\sqrt{1+y^2})^3) - 8y^3) + 4(\sqrt{1+y^2})^3)(\sqrt{1+y^2})^3 - 8)}{(1+y^2)^3 - (4y)^3},$$

$$m = \frac{m_0(\sqrt{x^2+y^2})^3) - 8y^3) + 4(\sqrt{x^2+y^2})^3)(\sqrt{x^2+y^2})^3 - 8)}{(x^2+y^2-4y)(x^4+4x^3y+y^4+4y+4y^3+16y^3)},$$

$$m = \frac{m_0(\sqrt{1+y^2} - 2y)(1+5y^2) + 2y(\sqrt{1+y^2})}{(1+y^2-4y)(1+2y^2+y^4+4y+4y^3+16y^2)}$$

$$+ \frac{4(\sqrt{1+y^2})^3)y^3(\sqrt{1+y^2})^3 - 2)(5+y^2+2(\sqrt{1+y^2}))}{(1+y^2-4y)(1+2y^2+y^4+4y+4y^3+16y^2)}.$$

Further simplify we obtain the following form,

$$m(y,m_0)=\frac{4(\sqrt{1+y^2})^3N_m(y,m_0)}{(1-4y+y^2)(1+4y+18y^2+4y^3+y^4)},$$

This is the required expression for m where,

$$N_m(y,m_0)=y^3(\sqrt{1+y^2})^3-2)(5+y^2+2(\sqrt{1+y^2})$$
$$+m_0(\sqrt{1+y^2}-2y)(1+5y^2+2y(\sqrt{1+y^2}).$$

Now on using equation (4.44) in (4.43) we get the expression for M,

$$m_0-\frac{2}{\sqrt{1+y^2}^3}\left(4y^3+4m_0-\frac{8My^3}{\sqrt{1+y^2}^3}\right)-\frac{M}{4}+1,$$

$$m_0-\frac{16My^3}{(1+y^2)^3}=m_0-\frac{8y^3}{(\sqrt{1+y^2})^3}-\frac{8m_0}{\sqrt{(1+y^2)^3}}+1,$$

$$M\left(\frac{(1+y^2)^3-64y^3}{4y^2(1+y^2)^3}\right)=\frac{m_0(\sqrt{1+y^2})^3-8y^3-8m_0+(\sqrt{1+y^2})^3}{(\sqrt{1+y^2})^3},$$

$$M\left(\frac{(1+y^2)^3-(4y)^3}{4y^2(1+y^2)^3}\right)=\frac{m_0(\sqrt{1+y^2})^3-8y^3-8m_0+(\sqrt{1+y^2})^3}{(\sqrt{1+y^2})^3},$$

$$M=\frac{m_0((\sqrt{1+y^2})^3)-2^3)+4(\sqrt{1+y^2})^3)(\sqrt{1+y^2})^3-(2y)^3)}{(1+y^2)^3-(4y)^3},$$

$$M(y,m_0)=\frac{4(\sqrt{1+y^2})^3N_M(y,m_0)}{(1-4y+y^2)(1+4y+18y^2+4y^3+y^4)}.$$

Similarly this is the required expression of M where,

$$N_M(y,m_0)=(-2y+\sqrt{1+y^2})(1+5y^2+2y(\sqrt{1+y^2})$$
$$+m_0\big(-2+\sqrt{1+y^2})(5+y^2+2(\sqrt{1+y^2})\big).$$

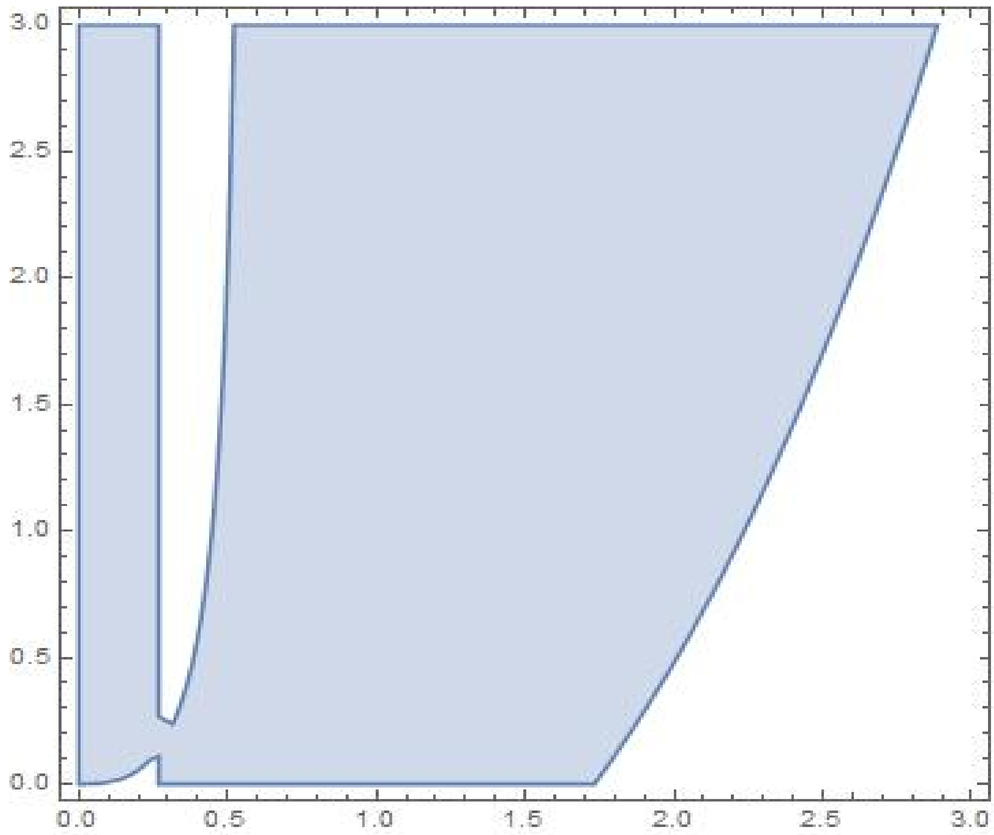

FIGURE 4.6: Region where masses are positive i.e., $m(y, m_0) > 0$. Since y are in x-axis and m_0 are in y-axis.

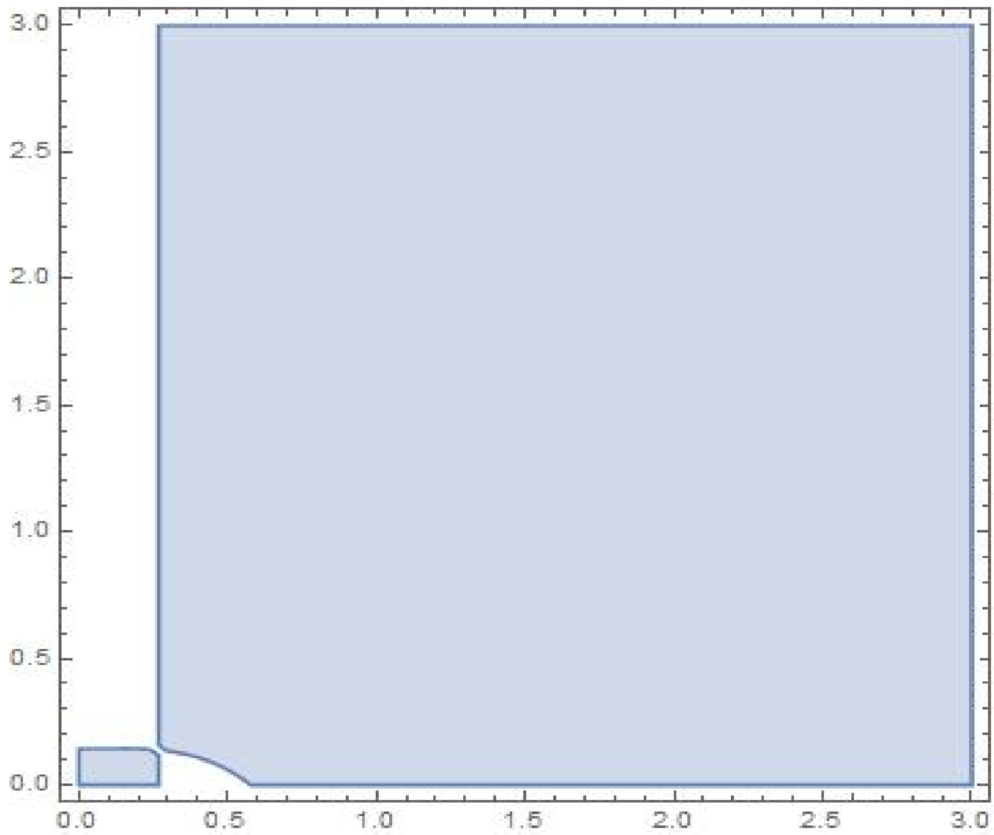

FIGURE 4.7: Region for positive masses i.e., $M(y, m_0) > 0$.

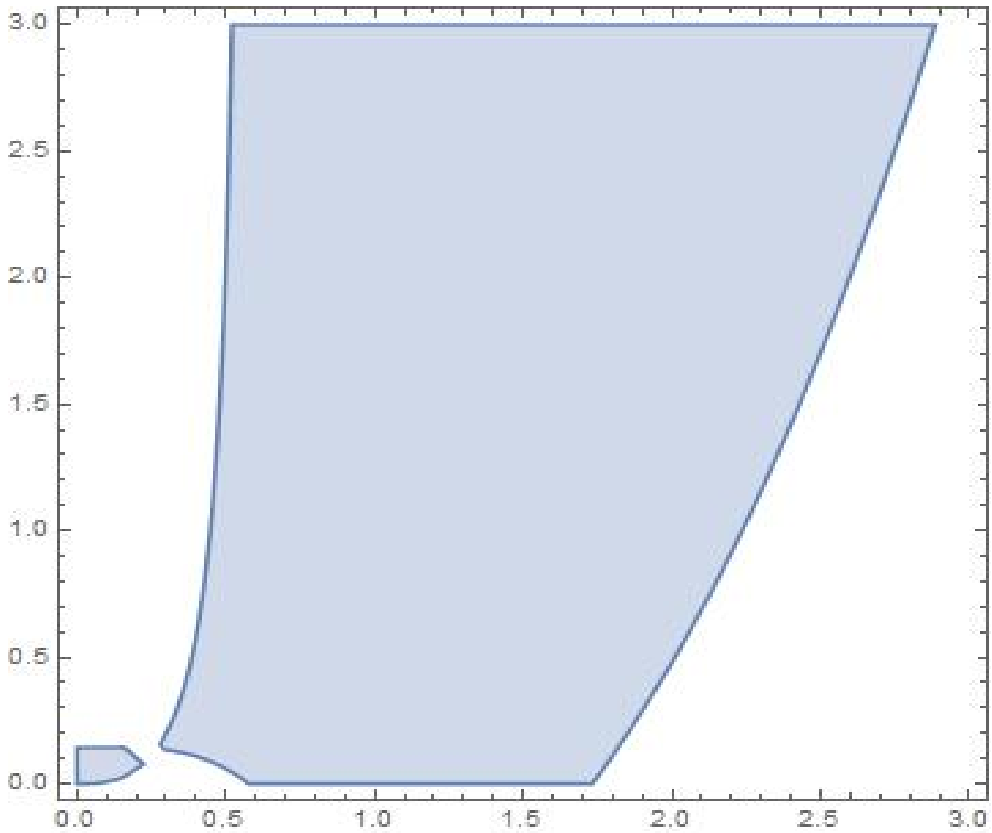

FIGURE 4.8: Region where both of them are positive i.e., $m(y, m_0) > 0$ and $M(y, m_0) > 0$.

The shaded region show the central configuration region. Actually this the region where masses are positive.

The $1-4y+y^2$ component in both m and M arithmetic denominator is positive for $y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)$ and negative for the $y \in (2-\sqrt{3}, 2+\sqrt{3})$ interval. That is the reason we should analyze the sign of m and M $N_m(y, m_0)$. The numerator of $m(y, m_0)$ has two factors $-2+\sqrt{(1+y^2)}$ and $-2y+\sqrt{(1+y^2)}$ can make $N_m(y, m_0)$ negative. The factor $-2+\sqrt{(1+y^2)}$ is positive for y belongs to $(\sqrt{3}, \infty)$ and $-2y+\sqrt{(1+y^2)}$ is positive when $y \in (0, \frac{1}{\sqrt{3}})$. Since $(0, \frac{1}{\sqrt{3}}) \cap (\sqrt{3}, \infty) = \phi$ (null set), therefore we must have following bound on the numerator of m_0 i.e, $N_m(y, m_0)$ to be positive,

$$m_0 > \frac{y^3(8-(1+y^2)^{\frac{3}{2}})}{8y^3+(1+y^2)^{\frac{3}{2}}}.$$

Hence for the following regions $m(y, m_0)$ will be positive,

$$R_{1m} = (y, m_0) \mid m_0 > \frac{y^3(8-(1+y^2)^{\frac{3}{2}})}{8y^3+(1+y^2)^{\frac{3}{2}}} \text{ and } y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty),$$

$$R^*_{1m} = (y, m_0) \mid m_0 > \frac{y^3(8-(1+y^2)^{\frac{3}{2}})}{8y^3+(1+y^2)^{\frac{3}{2}}} \text{ and } y \in (2-\sqrt{3}, 2+\sqrt{3}).$$

Similarly we can determine the positive mass for M which is given below,

$$R_{1M} = (y, m_0) \mid m_0 > \frac{(8y^3-(1+y^2)^{\frac{3}{2}}))}{-8+(1+y^2)^{\frac{3}{2}}} \text{ and } y \in (0, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty),$$

$$R^*_{1M} = (y, m_0) \mid m_0 > \frac{(8y^3-(1+y^2)^{\frac{3}{2}}))}{-8+(1+y^2)^{\frac{3}{2}}} \text{ and } y \in (2-\sqrt{3}, 2+\sqrt{3}).$$

Hence the region of CC is positive for both $m(x, y, m_0)$ and $M(x, y, m_0)$ in the following region.

$$R_1 = (R_{1m} \cup R_{1m}) \cap (R_{1M} \cup R_{1M}).$$

This is the required proof of theorem 4.2. CC regions are shown in Figures 4.6, 4.7 and 4.8.

Theorem 4.3. Let us suppose that five bodies located at $(0, 0), (-x, 0), (y, 0), (x, 0)$ and $(-y, 0)$ respectively. The m_0 mass is kept constant at the at the center of mass. Let us have $m_1 = m_3 = M, m_2 = m_4 = m$. There exist a region,

$$R_3 = ((R_d \cap R_{3m}) \cup (R_d^c \cap R_{3m}^c)) \cap (R_d^c \cap R_{3M}^c) \cup (R_d^c \cap R_{3M}^c),$$

Where it is possible to pick positive masses in the x, y plane which will allow central configuration. The required region where CC exist are,

$$R_{3m} = \{(x, y) | r(x, y) > 2y\sqrt[3]{\frac{m_0 + x^3}{m_0 + y^3}}, x > 0, y > 0, m_0 > 0\},$$

and,

$$R_{3M} = \{(x,y)|r(x,y) > 2x\sqrt[3]{\frac{m_0+x^3}{m_0+y^3}}, x>0, y>0, m_0>0\}.$$

There is no CC for $M, m, m_0 > 0$ in the complement of this region. We have to take five masses of bodies in such a manner that four of the masses kept at the vertices while the $5th$ mass remained constant in the center of mass of the system. The coordinates for the collinear five bodies are described below,

$$q_0 = (0,0), q_1 = (-x,0), q_2 = (y,0), q_3 = (x,0), q_4 = (-y,0).$$

By applying the above coordinate in the general classical equation to obtain the following central configuration equations,

$$\frac{m_0 q_1}{x^3} + \frac{m_2 q_{12}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{13}}{8x^3} + \frac{m_4 q_{14}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_1 - c),$$

$$\frac{m_0 q_2}{y^3} + \frac{m_1 q_{21}}{(\sqrt{x^2+y^2})^3} + \frac{m_3 q_{23}}{(\sqrt{x^2+y^2})^3} + \frac{m_4 q_{24}}{8y^3} = -\lambda(q_2 - c),$$

$$\frac{m_0 q_3}{x^3} + \frac{m_1 q_{31}}{8x^3} + \frac{m_2 q_{32}}{(\sqrt{x^2+y^2})^3} + \frac{m_4 q_{34}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_3 - c),$$

$$\frac{m_0 q_4}{y^3} + \frac{m_1 q_{41}}{(\sqrt{x^2+y^2})^3} + \frac{m_2 q_{42}}{8y^3} + \frac{m_3 q_{43}}{(\sqrt{x^2+y^2})^3} = -\lambda(q_4 - c).$$

4.4 Proof of Theorem 4.3

Using $m_1 = m_3 = M, m_2 = m_4 = m$ and $(-x,0), (y,0), (x,0), (-y,0)$ in the above equations to get the following equations are as under:

$$\frac{-m_0}{x^2} + \frac{2mx}{\left(\sqrt{x^2+y^2}\right)^3} + \frac{M}{4x^2} = \lambda x + \lambda c,$$

$$\frac{m_0}{y^2} - \frac{2My}{\left(\sqrt{x^2+y^2}\right)^3} - \frac{m}{4y^2} = -\lambda y + \lambda c,$$

$$\frac{m_0}{x^2} - \frac{2mx}{\left(\sqrt{x^2+y^2}\right)^3} - \frac{M}{4x^3} = -\lambda x + \lambda c,$$

$$\frac{-m_0}{y^2} + \frac{2My}{\left(\sqrt{x^2+y^2}\right)^3} + \frac{m}{4y^2} = \lambda y + \lambda c.$$

Subtracting equation (4.49) from (4.51) to get the following equation are as under,

$$\frac{m_0}{x^2} - \frac{M}{4x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} = -\lambda x.$$

Similarly subtract equation (4.50) from (4.52) to get the following result,

$$\frac{-m_0}{y^2} + \frac{m}{4y^2} + \frac{2My}{(\sqrt{x^2+y^2})^3} = -\lambda y.$$

Using $\lambda = 1$ in equation (4.53) and (4.54) which become as,

$$\frac{m_0}{x^2} - \frac{M}{4x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} = -x,$$

$$\frac{-m_0}{y^2} + \frac{m}{4y^2} + \frac{2My}{(\sqrt{x^2+y^2})^3} = y.$$

Eliminating M from equation (4.55) get the following result,

$$\frac{m_0}{x^2} - \frac{M}{4x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} + x = 0,$$

$$\frac{M}{4x^2} = \frac{m_0}{x^2} - \frac{2mx}{(\sqrt{x^2+y^2})^3} + x,$$

$$M = 4m_0 - \frac{8mx^3}{(\sqrt{x^2+y^2})^3} + 4x^3.$$

Using the value of M from equation (4.57) in (4.56) to find the value of m,

$$\frac{-m_0}{y^2} + \frac{2y}{(\sqrt{x^2+y^2})^3}\left(4m_0 - \frac{8mx^3}{(\sqrt{x^2+y^2})^3} + 4x^3\right) + \frac{m}{4y^2} - y = 0,$$

$$\frac{-m_0}{y^2} + \frac{8m_0 y}{(\sqrt{1+y^2})^3} - \frac{16mx^3y}{(\sqrt{1+y^2})^3} + \frac{8x^3y}{(\sqrt{1+y^2})^3} + \frac{m}{4y^2} - y = 0,$$

$$\frac{m}{4y^2} - \frac{16mx^3y}{(\sqrt{1+y^2})^3} = \frac{m_0}{y^2} - \frac{8m_0 y}{(\sqrt{1+y^2})^3} - \frac{8x^3y}{(\sqrt{1+y^2})^3} + y,$$

$$m\left(\frac{1}{4y^2} - \frac{16x^3y}{(x^2+y^2)^3}\right) = \frac{m_0(\sqrt{x^2+y^2})^3 - 8m_0y^3 - 8x^3y^3 + y^3(\sqrt{x^2+y^2})^3}{y^2(\sqrt{1+y^2})^3},$$

$$m\left(\frac{(x^2+y^2)^3 - 64y^3}{4y^2(x^2+y^2)^3}\right) = \frac{m_0(\sqrt{x^2+y^2})^3) - 8y^3) + y^3(\sqrt{x^2+y^2})^3 - 8x^3)}{y^2(\sqrt{1+y^2})^3},$$

$$m = \frac{m_0(\sqrt{x^2+y^2})^3) - 8y^3) + 4y^3(\sqrt{x^2+y^2})^3(\sqrt{x^2+y^2})^3 - 8x^3)}{(x^2+y^2)^3 - (4xy)^3},$$

$$m = \frac{m_0(\sqrt{x^2+y^2} - 8y^3y)}{(x^2-4xy+y^2)(x^4+4x^3y+18x^2y^2+4xy^3+y^4)},$$

$$m(x,y,m_0) = \frac{m_0(\sqrt{x^2+y^2} - 8y^3)}{(x^2-4xy+y^2)(x^4+4x^3y+18x^2y^2+4xy^3+y^4)}$$

$$+ \frac{4y^3(\sqrt{x^2+y^2})^3)(\sqrt{x^2+y^2})^3 - 8x^3)}{(x^2-4xy+y^2)(x^4+4x^3y+18x^2y^2+4xy^3+y^4)}.$$

This is the required expression for m, now we have found the value of m from equation (4.56) such that,

$$\frac{-m_0}{y^2} + \frac{m}{4y^2} + \frac{2My}{(\sqrt{x^2+y^2})^3} = y,$$

$$m = 4m_0 - \frac{8My^3}{(\sqrt{x^2+y^2})^3} + 4y^3.$$

Simplifying equation (4.55) to obtain the following result for m,

$$\frac{m_0}{x^2} - \frac{M}{4x^2} - \frac{2x}{(\sqrt{x^2+y^2})^3}\left(4m_0 - \frac{8My^3}{(\sqrt{x^2+y^2})^3} + 4y^3\right) + x = 0,$$

$$\frac{m_0}{x^2} - \frac{M}{4x^2} - \frac{8m_0x}{(\sqrt{x^2+y^2})^3} + \frac{16Mxy^3}{(\sqrt{x^2+y^2})^3} - \frac{8xy^3}{(\sqrt{x^2+y^2})^3} + x = 0,$$

$$\frac{M}{4x^2} - \frac{16Mxy^3}{(\sqrt{x^2+y^2})^3} = \frac{m_0}{x^2} - \frac{8m_0x}{(\sqrt{x^2+y^2})^3} - \frac{8xy^3}{(\sqrt{x^2+y^2})^3} + x,$$

$$M\left(\frac{1}{4x^2} - \frac{16xy^3}{(x^2+y^2)^3}\right) = \frac{m_0(\sqrt{x^2+y^2})^3 - 8m_0x^3 - 8x^3y^3 + x^3(\sqrt{x^2+y^2})^3}{x^2(\sqrt{1+y^2})^3},$$

$$M\left(\frac{(1+y^2)^3 - 64x^3y^3}{4x^2(x^2+y^2)^3}\right) = \frac{m_0(\sqrt{x^2+y^2})^3) - 8x^3) + x^3(\sqrt{x^2+y^2})^3 - 8y^3)}{x^2(\sqrt{x^2+y^2})^3},$$

$$M\left(\frac{(1+y^2)^3 - 64x^3y^3}{4x^2(x^2+y^2)^3}\right) = \frac{m_0(\sqrt{x^2+y^2})^3) - 8x^3)}{(\sqrt{x^2+y^2})^3 - (4xy)^3}$$

$$+ \frac{x^3(\sqrt{x^2+y^2})^3(\sqrt{x^2+y^2})^3 - 8y^3)}{(\sqrt{x^2+y^2})^3 - (4xy)^3},$$

$$M(x,y,m_0) = \frac{m_0(\sqrt{x^2+y^2} - 8x^3)}{(x^2-4xy+y^2)(x^4+4x^3y+18x^2y^2+4xy^3+y^4)}$$

$$+ \frac{4x^3(\sqrt{x^2+y^2})^3)(\sqrt{x^2+y^2})^3 - 8y^3)}{(x^2-4xy+y^2)(x^4+4x^3y+18x^2y^2+4xy^3+y^4)},$$

This is the required expression for M.

Chapter 5

Conclusions

In this thesis I have discussed different cases for the existence and non existence of central configuration region, and we know that the CC region exist in all those cases in which there have some positive masses. Actually CC region exist for positive mases. We have clearly show CC region graphically in chapter 4. We model a general central configuration in a symmetric 5BP, in which four of the bodies are arranged on a line and the $5th$ body placed stationary at the center of mass of the system. For m and M, we form expression in x, y and m_0, which give CC in the collinear five-body problem. Central configuration exist for positive masses as shown in each case. We also used the numerical and analytical tools for investigation of CC. There have some relationship between the masses which are placed at the center of mass of the system and remaining four masses. Here we also have discussed number of masses which have fixed arrangement of masses for which there exist a unique CC, and the similar question asked i.e., for a certain set of masses, it is possible to find positive masses whose central configuration exists or does not exist.

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