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Fixed Point Theorems on Extended b-metric-like Space with Expansive Mappings

by

Muhammad Tahir Afzal

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Dedicated

to my Parents

For their continuous support throughout my life. They always encouraged me to seek ahead and provided me with key support at every moment of life. Nothing was possible for me in life without their guidance and support. May God bless them with happiness and marvellous health.

CERTIFICATE OF APPROVAL

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Acknowledgement

In the name of Allah, the most beneficent and the most merciful

By the grace of Almighty Allah, who is the creator of the whole universe and all praises to Allah, who helped me in completing this thesis. I can never forget the efforts of my supervisor, Dr. Dur e Shehwar Sagheer, and her full support and motivation. I would like to pay my warmest thanks to her for all the invaluable and inspirational supervision. She always helped me and provided me with guidance and necessary material required to complete this task. I am sure that I would not have been able to complete this work without her guidance. I am also very thankful to the Department of Mathematics, Capital University of Science and Technology, specially to Head of department Dr. Muhammad Sagheer for providing us with such a productive learning environment.

(Muhammad Tahir Afzal)

Abstract

The idea of b-metric-like space is presented first which generalizes the notion of partial metric space, metric-like space and b-metric space. Some fixed point theorems which are established in this space are reviewed and elaborated. Intrigued by the idea of extended b-metric space provided in 2017, extended b-metric-like space is introduced. Some fixed point results are established in this new notion. An example is also provided to validate the main result.

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Abbreviations

***b*-MLS**	b-Metric-Like Space
***b*-MS**	b-Metric Space
BCP	Banach Contraction Principle
FPT	Fixed Point Theory
G.L.B	Greatest Lower Bound
inf	Infimum
L.U.B	Least Upper Bound
max	Maximum
MLS	Metric-Like Space
PMS	Partial Metric Space
PO	Partially Ordered
sup	Supremum

Symbols

X	A Non-Empty Set X
(X, ρ)	Partial Metric Space
(X, d)	Metric Space
ℓ^{∞}	Sequence Space
ℓ^{2}	Hilbert Sequence Space
$\mathbb{R}$	Set of Real Numbers
$\mathbb{N}$	Set of Natural Number
$\mathbb{Z}$	Set of Integers
(X, φ)	Metric Like Space
(X, d_{β})	b-Metric Space
(X, ϱ)	b-Metric-Like Space
$\{\alpha_n\}$	Sequence
$\subseteq$	Sub/Equal Set
$\supseteq$	Super/Equal Set
$\subset$	Subset
$\supset$	Superset
A^c	Complement of Set A
(X, d_{ϑ})	Extended b-Metric-Like Space

Chapter 1

Introduction

In the early decades of twentieth century, functional analysis is originated from classical analysis. Mainly, vector space and different operators are focused in functional analysis. It is also related to topology, abstract linear algebra and modern geometry. Its impetus originated from approximation theory, calculus of variations, ordinary and partial linear differential equations and linear integral equations has great impact on the development of modern ideas. At its earliest stage, it was used to solve differential equations and has many wide applications for non-linear problems. Recently, functional analytic methods are very useful in different areas of mathematics.

Fixed point theory is one of those branches in functional analysis which are very useful in the modern era. The fixed point theory provides us with very constructive tools to study different problems in mathematics. In the last couple of decades, it has become the main focus and interest for researchers in mathematics. Its origin from the late 19th century is in sequential and unbroken estimation to construct the existence, as well as uniqueness of results. This technique is used by many recognized mathematicians.

In 1912, Brouwer [1] elaborated fixed point result for a square and a sphere including their n-dimensional counter parts. In 1922, a very strong result in the history of mathematics was given by a Polish mathematician, Stefan Banach, [2] known as "Banach Contraction Principle". According to this "every contraction has a unique fixed point in a complete metric space". Mathematically we can say "If (X, d) is a metric space and $T : X \to X$ be a self map, which satisfies the following inequality for $\alpha_0 \in [0, 1)$,

$$d(T\alpha, T\beta) \leq \alpha_0 d(\alpha, \beta) \quad \forall \quad \alpha, \beta \in X,$$

then T has unique fixed point".

In 1961, by taking the compact space and considering $\alpha = 1$, Edelstein [3] generalized the Banach contraction principle. In 1962, E Rakotch [4] worked on the contractive mappings. He proved the existence of fixed point in complete metric space using contractive mappings. While in 1964, Micheal Edelstein [5] worked on the extension of BCP, using the non-expansive mappings.

In 1969, Kannan [6] extended Banach Contraction Principle and proved the existence and uniquesness of fixed point.

In 1975, Dass et al. [7] extended Banach contraction principle using rational expressions, which later on, further extended by Dulhare [8] using the self mapping in b-metric spaces (b-MS). In some generalizations, the contraction mapping is weakened by changing the contraction conditions (for example, see[4, 9–11]).

The fixed point theory (FPT) was considered an analytical theory at its earlier stage, but it is further divided into many other fields like topological and metric FPT.

The idea of metric space was given by Maurice Frechet [12] in 1905. A number of the generalizations of metric spaces exists in literature. Initially, the generalization of metric spaces was done by Matthews [13] as partial metric space (PMS) and by Czerwik [14] as b-metric space in the last decade of 20th century.

In 2014, another mathematician named Shukla [15] worked on these two spaces and introduced another space named as partial b-metric space. In this dissertation, b-metric-like space (b-MLS) will be focused on further useful results. There are many results of fixed point in b-MLS (e.g. see[16, 17]).

Similarly, many other mathematicians have worked on these spaces for example in 2012, Amini-Harandi [18] an Iranian mathematician presented the idea of metric-like spaces (MLS) from PMS. He also mentioned the convergence, completeness and Cauchy criteria for MLS to prove the fixed point results.

In 2013, on the basis of the concepts of b-MS, PMS and MLS, Alghamdi et al. [19] introduced b-metric-like spaces. By providing some supportive results, authors proved fixed point results for expansive mappings. They also worked on the b-MLS which are partially ordered and proved fixed point theorems.

In 2014, Zhu et al. [20] introduced the notion of qausi b-metric-like spaces. He also gives the criteria for the convergence and completeness, and proved some results showing fixed points in qausi b-metric-like space. While in 2015, Chen et al. [21] also worked on b-MLS. He generalizes many related results.

In 2017, Hammache et al. [22] also investigated b-MLS with weak contractions. In the same year, Aydi et al. [17] established some common fixed point results using implicit contractions. Many mathematicians [23–26] worked on b-MLS and qausi b-metric-like spaces from 2013 to date.

In this dissertation, the main focus is to work on b-MLS. The background of b-MLS, such as its definition, examples, completeness, convergence are adopted as given in Alghamdi's article. The detailed review of this article is presented in this thesis. Alghamdi et al. also proved some basic results in b-metric-like space to support the uniqueness of fixed point with expansive mappings. An article of Kamran et al. [27] is the source of motivation to extend some fixed point results in extended b-MLS.

The details for the rest of thesis work is as under:

Chapter 2:

This chapter includes seven sections. First section contains the definition and examples of metric space. Second to fourth sections includes the definitions and examples of different spaces. Section five includes some mappings on metric space. Section six includes the theory of fixed point and last section contains some fixed point results with different contractions.

Chapter 3:

This chapter contains the detailed review of "Fixed point and coupled fixed point theorems on b-metric-like space" by Alghamdi et al. [19].

Chapter 4:

In this chapter the idea of extended b-MLS is presented. Examples are also constructed for better understanding. Some fixed point results are also established in this new space.

Chapter 5:

Provides conclusion of the dissertation.

Chapter 2

Fundamental Material

This chapter includes necessary material required for the dissertation. Different sections are organized for the proper definitions and examples of different terminologies. Also the sections for different mappings in metric spaces and fixed point theory are organized to elaborate these terminologies clearly.

2.1 Metric Spaces

In 1905, Maurice Frechet [12] who was a french mathematician, gave the idea of metric space. This idea provides a foundation for metric fixed point theory. This section includes the definitions and examples of metric space.

Definition 2.1.1.

"A **metric space** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a real valued function defined on $X \times X \to \mathbb{R}$ such that for all a_1, $a_2 \in X$ we have:

(M1) d is real valued, finite and non negative.

(M2) $d(a_1, a_2) = 0$ if and only if $a_1 = a_2$.

(M3) $d(a_1, a_2) = d(a_2, a_1)$ (Symmetry).

(M4) $d(a_1, a_2) \leq d(a_1, a_3) + d(a_3, a_2)$ (Triangle inequality)." [28]

Example 2.1.1.

1. A distance defined on $\mathbb{R}$ as $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$d(a_1, a_2) = |a_1 - a_2|$$

is a metric and $(\mathbb{R}, d)$ is known as usual or standard metric space.

2. A metric in the n-dimensional space $\mathbb{R}^n$ with respect to the function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$d(\alpha, \beta) = \left\{ \sum_{i=1}^{n} (\alpha_i - \beta_i)^2 \right\}^{\frac{1}{2}}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, \beta_3, ..., \beta_n)$ belongs to $\mathbb{R}^n$.

Example 2.1.2.

Let $X = \mathbb{R}^2$, the set of all points in the coordinate plane. For $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ in X, define

$$d(\alpha, \beta) = \begin{cases} |\alpha_1 - \beta_1| & \alpha_2 = \beta_2 \\ |\alpha_1| + |\beta_1| + |\alpha_2 - \beta_2| & \alpha_2 \neq \beta_2 \end{cases}$$

(M1) By definition it is clear that $d(\alpha, \beta) \geq 0$.

(M2) if $\alpha_2 = \beta_2$, then $d(\alpha, \beta) = 0 \Leftrightarrow |\alpha_1 - \beta_1| = 0$

$\Leftrightarrow \alpha_1 - \beta_1 = 0 \Leftrightarrow \alpha_1 = \beta_1 \Leftrightarrow (\alpha_1, \alpha_2) = (\beta_1, \beta_2) \Leftrightarrow \alpha = \beta$

If $\alpha_2 \neq \beta_2$, then $d(\alpha, \beta) = |\alpha_1| + |\beta_1| + |\alpha_2 - \beta_2| \neq 0$.

(M3)

$$d(\alpha, \beta) = \begin{cases} |\alpha_1 - \beta_1| & \alpha_2 = \beta_2 \\ |\alpha_1| + |\beta_1| + |\alpha_2 - \beta_2| & \alpha_2 \neq \beta_2 \end{cases}$$

$$= \begin{cases} |\beta_1 - \alpha_1| & \beta_2 = \alpha_2 \\ |\beta_1| + |\alpha_1| + |\beta_2 - \alpha_2| & \beta_2 \neq \alpha_2 \end{cases}$$

$$= d(\beta, \alpha)$$

(M4)

Let $\gamma = (\gamma_1, \gamma_2)$. It is clear by definition of d that

$$|\alpha_1 - \beta_1| \leq d(\alpha, \beta), \quad |\beta_1 - \gamma_1| \leq d(\beta, \gamma) \tag{2.1}$$

If $\alpha_2 = \gamma_2$,

$$d(\alpha, \gamma) = |\alpha_1 - \beta_1| = |\alpha_1 - \beta_1 + \beta_1 - \gamma_1|$$

By (2.5)

$$\leq |\alpha_1 - \beta_1| + |\beta_1 - \gamma_1| \leq d(\alpha, \beta) + d(\beta, \gamma)$$

$$\Rightarrow d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma).$$

If $\alpha_2 \neq \gamma_2$,

then β_2 cannot be equal to both α_2, γ_2.

Let $\beta_2 \neq \alpha_2$,

then $d(\alpha, \gamma) = |\alpha_1| + |\gamma_1| + |\alpha_2 - \gamma_2|, \because \alpha_2 \neq \gamma_2$

$\Rightarrow d(\alpha, \gamma) = |\alpha_1| + |-\gamma_1| + |\alpha_2 - \gamma_2|, \because |\gamma_1| = |-\gamma_1|$

$\Rightarrow d(\alpha, \gamma) = |\alpha_1| + |-\beta_1 + \beta_1 - \gamma_1| + |\alpha_2 - \beta_2 + \beta_2 - \gamma_2|$

$\leq |\alpha_1| + |-\beta_1| + |\beta_1 - \gamma_1| + |\alpha_2 - \beta_2| + |\beta_2 - \gamma_2|$

$$= \begin{cases} |\alpha_1| + |\beta_1| + |\beta_1 - \gamma_1| + |\alpha_2 - \beta_2| & \beta_2 = \gamma_2 \\ |\alpha_1| + |\beta_1| + |\beta_1 - \gamma_1| + |\alpha_2 - \beta_2| + |\beta_2 - \gamma_2| & \beta_2 \neq \gamma_2 \end{cases}$$

$$\leq \begin{cases} |\alpha_1| + |\beta_1| + |\beta_1 - \gamma_1| + |\alpha_2 - \beta_2| & \beta_2 = \gamma_2 \\ |\alpha_1| + |\beta_1| + |\alpha_2 - \beta_2| + |\beta_1| + |\gamma_1| + |\beta_2 - \gamma_2| & \beta_2 \neq \gamma_2 \end{cases}$$

$\Rightarrow d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma), \forall \alpha, \beta, \gamma \in \mathbb{R}^2$

If $\beta_2 = \gamma_2$, then the above inequality can be proved similarly. Hence d is a metric on $\mathbb{R}^2$. This is known as *Lift Metric*.

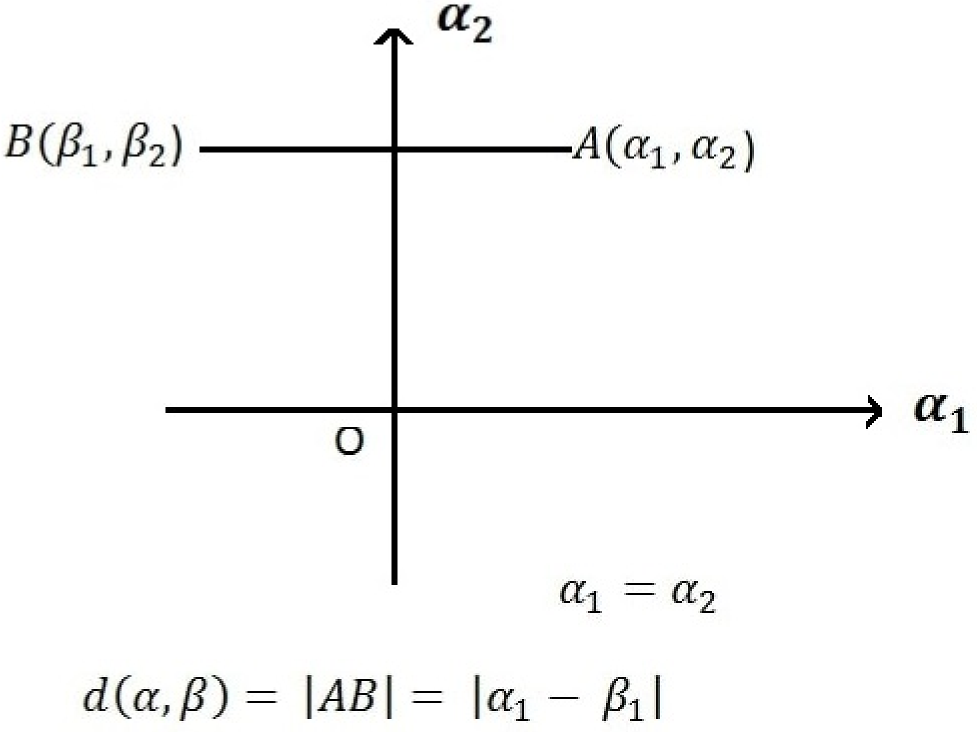

FIGURE 2.1: Case-I

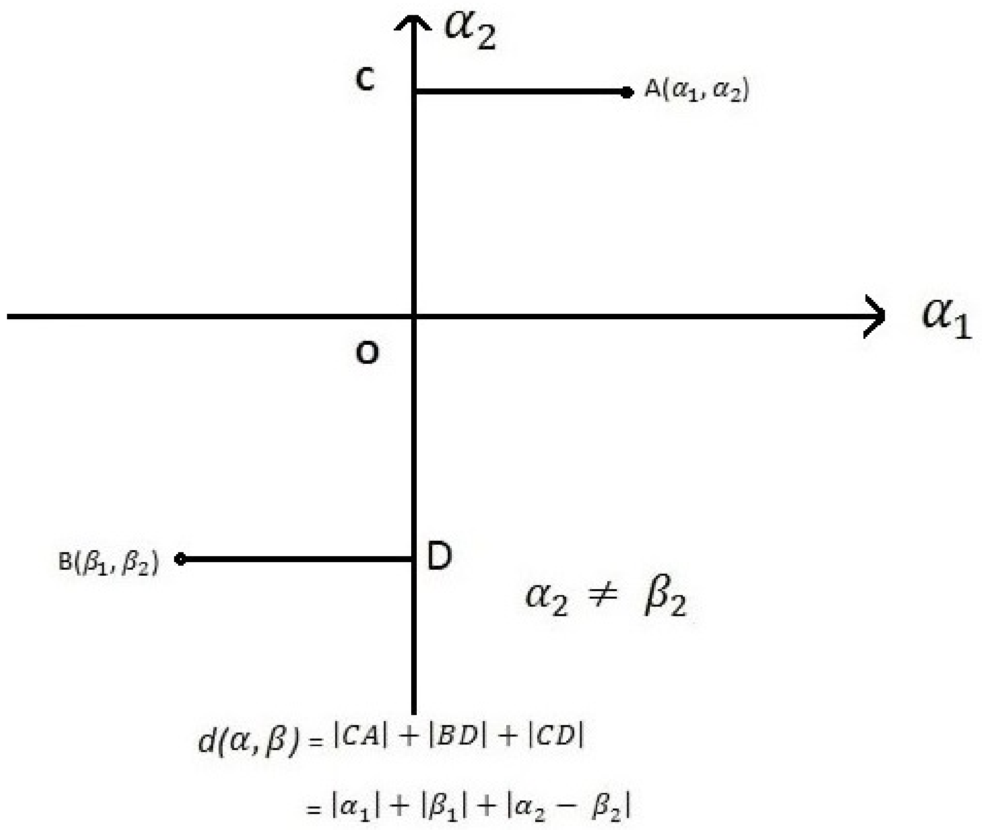

FIGURE 2.2: Case-II

Definition 2.1.2.

"Let $X = (X, d)$ be a metric space. A point $\alpha_0 \in X$ and a real number $r > 0$, we define the **open ball** as

$$\mathcal{B}(\alpha_0; r) = \{\alpha \in X \quad | d(\alpha, \alpha_0) < r\},$$

and **closed ball** can be defined as

$$\mathcal{B}(\alpha_0; r) = \{\alpha \in X \quad | d(\alpha, \alpha_0) \leq r\}." \quad [28]$$

Definition 2.1.3.

"A subset $\mathcal{K}$ of a metric space (X, d) is said to be **open set** if it contains a ball about each of its point which is contained in $\mathcal{M}$." [28]

Definition 2.1.4.

"A subset $\mathcal{M}$ of a metric space (X, d) is said to be **closed set** if its compliment (in X) is open, that is, $\mathcal{M}^c = X - \mathcal{K}$." [28]

Example 2.1.3.

From complex numbers we take X as the set of all bounded sequences so that each element of X is a complex sequence

$$\alpha = (\eta_1, \eta_2, \eta_3, ...)$$

In short $\alpha = (\eta_k)$

so for all $k = 1, 2, 3, ...,$

we have

$$|\eta_k| \leq c_\alpha,$$

where the real number c_α is dependent on α, but not on k. We take the metric defined as

$$d(\alpha, \beta) = \sup_{k \in \mathbb{N}} |\eta_k - \xi_k|.$$

Where $\beta = (\xi_k) \in X$ and $\mathbb{N} = \{1, 2, 3, ...\}$. Then the space of all such sequences is known as **sequence space**.

Example 2.1.4.

Suppose that ℓ^1 representing the set of all sequences $\{\alpha_n\}$ in $\mathbb{R}$ so that the series $\sum_{n=1}^{\infty} |\alpha_n|$ is convergent. Define a function $d : \ell^1 \times \ell^1 \to \mathbb{R}$ as

$$d(\alpha, \beta) = \sum_{n=1}^{\infty} |\alpha_n - \beta_n|, \quad \forall \alpha = \{\alpha_n\}, \beta = \{\beta_n\} \in \ell^1$$

is a *metric space.*

(M1) to (M3) are obvious.

(M4)

$$\begin{aligned} d(\alpha, \gamma) &= \sum_{n=1}^{\infty} |\alpha_n - \gamma_n| \\ &= \sum_{n=1}^{\infty} |\alpha_n - \beta_n + \beta_n - \gamma_n| \\ &\leq \sum_{n=1}^{\infty} [|\alpha_n - \beta_n| + |\beta_n - \gamma_n|] \\ &= \sum_{n=1}^{\infty} |\alpha_n - \beta_n| + \sum_{n=1}^{\infty} |\beta_n - \gamma_n| \end{aligned}$$

$$\Rightarrow d(\alpha, \gamma) \leq d(\alpha, \beta) + d(\beta, \gamma), \quad \text{for all} \quad \alpha, \beta, \gamma \in \ell^1.$$

Hence (ℓ^1, d) is proved as a *metric space.*

Definition 2.1.5.

"A sequence $\{\alpha_n\}$ in a metric space (X, d) is said to converge or to be **convergent** if there is an $\alpha \in X$,

such that

$$\lim_{n\to\infty} d(\alpha_n, \alpha) = 0,$$

α is called the limit of $\{\alpha_n\}$ and we write,

$$\lim_{n\to\infty} \alpha_n = \alpha,$$

or simply, $\alpha_n \to \alpha$." [28]

Example 2.1.5.

Let $X = \mathbb{R}$, and consider a sequence $\{\alpha_n\} = \frac{1}{n}$, with

$$d(\alpha, \beta) = |\alpha - \beta|,$$

$\{\alpha_n\}$ is convergent and $\lim_{n\to\infty} d(\alpha_n, 0) = 0$.

Definition 2.1.6.

"A sequence $\{\alpha_n\}$ in a metric space (X, d) is said to be **Cauchy** (or fundamental) if for every $\varepsilon > 0$ there is an $\mathbb{N} = \mathbb{N}(\varepsilon)$ such that

$$d(\alpha_m, \alpha_n) < \varepsilon,$$

for all $m, n > \mathbb{N}$." [28]

Remark 2.1.1.

Every convergent sequence is known as Cauchy sequence but converse is not true.

Definition 2.1.7.

"The space X is said to be **complete** if every Cauchy sequence in X converges (that is, has a limit which is an element of X)." [28]

Remark 2.1.2.

The complex plane $\mathbb{C}$ and the real line $\mathbb{R}$ are the examples of complete metric space.

Definition 2.1.8.

For the comparison of two binary data strings, we can use the hamming distance. We can achieve this purpose by comparing two such binary strings which are equal in length. Hamming distance is defined on the basis of bit positions describing where the the two bits differ by each other. For the two strings p and q, Hamming distance is denoted by $d(p, q)$. It is calculated by performing XOR operation between the two strings.

Example 2.1.6.

Let X be the set of all binary strings of length 9. Consider the two strings 100110110 and 101110011.

$$100110110 \oplus 101110011 = 001000101$$

As it contains three 1's. So, its Hamming distance is described as

$$d(100110110, 101110011) = 3.$$

Example 2.1.7.

Let X be the set of all such functions $\alpha, \beta, ...$ which are real-valued functions. These are functions of real variable s which is independent and also the functions are continuous and defined on a closed interval $K = [p, q]$. Choosing a metric defined as

$$d(\alpha, \beta) = \max_{s \in K} |\alpha(s) - \beta(s)|,$$

we get a metric space denoted by $C[p, q]$. This space is known as **function space** because each of its point is a function.

2.2 Partial Metric Space

This section is dedicated to the notion of PMS. In 1980, the idea of PMS is presented by Steve Matthews [13]. Matthews was working in the field of Computer Science. For his studies, he had to encounter the self distances which are non-zero. Matthews gave a new idea of metric space in which the self distances are non-zero. His work was first published in 1994. This section includes the definition and examples of PMS.

Definition 2.2.1.

"A partial metric on a (nonempty) set X is a function $\rho : X \times X \to \mathbb{R}_+$ such that for all a_1, a_2, $a_3 \in X$:

(P1) $a_1 = a_2 \Leftrightarrow \rho(a_1, a_1) = \rho(a_1, a_2) = \rho(a_2, a_2)$;

(P2) $\rho(a_1, a_1) \leq \rho(a_1, a_2)$;

(P3) $\rho(a_1, a_2) = \rho(a_2, a_1)$;

(P4) $\rho(a_1, a_3) \leq \rho(a_1, a_2) + \rho(a_2, a_3) - \rho(a_2, a_2)$.

A **partial metric space** is a pair (X, ρ) such that X is a nonempty set and ρ is a partial metric on X." [29]

Example 2.2.1.

Consider any $X = \mathbb{R}_+$ and let $\rho : X \times X \to \mathbb{R}_+$ which is defined as

$$\rho(\alpha, \beta) = \max\{\alpha, \beta\},$$

this gives (X, ρ) as PMS.

$(P1)$ to $(P3)$ are obvious.

(P4)

$$\begin{aligned}
\rho(\alpha, \gamma) &= \max\{\alpha, \gamma\} \\
&\leq \max\{\alpha + \beta - \beta, \gamma + \beta - \beta\} \\
&= \max\{\alpha, \beta\} + \max\{\beta, \gamma\} - \max\{\beta, \beta\} \\
&= \rho(\alpha, \beta) + \rho(\beta, \gamma) - \rho(\beta, \beta). \\
\Rightarrow \rho(\alpha, \gamma) &\leq \rho(\alpha, \beta) + \rho(\beta, \gamma) - \rho(\beta, \beta).
\end{aligned}$$

This shows that the given function is a PMS.

Definition 2.2.2.

"Let (X, ρ) be a partial metric space.

(i) A sequence $\{\alpha_n\}$ in a partial metric space (X, ρ) is said to be **convergent** to a point $\alpha \in X$ if and only if

$$\rho(\alpha, \alpha) = \lim_{n \to \infty} \rho(\alpha, \alpha_n).$$

(ii) A sequence $\{\alpha_n\}$ in a partial metric space (X, ρ) is called a **Cauchy sequence** if there exist (and is finite)

$$\lim_{m,n \to \infty} \rho(\alpha_m, \alpha_n).$$

(iii) A partial metric space (X, ρ) is called **complete** if and only if every Cauchy sequence $\{\alpha_n\}$ in X converges to a point $\alpha \in X$

such that

$$\rho(\alpha, \alpha) = \lim_{m,n \to \infty} \rho(\alpha_m, \alpha_n)." \quad [30]$$

2.3 b-Metric Space

The notion of b-metric space was firstly presented by Bakhtin [31] in 1989. Also in 1993, Czerwik [14] gave its formal definition. Another mathematician Bourbaki [32] also worked on this idea. This section includes the definition and examples of the said space.

Definition 2.3.1.
"Let X be a set, and $\beta \geq 1$ be a real number. A function $d_\beta : X \times X \to \mathbb{R}^+$ is said to be a b-metric on X, and the pair (X, d_β) is called a ***b*-metric space** if, for all a_1, a_2, $a_3 \in X$,

(BM1) $d_\beta(a_1, a_2) = 0$ if and only if $a_1 = a_2$,

(BM2) $d_\beta(a_1, a_2) = d_\beta(a_2, a_1)$,

(BM3) $d_\beta(a_1, a_3) \leq \beta(d_\beta(a_1, a_2) + d_\beta(a_2, a_3))$." [33]

Remark 2.3.1.
i. As b-MS is originated from metric space, so if we choose $\beta = 1$, then the above condition leads towards the metric space.

ii. The set of b-MS is bigger than the set of metric spaces.

Example 2.3.1.
Let (X, d) be a metric space. Then for a real number $m > 1$. we define a function $d_\beta : X \times X \to \mathbb{R}^+$ by

$$d_{\beta 1}(\alpha, \beta) = (d_\beta(\alpha, \beta))^m,$$

this gives $d_{\beta 1}$ as a b-MS with its coefficient $\beta = 2^{m-1}$.

For proof we will use the inequality

$$\left(\frac{\alpha+\beta}{2}\right)^m \leq \frac{\alpha^m+\beta^m}{2}$$

$$\frac{(\alpha+\beta)^m}{2^m} \leq \frac{\alpha^m+\beta^m}{2}$$

$$(\alpha+\beta)^m \leq 2^{m-1}\left(\alpha^m+\beta^m\right).$$

Since for every $\alpha, \beta, \gamma \in X$ we get

(BM1)

$$\begin{aligned} d_{\beta 1}(\alpha, \beta) = (d_\beta(\alpha,\beta))^m &= 0. \\ &\Rightarrow d_\beta(\alpha,\beta) = 0. \\ &\Rightarrow \alpha = \beta. \end{aligned}$$

(BM2) Clearly, it holds.

(BM3)

$$\begin{aligned} d_{\beta 1}(\alpha,\gamma) = (d_\beta(\alpha,\gamma))^m &\leq \left[d_\beta(\alpha,\beta) + d_\beta(\beta,\gamma)\right]^m. \\ &\leq 2^{m-1}\left[d_\beta(\alpha,\beta)^m + d_\beta(\beta,\gamma)^m\right]. \\ &\leq 2^{m-1}\left[d_{\beta 1}(\alpha,\beta) + d_{\beta 1}(\beta,\gamma)\right]. \\ \Rightarrow d_\beta(\alpha,\gamma) &\leq \beta\left[d_\beta(\alpha,\beta) + d_\beta(\beta,\gamma)\right]. \end{aligned}$$

Hence the given function $d_{\beta 1}$ represents a b-MS having coefficient 2^{m-1}.

Example 2.3.2.

Consider the set $X = [0, \infty)$ and define $d_\beta : X \times X \to [0,\infty)$ by

$$d_\beta(p,q) = \mid p-q \mid^2, \quad \forall \quad p,q \in X.$$

Then (X, d_β) is a b-MS with $\beta = 2$.

$(BM1)$ and $(BM2)$ are obvious.

(BM3)

$$\begin{aligned} d_\beta\,(p,r) &= \mid p-r\mid^2 \\ &= \mid p-q+q-r\mid^2 \\ &= \mid p-q\mid^2 + \mid q-r\mid^2 + 2\mid p-q\mid\mid q-r\mid \\ &\leq 2\left[\mid p-q\mid^2 + \mid q-r\mid^2\right] \end{aligned}$$

$$\Rightarrow d_\beta\,(p,r) = 2\left[d_\beta\,(p,q) + d_\beta\,(q,r)\right]$$

Hence the given function represents a b-MS having coefficient 2.

Remark 2.3.2.

"Let (X, d_β) be a b-metric space. Then in general b-metric is not continuous." [34]

Following example illustrates the above remark.

Example 2.3.3.

Suppose $X = \mathbb{N} \cup \{\infty\}$. Define a function $d_\beta : X \times X \to \mathbb{R}$ as

$$d_\beta(s,t) = \begin{cases} 0 & \text{if} \quad s = t, \\ |\frac{1}{s} - \frac{1}{t}| & \text{if one of the } s,t \text{ is even and the other is also even or } = \infty, \\ 7 & \text{if one of the } s,t \text{ is odd and others is also odd } s \neq t \text{ or } \infty \\ 2 & \text{otherwise} \end{cases}$$

Now we can check that for every $s, t, r \in X$, we get

$$d_\beta(s,r) \leq \frac{7}{2}\,(d_\beta(s,t) \ + \ d_\beta(t,r)).$$

So (X, d_β) is a b-metric space with coefficient $(\beta = \frac{7}{2})$. Suppose $x_t = 2t$ for every $t \in \mathbb{N}$, then

$$d_\beta(2t, \infty) = \frac{1}{2t} \ \to \ 0 \quad \text{as} \quad t \ \to \ \infty,$$

now, $\{x_t\} \to \infty$, but $d(x_t, 1) = 2 \neq 7 = d(\infty, 1)\ as\ t \to \infty$.

Definition 2.3.2.

"Let (X, d_β) be a b-metric space. A sequence $\{\alpha_n\}$ in X is said to be:

(i) **Cauchy** if and only if

$$\lim_{m,n\to\infty} d_\beta(\alpha_m, \alpha_n) \to 0 \quad as \quad m, n \to \infty.$$

(ii) **Convergent** if and only if there exist $\alpha \in X$ such that $d_\beta(\alpha_n, \alpha) \to 0$ as $n \to \infty$ and we write

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

(iii) The b-metric space (X, d_β) is **complete** if every Cauchy sequence is convergent." [27]

2.4 Metric-Like Space

The generalized form of PMS is MLS. In 2012, the idea of MLS was presented by Amini-Harandi [18]. This section includes the definitions and examples of MLS.

Definition 2.4.1.
"A mapping $\varphi : X \times X \to \mathbb{R}_+$, where X is non empty set, is said to be a metric-like on X if for any a_1, a_2, $a_3 \in X$, the following three conditions hold true for the given mapping:

(ML1) $\varphi(a_1, a_2) = 0 \Rightarrow a_1 = a_2$;

(ML2) $\varphi(a_1, a_2) = \varphi(a_2, a_1)$;

(ML3) $\varphi(a_1, a_3) \leq \varphi(a_1, a_2) + \varphi(a_2, a_3)$;

The pair (X, φ) is called a **metric-like space**." [18]

Example 2.4.1.
Consider the set $X = [0, \infty)$, and $\varphi : X \times X \to \mathbb{R}$ by

$$\varphi(a_1, a_2) = \max\{a_1, a_2\},$$

we claim that φ is a metric-like space as:

(ML1)

$$\varphi(a_1, a_2) = \max\{a_1, a_2\} = 0$$

$$\Rightarrow a_1 = a_2$$

If the *maximum* is 0, then the other values of this function should must be less than 0, which is not possible due to the given domain $X = [0, \infty)$. So, the other values will also be 0.

(ML2)

$$\varphi(a_1, a_2) = \max\{a_1, a_2\} = \max\{a_2, a_1\} = \varphi(a_2, a_1)$$

(ML3)

$$\begin{aligned}\varphi(a_1, a_3) &= \max\{a_1, a_3\} \\ &\leq \max\{a_1, a_2, a_3\} \\ &\leq \max\{a_1, a_2\} + \max\{a_2, a_3\}\end{aligned}$$

So,

$$\varphi\{a_1, a_3\} \leq \varphi\{a_1, a_2\} + \varphi\{a_2, a_3\}$$

Definition 2.4.2.

"A sequence $\{\alpha_n\}$ in a metric-like space (X, φ) is said to be **convergent** to a point $\alpha \in X$ if and only if

$$\lim_{n \to +\infty} \varphi(\alpha_n, \alpha) = \varphi(\alpha, \alpha),$$

exists." [18]

Definition 2.4.3.

"A sequence $\{\alpha_n\}$ of elements of X is called **Cauchy** if the limit $\lim_{m,n \to +\infty} \varphi(\alpha_m, \alpha_n)$ exists and is finite." [18]

Definition 2.4.4.

"The metric-like space (X, φ) is called **complete** if for each φ-Cauchy sequence

$\{\alpha_n\}$, There is some $\alpha \in X$ such that

$$\lim_{n\to+\infty} \varphi(\alpha_n, \alpha) = \varphi(\alpha, \alpha) = \lim_{m,n\to+\infty} \varphi(\alpha_m, \alpha_n).\text{''} \quad [18]$$

2.5 Mappings on Metric Spaces

This section addresses some important mappings on metric space. These mappings play a fundamental role in the field of metric FPT.

Definition 2.5.1.

"Let (X, d) and $(\mathcal{Y}, d^a)$ be metric spaces. A mapping $\mathcal{T} : X \to \mathcal{Y}$ is said to be **continuous** at a point $\alpha_0 \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d^a(\mathcal{T}_\alpha, \mathcal{T}_{\alpha_0}) < \varepsilon \quad \text{for all} \quad \alpha \text{ satisfying} \quad d(\alpha, \alpha_0) < \delta.\text{''} \quad [28]$$

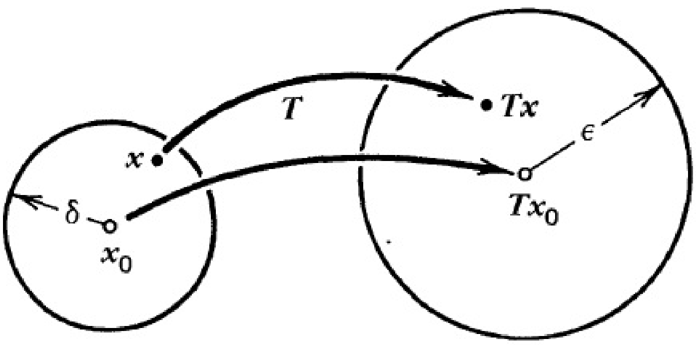

FIGURE 2.3: Continuous Mapping

Theorem 2.5.2

"Consider a mapping $\mathcal{T} : X \to \mathcal{Y}$ of a metric space (X, d) into another metric space (X, d^a). The mapping is said to be continuous at a point $\alpha_0 \in X$ iff

$$\alpha_n \to \alpha_0 \quad \text{yields} \quad \mathcal{T}\alpha_n \to \mathcal{T}\alpha_0.\text{''} \quad [28]$$

Proof.

Let the mapping $\mathcal{T}$ be continuous at α_0. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(\alpha, \alpha_0) < \delta \quad \text{yields} \quad d(\mathcal{T}\alpha, \mathcal{T}\alpha_0) < \varepsilon.$$

Consider $\alpha_n \to \alpha$. So, an $\mathbb{N}$ exists there such that for each $n > \mathbb{N}$,

implies

$$d(\alpha_n, \alpha_0) < \delta.$$

Therefore, for every $n > \mathbb{N}$,

$$d(\mathcal{T}\alpha_n, \mathcal{T}\alpha_0) < \varepsilon.$$

By definition, this yields

$$\mathcal{T}\alpha_n \to \mathcal{T}\alpha_0.$$

Conversely we consider

$$\alpha_n \to \alpha.$$

This yields

$$\mathcal{T}\alpha_n \to \mathcal{T}\alpha.$$

To prove $\mathcal{T}$ is continuous at α_0. Assume that this is not true. So there exists an $\varepsilon > 0$ so that for every $\delta > 0$ there exists an $\alpha \neq \alpha_0$,

satisfying

$$d(\alpha, \alpha_n) < \delta \quad \text{but} \quad d^a(\mathcal{T}\alpha, \mathcal{T}\alpha_n) \geq \varepsilon.$$

Particularly, for $\delta = \dfrac{1}{n}$ there exists an α_n satisfying

$$d(\alpha, \alpha_0) < \frac{1}{n} \quad \text{but} \quad d^a(\mathcal{T}\alpha, \mathcal{T}\alpha_0) \geq \varepsilon.$$

Now, $\alpha_n \to \alpha$ but $(\mathcal{T}\alpha_n)$ not converges to $(\mathcal{T}\alpha_0)$. This contradicts our supposition. So, $\mathcal{T}\alpha_n \to \mathcal{T}\alpha_0$. $\square$

Definition 2.5.3.

"Let (X, d) be a metric space. A mapping $\mathcal{F} : X \to X$ is said to be **Lipschitzian** if there exist a constant $\varepsilon \geq 0$ such that

$$d(\mathcal{F}_\alpha, \mathcal{F}_\beta) \leq \varepsilon d(\alpha, \beta),$$

for all $\alpha, \beta \in X$.

The smallest number ε for which the above inequality is true is called Lipschitzian constant." [35]

Example 2.5.1.

Consider the set $X = \mathbb{R}^2$ consisting of all column vectors, and let a metric space d is defined as

$$d(\alpha, \beta) = \sqrt{(\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2}.$$

Assume $\mathcal{F} : X \to X$ be a mapping which is defined as

$$\mathcal{F}(s) = As, \quad \forall \quad s \in X.$$

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, s = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \text{ and } u = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}.$$

$$\begin{aligned} \mathcal{F}(s) &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \\ &= \begin{pmatrix} 3\alpha_1 \\ 3\beta_1 \end{pmatrix} \\ &= 3 \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \\ &= 3s \end{aligned}$$

$$\begin{aligned} d(\mathcal{F}(s), \mathcal{F}(u)) &= d(3s, 3u) \\ &= \sqrt{(3\alpha_1 - 3\beta_1)^2 + (3\alpha_2 - 3\beta_2)^2} \\ &= 3d(s, u) \end{aligned}$$

So, $\mathcal{F}$ is Lipschitzian.

Example 2.5.2.

Consider a set $X = \mathbb{R}$ coupled with the usual metric. A mapping $\mathcal{F} : X \to X$ defined by $\mathcal{F}(s) = 2s$. Then

$$\begin{aligned} d(\mathcal{F}(s), \mathcal{F}(t)) &= |\mathcal{F}(s) - \mathcal{F}(t)| \\ &= |2s - 2t| \\ &= 2|s - t| \\ &= 2d(s, t) \\ \Rightarrow \mathcal{F}(s) &= 2s. \end{aligned}$$

So, $\mathcal{F}$ is Lipschitzian and its Lipschitz constant is 2.

Definition 2.5.4.

"Let (X, d) be a metric space. A mapping $\mathcal{F} : X \to X$ is said to be **contraction** if there exist a constant $\varepsilon \in [0, 1)$, such that for all $\alpha, \beta \in X$

$$d(\mathcal{F}_\alpha, \mathcal{F}_\beta) \leq \varepsilon d(\alpha, \beta),$$

where ε is called contraction constant." [35]

Remark 2.5.1.

Geometrically, by a contraction, we mean any points $\alpha, \beta \in X$ have images $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ are more closer than those points.

Example 2.5.3.

Consider $X = [0, 1]$ and the usual metric . Also let $\mathcal{F} : X \to X$ defined as

$$\mathcal{F}(\alpha) = \frac{1}{b + \alpha} \quad \text{given} \quad (b > 1).$$

So,

$$\begin{aligned} d[\mathcal{F}(\alpha_1), \mathcal{F}(\alpha_2)] &= d(\frac{1}{b + \alpha_1}, \frac{1}{b + \alpha_2}) \\ &= |\frac{1}{b + \alpha_1} - \frac{1}{b + \alpha_2}| \end{aligned}$$

$$
\begin{aligned}
&= |\frac{(b+\alpha_2)-(b+\alpha_1)}{(b+\alpha_1)(b+\alpha_2)}| \\
&= |\frac{\alpha_2-\alpha_1}{(b+\alpha_1)(b+\alpha_2)}| \\
&= |\alpha_1-\alpha_2|\frac{1}{|(b+\alpha_1)(b+\alpha_2)|} \\
&< |\alpha_1-\alpha_2|\frac{1}{|(b+0)(b+0)|} \\
&= \frac{1}{b^2}|\alpha_1-\alpha_2| \\
&= \varepsilon d(\alpha_1,\alpha_2) \quad \text{where} \quad \varepsilon = \frac{1}{b^2}
\end{aligned}
$$

$$\Rightarrow d[\mathcal{F}(\alpha_1),\mathcal{F}(\alpha_2)] < \varepsilon d(\alpha_1,\alpha_2).$$

Hence, the given mapping is a contraction.

Example 2.5.4.

Consider $\mathcal{F}:\mathbb{R}^n \to \mathbb{R}^n$ be any linear mapping with matrix

$$M = (m_{ij})_{i,j=1,....,n}$$

such that

$$\sum_{j=1}^{n} |m_{ij}| < 1$$

for each $i = 1, ..., n$. It is a contraction with respect to the metric

$$d(\xi,\eta) = d\left((\xi_1,\xi_2,...,\xi_n),(\eta_1,\eta_2,...,\eta_n)\right) = \max_{1\le i\le n} |\xi_i-\eta_i|.$$

Consider $\xi = (\xi_1,\xi_2,\xi_3,\xi_4,...,\xi_n)$ and $\eta = (\eta_1,\eta_2,\eta_3,\eta_4,...,\eta_n)$ where both $\xi,\eta \in \mathbb{R}^n$ and

$$\varepsilon = \max_{1\le i\le n} \sum_{j=1}^{n} |m_{ij}| < 1.$$

Now $0 < \varepsilon < 1$ and assuming $\mathcal{F} \neq 0$.

Then

$$d(\mathcal{F}\xi,\mathcal{F}\eta) = d\left(\left(\sum_{j=1}^{n} m_{ij}\xi_j, ..., \sum_{j=1}^{n} m_{nj}\xi_j\right), \left(\sum_{j=1}^{n} m_{ij}\eta_j, ..., \sum_{j=1}^{n} m_{nj}\eta_j\right)\right)$$

$$
\begin{aligned}
&= \max_{1\le i\le n} |\sum_{j=1}^{n} m_{ij}\xi_j - \sum_{j=1}^{n} m_{ij}\eta_j| \\
&= \max_{1\le i\le n} |\sum_{j=1}^{n} m_{ij}(\xi_j - \eta_j)| \\
&\le \left(\max_{1\le i\le n} \sum_{j=1}^{n} |m_{ij}|\right) |(\xi_j - \eta_j)| \\
&\le \varepsilon d(\xi, \eta).
\end{aligned}
$$

Hence $\mathcal{F}$ is a contraction.

Definition 2.5.5.

"Let (X, d) be a metric space and $\mathcal{F}$ be a self map, $\mathcal{F}$ is called a **contractive** mapping if, for all $\alpha, \beta \in X$

$$d(\mathcal{F}(\alpha), \mathcal{F}(\beta)) < d(\alpha, \beta),$$

where $\alpha \neq \beta$." [35]

Example 2.5.5.

Let a usual metric space (X, d), and $X = \mathbb{R}$. Consider $\mathcal{F} : X \to X$ be a mapping which is defined by

$$\mathcal{F}(t) = \frac{1}{t} \quad \text{given} \quad (t > 1),$$

So,

$$
\begin{aligned}
d[\mathcal{F}(t_1), \mathcal{F}(t_2)] &= d(\frac{1}{t_1}, \frac{1}{t_2}) \\
&= |\frac{1}{t_1} - \frac{1}{t_2}| \\
&= |\frac{t_2 - t_1}{t_1 t_2}| \\
&= |\frac{t_1 - t_2}{t_1 t_2}| \\
&= |t_1 - t_2||\frac{1}{t_1 t_2}| \\
&< |t_1 - t_2| \\
&= d(t_1, t_2)
\end{aligned}
$$

$$\Rightarrow d[\mathcal{F}(t_1), \mathcal{F}(t_2)] < d(t_1, t_2).$$

Hence, the given mapping is contractive.

Example 2.5.6.

Consider the set $X = [1, \infty)$ with the usual metric, and define $\mathcal{F} : X \to X$ as $\mathcal{F}(s) = s + \frac{1}{s}$.

$$\begin{aligned} d(\mathcal{F}(s), \mathcal{F}(q)) &= d\left(s + \frac{1}{s}, q + \frac{1}{q}\right) \\ &= |(s + \frac{1}{s}) - (q + \frac{1}{q})| \\ &= |s - q + \frac{1}{s} - \frac{1}{q}| \\ &= |s - q + \frac{q - s}{sq}| \\ &= |(s - q)(1 - \frac{1}{sq})| \\ &= |s - q||1 - \frac{1}{sq}| \\ &< |s - q| \end{aligned}$$

$\Rightarrow \mathcal{F}$ is contractive.

Definition 2.5.6.

"Let $\mathcal{F} : X \to X$ be a mapping on metric space (X, d) into itself. We call $\mathcal{F}$ **non-expansive** if,

$$d(\mathcal{F}_\alpha, \mathcal{F}_\beta) \leq d(\alpha, \beta),$$

for all $\alpha, \beta \in X$." [5]

Example 2.5.7.

Consider a set $X = \mathbb{R}$ coupled with the usual metric, a mapping $\mathcal{F} : X \to X$ defined as $\mathcal{F}\alpha = \alpha$.

Then

$$\begin{aligned} d(\mathcal{F}\alpha, \mathcal{F}\beta) &= |\mathcal{F}\alpha - \mathcal{F}\beta| \\ &= |\alpha - \beta| \\ &= d(\alpha, \beta) \end{aligned}$$

Hence $\mathcal{F}$ is non-expansive.

Remark 2.5.2.

Contraction $\Rightarrow$ Contractive $\Rightarrow$ Non-Expansive $\Rightarrow$ Lipschitzian.

Definition 2.5.7.

"Let (X, d) be a metric space. A mapping $\mathcal{F} : X \to X$ on (X, d) such that

$$\forall \quad \alpha, \beta \in X : d(\mathcal{F}\alpha, \mathcal{F}\beta) \geq d(\alpha, \beta),$$

is called an **expansive mapping**." [36]

Definition 2.5.8. Types of Expansive Mappings

"Let (X, d) be a metric space.

i. An expansion $\mathcal{F} : X \to X$ on (X, d) such that

$$\forall \quad \alpha, \beta \in X : d(\mathcal{F}\alpha, \mathcal{F}\beta) = d(\alpha, \beta),$$

is called an isometry, which is the weakest form of expansive mappings.

ii. An expansion $\mathcal{F} : X \to X$ on (X, d) such that

$$\exists \quad \alpha, \beta \in X, \alpha \neq \beta : d(\mathcal{F}\alpha, \mathcal{F}\beta) > d(\alpha, \beta),$$

we call it a proper expansion.

iii. An expansion $\mathcal{F} : X \to X$ on (X, d) such that

$$\forall \quad \alpha, \beta \in X, \alpha \neq \beta : d(\mathcal{F}\alpha, \mathcal{F}\beta) > d(\alpha, \beta),$$

we call it a strict expansion.

iv. Finally, an expansion $\mathcal{F} : X \to X$ on (X, d) such that

$$\exists \quad E > 1 \quad \forall \quad \alpha, \beta \in X : d(\mathcal{F}\alpha, \mathcal{F}\beta) > Ed(\alpha, \beta),$$

we call it an anticontraction with expansion constant E." [36]

2.6 Fixed Point

Fixed point is a useful tool in mathematics which can be used to prove the existence of solution of a differential equation, integral equation and eigen value equation. Present section is providing the definition and examples of fixed point.

Definition 2.6.1.
"A fixed point of a mapping $\mathcal{F} : X \to X$ of a set X into itself is an $\alpha_0 \in X$ which is mapped onto itself (is kept fixed by $\mathcal{F}$),

that is,

$$\mathcal{F}(\alpha_0) = \alpha_0,$$

the image $\mathcal{F}(\alpha_0)$ coincides with α_0."[28]

Example 2.6.1.
i. Consider $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ be a mapping which is defined as

$$\mathcal{F}(p) = p^2 - 3p + 3,$$

then $p = 1, 3$ are the fixed point of the given mapping. (see Figure 2.4)

ii. Consider $\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping which is defined as $\mathcal{F}(\alpha) = A\alpha$,

where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $\mathcal{F}(\alpha) = A\alpha = \alpha$ for all $\alpha \in \mathbb{R}^2$. Here $\mathcal{F}$ has infinite fixed points. Geometrically, this function rotates the point at the angle 2π.

iii. Consider $\mathcal{F} : \mathcal{C}[0, \frac{1}{2}] \to \mathcal{C}[0, \frac{1}{2}]$ be a mapping which is defined as

$$\mathcal{F}(\varepsilon(t)) = t(\varepsilon(t) + 1) \quad \forall \quad \varepsilon(t) \in \mathcal{C}[0, \frac{1}{2}],$$

then $\varepsilon(t) = \dfrac{t}{1-t}$ is a fixed point of $\mathcal{F}$.

iv. Consider $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ be a mapping which is defined as

$$\mathcal{F}(\varepsilon) = \varepsilon - \frac{e^{\varepsilon}}{1 + e^{\varepsilon}},$$

has no fixed point. (see Figure 2.5)

If we consider the real valued function, fixed point is the point of intersection of the curve $y = f(x)$ and the line $y = x$. This fact is shown by the following graphs of different functions.

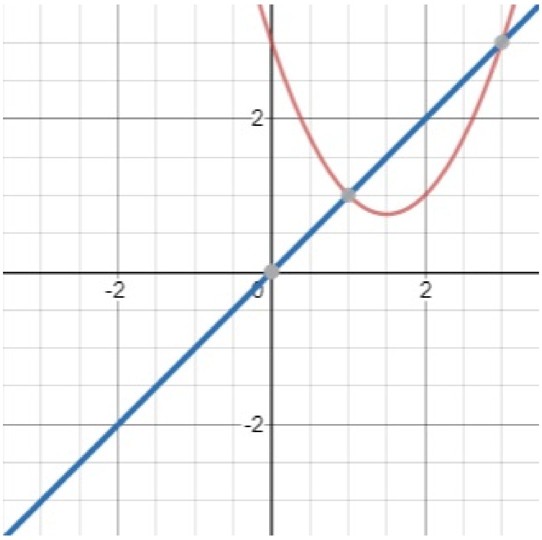

FIGURE 2.4: Graph of Function $\mathcal{F}(p) = p^2 - 3p + 3$.

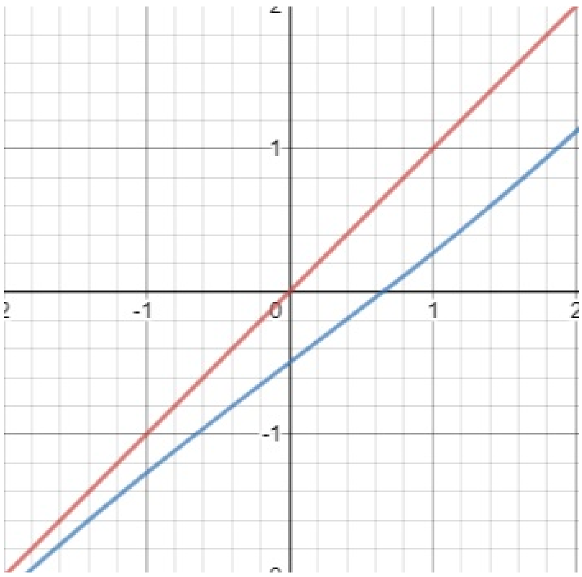

FIGURE 2.5: Graph of Function $\mathcal{F}(\alpha) = \alpha - \dfrac{e^{\alpha}}{1 + e^{\alpha}}$.

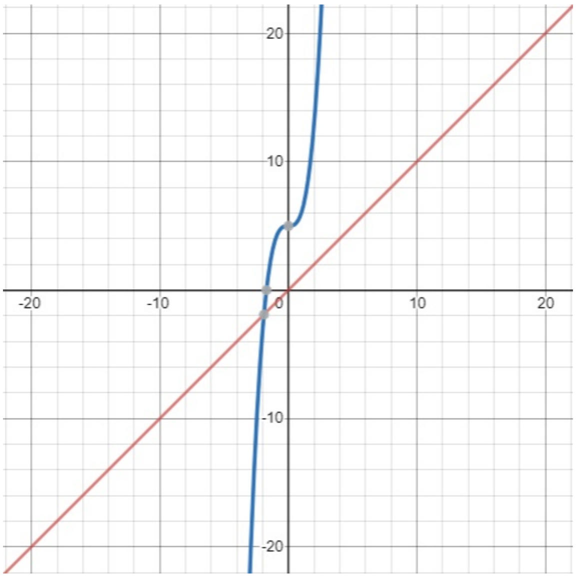

FIGURE 2.6: Graph of Function $\mathcal{F}(p) = p^3 + 1$.

2.7 Some Classical Fixed Point Results

In this section, some fixed point results in different mappings are mentioned. All of these results have their own importance in the history of mathematics.

Theorem 2.7.1 Banach Contraction Principle

"Consider a metric space $X = (X.d)$, where $X \neq \phi$. Let X is complete and $F : X \to X$ be a contraction on X. Then F has precisely one fixed point."[28]

Proof.

Choose $\alpha_0 \in X$ and define $\{\alpha_n\}$ inductively by iterating process

$$\alpha_{n+1} = F\alpha_n. \tag{2.2}$$

From (2.6), starting with α_0, we have

$$\begin{aligned} \alpha_1 = F\alpha_0; \quad & \alpha_2 = F\alpha_1 = F(F\alpha_0) = F^2\alpha_0. \\ & \alpha_3 = F\alpha_2 = F(F^2\alpha_0) = F^3\alpha_0. \\ & . \\ & . \\ & \alpha_n = F^n\alpha_0. \end{aligned}$$

Since, F is a contraction, it follows that

$$\begin{aligned}
d(\alpha_n, \alpha_{n+1}) &= D(F\alpha_{n-1}, F\alpha_n) \leq \varepsilon d(\alpha_{n-1}, \alpha_n) \\
&= d(F\alpha_{n-2}, F\alpha_{n-1}) \\
&\leq \varepsilon(d(F\alpha_{n-2}, F\alpha_{n-1})) = \varepsilon^2 d(\alpha_{n-2}, \alpha_{n-1}) \\
&\leq \varepsilon^3 d(\alpha_{n-3}, \alpha_{n-2}) \\
&. \\
&. \\
&\leq \varepsilon^n d(\alpha_0, \alpha_1). \\
\Rightarrow d(\alpha_n, \alpha_{n+1}) &\leq \varepsilon^n d(\alpha_0, \alpha_1).
\end{aligned}$$

Now suppose that $m > n$ and then by triangular inequality, we have

$$\begin{aligned}
d(\alpha_n, \alpha_m) &\leq d(\alpha_n, \alpha_{n+1} + ... + d(\alpha_{m-1}, \alpha_m)). \\
&\leq \varepsilon^n d(\alpha_0, \alpha_1) + \varepsilon^{n+1} d(\alpha_0, \alpha_1) + ... + \varepsilon^{m-1} d(\alpha_0, \alpha_1). \\
&\leq (\varepsilon^n + \varepsilon^{n+1} + ...) d(\alpha_0, \alpha_1). \\
&= \left(\frac{\varepsilon^n}{1-\varepsilon}\right) d(\alpha_0, \alpha_1).
\end{aligned}$$

Since, $\varepsilon < 1 \quad \Rightarrow \dfrac{\varepsilon^n}{1-\varepsilon} \to 0$ as $n \to \infty$.

It follows that the sequence $\{\alpha_n\}$ is Cauchy. But X is also complete. Then, an $\alpha \in X$ exists there such that $\alpha_n \to \alpha$.

then we claim the point $'\alpha'$ is a fixed point of the mapping F.

Since

$$\lim_{n\to\infty} \alpha_n = \alpha,$$

and

$$\lim_{n\to\infty} \alpha_{n+1} = \alpha.$$

Now,

$$F\alpha = F(\lim_{n\to\infty} \alpha_n) = \lim_{n\to\infty} F\alpha_n = \lim_{n\to\infty} \alpha_{n+1} = \alpha.$$

Hence, α is a fixed point of F.

For the uniqueness, suppose that $\beta \in X$ is another fixed point of F, so that $\alpha \neq \beta$ and $F(\beta) = \beta$.

$$
\begin{aligned}
d(\alpha, \beta) &= d(F\alpha, F\beta) \\
&\leq \varepsilon d(\alpha, \beta) \\
&< d(\alpha, \beta) \quad \because (0 \leq \varepsilon < 1)
\end{aligned}
$$

This can be possible only when

$$d(\alpha, \beta) = 0 \quad \Rightarrow \quad \alpha = \beta.$$

Hence F has a unique fixed point. $\square$

This result doesn't only provide the criteria for the existence and uniqueness of fixed point, but it also provides the technique to find that fixed point. The most interesting fact is that it gives the error estimations.

Theorem 2.7.2 Extension of BCP on Partial Metric Space

"Let (X, ρ) be a complete partial metric space, $\varepsilon \in [0, 1)$ and $F : X \to X$ be a given mapping. Suppose for each $\alpha, \beta \in X$ the following condition holds:

$$\rho(F\alpha, F\beta) \leq \max\{\varepsilon\rho(\alpha, \beta), \rho(\alpha, \alpha), \rho(\beta, \beta)\}.$$

Then:

1. the set X_ρ is nonempty.

2. there is a unique $\alpha_0 \in X_\rho$ such that $F\alpha_0 = \alpha_0$.

3. for each $\alpha \in X_\rho$ the sequence $\{F^n\alpha\}_{n\geq 1}$ converges w.r.t the metric ρ^s to α_0."[37]

Remark 2.7.1.

"If (X, ρ) is a partial metric space, then $\rho^s(\alpha, \beta) = 2\rho(\alpha, \beta) - \rho(\alpha, \alpha) - \rho(\beta, \beta)$ is a metric on X for all $\alpha, \beta \in X$. "[37]

Theorem 2.7.3 Extension of BCP on b-Metric Space

"Let (X, d_β) be a complete b-metric space and let $F : X \to X$ satisfy

$$d[F(\alpha), F(\beta)] \leq \Psi[d(\alpha, \beta)],$$

for all $\alpha, \beta \in X$, where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing function such that

$$\lim_{n\to\infty} \Psi^n(s) = 0,$$

for each fixed > 0. Then F has exactly one fixed point z and

$$\lim_{n\to\infty} d[F^n(\alpha), z] = 0,$$

for each $\alpha \in X$."[14]

Theorem 2.7.4 Extension of BCP on Metric-Like Space

"Let (X, φ) be a complete metric-like space, and let $F : X \to X$ be a map such that

$$\varphi(F\alpha, F\beta) \leq \xi(\mathcal{M}(\alpha, \beta)),$$

for all $\alpha, \beta \in X$, where

$$\mathcal{M}(\alpha,\beta) = \max\{\varphi(\alpha,\beta), \varphi(\alpha,F\alpha), \varphi(\beta,F\beta), \varphi(\alpha,F\beta), \varphi(\beta,F\alpha), \varphi(\alpha,\alpha), \varphi(\beta,\beta)\},$$

where $\xi : [0, \infty) \to [0, \infty)$ is a non-decreasing function satisfying

$$\xi(s) < s \quad \text{for all} \quad s > 0, \quad \lim_{p\to s^+} \xi(p) < s \quad \text{for all} \quad s > 0 \text{ and } \lim_{s\to\infty} (s - \xi(s)) = \infty.$$

This F has a fixed point."[18]

Chapter 3

Fixed Point Results on b-metric-like Spaces

In 2013, the idea of b-MLS was prsented by Alghamdi et al. [19] in which authors generalized the concepts of PMS, b-MS and MLS. Some other authors also worked on the existence as well as the uniqueness of fixed point in newly introduced b-MLS and proved the uniqueness of fixed points as well [21]. To provide the detailed review of this article, some definitions are presented first to give a good elaboration of the main result.

3.1 Some Basic Tools

The idea of b-MLS is originated form the concepts of PMS with MLS and b-MS. b-MLS is the generalized form of many spaces.

Definition 3.1.1.
Consider a non-empty set X, a function $\varrho : X \times X \to [0, \infty)$ is said to be a b-ML such that for each a_1, a_2, $a_3 \in X$ and a constant $\kappa \geq 1$, it holds the following.

(BM1) $\varrho(a_1, a_2) = 0 \Rightarrow a_1 = a_2$;

(BM2) $\varrho(a_1, a_2) = \varrho(a_2, a_1)$;

(BM3) $\varrho(a_1, a_3) \leq \kappa(\varrho(a_1, a_2) + \varrho(a_2, a_3))$.

And (X, ϱ) is known as ***b*-metric-like space**. [19]

Example 3.1.1.

We take $X = [0, \infty)$. Consider $\varrho : X^2 \to [0, \infty)$, Define a function by

$$\varrho(a_1, a_2) = (a_1 + a_2)^2,$$

then it is a b-MLS and its constant is 2.

(BM1)and (BM2) are obvious.

(BM3)

$$\begin{aligned}
\varrho(a_1, a_2) &= (a_1 + a_2)^2 \\
&\leq (a_1 + a_3 + a_3 + a_2)^2 \\
&= (a_1 + a_3)^2 + (a_3 + a_2)^2 + 2(a_1 + a_3)(a_3 + a_2) \\
&\leq 2[(a_1 + a_3)^2 + (a_3 + a_2)^2] \\
&= 2[\varrho(a_1, a_3) + \varrho(a_3, a_2)] \\
\Rightarrow \quad \varrho(a_1, a_2) &\leq \kappa[\varrho(a_1, a_3) + \varrho(a_3, a_2)].
\end{aligned}$$

Hence, the given function is a b-metric-like space.

Remark 3.1.1.

Note that the above mentioned example is a b-MLS but clearly, it is not a b-MS or a MLS.

Example 3.1.2.

Let $X = [0, \infty)$. Consider $\varrho : X^2 \to [0, \infty)$. Then we take a function

$$\varrho(\alpha, \beta) = (\max\{\alpha, \beta\})^2$$

then (X, ϱ) is a b-MLS and its constant is 2.

(BM1)and (BM2) are obvious.

(BM3)

$$\begin{aligned}
\varrho(\alpha,\beta) &= (\max\{\alpha,\beta\})^2 \\
&\leq (\max\{\alpha,\beta,\gamma\})^2 \\
&\leq (\max\{\alpha,\beta\})^2 + (\max\{\beta,\gamma\})^2 \\
&\leq 2[(\max\{\alpha,\beta\})^2 + (\max\{\beta,\gamma\})^2] \\
\Rightarrow \varrho(\alpha,\beta) &\leq \kappa[\varrho(\alpha,\beta) + \varrho(\beta,\gamma)].
\end{aligned}$$

Hence, the given function is a b-MLS.

Definition 3.1.2.

Consider a b-MLS (X,ϱ). Also let an $\alpha \in X$ and some $r > 0$, so

$$\mathcal{B}(\alpha, r) = \{\beta \in X : |\varrho(\alpha,\beta) - \varrho(\alpha,\alpha)| < r\}$$

is an **open ball** whose radius is r and centered at α.

Definition 3.1.3.

Consider a b-MLS (X,ϱ). Also consider a sequence $\{\alpha_n\}$ containing the points of the set X. Then any $\alpha \in X$ is said to be the limit point of $\{\alpha_n\}$ if,

$$\lim_{n\to\infty} \varrho(\alpha,\alpha_n) = \varrho(\alpha,\alpha),$$

and the sequence $\{\alpha_n\}$ is said to **convergent**.

Definition 3.1.4.

Consider a sequence $\{\alpha_n\}$ in a b-MLS (X,ϱ), it is known as **Cauchy sequence** iff,

$$\lim_{m,n\to\infty} \varrho(\alpha_n,\alpha_m),$$

is a finite number.

Definition 3.1.5.

If every Cauchy sequence $\{\alpha_n\}$ is convergent to $\alpha \in X$, then the b-MLS is called **complete** b-MLS.

Proposition 3.1.1.

Consider (X, ϱ) is a b-MLS. Let $\kappa \geq 1$ and $\{\alpha_n\}$ be a sequence in the set X such that

$$\lim_{n\to\infty} \varrho(\alpha_n, \alpha) = 0,$$

then for all $\alpha \in X$.

1. α is unique.
2. $\frac{1}{\kappa}\varrho(\alpha, \beta) \leq \lim_{n\to\infty} \varrho(\alpha_n, \beta) \leq \kappa\varrho(\alpha, \beta) \quad$ for all $\quad \beta \in X$.

Proof.

For the proof of (1.)

Using the assumption that there exists another $\beta \in X$ such that

$$\lim_{n\to\infty} \varrho(\alpha_n, \beta) = 0,$$

$$\Rightarrow 0 \leq \varrho(\beta, \alpha) \leq \kappa(\lim_{n\to\infty} \varrho(\alpha_n, \beta) + \lim_{n\to\infty} \varrho(\alpha_n, \alpha)) = 0,$$

this implies

$$\varrho(\alpha, \beta) = 0 \quad \Rightarrow \quad \alpha = \beta.$$

Now, to prove (2.)

Since

$$\varrho(\alpha, \gamma) \leq \kappa[\varrho(\alpha, \beta) + \varrho(\beta, \gamma)].$$

Therefore,

$$\begin{aligned} &\frac{1}{\kappa}\varrho(\alpha, \beta) - \lim_{n\to\infty} \varrho(\alpha_n, \alpha) \leq \lim_{n\to\infty} \varrho(\alpha_n, \beta) \leq \kappa(\varrho(\alpha, \beta) + \lim_{n\to\infty} \varrho(\alpha_n, \alpha)) \\ \Rightarrow &\frac{1}{\kappa}\varrho(\alpha, \beta) \leq \lim_{n\to\infty} \varrho(\alpha_n, \beta) \leq \kappa\varrho(\alpha, \beta) \lim_{n\to\infty} \varrho(\alpha_n, \alpha) \quad \forall \quad \beta \in X. \end{aligned}$$

Hence proved. □

Definition 3.1.6.

Consider a b-MLS (X, ϱ) and also assume that $\mathcal{U}$ is a subset of the set X. Then

$\mathcal{U}$ is said to be **open** subset if for each $\alpha \in \mathcal{U}$ there exists some $r > 0$ such that $\mathcal{U} \supseteq \mathcal{B}(\alpha, r)$. [19]

Definition 3.1.7.

The set $\mathcal{V}$ is said to be **closed** subset of X if $\mathcal{V}^c$ is open in X. [19]

Proposition 3.1.2.

Consider a b-MLS (X, ϱ) and let the set $\mathcal{V}$ be any subset of the set X. Then for any sequence $\{\alpha_n\}$ contained in $\mathcal{V}$ is closed if and only if $\{\alpha_n\}$ converges to α for any $\alpha \in \mathcal{V}$.

Proof.

Firstly, we assume $\mathcal{V}$ to be closed and $\alpha \notin \mathcal{V}$. Then $\mathcal{V}^c$ is open. So, an $r > 0$ exists there for which $\mathcal{V} \supseteq \mathcal{B}(\alpha, r)$. Also $\alpha_n \to \alpha$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} |\varrho(\alpha_n, \alpha) - \varrho(\alpha, \alpha)| = 0.$$

So, for all $n \geq n_0$, there exists an $n_0 \in \mathbb{N}$, which gives

$$|\varrho(\alpha_n, \alpha) - \varrho(\alpha, \alpha)| < r.$$

This leads to a contradiction, because for every $n \geq n_0$ there exists,

$$\mathcal{V}^c \supseteq \mathcal{B}(\alpha, r) \supseteq \{\alpha_n\}.$$

As for every $n \in \mathbb{N}$, $\{\alpha_n\}$ is subset of $\mathcal{V}$.

Conversely, we assume $\{\alpha_n\}$ be any sequence in $\mathcal{V}$ convergent to α, this gives $\alpha \in \mathcal{V}$. Letting $\beta \notin \mathcal{V}$ to prove that there exists any $r_0 > 0$,

which implies,

$$\mathcal{V} \cap \mathcal{B}(\beta, r_0) = \phi.$$

Assuming contrarily that for every $r > 0$ implies

$$\mathcal{V} \cap \mathcal{B}(\beta, r) \neq \phi.$$

Therefore, for every $n \in \mathbb{N}$ take,

$$\alpha_n \in \mathcal{V} \cap \mathcal{B}(\beta, \frac{1}{n}) \neq \phi.$$

This implies,

$$|\varrho(\alpha_n, \beta) - \varrho(\beta, \beta)| < \frac{1}{n} \quad \forall \quad n \in \mathbb{N}.$$

Therefore, $\alpha_n \to \beta$ as $n \to \infty$. Which contradicts to our supposition that $\mathcal{V}$ gives $\beta \in \mathcal{V}$. Therefore, there exists $r_0 > 0$ for every $\beta \notin \mathcal{V}$ such that $\mathcal{V} \cap \mathcal{B}(\beta, r_0) = \phi$. Which confirms that, $\mathcal{V}$ is closed. □

Lemma 3.1.3.

Consider a b-MLS (X, ϱ). Also let $X \supset \{\alpha_j\}_{j=0}^{m}$. Then

$$\varrho(\alpha_m, \alpha_0) \leq \mathcal{K}\varrho(\alpha_0, \alpha_1) + ... + \mathcal{K}\varrho(\alpha_{m-2}, \alpha_{m-1}) + \mathcal{K}^{m-1}\varrho(\alpha_{m-1}, \alpha_n).$$

Lemma 3.1.4.

Consider a b-MLS (X, ϱ), and consider $\{\beta_n\}$ be any sequence in it. If for some μ and $\mathcal{K} > 1$, $0 < \mu < \dfrac{1}{\mathcal{K}}$, and for every $n \in \mathbb{N}$ such that

$$\varrho(\beta_n, \beta_n + 1) \leq \mu\varrho(\beta_{n-1}, \beta).$$

This leads to $\lim\limits_{m,n\to\infty} \varrho(\beta_m, \beta_n) = 0$.

Definition 3.1.8.

Consider a b-MLS (X, ϱ) and define a mapping $D^s : X \times X \to \mathbb{R}^+$ by

$$D^s(\zeta_1, \zeta_2) = |2\varrho(\zeta_1, \zeta_2) - \varrho(\zeta_1, \zeta_1) - \varrho(\zeta_2, \zeta_2)|.$$

Here, $D^s(\zeta_1, \zeta_1) = 0 \quad \forall \quad \zeta_1 \in X$.

Definition 3.1.9.

Let $\psi_{\mathcal{E}}^{\mathcal{I}}$ denote the class of all functions $\mathcal{E} : (0, \infty) \to (\mathcal{I}^2, \infty)$ holding the condition

$$\mathcal{E}(u_n) \to (\mathcal{I}^2)^+ \Rightarrow u_n \to 0, \tag{3.1}$$

and $\mathcal{I} > 0$.

3.2 Fixed point Results

The idea of b-MLS is widely used by different researchers for the existence and uniqueness of fixed point (See for example [38–42]). This section includes some results showing fixed points in b-MLS endowed with expansive mappings. These results are generalization and extension of some old fixed point theorems. Some useful examples are also provided to validate the results.

Theorem 3.2.1

Consider a b-MLS (X, d_ϱ) which is complete. Assume $T : X \to X$ be any mapping which is onto and it satisfies;

$$\varrho(T\alpha, T\beta) \geq [Q + M \min\{D^s(\alpha, T\alpha), D^s(\beta, T\beta), D^s(\alpha, T\beta), D^s(\beta, T\alpha)\}]\varrho(\alpha, \beta) \tag{3.2}$$

$\forall \quad \alpha, \beta \in X$, where $Q > \mathcal{K}, M \geq 0$. Then the mapping T has a fixed point.

Proof.

Assume any $\alpha_0 \in X$, as T is onto, then $\alpha_1 \in X$ exists there,

such that

$$\alpha_0 = T\alpha_1.$$

Applying this process further,

we have

$$\alpha_n = T\alpha_{n+1} \quad \text{for all} \quad n \in \mathbb{N}U\{0\}.$$

So that if

$$\alpha_{n_0} = \alpha_{n_0+1} \quad \text{for each} \quad n_0 \in \mathbb{N}U\{0\},$$

then obviously α_{n_0} is a fixed point.

Now by assuming

$$\alpha_n \neq \alpha_{n+1},$$

for every n.

Using $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$ in (3.2) gives

$$\begin{aligned}\varrho(T\alpha_n, T\alpha_{n+1}) \geq &[Q + M\min\{D^s(\alpha_n, T\alpha_n), D^s(\alpha_{n+1}, T\alpha_{n+1}),\\ &D^s(\alpha_n, T\alpha_{n+1}), D^s(\alpha_{n+1}, T\alpha_n)\}]\varrho(\alpha_n, \alpha_{n+1})\\ \Rightarrow \varrho(\alpha_{n-1}, \alpha_n) \geq &[Q + M\min\{D^s(\alpha_n, \alpha_{n-1}), D^s(\alpha_{n+1}, \alpha_n),\\ &D^s(\alpha_n, \alpha_n), D^s(\alpha_{n+1}, \alpha_{n-1})\}]\varrho(\alpha_n, \alpha_{n+1})\\ =&Q\varrho(\alpha_n, \alpha_{n+1}).\end{aligned}$$

This implies

$$\varrho(\alpha_n, \alpha_{n+1}) \leq \hbar\varrho(\alpha_{n-1}, \alpha_n),$$

where $\hbar = \dfrac{1}{Q} < \dfrac{1}{\mathcal{K}}$.

Therefore Lemma 3.1.4 yields

$$\lim_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = 0.$$

Now, As $\lim\limits_{m,n\to\infty}\varrho(\alpha_n, \alpha_m) = 0$ exists finitely, which implies that the sequence $\{\alpha_n\}$ is Cauchy. As (X, d_ϑ) is a b-metric-like space and is also complete, this implies that the sequence $\{\alpha_n\}$ in X is convergent to some $p \in X$,

such that

$$\lim_{m,n\to\infty} \varrho(\alpha_n, p) = \varrho(p, p) = \lim_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) = 0.$$

Since T is onto, so that some $\omega \in X$ exists there which implies

$$p = T\upsilon.$$

From (3.2), we have the following

$$\begin{aligned}\varrho(\alpha_n, \upsilon) =&\varrho(T\alpha_{n+1}, T\upsilon)\\ \geq&[Q + M\min\{D^s(\alpha_{n+1}, T\alpha_{n+1}), D^s(\upsilon, T\upsilon), D^s(\alpha_{n+1}, T\upsilon),\\ &D^s(\upsilon, T\alpha_{n+1})\}]\varrho(\alpha_{n+1}, \upsilon)\\ =&[Q + M\min\{D^s(\alpha_{n+1}, \alpha_n), D^s(\upsilon, p), D^s(\alpha_{n+1}, p),\end{aligned}$$

$$D^s(\upsilon,\alpha_n)\}]\varrho(\alpha_{n+1},\upsilon).$$

Using limit $n \to \infty$, we have

$$0 = \lim_{n\to\infty} \varrho(\alpha_n, p) \geq Q \lim_{n\to\infty} \varrho(\alpha_{n+1}, \upsilon).$$

Hence we have,

$$\lim_{n\to\infty} \varrho(\alpha_{n+1}, \upsilon) = 0.$$

we get $p = \upsilon$ which implies $p = Tp$. □

Corollary 1.

Consider (X, ϱ) be a b-MLS which is also complete. Assume $T : X \to X$ which is onto and it satisfies

$$\varrho(T\alpha, T\beta) \geq Q\varrho(\alpha, \beta) \tag{3.3}$$

for each $\alpha, \beta \in X$, and $Q > \mathcal{K}$. This gives the fixed point of mapping T.

Example 3.2.1.

Consider $X = [0, \infty)$ and $\varrho : X \times X \to [0, \infty)$ which is defined as

$$\varrho(a_1, a_2) = (a_1 + a_2)^2.$$

It is obvious that, (X, ϱ) is b-MLS with constant 2 and it is also complete. Define a mapping $T : X \to X$ as

$$Ta_1 = \begin{cases} 7a_1 & if a_1 \in [0, 1), \\ 6a_1 + 2 & if a_1 \in [1, 2), \\ 5a_1 + 4 & if a_1 \in [2, \infty). \end{cases}$$

Obviously, T is onto. Now, we check the following cases:

i. Assume that $a_1, a_2 \in [1, 2)$, So

$$\varrho(Ta_1, Ta_2) = (7a_1 + 7a_2)^2 = 49(a_1 + a_2)^2 \geq 3(a_1 + a_2)^2 = 3\varrho(a_1, a_2)$$

ii. Assume that $a_1, a_2 \in [1, 2)$, So

$$\varrho(Ta_1, Ta_2) = (6a_1+6a_2+2)^2 \geq (6a_1+6a_2)^2 = 36(a_1+a_2)^2 \geq 3(a_1+a_2)^2 = 3\varrho(a_1, a_2)$$

iii. Assume that $a_1, a_2 \in [2, \infty)$, So

$$\varrho(Ta_1, Ta_2) = (5a_1+5a_2+4)^2 \geq (5a_1+5a_2)^2 = 25(a_1+a_2)^2 \geq 3(a_1+a_2)^2 = 3\varrho(a_1, a_2)$$

iv. Assume that $a_1 \in [0, 1)$ and $a_2 \in [1, 2)$, So

$$\varrho(Ta_1, Ta_2) = (7a_1+6a_2+2)^2 \geq (6a_1+6a_2)^2 = 36(a_1+a_2)^2 \geq 3(a_1+a_2)^2 = 3\varrho(a_1, a_2)$$

v. Assume that $a_1 \in [0, 1)$ and $a_2 \in [2, \infty)$, So

$$\varrho(Ta_1, Ta_2) = (7a_1+5a_2+4)^2 \geq (5a_1+5a_2)^2 = 25(a_1+a_2)^2 \geq 3(a_1+a_2)^2 = 3\varrho(a_1, a_2)$$

vi. Assume that $a_1 \in [1, 2)$ and $a_2 \in [2, \infty)$, So

$$\varrho(Ta_1, Ta_2) = (6a_1+5a_2+4)^2 \geq (5a_1+5a_2)^2 = 25(a_1+a_2)^2 \geq 3(a_1+a_2)^2 = 3\varrho(a_1, a_2)$$

So, $\varrho(Ta_1, Ta_2) \geq Q\varrho(a_1, a_2)$ for each $a_1, a_2 \in X$, and $Q = 3 > 2 = \kappa$. So it satisfies the conditions in Corollary (3.1). Hence $a_1 = 0$ is the fixed point of the mapping T.

Theorem 3.2.2

Consider a b-MLS (X, ϱ) which is complete. Using the assumption that $T : X \to X$ is a self mapping which is onto and it satisfies

$$\varrho(T\alpha, T\beta) \geq \mathcal{E}(\varrho(\alpha, \beta))\varrho(\alpha, \beta), \tag{3.4}$$

for every $\alpha, \beta \in X$, and $\mathcal{E} \in \psi^{\kappa}_{\mathcal{E}}$. This leads to a fixed point of the mapping T.

Proof.

Consider any $\alpha_0 \in X$, As in the given mapping, T is onto, This implies that

$\alpha_1 \in X$ exists there such that

$$\alpha_0 = T\alpha_1.$$

Continuing the process further, we have

$$\alpha_n = T\alpha_{n+1} \quad \text{for every} \quad n \in \mathbb{N}U\{0\}.$$

For some case if

$$\alpha_{n_0} = \alpha_{n_0+1} \quad \text{for any} \quad n_0 \in \mathbb{N}U\{0\},$$

then obviously α_{n_0} is a point which is the fixed point of T. Then by assuming that $\alpha_n \neq \alpha_{n+1}$ for every n. Taking $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$ and using in (3.4) it follows;

$$\begin{aligned}\varrho(\alpha_{n-1}, \alpha_n) = \varrho(T\alpha_n, T\alpha_{n+1}) &\geq \mathcal{E}(\varrho(\alpha_n, \alpha_{n+1}))\varrho(\alpha_n, \alpha_{n+1}) \\ &\geq \kappa^2\varrho(\alpha_n, \alpha_{n+1}) \geq \varrho(\alpha_n, \alpha_{n+1}).\end{aligned}$$

Therefore, the sequence $\{\varrho(\alpha_n, \alpha_{n+1})\}$ in $\mathbb{R}^+$ is decreasing sequence. So, for any $r > 0$ so that

$$\lim_{n\to\infty} \varrho(\alpha_n, \alpha_{n+1}) = r.$$

Then by taking the supposition contrarily that $r > 0$, Then by the above expression, we have

$$\kappa^2 \frac{\varrho(\alpha_{n-1}, \alpha_n)}{\varrho(\alpha_n, \alpha_{n+1})} \geq \frac{\varrho(\alpha_{n-1}, \alpha_n)}{\varrho(\alpha_n, \alpha_{n+1})} \geq \mathcal{E}(\varrho(\alpha_n, \alpha_{n+1})) \geq \kappa^2$$

Now as $n \to \infty$, by using limit, it leads to

$$\lim_{n\to\infty} \mathcal{E}(\varrho(\alpha_n, \alpha_{n+1})) = \kappa^2.$$

Therefore,

$$0 = \lim_{n\to\infty} \mathcal{E}(\varrho(\alpha_n, \alpha_{n+1})) = r.$$

which leads to a contradiction, so $r = 0$. Now we claim

$$\limsup_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) = 0.$$

Assume on contrary

$$\limsup_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) > 0.$$

Therefore, from (3.4), this implies

$$\varrho(\alpha_n, \alpha_m) = \varrho(T\alpha_{n+1}, T\alpha_{m+1}) \geq \mathcal{E}(\varrho(\alpha_{n+1}, \alpha_{m+1}))\varrho(\alpha_{n+1}, \alpha_{m+1}).$$

This implies,

$$\frac{\varrho(\alpha_n, \alpha_m)}{\mathcal{E}(\varrho(\alpha_{n+1}, \alpha_{m+1}))} \geq \varrho(\alpha_{n+1}, \alpha_{m+1}).$$

Now, from the third property of b-MLS, it gives

$$\begin{aligned}\varrho(\alpha_n, \alpha_m) &\leq \kappa\varrho(\alpha_n, \alpha_{n+1}) + \kappa^2\varrho(\alpha_{n+1}, \alpha_{m+1}) + \kappa^2\varrho(\alpha_{m+1}, \alpha_m)\\ &\leq \kappa\varrho(\alpha_n, \alpha_{n+1}) + \frac{\varrho(\alpha_n, \alpha_m)}{\mathcal{E}(\varrho(\alpha_{n+1}, \alpha_{m+1}))} + \kappa^2\varrho(\alpha_{m+1}, \alpha_m).\end{aligned}$$

Hence,

$$\varrho(\alpha_n, \alpha_m) \leq [1 - \frac{\kappa^2}{\mathcal{E}(\varrho(\alpha_{n+1}, \alpha_{m+1}))}]^{-1}(\kappa\varrho(\alpha_n, \alpha_{n+1}) + \kappa^2\varrho(\alpha_{m+1}, \alpha_m)).$$

Taking $m, n \to \infty$ and using the facts that

$$\limsup_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) > 0,$$

and

$$0 = \lim_{n\to\infty} \varrho(\alpha_n, \alpha_{n+1}) = r,$$

we have

$$\limsup_{m,n\to\infty} [1 - \frac{\kappa^2}{\mathcal{E}(\varrho(\alpha_{n+1}, \alpha_{m+1}))}]^{-1} = \infty,$$

this implies

$$\limsup_{m,n\to\infty} \mathcal{E}(\varrho(\alpha_{n+1}, \alpha_{m+1})) = (\kappa^2)^+,$$

therefore

$$\limsup_{m,n\to\infty} \varrho(\alpha_{n+1}, \alpha_{m+1}) = 0,$$

which again contradicts.

Therefore,

$$\limsup_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) = 0.$$

Now, as $\limsup\limits_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) = 0$, exists finitely, Hence it comes that $\{\alpha_n\}$ is a Cauchy sequence. As (X, d) is a b-MLS and is also complete. Also, the sequence $\{\alpha_n\}$ is convergent in the set X to some $q \in X$. Therefore,

$$\lim_{m,n\to\infty} \varrho(\alpha_n, q) = \varrho(q, q) = \lim_{m,n\to\infty} \varrho(\alpha_n, \alpha_m) = 0.$$

For the onto mapping T, some ω belongs to X so that $q = T\omega$. To prove $\omega = q$, let us assume on contrary that $q \neq \omega$. Then the inequality (3.4) yields

$$\varrho(\alpha_n, q) = \varrho(T\alpha_{n+1}, T\omega) \geq \mathcal{E}(\varrho(\alpha_{n+1}, \omega))\varrho(\alpha_{n+1,\omega}).$$

Now in the above mentioned inequality, by proposition (3.1.1)(2) and limit $n \to \infty$, this implies

$$\begin{aligned}
\frac{1}{\kappa} \lim_{n\to\infty} \mathcal{E}(\varrho(\alpha_{n+1}, q))\varrho(q, \omega) &\leq \lim_{n\to\infty} \mathcal{E}(\varrho(\alpha_{n+1}, \omega)) \lim_{n\to\infty} \varrho(\alpha_{n+1}, \omega) \\
&\leq \lim_{n\to\infty} \varrho(\alpha_n, q) \\
&= 0.
\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \mathcal{E}(\varrho(\alpha_{n+1}, q)) = 0,$$

which leads to a contradiction.

So, $\kappa^2 \leq \lim_{n\to\infty} \mathcal{E}(\varrho(\alpha_{n+1}, q))$. As $\kappa^2\mathcal{E}(u)$ for each $u \in [0, \infty)$.

Hence

$$q = \omega$$

Therefore,

$$q = T\omega = Tq.$$

□

Corollary 2.

Consider a PMS (X,ρ) which is also complete. Using the assumption that $T: X \to X$ be a self mapping which is onto and it satisfies

$$\rho(T\alpha, T\beta) \geq \mathcal{E}(\rho(\alpha,\beta))\rho(\alpha,\beta), \tag{3.5}$$

for every $\alpha, \beta \in X$, and $\mathcal{E} \in \psi^1_{\mathcal{E}}$. This leads to a fixed point of T.

Corollary 3.

Consider a MLS (X,φ) which is also complete. Using the assumption that $T: X \to X$ be a self mapping which is onto and it satisfies

$$\varphi(T\zeta, T\eta) \geq \mathcal{E}(\varphi(\zeta,\eta))\varphi(\zeta,\eta), \tag{3.6}$$

for every $\zeta, \eta \in X$, and $\mathcal{E} \in \psi^1_{\mathcal{E}}$. This leads to a fixed point of T.

Corollary 4.

Consider a b-MS (X,d_β) which is also complete. Using the assumption that $T: X \to X$ be a self mapping which is onto and it satisfies

$$d_\beta(T\alpha, T\beta) \geq \mathcal{E}(d_\beta(\alpha,\beta))d_\beta(\alpha,\beta), \tag{3.7}$$

for every $\alpha, \beta \in X$, and $\mathcal{E} \in \psi^\kappa_{\mathcal{E}}$. This leads to a fixed point of T.

Definition 3.2.3.

Let $\psi^{\mathcal{L}}_{\mathcal{J}}$ denote the family of all functions $\mathcal{J} : (0,\infty) \to (0, \frac{1}{\mathcal{L}^2})$ holding the condition

$$\mathcal{J}(u_n) \to (\frac{1}{\mathcal{L}^2})^+ \Rightarrow u_n \to 0, \tag{3.8}$$

and $\mathcal{L} > 0$.

Theorem 3.2.4 Consider a partially ordered b-MLS (X,ϱ) which is also complete. Also consider a non-decreasing mapping $\mathcal{N} : X \to X$ such that

$$\varrho(\mathcal{N}\alpha, \mathcal{N}\beta) \leq \mathcal{J}(\mathcal{P}(\alpha,\beta))\mathcal{P}(\alpha,\beta) + \mathcal{I}(\mathcal{Q}(\alpha,\beta))\mathcal{Q}(\alpha,\beta) \tag{3.9}$$

for each $\alpha, \beta \in X$ and $\alpha \preceq \beta$, where $\mathcal{J} \in \psi^{\kappa}_{\mathcal{J}}$, $\quad \mathcal{I} : [0, \infty) \to [0, \infty)$ is any bounded function and

$$\mathcal{P}(\alpha, \beta) = \max\left\{\varrho(\alpha, \beta), \varrho(\alpha, \mathcal{N}\alpha), \varrho(\beta, \mathcal{N}\beta), \frac{\varrho(\alpha, \mathcal{N}\beta), \varrho(\beta, \mathcal{T}\alpha)}{6\kappa}\right\}$$

and

$$\mathcal{Q}(\alpha, \beta) = \min\left\{D^s(\alpha, \mathcal{N}\alpha), D^s(\beta, \mathcal{N}\beta), D^s(\alpha, \mathcal{N}\beta), D^s(\beta, \mathcal{N}\alpha)\right\}.$$

Then it holds the following:

(i) $\alpha_0 \in X$ exists there such that $\alpha_0 \preceq \mathcal{N}\alpha_0$;

(ii) for any sequence $\{\alpha_n\} \subset X$ which is increasing and is convergent to $\alpha \in X$, then we have

$$\alpha_n \preceq \alpha$$

for each $n \in \mathbb{N}$; this leads to the fixed point of $\mathcal{N}$.

Corollary 5. Consider a partially ordered PMS (X, ρ) which is also complete. Also consider a non-decreasing mapping $\mathcal{N} : X \to X$ such that

$$\rho(\mathcal{N}\alpha, \mathcal{N}\beta) \leq \mathcal{J}(\mathcal{P}(\alpha, \beta))\mathcal{P}(\alpha, \beta) + \mathcal{I}(\mathcal{Q}(\alpha, \beta))\mathcal{Q}(\alpha, \beta)$$

for each $\alpha, \beta \in X$ and $\alpha \preceq \beta$,

where $\mathcal{J} \in \psi^{1}_{\mathcal{J}}$, $\quad \mathcal{I} : [0, \infty) \to [0, \infty)$ is any bounded function and

$$\mathcal{P}(\alpha, \beta) = \max\left\{\rho(\alpha, \beta), \rho(\alpha, \mathcal{N}\alpha), \rho(\beta, \mathcal{N}\beta), \frac{\rho(\alpha, \mathcal{N}\beta), \rho(\beta, \mathcal{N}\alpha)}{6}\right\}$$

and

$$\mathcal{Q}(\alpha, \beta) = \min\left\{\rho^s(\alpha, \mathcal{N}\alpha), \rho^s(\beta, \mathcal{N}\beta), \rho^s(\alpha, \mathcal{N}\beta), \rho^s(\beta, \mathcal{N}\alpha)\right\}.$$

Then it holds the following:

(i) $\alpha_0 \in X$ exists there such that $\alpha_0 \preceq \mathcal{N}\alpha_0$;

(ii) for any sequence $\{\alpha_n\} \subset X$ which is increasing and is convergent to $\alpha \in X$, then we have $\alpha_n \preceq \alpha$ for each $n \in \mathbb{N}$; this leads to the fixed point of $\mathcal{N}$.

Corollary 6. Consider a partially ordered b-MS (X, d_β) which is also complete. Also consider a non-decreasing mapping $\mathcal{N} : X \to X$ such that

$$d_\beta(\mathcal{N}\alpha, \mathcal{N}\beta) \leq \mathcal{J}(\mathcal{P}(\alpha, \beta))\mathcal{P}(\alpha, \beta) + \mathcal{I}(\mathcal{Q}(\alpha, \beta))\mathcal{Q}(\alpha, \beta)$$

for each $\alpha, \beta \in X$ and $\alpha \preceq \beta$, where $\mathcal{J} \in \psi^{\kappa}_{\mathcal{J}}$, $\quad \mathcal{I} : [0, \infty) \to [0, \infty)$ is any bounded function and

$$\mathcal{P}(\alpha, \beta) = \max\left\{d_\beta(\alpha, \beta), d_\beta(\alpha, \mathcal{N}\alpha), d_\beta(\beta, \mathcal{N}\beta), \frac{d_\beta(\alpha, \mathcal{N}\beta), d_\beta(\beta, \mathcal{N}\alpha)}{4\kappa}\right\}$$

and

$$\mathcal{Q}(\alpha, \beta) = 2\min\left\{d_\beta(\alpha, \mathcal{N}\alpha), d_\beta(\beta, \mathcal{N}\beta), d_\beta(\alpha, \mathcal{N}\beta), d_\beta(\beta, \mathcal{N}\alpha)\right\}.$$

Then it holds the following:

(i) $\alpha_0 \in X$ exists there such that $\alpha_0 \preceq \mathcal{N}\alpha_0$;

(ii) for any sequence $\{\alpha_n\} \subset X$ which is increasing and is convergent to $\alpha \in X$, then we have $\alpha_n \preceq \alpha$ for each $n \in \mathbb{N}$; this leads to the fixed point of $\mathcal{N}$.

Chapter 4

Extended b-metric-like space

In 2013, Alghamdi et al. [19] investigated the idea of b-MLS. Providing necessary definitions and examples, Alghamdi et al. proved the existence and uniqueness of fixed point on expansive mappings in b-MLS. This chapter includes the extension of b-MLS with necessary definitions, examples and a fixed point result in extended b-MLS.

4.1 Extended b-metric-like space

This section includes the definitions and examples of extended b-MLS.

Definition 4.1.1.
Consider a set X which is non-empty and $\vartheta : X \times X \to [1, \infty)$. A function $d_\vartheta : X \times X \to [0, \infty)$ is said to be an extended b-ML if for all $a_1, a_2, a_3 \in X$. It follows:

($d_\vartheta 1$) $d_\vartheta(a_1, a_2) = 0 \Rightarrow a_1 = a_2$;

($d_\vartheta 2$) $d_\vartheta(a_1, a_2) = d_\vartheta(a_2, a_1)$;

($d_\vartheta 3$) $d_\vartheta(a_1, a_3) \leq \vartheta(a_1, a_3)[d_\vartheta(a_1, a_2) + d_\vartheta(a_2, a_3)]$

Then (X, ϑ) is known as **extended b-metric-like space**.

Remark 4.1.1.

i. In above definition b-MLS is a special case of extended b-MLS when $\vartheta(a_1, a_2) = S$ with $S \geq 1$.

ii. Further, MLS is a special case of extended b-MLS when $\vartheta(a_1, a_2) = S$ with $S = 1$.

Example 4.1.1.

Let $X = \{1, 2, 3, ...\}$ and $d_\vartheta : X \times X \to [0, \infty)$ defined as

$$d_\vartheta(p_1, p_2) = (p_1 + p_2)^2.$$

Consider a function $\vartheta : X \times X \to [1, \infty)$ defined as

$$\vartheta(p_1, p_2) = \frac{p_1 + p_2 + 2}{p_1 + p_2}.$$

Then (X, d_ϑ) is an extended b-MLS.

$(d_\vartheta 1)$ and $(d_\vartheta 2)$ are obvious.

$(d_\vartheta 3)$ To prove $d_\vartheta(p_1, p_3) \leq \vartheta(p_1, p_3)[d_\vartheta(p_1, p_2) + d_\vartheta(p_2, p_3)]$,

we proceed as follows.

$$\begin{aligned}
d_\vartheta(p_1, p_3) &= (p_1 + p_3)^2 \\
&\leq [(p_1 + p_2)^2 + (p_2 + p_3)^2] \\
&\leq \frac{p_1 + p_3 + 2}{p_1 + p_3}[(p_1 + p_2)^2 + (p_2 + p_3)^2] \\
&= \vartheta(p_1 + p_3)[d_\vartheta(p_1, p_2) + d_\vartheta(p_2, p_3)] \\
\Rightarrow d_\vartheta(p_1, p_3) &\leq \vartheta(p_1 + p_3)[d_\vartheta(p_1, p_2) + d_\vartheta(p_2, p_3)].
\end{aligned}$$

Hence proved that the given mapping is an extended b-MLS.

Example 4.1.2.

Consider a set $X = \{0, 1, 2, 3, ...\} = [0, \infty)$ and $d_\vartheta : X \times X \to [0, \infty)$ and is defined as

$$d_\vartheta(p_1, p_2) = [\max\{p_1, p_2\}]^2.$$

Consider a function $\vartheta : X \times X \to [1, \infty)$ defined as

$$\vartheta(p_1, p_2) = 2p_1 + p_2 + 2.$$

Then (X, d_ϑ) is an extended b-MLS.

$(d_\vartheta 1)$ and $(d_\vartheta 2)$ are obvious.

$(d_\vartheta 3)$ To prove $d_\vartheta(p_1, p_3) \leq \vartheta(p_1, p_3)[d_\vartheta(p_1, p_2) + d_\vartheta(p_2, p_3)]$,

we proceed as follows.

$$\begin{aligned}
\max\{p_1, p_3\}^2 &\leq \max\{p_1, p_2, p_3\}^2 \\
&\leq \max\{p_1, p_2\}^2 + \max\{p_2, p_3\}^2 \\
&\leq (2p_1 + p_3 + 2)[\max\{p_1, p_2\}^2 + \max\{p_2, p_3\}^2] \quad (\because 2p_1 + p_2 + 2 > 1). \\
&= \vartheta(p_1, p_3)[\vartheta(p_1, p_2) + \vartheta(p_2, p_3)] \\
\Rightarrow d_\vartheta(p_1, p_3) &\leq \vartheta(p_1, p_3)[\vartheta(p_1, p_2) + \vartheta(p_2, p_3)].
\end{aligned}$$

Hence proved that it is an extended b-MLS.

Definition 4.1.2.

Consider an extended b-MLS (X, d_ϑ). It induces a topology τ_{d_ϑ} on X based on the family of open d_ϑ-balls

$$\mathcal{B}_{d_\vartheta}(\alpha, r) = \{\beta \in X : |d_\vartheta(\alpha, \beta) - d_\vartheta(\alpha, \alpha)| < r\},$$

for all $r > 0$ and $\alpha \in X$.

Definition 4.1.3.

Assume that (X, d_ϑ) is an extended b-MLS having coefficient $\vartheta(\alpha, \beta)$, and let a sequence $\{\alpha_n\}$ in X and $\alpha \in X$, then.

i. $\{\alpha_n\}$ is **convergent** if for $\alpha \in X$, we have

$$\lim_{n \to +\infty} d_\vartheta(\alpha, \alpha_n) = d_\vartheta(\alpha, \alpha).$$

ii. $\{\alpha_n\}$ is a **Cauchy sequence** if and only if

$$\lim_{n\to\infty} d_\vartheta(\alpha_n, \alpha_m),$$

exists finitely.

iii. An extended b-MLS (X, d_ϑ) is called **complete** iff each sequence $\{\alpha_n\}$ in X which is Cauchy in X is convergent to $\alpha \in X$ that is

$$\lim_{n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = d_\vartheta(\alpha, \alpha) = \lim_{n\to\infty} d_\vartheta(\alpha_n, \alpha).$$

Definition 4.1.4.

Assume that (X, d_ϑ) be an extended b-MLS and also assume $\mathcal{U}$ is subset of X. $\mathcal{U}$ is said to be **open** subset in the set X if for every $\alpha \in \mathcal{U}$ some $r > 0$ exists there such that $\mathcal{U} \supseteq \mathcal{B}(\alpha, r)$.

Definition 4.1.5.

Assume that (X, d_ϑ) be an extended b-MLS and also assume $\mathcal{V}$ is a subset of X. $\mathcal{V}$ is said to be **closed** subset of X if $\mathcal{V}^c$ is open in the set X if for every $\alpha \in \mathcal{U}$ there exists some $r > 0$ such that $\mathcal{U} \supseteq \mathcal{B}(\alpha, r)$.

Proposition 4.1.1.

Assume that (X, d_ϑ) be an extended b-MLS, and consider a sequence $\{\alpha_n\}$ in the set X such that

$$\lim_{n\to+\infty} d_\vartheta(\alpha_n, \alpha) = 0.$$

Then $'\alpha'$ is unique.

Proof.

Consider there is a $\beta \in X$, we have

$$\lim_{n\to 0} d_\vartheta(\alpha, \beta) = 0,$$

then

$$0 \leq d_\vartheta(\beta, \alpha) \leq \vartheta(\alpha, \beta)[\lim_{n\to\infty} d_\vartheta(\alpha_n, \beta) + \lim_{n\to\infty} d_\vartheta(\alpha_n, \alpha)] = 0.$$

Therefore from $(d_\vartheta.1)$, we get

$$\beta = \alpha.$$

□

Definition 4.1.6.

Consider an extended b-MLS (X, d_ϑ). we define $D^s : X \times X \to [0, \infty)$ by

$$D^s(\zeta, \eta) = |2d_\vartheta(\zeta, \eta) - d_\vartheta(\zeta, \zeta) - d_\vartheta(\eta, \eta)|.$$

Obviously, $D^s(\zeta, \zeta) = 0$ for all $\zeta \in X$.

Proposition 4.1.2.

Consider an extended b-MLS (X, d_ϑ) and the set $\mathcal{V}$ be any subset of the set X. Then for any sequence $\{\alpha_n\}$ contained in $\mathcal{V}$ is closed if and only if $\{\alpha_n\}$ converges to α for any $\alpha \in \mathcal{V}$.

Proof.

Firstly, we assume the set $\mathcal{V}$ to be closed and $\alpha \notin \mathcal{V}$. Then $\mathcal{V}^c$ is open in the set X. An $r > 0$ exists there for which $\mathcal{V} \supseteq \mathcal{B}_{d_\vartheta}(\alpha, r)$.

Also

$$\alpha_n \to \alpha \quad \text{as} \quad n \to \infty.$$

Therefore,

$$\lim_{n\to\infty} |d_\vartheta(\alpha_n, \alpha) - d_\vartheta(\alpha, \alpha)| = 0.$$

So, for all $n \geq n_0$,

there exists an $n_0 \in \mathbb{N}$, which implies

$$|d_\vartheta(\alpha_n, \alpha) - d_\vartheta(\alpha, \alpha)| < r.$$

This leads to a contradiction, because for every $n \geq n_0$ there exists,

$$\mathcal{V}^c \supseteq \mathcal{B}_{d_\vartheta}(\alpha, r) \supseteq \{\alpha_n\}.$$

As for every $n \in \mathbb{N}$, $\{\alpha_n\}$ is subset of $\mathcal{V}$.

Conversely, we assume $\{\alpha_n\}$ be any sequence in $\mathcal{V}$ convergent to α, this gives $\alpha \in \mathcal{V}$. Letting $\beta \notin \mathcal{V}$ to prove that there exists any $r_0 > 0$,

which implies
$$\mathcal{V} \cap \mathcal{B}_{d_\vartheta}(\beta, r_0) = \phi.$$

Assuming contrarily for every $r > 0$, we get

$$\mathcal{V} \cap \mathcal{B}_{d_\vartheta}(\beta, r) \neq \phi.$$

Therefore, for every $n \in \mathbb{N}$ take,

$$\alpha_n \in \mathcal{V} \cap \mathcal{B}_{d_\vartheta}(\beta, \frac{1}{n}) \neq \phi.$$

This implies,
$$|d_\vartheta(\alpha_n, \beta) - d_\vartheta(\beta, \beta)| < \frac{1}{n} \quad \forall \quad n \in \mathbb{N}.$$

Therefore,
$$\alpha_n \to \beta \quad \text{as} \quad n \to \infty.$$

Which contradicts to our supposition that $\mathcal{V}$ gives $\beta \in \mathcal{V}$. Therefore, there exists $r_0 > 0$ for every $\beta \notin \mathcal{V}$ such that $\mathcal{V} \cap \mathcal{B}_{d_\vartheta}(\beta, r_0) = \phi$. Which confirms that, $\mathcal{V}$ is closed. □

Lemma 4.1.3.

Consider (X, d_ϑ) be an extended b-MLS and is also complete such that d_ϑ is a continuous functional and $\{\alpha_n\}$ be a sequence in X. Let $\alpha_0 \in X$ be an arbitrary element of X. Consider $\{\alpha_n\} = \{T^n(\alpha_0)\}$. If there is a mapping $T : X \to X$, it satisfies;

$$d_\vartheta(\alpha_n, \alpha_{n+1}) \leq \mu d_\vartheta(\alpha_{n-1}, \alpha_n), \tag{4.1}$$

for any $0 < \mu < \dfrac{1}{\kappa}$, and $\kappa \in [0, 1)$. So, for any $\alpha_0 \in X$,

$$\lim_{m,n \to \infty} \vartheta(\alpha_n, \alpha_m) < \frac{1}{\mu}.$$

Then it is a Cauchy sequence.

Proof.

Consider any $\alpha_0 \in X$ arbitrarily, then we define an iterative sequence $\{\alpha_n\}$ by,

$$\alpha_0, T\alpha_0 = \alpha_1, \alpha_2 = T\alpha_1 = T(T\alpha_0) = T^2(\alpha_0), ..., \alpha_n = T^n(\alpha_0), ...$$

Then by successively applying (4.1), we get

$$d_\vartheta(\alpha_n, \alpha_{n+1}) \leq \mu^n d_\vartheta(\alpha_0, \alpha_1). \tag{4.2}$$

Then by (4.2) and triangular inequality,

for some $m > n$, we get

$$\begin{aligned}
d_\vartheta(\alpha_n, \alpha_m) \leq & \vartheta(\alpha_n, \alpha_m)\mu^n d_\vartheta(\alpha_0, \alpha_1) + \vartheta(\alpha_n, \alpha_m)\vartheta(\alpha_{n+1}, \alpha_m)\mu^{n+1} d_\vartheta(\alpha_0, \alpha_1) \\
& + ... + \vartheta(\alpha_n, \alpha_m)\vartheta(\alpha_{n+1}, \alpha_m)\vartheta(\alpha_{n+2}, \alpha_m).......\vartheta(\alpha_{m-2}, \alpha_m) \\
& \vartheta(\alpha_{m-1}, \alpha_m)\mu^{m-1} d_\vartheta(\alpha_0, \alpha_1) \\
& \leq d_\vartheta(\alpha_0, \alpha_1)[\vartheta(\alpha_1, \alpha_m)\vartheta(\alpha_2, \alpha_m)....\vartheta(\alpha_{n-1}, \alpha_m)\vartheta(\alpha_n, \alpha_m)\mu^n \\
& + \vartheta(\alpha_1, \alpha_m)\vartheta(\alpha_2, \alpha_m)....\vartheta(\alpha_n, \alpha_m)\vartheta(\alpha_{n+1}, \alpha_m)\mu^{n+1} \\
& + ... + \vartheta(\alpha_1, \alpha_m)\vartheta(\alpha_2, \alpha_m)....\vartheta(\alpha_n, \alpha_m)\vartheta(\alpha_{m-1}, \alpha_m) \\
& ....\vartheta(\alpha_{m-2}, \alpha_m)\vartheta(\alpha_{m-1}, \alpha_m)\mu^{m-1}]
\end{aligned}$$

Since,

$$\lim_{m,n\to\infty} \vartheta(\alpha_{n+1}, \alpha_m)\mu < 1,$$

so this implies that the series $\sum_{k=1}^{\infty} \mu^n \prod_{i=1}^{n} \vartheta(\alpha_i, \alpha_m)$ is convergent by ratio test for any $m \in \mathbb{N}$.

Let,

$$S = \sum_{k=1}^{\infty} \mu^n \prod_{i=1}^{n} \vartheta(\alpha_i, \alpha_m),$$

and

$$S_n = \sum_{j=1}^{\infty} \mu_j \prod_{i=1}^{j} \vartheta(\alpha_i, \alpha_m).$$

For $m > n$, it follows from the above inequality

$$\begin{aligned}
& d_\vartheta(\alpha_n, \alpha_m) \leq d_\vartheta(\alpha_0, \alpha_1)[S_{m-1} - S_{n-1}]. \\
\Rightarrow & \lim_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) \leq d_\vartheta(\alpha_0, \alpha_1)[S_{m-1} - S_{n-1}] = 0. \\
\Rightarrow & \lim_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = 0.
\end{aligned}$$

As $\lim_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = 0$ is finite. Hence the sequence $\{\alpha_n\}$ is Cauchy. □

Theorem 4.1.7

Consider (X, d_ϑ) be an extended b-MLS which is also complete. Assume $T : X \to X$ be a mapping which is onto and it satisfies;

$$d_\vartheta(T\alpha, T\beta) \geq [P + N \min\{D^s(\alpha, T\alpha), D^s(\beta, T\beta), D^s(\alpha, T\beta), D^s(\beta, T\alpha)\}]d_\vartheta(\alpha, \beta), \tag{4.3}$$

for each $\alpha, \beta \in X$, where $P > \kappa, N \geq 0$. This leads to a fixed point of the mapping T.

Proof.

Assume any $\alpha_0 \in X$, as T is onto, then $\alpha_1 \in X$ exists there such that

$$\alpha_0 = T\alpha_1.$$

Applying this process further, we have

$$\alpha_n = T\alpha_{n+1} \quad \text{for every} \quad n \in \mathbb{N}U\{0\}.$$

For some case if

$$\alpha_{n_0} = \alpha_{n_0+1} \quad \text{for any} \quad n_0 \in \mathbb{N}U\{0\},$$

then obviously α_{n_0} is a fixed point of the mapping T.

Now by assuming

$$\alpha_n \neq \alpha_{n+1} \quad \text{for every} \quad n.$$

Using $\alpha = \alpha_n$ and $\beta = \alpha_{n+1}$ in (4.3), we have

$$
\begin{aligned}
d_\vartheta(T\alpha_n, T\alpha_{n+1}) \geq & [P + N \min\{D^s(\alpha_n, T\alpha_n), D^s(\alpha_{n+1}, T\alpha_{n+1}), \\
& D^s(\alpha_n, T\alpha_{n+1}), D^s(\alpha_{n+1}, T\alpha_n)\}] d_\vartheta(\alpha_n, \alpha_{n+1}).
\end{aligned}
$$

This implies,

$$
\begin{aligned}
d_\vartheta(\alpha_{n-1}, \alpha_n) \geq & [P + N \min\{D^s(\alpha_n, \alpha_{n-1}), D^s(\alpha_{n+1}, \alpha_n), D^s(\alpha_n, \alpha_n), \\
& D^s(\alpha_{n+1}, \alpha_{n-1})\}] d_\vartheta(\alpha_n, \alpha_{n+1}) \\
& = P d_\vartheta(\alpha_n, \alpha_{n+1}).
\end{aligned}
$$

This implies

$$d_\vartheta(\alpha_n, \alpha_{n+1}) \leq l d_\vartheta(\alpha_{n-1}, \alpha_n),$$

Where $l = \dfrac{1}{P} < \dfrac{1}{\kappa}$.

Then by Lemma 4.1.3,

we get $\lim\limits_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = 0$.

Now, As $\lim\limits_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = 0$. exists finitely, which implies that the sequence $\{\alpha_n\}$ is a Cauchy sequence. As (X, d_ϑ) is an extended b-MLS and is also complete, this implies $\{\alpha_n\}$ in the set X is convergent to some $q \in X$, such that

$$\lim_{m,n\to\infty} d_\vartheta(\alpha_n, q) = d_\vartheta(q, q) = \lim_{m,n\to\infty} d_\vartheta(\alpha_n, \alpha_m) = 0.$$

Since the mapping T is onto, so $\omega \in X$ exists there which implies $q = T\omega$. From (4.3), we have the following

$$
\begin{aligned}
d_\vartheta(\alpha_n, \omega) = & d_\vartheta(T\alpha_{n+1}, T\omega) \\
\geq & [P + N \min\{D^s(\alpha_{n+1}, T\alpha_{n+1}), D^s(\omega, T\omega), D^s(\alpha_{n+1}, T\omega), \\
& D^s(\omega, T\alpha_{n+1})\}] d_\vartheta(\alpha_{n+1}, \omega). \\
= & [P + N \min\{D^s(\alpha_{n+1}, \alpha_n), D^s(\omega, q), D^s(\alpha_{n+1}, q), \\
& D^s(\omega, \alpha_n)\}] d_\vartheta(\alpha_{n+1}, \omega).
\end{aligned}
$$

Using limit $n \to \infty$, we have

$$0 = \lim_{n\to\infty} d_\vartheta(\alpha_n, q) \geq P \lim_{n\to\infty} d_\vartheta(\alpha_{n+1}, \omega).$$

This implies,

$$\lim_{n\to\infty} d_\vartheta(\alpha_{n+1}, \omega) = 0.$$

we have $q = \omega$,

which implies $q = Tq$. □

In the above mentioned theorem if we consider $N = 0$. This leads to the following.

Corollary 7.

Consider (X, d_ϑ) is an extended b-MLS which is also complete. Assume that $T : X \to X$ be a mapping which is onto and it satisfies

$$d_\vartheta(T\zeta, T\eta) \geq P d_\vartheta(\zeta, \eta), \tag{4.4}$$

for every $\alpha, \beta \in X$, where $P > \kappa$. This leads to the fixed point of the mapping T.

Example 4.1.3. Consider a set $X = [0, \infty)$ and $d_\vartheta : X \times X \to [0, \infty)$ defined as

$$d_\vartheta(\alpha, \beta) = (\alpha + \beta)^2.$$

And $\vartheta : X \times X \to [1, \infty)$ defined as

$$\vartheta(\alpha, \beta) = \frac{\alpha + \beta + 2}{\alpha + \beta}.$$

Clearly, (X, d_ϑ) is an extended b-MLS. Define $T : X \to X$ as

$$T\alpha = \begin{cases} 5\alpha & if \alpha \in [0, 1) \\ 6\alpha + 2 & if \alpha \in [1, 2), \\ 5\alpha + 4 & if \alpha \in [2, \infty). \end{cases}$$

As T is onto. So, now we check the following cases:

i. Assume that $\alpha, \beta \in [0,1)$, so

$$d_\vartheta(T\alpha, T\beta) = (5\alpha + 5\beta)^2 = 25(\alpha + \beta)^2 \geq 3(\alpha + \beta)^2 = 3d_\vartheta(\alpha, \beta).$$

ii. Assume that $\alpha, \beta \in [1,2)$, so

$$d_\vartheta(T\alpha, T\beta) = (6\alpha+6\beta+2)^2 \geq (6\alpha+6\beta)^2 = 36(\alpha+\beta)^2 \geq 3(\alpha+\beta)^2 = 3d_\vartheta(\alpha, \beta).$$

iii. Assume that $\alpha, \beta \in [2,\infty)$, so

$$d_\vartheta(T\alpha, T\beta) = (5\alpha+5\beta+4)^2 \geq (5\alpha+5\beta)^2 = 25(\alpha+\beta)^2 \geq 3(\alpha+\beta)^2 = 3d_\vartheta(\alpha, \beta).$$

iv. Assume that $\alpha \in [0,1)$ and $\beta \in [1,2)$, so

$$d_\vartheta(T\alpha, T\beta) = (5\alpha+6\beta+2)^2 \geq (5\alpha+5\beta)^2 = 25(\alpha+\beta)^2 \geq 3(\alpha+\beta)^2 = 3d_\vartheta(\alpha, \beta).$$

v. Assume that $\alpha \in [0,1)$ and $\beta \in [2,\infty)$, so

$$d_\vartheta(T\alpha, T\beta) = (5\alpha+5\beta+4)^2 \geq (5\alpha+5\beta)^2 = 25(\alpha+\beta)^2 \geq 3(\alpha+\beta)^2 = 3d_\vartheta(\alpha, \beta).$$

vi. Assume that $\alpha \in [1,2)$ and $\beta \in [2,\infty)$, so

$$d_\vartheta(T\alpha, T\beta) = (6\alpha+5\beta+4)^2 \geq (5\alpha+5\beta)^2 = 25(\alpha+\beta)^2 \geq 3(\alpha+\beta)^2 = 3d_\vartheta(\alpha, \beta).$$

So, $d_\vartheta(T\alpha, T\beta) \geq Pd_\vartheta(\alpha, \beta)$ for each $\alpha, \beta \in X$, and $P = 3 > 2 = \kappa$.

So it satisfies the conditions in Corollary 7. Hence $\alpha = 0$ is a fixed point of the mapping T.

Chapter 5

Final Remarks

The dissertation comes to its end in the following manner:

- The dissertation is started with brief introduction, pointing out the related history and work done by many mathematicians.
- As supportive material, some abstract spaces like metric space, partial metric space, b-metric space and metric-like space are elaborated with proper examples, convergence, completeness and Cauchy criteria.
- A section is mentioned for brief discussion on fixed point theory. This helps to understand the existence and uniqueness of the fixed point in main results.
- Different mappings are also elaborated for better understanding of expansive mappings, that are used in the main results.
- This research is based on b-metric-like space. A detailed review of the paper "Fixed point and coupled fixed point point on b-metric-like spaces". Definition and examples of b-metric-like space are elaborated. Some basic results are also mentioned which are helpful for the main result. The main result is based on expansive mappings providing the existence and uniqueness of fixed point.
- Inspired from the paper of Alghamdi et al. and taking inspiration from the paper of Kamran et al. an extended b-metric-like space is investigated.

Some basic tools such as convergence and completeness are provided. Some examples are verified for better understanding.

- Providing some basic results, the main result is elaborated which is based on expansive mapping and showing the existence and uniqueness of fixed point. An example is also provided to validate the result.

- In future,

    i. the application of given result can be provided.

    ii. using the idea of extended b-metric-like space, one can establish further results.

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