CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Wave Propagation through Expansion Chamber having Arbitrary Configuration

by

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DEDICATED

TO

MY BELOVED PARENTS,

BALQES KHANUM, M. HANIF KHAN(LATE)

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CERTIFICATE OF APPROVAL

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Abstract

The current thesis addresses a class of problem arising in the modeling of scattering of acoustic waves through an expansion chamber of arbitrary configuration. The physical problem is governed by Helmholtz equation and has bounding wall properties of guiding channel to be soft-soft, rigid-rigid and soft-rigid. The Multimodal technique is applied to solve the governing boundary value problem. The solution is projected on the local transverse modes to produce the coupled mode equations. The admittance matrix is introduced to convert the coupled mode equations to Riccati equation which is integrated by using the Magnus-Möbius expansion method.

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Abbreviations

MG2 Second Order Magnus Series Expansion Method

MG4 Fourth Order Magnus Series Expansion Method

Symbols

$A(t)$	Coefficient matrix
Ad_A	Adjoint representation
ad_X	Adjoint operator
B_j	Bernoulli numbers
$d\exp$	Derivation of exponential transformation
$d\exp^{-1}$	Inverse of derivation of the exponential transformation
exp	Exponential transformation
$expm$	Matlab matrix exponential map
G	Lie group
$\mathfrak{g}$	Lie algebra
h	Step range
Id	The identity matrix
log	Inverse of exponential transformation
$\mathcal{M}$	Manifolds
$\mathbb{N}$	Natural number set
$\mathbb{R}$	Set of real numbers
$\mathbb{Z}$	The set of integers
ρ	Density
γ_n	Eigen value
$\Pi(t)$	Magnus series expansion method
Φ	Phi
ψ	Psi

Chapter 1

Introduction

The computation of wave propagation in waveguides is a classic topic in many domains of physics including electromagnetics and acoustics [1] with increasing interest in the recent two decades due to the development of quantum waveguides [2, 3] and of elastic waveguides [4, 5]. The solutions of the Helmholtz equation in waveguides can be calculated using traditional numerical approaches such as finite element methods or boundary element methods and analytic approaches, for instance see [6–34]. These approaches work by projecting the solution to the waveguide local transverse modes and then solving the coupled mode equations by using some numerical technique. When we solve coupled mode equations two issues arise; first, they are numerically unstable due to the presence of evanescent modes and second the emerging initial value problems are not well posed because of radiation conditions.

In the multimodal context, this impossibility leads to the introduction of an admittance matrix [35, 36] which corresponds to the Dirichlet-to-Neumann (DtN) operator. The radiation condition (the outlet boundary condition) is represented by this matrix which is regulated by the Riccati equation. It allows us to solve the Helmholtz equation in waveguides with an efficient and stable numerical technique [4, 5, 35–38]. At high frequencies the Riccati equation has a lot of quasi-singularities, because of a "Magnus-Möbius scheme" [39] that we conduct with a Magnus exponential approach [40] numerical integration over singularities is conceivable.

An exponential representation $x(t) = e^{\Pi(t)}x_0$ of solution of a first order linear differential equation revealed by Magnus in 1954 [41]. His work was later called Magnus series expansion. Since then, the Magnus series expansion has been successfully applied to many fields to explain different physical situation [42–44]. However, Magnus has not proved convergence and he has not illuminated the general form of the $\Pi(t)$ expansion.

In 1997, Iserles and Norsett [45] have successffully completed the both tasks. Some of the work on the convergence of Magnus series examined by Moan and Niesen [46]. Magnus series have taken attention in the theory of differential equation [47] and control theory [48]. In 1999, Iserles et al. [49] made a study on the solution of linear differential equation using Lie groups and investigated the solution of the first order linear homogenous differential equation $x' = H(t)x$.

In 2006, Cases and Iserles [50] examined and introduced the algorithim for nonlinear differential equation. In 2012, Magnus series expansion method are used to solve the initial value problem which are given by Blanes and ponsoda [51]. In 2015, Atay et al. [52] applied Magnus series expansion method to homogeneous linear stiff ordinary differential systems. Later in 2016, Atay et al. [53] conducted a study on the Magnus series expansion method for inhomogeneous linear stiff ordinary differential systems . Köme et al. [54] applied the Magnus series expansion method to the nonlinear Liénard differential system and the isothermal gas sphere equation system . The aim of this thesis is to discuss acoustic scattering through a waveguide having expansion chamber with arbitrary configuration. The boundary wall conditions may be rigid-rigid, rigid-soft or soft-soft. The study in continuation of the work carried by Pagneux [55] with addition to soft-soft and rigid-soft settings.

Chapter 2

Basic Definitions

The aim of this chapter is to describe some basic concepts that are useful to understand the work done in rest of the chapters.

Definition 2.1. Topological Manifold

"A *topological manifold* is a Hausdorff, second countable, locally Euclidean Hausdorff space. It is said to be of *dimension* n if it is locally Euclidean of dimension n."

This definition is taken from [56].

Definition 2.2. Differentiable Manifold

" *Let M be a differentiable manifold. A Set $A \subset M$ is called **open** if for each $a \in A$ there is an admissible local chart (U, ϕ) such that $a \in U$ and $U \subset A$.*"

This definition is taken from [57].

Definition 2.3. Lie Group

" A **Lie group** is a smooth manifold G which is also a group and such that the group product

$$G \times G \to G$$

and the inverse map $G \times G$."

This definition is taken from [58].

Definition 2.4. Lie Algebra

" A **finite-dimensional real or complex Lie algebra** is a finite-dimensional real or complex vector space $\mathfrak{g}$, togather with a map $[\cdot,\cdot]$ from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$, with the following properties:

1. $[\cdot,\cdot]$ is bilinear.
2. $[\cdot,\cdot]$ is skew symmetric: $[X, Y] = -[Y, X] \quad \text{for all } X, Y \in \mathfrak{g}$.
3. The **Jacobi identity** holds:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$".
This definition is taken from [58].

Definition 2.5. Lie Group Homomorphism and Isomorphism

" Let G and H be matrix Lie groups. A map ϕ from G to H is called a **Lie group homomorphism** if

1. ϕ is a group homomorphism
2. ϕ is continous.

If, in addition, ϕ is one-to-one and onto and the invers map ϕ^{-1} is continuous, then ϕ is called a **Lie group isomorphism**."
This definition is taken from [58].

Definition 2.6. Lie Algebra Homomorphism and Isomorphism

" If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. If, in addition, ϕ is one-to-one and onto, then ϕ is called a **Lie algebra isomorphism**."
This definition is taken from [58].

Definition 2.7. Adjoint Operator

" If $\mathfrak{g}$ is a Lie algebra and X is element of $\mathfrak{g}$, define a linear map $ad_X : \mathfrak{g} \to \mathfrak{g}$ by

$$ad_X(Y) = [X, Y].$$

The map $X \to ad_X$ is the **adjoint map** or **adjoint operator**".

This definition is taken from [58].

Definition 2.8. Adjoint Notation

" Let G be a matrix Lie group, with Lie algebra $\mathfrak{g}$. Then for each $A \in G$, define a linear map $Ad_A : \mathfrak{g} \to \mathfrak{g}$ by the formula

$$Ad_A(X) = AXA^{-1}."$$

This definition is taken from [58].

Definition 2.9. Derivative of Exponential Mapping

The differential of the exponential map can defined as

$$\frac{d}{dt}\exp(A(t)) = d\exp_{A(t)} A'(t)\exp(A(t)), \tag{2.1}$$

where

$$d\exp_A = \frac{\exp(ad_A) - I}{ad_A}. \tag{2.2}$$

Here I denotes identity matrix and ad_A stands for the adjoint of matrix A. Accordingly, inverse of $d\exp_A$, denoted by $d\exp_A^{-1}$ is defined as

$$d\exp_A^{-1} = \frac{ad_A}{\exp(ad_A) - I}.$$

As we know

$$\frac{e^x - 1}{x} = \sum_{j=0}^{\infty} \frac{x^j}{(j+1)!}.$$

Replacing x with ad_A and 1 with I

$$\frac{e^{ad_A}-1}{ad_A} = \sum_{j=0}^{\infty} \frac{ad_A^j}{(j+1)!}. \tag{2.3}$$

On using (2.3) into (2.2)

$$d\exp_A = \sum_{j=0}^{\infty} \frac{ad_A^j}{(j+1)!}.$$

For any matrix C

$$\begin{aligned} d\exp_A(C) &= \sum_{j=0}^{\infty} \frac{ad_A^j}{(j+1)!}(C) \\ &= C + \frac{1}{2!}ad_A C + \frac{1}{3!}ad_A^2 C + \frac{1}{4!}ad_A^3 C + \cdots \\ &= C + \frac{1}{2!}[A,C] + \frac{1}{3!}[A,[A,C]] + \frac{1}{4!}[A,[A,[A,C]]] + \cdots . \end{aligned} \tag{2.4}$$

For $d\exp_A^{-1}$, we note

$$\frac{x}{e^x-1} = 1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots ,$$

which gives

$$\frac{ad_A}{\exp(ad_A)-I} = 1 - \frac{1}{2!}ad_A + \frac{1}{12}ad_A^2 - \frac{1}{720}ad_A^4 + \cdots .$$

Therefore, for matrix C

$$\begin{aligned} d\exp_A^{-1}(C) &= \frac{ad_A}{\exp(ad_A)-I}(C) \\ &= C - \frac{1}{2}[A,C] + \frac{1}{12}[A,[A,C]] - \frac{1}{720}[A,[A,[A,C]]] + \cdots \\ &= \sum_{j=0}^{\infty} \frac{B_j}{j!} ad_A^j C, \end{aligned} \tag{2.5}$$

where $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}$, are the Bernoulli numbers [60].

This definition is taken from [59].

Chapter 3

Magnus Expansion Scheme for Initial Value Problems

This chapter includes a comprehensive detail of Magnus method taken from different sources [38, 40, 45, 49]. To understand the application of this scheme some initial value problems are discussed.

3.1 Motivation

Here, our interest is to obtain the solution of a linear Lie group differential equation of the form

$$X' = A(t)X, \quad t \geq 0, \quad X(0) = X_0 \in G, \tag{3.1}$$

where $X \in G$ and $A \in \mathfrak{g}$ are matrices.

Equivalently to (3.1), the scalar formulation can be written as

$$x' = a(t)x, \quad t \geq 0, \quad x(0) = x_0, \tag{3.2}$$

where $x(t)$ and $a(t)$ are scalar functions corresponding to matrix functions $X(t)$ and $A(t)$.

The solution of (3.2), can be written as

$$x(t) = \exp\left(\int_0^t a(s)\,ds\right) x(0). \tag{3.3}$$

However it is impossible to solve matrix function into scalar function because $A(t_1)$ and $A(t_2)$ can not commute with each other for $t_1, t_2 \geq 0$.
Solution of (3.1) is same as in the scalar form, then

$$X(t) = \exp\left(\int_0^t A(s)\,ds\right) X(t_0). \tag{3.4}$$

For t_1 and t_2 (3.4) yields

$$X(t_1) = \exp\left(\int_{t_0}^{t_1} A(s)\,ds\right) X(t_0), \tag{3.5}$$

and

$$X(t_2) = \exp\left(\int_{t_1}^{t_2} A(s)\,ds\right) X(t_1), \tag{3.6}$$

by using (3.5) into (3.6), we get

$$X(t_2) = \exp\left(\int_{t_1}^{t_2} A(s)\,ds\right) \exp\left(\int_{t_0}^{t_1} A(s)\,ds\right) X(t_0),$$

$$X(t_2) = CB, \tag{3.7}$$

where

$$B = \exp\left(\int_{t_0}^{t_1} A(s)\,ds\right) X(t_0),$$

and

$$C = \exp\left(\int_{t_1}^{t_2} A(s)\,ds\right) X(t_0),$$

are matrices.
Now, we break (3.4), into two intervals as

$$X(t_2) = \exp\left(\int_{t_0}^{t_1} A(s)\,ds\right) \exp\left(\int_{t_1}^{t_2} A(s)\,ds\right) X(t_0),$$

$$X(t_2) = BC. \tag{3.8}$$

On comparing (3.7) and (3.8), we investigate that $BC = CB$, but in general $BC \neq CB$, which implies that solution of scalar differentional equation is not suitable for matrix differential equation.

3.2 Magnus Series Expansion Method for Homogeneous Linear Differential Systems

The matrix form of differential equation can be solved by using Magnus series expansion method. This method gives approximate solution. To explain the procedure of Magnus series expansion method, we consider

$$X'(t) = A(t)X(t), \quad t \geq 0, \quad X(0) = X_o \in G, \tag{3.9}$$

where $A \in \mathfrak{g}$ and $X \in G$.

Assume the solution of (3.9) as

$$X = \exp(\Pi(t))X_0, \tag{3.10}$$

where Π is matrix and is unknown.

Differentiating (3.10) with respect to variable t,

$$X' = (\exp(\Pi))' X_o. \tag{3.11}$$

Applying the definition of exponential map (2.1), we get

$$X' = d\exp_{\Pi}(\Pi'(t))X. \tag{3.12}$$

On comparing (3.9) and (3.12)

$$(d\exp_{\Pi}(\Pi') - A(t))X = 0.$$

For non-trival solution $X \neq 0$, thus

$$d\exp_{\Pi}(\Pi') = A(t). \tag{3.13}$$

On multiplying $d\exp_{\Pi}^{-1}$ on both side of (3.13), leads to

$$\Pi' = d\exp_{\Pi}^{-1}(A(t)). \tag{3.14}$$

On applying the definition (2.5), $d\exp_{\Pi}^{-1}$ can be expressed as

$$\Pi' = \sum_{j=0}^{\infty} \frac{B_j}{j!} ad_{\Pi}^{j}(A(t)). \tag{3.15}$$

Integration of (3.15) will yield the value of Π. To integrate this Picard iteration method will be applied.

3.3 Picard Iteration

Reconsider

$$X'(t) = A(t)X(t), \quad X(0) = I, \quad 0 \leq \tau \leq t. \tag{3.16}$$

Integrating over $0 \leq \tau \leq t$,

$$\frac{d}{d\tau} \int_0^t X(\tau)\, d\tau = \int_0^t A(\tau)X(\tau)\, d\tau. \tag{3.17}$$

From the fundamental theorem of calculus, we know

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

On applying fundamental theorem, (3.17) yields

$$X(t) = I + \int_0^t A(\tau)X(\tau)\, d\tau. \tag{3.18}$$

For $m = 1, 2, 3\cdots$, the value can be generaly written as:

$$X_{m+1}(t) = I + \int_0^t A(\tau)X_m(\tau)\,d\tau, \quad 0 \leq \tau \leq t. \tag{3.19}$$

Equation (3.19) is called Picard iteration.

On integration (3.15) over $0 \leq \tau \leq t$, we yield the value of Π as

$$\int_0^t \Pi'(t)\,dt = \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_0^t ad^j_{\Pi^{[m]}(s)} A(s)\,ds,$$

$$\Pi(t) - \Pi(0) = \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_0^t ad^j_{\Pi^{[m]}(s)} A(s)\,ds,$$

$$\Pi = O + \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_0^t ad^j_{\Pi^{[m]}(s)} A(s)\,ds. \tag{3.20}$$

We can write (3.20) in general form

$$\Pi^{[0]} \equiv O,$$

$$\Pi^{[m+1]} = \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_0^t ad^j_{\Pi^{[m]}(s)} A(s)\,ds. \tag{3.21}$$

We can find $\Pi^{[1]}, \Pi^{[2]}, \Pi^{[3]}\cdots$, by putting $m = 0, 1, 2\cdots$, into (3.21).

For $m = 0$, (3.21) gives

$$\Pi^{[1]} = \frac{B_0}{0!} \int_0^t ad^0_{\Pi^{[0]}(s)} A(s)\,ds,$$

or

$$\Pi^{[1]} = \int_0^t ad^0_O A(s)\,ds,$$

or

$$\Pi^{[1]} = \int_0^t A(s_1)\,ds_1. \tag{3.22}$$

Likewise for $m = 1$, (3.21) gives

$$\begin{aligned}\Pi^{[2]} =& \frac{B_0}{0!}\int_0^t ad^0_{\Pi^{[1]}(s)}A(s_1)\,ds_1 + \frac{B_1}{1!}\int_0^t ad^1_{\Pi^{[1]}(s)}A(s_1)\,ds_1 \\ &+ \frac{B_2}{2!}\int_0^t ad^2_{\Pi^{[1]}(s)}A(s_1)\,ds_1 + \cdots,\end{aligned}$$

or

$$\begin{aligned}\Pi^{[2]} =& \int_0^t A(s_1)\,ds_1 - \frac{1}{2}\int_0^t [\Pi^{[1]}, A(s_1)]\,ds_1 \\ &+ \frac{1}{12}\int_0^t [\Pi^{[1]}, [\Pi^{[1]}, A(s_1)]]\,ds_1 + \cdots,\end{aligned}$$

or

$$\begin{aligned}\Pi^{[2]} =& \int_0^t A(s_1)\,ds_1 - \frac{1}{2}\int_0^t [\int_0^{s_1} A(s_2)\,ds_2, A(s_1)]\,ds_1 \\ &+ \frac{1}{12}\int_0^t [\int_0^{s_1} A(s_2)\,ds_2, [\int_0^{s_1} A(s_2)\,ds_2, A(s_1)]]\,ds_1 \cdots,\end{aligned} \tag{3.23}$$

and for $m = 2$, (3.21) corresponds

$$\begin{aligned}\Pi^{[3]} =& \int_0^t A(s_1)\,ds_1 - \frac{1}{2}\int_0^t [\int_0^{s_1} A(s_2)\,ds_2, A(s_1)]\,ds_1 \\ &+ \frac{1}{12}\int_0^t [\int_0^{s_1} A(s_2)\,ds_2, [\int_0^{s_1} A(s_2)\,ds_2, A(s_1)]]\,ds_1 \\ &- \frac{1}{24}\int_0^t [\int_0^{s_1} [\int_0^{s_2} A(s_3)\,ds_3, [\int_0^{s_2} A(s_3)\,ds_3, A(s_2)]]\,ds_2, A(s_1)]\,ds_1 \\ &- \frac{1}{24}\int_0^t [\int_0^{s_1} A(s_2)\,ds_2, [\int_0^{s_1} [\int_0^{s_2} A(s_3)\,ds_3, A(s_2)]\,ds_2, A(s_1)]]\,ds_1 \\ &- \frac{1}{24}\int_0^t [\int_0^{s_1} A(s_2)\,ds_2, [\int_0^{s_1} A(s_2)\,ds_2, [\int_0^{s_1} A(s_2)\,ds_2, A(s_1)]]]\,ds_1 \\ &+ \frac{1}{4}\int_0^t [\int_0^{s_1} [\int_0^{s_2} A(s_3)\,ds_3, A(s_2)]\,ds_2, A(s_1)]\,ds_1 + \cdots.\end{aligned} \tag{3.24}$$

The terms $\Pi^{[1]}$, $\Pi^{[2]}$, $\Pi^{[3]} \cdots$ obtained by Picard iteration are rearranged according to the number of integrals and commutators. To obtain H_j, $(j = 0, 1, 2 \dots)$, it can be written as

$$\Pi = \sum_{j=0}^{\infty} H_j(t), \tag{3.25}$$

where each H_j include exactly $(j + 1)$ integrals and j commutators [59]. Thus

$$H_0(t) = \int_0^t A(s_1)\,ds_1, \tag{3.26}$$

$$H_1(t) = -\frac{1}{2}\int_0^t \left[\int_0^{t_1} A(s_2)\,ds_2, A(s_1)\right] ds_1, \tag{3.27}$$

$$\begin{aligned} H_2(t) =& \frac{1}{12}\int_0^t \left[\int_0^{t_1} A(s_2)\,ds_2, \left[\int_0^{t_1} A(s_2)\,ds_2, A(s_1)\right]\right] ds_1 \\ &+ \frac{1}{4}\int_0^t \left[\int_0^{t_1} \left[\int_0^{t_2} A(s_3)\,ds_3, A(s_2)\right] ds_2, A(s_1)\right] ds_1, \end{aligned} \tag{3.28}$$

$$\begin{aligned} H_3(t) = &-\frac{1}{24}\int_0^t \left[\int_0^{s_1} \left[\int_0^{s_2} A(s_3)\,ds_3, \left[\int_0^{s_2} A(s_3)\,ds_3, A(s_2)\right]\right] ds_2, A(s_1)\right] ds_1 \\ &- \frac{1}{24}\int_0^t \left[\int_0^{s_1} \left[\int_0^{s_2} A(s_3)\,ds_3, A(s_2)\right] ds_2, \left[\int_0^{s_1} A(s_2)\,ds_2, A(s_1)\right]\right] \\ &- \frac{1}{24}\int_0^t \left[\int_0^{s_1} A(s_2)\,ds_2, \left[\int_0^{s_1} \left[\int_0^{s_2} A(s_3)\,ds_3, A(s_2)\right] ds_2, A(s_1)\right]\right] ds_1 \\ &- \frac{1}{8}\int_0^t \left[\int_0^{s_1} A(s_2)\,ds_2, \left[\int_0^{s_1} A(s_2)\,ds_2, \left[\int_0^{s_1} A(s_2)\,ds_2, A(s_1)\right]\right]\right] ds_1, \end{aligned} \tag{3.29}$$

and so on. Equation (3.25) is a value of the approximate solution of the system of differential equations (3.9).

3.3.1 Multivariate Quadrature

In previous sections, we applied Magnus expansion technique and obtained unknowns in terms of integrals. These integrals can be approximated using quadrature technique. The details concerning Gauss-Legendre quadrature formulation of integrals are explained below.

Gauss-Legendre Quadrature

The Legendre polynomial of degree n, denoted $P_n(x)$, is usually defined on the symmetric interval $[-1, 1]$. However, if we shift this interval to $[0, 1]$, then the Legendre polynomial, denoted by $P_n^*(x)$, is shifted by $2x - 1$ in $P_n(x)$, so that

$$P_n^*(x) = P_n(2x - 1).$$

First few value of $P_n^*(x)$ are [60]

$$
\begin{aligned}
P_0^*(x) &= 1 \\
P_1^*(x) &= 2x - 1 \\
P_2^*(x) &= 6x^2 - 6x + 1 \\
P_3^*(x) &= 20x^3 - 30x^2 + 12x - 1.
\end{aligned}
$$

Note that $P_1^*(x)$, the Legendre polynomial of degree one is zero at

$$x = \frac{1}{2}, \tag{3.30}$$

and $P_2^*(x)$, the Legendre polynomial of degree two is zero at

$$x_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}, \quad x_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}. \tag{3.31}$$

The values given in (3.30) and (3.31) are knowns as the roots of Legendre polynomials of degree one and two, respectively. Likewise, we can find the roots of $P_3^*(x)$.

3.3.2 Second Order Magnus Series Expansion Method (MG2)

Let us relabel the multiple integrals as $H_i(t)$ for $i = 1, 2, \cdots$, so we have

$$\Pi = H_1, \tag{3.32}$$

where the first term is

$$H_1(t) = \int_0^t A(s_1)\, ds_1.$$

We want to use the multivariate quadrature formulae to approximate the first integral $H_1(t)$, in the Magnus series expansion. The details are given below.
First, we find of equation $P_1^*(x) = 0$ which is $c_1 = \frac{1}{2}$, then, the cardinal Lagrange interpolation polynomial will be applied.

Definition 3.1. Suppose that x_i, for $i = 1, 2, 3...., n$ are the roots of an n-degree Legender polynomial $P_n^*(x)$, and that for each $i = 1, 2, 3....., n$ the weights b_i are defined by

$$b_i = \int_0^1 l_j(x), \tag{3.33}$$

where

$$l_j(x) = \Pi_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - c_i}{c_i - c_j}, \tag{3.34}$$

are called lagrange interpolation polynomial.

If $\Phi(x)$ is any polynomial of degree less than $2n$, then the Gauss-Legendre integral formula

$$\int_0^1 \Phi(x)\, ds = \sum_{i=1}^{n} b_i \Phi(c_i), \tag{3.35}$$

is defined in [61].

So, the cardinal lagrange interpolation polynomial in (3.34) gives

$$l_1(x) = \frac{x - c_2}{c_1 - c_2} = 2x. \tag{3.36}$$

Now, we calculate the approximation $A_i = hA(c_i h)$, for $i = 1, 2, \cdots, v$ and form the quadrature

$$H(t) = \sum_{i \in C_s^n} b_i L(A_{i_1}, A_{i_2}, ...A_{i_s}). \tag{3.37}$$

The approximation for the function $A(t)$ at the root c_1 is given by $A_1 = hA(\frac{h}{2})$. For the multivariate quadrature of the first integral $H_1(t)$, the function L is $L(A(s_1)) = A(s_1)$. The weights b_i can be found by integration according to (3.33), that is

$$b_1 = \int_0^1 l_1(s_1)\, ds_1 = \frac{1}{2}. \tag{3.38}$$

Hence, the multivariate quadrature by using Gauss-Legendre points for $H_1(t)$ becomes

$$H_1(t) = b_1 L(A_1) = A_1. \tag{3.39}$$

By using (3.39) into (3.32), we find

$$\Pi = hA(\frac{h}{2}). \tag{3.40}$$

As a result, the second order Magnus series expansion method, with $x(t_n) = x_n$ and $h = t_{n+1} - t_n$ is as [45]

$$\begin{aligned} A_1 &= hA(t_n + \frac{h}{2}), \\ \Pi^{[2]} &= hA(t_n + \frac{h}{2}), \\ x_{n+1} &= \exp(\Pi^{[2]})x_n. \end{aligned} \tag{3.41}$$

3.3.3 Fourth Order Magnus Series Expansion Method (MG4)

Similarly, for fourth order Magnus expansion method, we take multiple integrals as $H_i(t)$ for $i = 1, 2, \cdots$, so we have

$$\Pi = H_1 - \frac{1}{2}H_2 + \frac{1}{12}H_3 + \frac{1}{4}H_4. \tag{3.42}$$

We want to use the multivariate quadrature formulae to approximate the first four integrals $H_i(t)$, $i = 1, 2, 3, 4$ in the Magnus series expansion. The roots in the Legendre polynomial $P_2^*(x)$ are $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$ and $c_2 = \frac{1}{2} + \frac{\sqrt{3}}{6}$. So, the cardinal lagrange interpolation polynomial in (3.34) are

$$l_1(x) = \frac{x - c_2}{c_1 - c_2} = -\sqrt{3}x + \frac{1}{2}(\sqrt{3} + 1), \tag{3.43}$$

and

$$l_2(x) = \frac{x - c_1}{c_2 - c_1} = \sqrt{3}x - \frac{1}{2}(\sqrt{3} - 1). \tag{3.44}$$

Furthermore, the approximation for the function $A(t)$ at the two roots c_1 and c_2 are given by $A_1 = hA\left((\frac{1}{2} - \frac{\sqrt{3}}{6})h\right)$ and $A_2 = hA\left((\frac{1}{2} + \frac{\sqrt{3}}{6})h\right)$. For the multivariate quadrature of the first integral $H_1(t)$, the function L is $L(A(s_1)) = A(s_1)$, and the set of combination of s tuples value is given by $C_s^v = C_1^2 = \{(1), (2)\}$. The weights b_i can now be found by integration according to (3.33), as

$$
\begin{aligned}
b_1 &= \int_0^1 l_1(s_1)\, ds_1 = \frac{1}{2},\\
b_2 &= \int_0^1 l_2(s_2)\, ds_2 = \frac{1}{2}.
\end{aligned}
$$

Hence, from (3.37) the multivariate quadrature by using Gauss-Legendre points for $H_1(t)$ leads to

$$
H_1(t) = b_1 L(A_1) + b_2 L(A_2) = \frac{1}{2} A_1 + \frac{1}{2} A_2. \tag{3.45}
$$

Similarly, by using (3.37) for the second integral $H_2(t)$, the multivariate quadrature can be found as

$$
H_2(t) = b_{(1,2)} L(A_1, A_2) + b_{(2,1)} L(A_2, A_1) = b_{(1,2)} [A_2, A_1] + b_{(2,1)} [A_1, A_2],
$$

or

$$
H_2(t) = (b_{(1,2)} - b_{(2,1)}) [A_2, A_1] = -\frac{\sqrt{3}}{6} [A_2, A_1]. \tag{3.46}
$$

Accordingly, third integral $H_3(t)$, we get

$$
\begin{aligned}
H_3(t) &= b_{(1,1,2)} L(A_1, A_1, A_2) + b_{(1,2,1)} L(A_1, A_2, A_1) + b_{(2,1,1)} L(A_2, A_1, A_1)\\
&\quad + b_{(2,2,1)} L(A_2, A_2, A_1) + b_{(2,1,2)} L(A_2, A_1, A_2) + b_{(1,2,2)} L(A_1, A_2, A_2)\\
&= (b_{(2,1,1)} - b_{(1,1,2)}) [[A_2, A_1], A_1] + (b_{(2,2,1)} - b_{(1,2,2)}) [[A_2, A_1], A_2],
\end{aligned}
$$

or

$$
H_3(t) = \left(\frac{\sqrt{3}}{16} + \frac{3}{80} \right) [[A_2, A_1], A_1] + \left(\frac{\sqrt{3}}{16} - \frac{3}{80} \right) [[A_2, A_1], A_2]. \tag{3.47}
$$

Finally, the integral for $H_4(t)$ reveals

$$
H_4(t) = (b_{(1,2,1)} - b_{(1,1,2)}) [A_1, [A_2, A_1]] + (b_{(2,2,1)} - b_{(2,1,2)}) [A_2, [A_2, A_1]],
$$

or

$$
H_4(t) = \left(\frac{\sqrt{3}}{48} - \frac{3}{80} \right) [A_1, [A_2, A_1]] + \left(\frac{\sqrt{3}}{48} + \frac{3}{80} \right) [A_2, [A_2, A_1]]. \tag{3.48}
$$

On using (3.45), (3.46), (3.47) and (3.48) into (3.42), we obtain

$$\Pi = \frac{1}{2}(A_1 + A_2) + \frac{\sqrt{3}}{12}[A_2, A_1] + \frac{1}{80}[[A_2, A_1], A_1] - \frac{1}{80}[[A_2, A_1], A_2] \cdots . \quad (3.49)$$

The linear combinations of integrals $H_3(t)$ and $H_4(t)$ do not affect the result of the fourth order Magnus series expansion method. Thus, if the last two terms in the expansion are neglected, the 4th order Magnus series expansion is obtained as follows:

$$\Pi^{[4]} = \frac{1}{2}(A_1 + A_2) + \frac{\sqrt{3}}{12}[A_2, A_1] + O(t^5).$$

As a result, the fourth order Magnus series expansion method, with $x(t_n) = x_n$ and $h = t_{n+1} - t_n$ is as [45]

$$\begin{aligned}
A_1 &= A\left(t_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h\right), \\
A_2 &= A\left(t_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h\right), \\
\Pi^{[4]} &= \frac{1}{2}h(A_1 + A_2) + \frac{\sqrt{3}}{12}h^2[A_2, A_1], \\
x_{n+1} &= \exp(\Pi^{[4]})x_n.
\end{aligned} \quad (3.50)$$

3.4 Magnus Series Expansion Method for Non -Homogeneous Linear Differential Systems

In this section, the transformation given for non-homogeneous differential system by Blanes and Ponsoda [51] will be focused .

Consider a non-homogeneous linear differential system

$$x'(t) = H(t)x(t) + g(t), \quad x(0) = x_0, \quad (3.51)$$

where $x(t), g(t)$ are column matrices. We solve this system by using Magnus expansion technique. We rewrite the above system as

$$\begin{bmatrix} x(t) \\ 1 \end{bmatrix}' = \begin{bmatrix} H(t) & g(t) \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ 1 \end{bmatrix}, \tag{3.52}$$

subject to conditions

$$\begin{bmatrix} x_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 \\ 1 \end{bmatrix},$$

where $A(t)$

$$A(t) = \begin{bmatrix} H(t) & g(t) \\ 0^T & 0 \end{bmatrix}. \tag{3.53}$$

Example 3.1 Consider a system of equations with initial conditions as discussed in [62]

$$\begin{aligned} x_1'(t) &= x_2(t), \\ x_2'(t) &= -0.9999x_1(t) - 100x_2(t), \\ x_1(0) &= 1, x_2(0) = 0. \end{aligned} \tag{3.54}$$

We solve this system using Magnus expansion technique and then compare the solution with exact solution. The exact solution of this system is

$$x_1(t) = -0.000100020004000080088e^{-99.99t} + 1.000100020004001e^{-0.009999999999990905t},$$
$$x_2(t) = 0.010001000200040078e^{-99.99t} - 0.010001000200040078e^{-0.009999999999990905t}.$$

To apply Magnus expansion method, first we rewrite the above system as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.9999 & -100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3.55}$$

subject to conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which imply

$$X' = AX,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -0.9999 & -100 \end{bmatrix},$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Now applying the MG2, we have

$$x_{n+1} = \exp(\Pi^{[2]})x_n,$$

where

$$\Pi^{[2]} = hA(t_n + \frac{h}{2}),$$

in which $h = t_{n+1} - t_n$ and $x_{n+1} = x(t_{n+1})$.

By taking different values of $t \in [0, 1]$, such that $t_n = t_0 + nh$, the follwoing tables 3.1 and 3.2 are genrated.

TABLE 3.1: For h= 0.0625, iteration of Example 3.1 by MG2 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} 0 & 0.0625 \\ -1 & -0.625 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.0625 \\ -1 & -0.625 \end{bmatrix}$
$\exp(\Pi^{[2]})$	$\begin{bmatrix} 0.97448 & 0.04599 \\ -0.73590 & 0.51454 \end{bmatrix}$	$\begin{bmatrix} 0.97448 & 0.04599 \\ -0.73590 & 0.51454 \end{bmatrix}$
x_{n+1} $= \exp(\Pi^{[2]})x_n$	$x_1 = \exp(\Pi^{[2]})x_0$	$x_2 = \exp(\Pi^{[2]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.48724 \\ -0.36795 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.45788 \\ -0.54789 \end{bmatrix}$

TABLE 3.2: For h=0.125, iteration of Example 3.1 by MG2 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} 0 & 0.125 \\ -2 & -1.25 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.125 \\ -2 & -1.25 \end{bmatrix}$
$\exp(\Pi^{[2]})$	$\begin{bmatrix} 0.91577 & 0.06848 \\ -1.09579 & 0.23090 \end{bmatrix}$	$\begin{bmatrix} 0.91577 & 0.06848 \\ -1.09579 & 0.23090 \end{bmatrix}$
x_{n+1} $= \exp(\Pi^{[2]})x_n$	$x_1 = \exp(\Pi^{[2]})x_0$	$x_2 = \exp(\Pi^{[2]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.45789 \\ -0.5479 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.38179 \\ -0.62827 \end{bmatrix}$

Now by using these $t_1, t_2, \cdots, t_5$ in exact solution. The comparison is given in the tables 3.3 and 3.4.

TABLE 3.3: For $h = 0.0625$, the approximate and real solutions and absolute error values of Example 3.1 obtained from the MG2 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	Error
0.0625	0.48724	0.48724	0
0.125	0.45788	0.45788	0
0.1875	0.42100	0.42100	0
0.25	0.38179	0.38179	0
0.3125	0.34316	0.34316	0

TABLE 3.4: For $h = 0.125$, the approximate and real solutions and absolute error values of Example 3.1 obtained from the MG2 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	Error
0.125	0.45788	0.45788	0
0.25	0.38179	0.38179	0
0.375	0.30661	0.30661	0
0.5	0.24220	0.24220	0
0.625	0.18988	0.18988	0

Example 3.2 Consider a differential equation with initial conditions as discussed in [63]

$$\begin{gathered} x''(t) + 10x'(t) + 16x(t) = 0, \\ x_1(0) = 0.5, x_2(0) = 0. \end{gathered} \tag{3.56}$$

We solve this system using Magnus expansion technique and then compare the solution with exact solution. The exact solution of this system is

$$x = \frac{2}{3}e^{-2t} - \frac{1}{6}e^{-8t}.$$

To apply Magnus expansion method, first we rewrite (3.56) as

$$\begin{aligned} x &= x_1, \\ x_1' &= x_2, \\ x_2' &= -10x_2 - 16x_1. \end{aligned} \tag{3.57}$$

Equation (3.57) can be written as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -16 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{3.58}$$

subject to conditions

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

which imply

$$X' = AX,$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -16 & -10 \end{bmatrix},$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Now applying the MG2, we have

$$x_{n+1} = \exp(\Pi^{[2]})x_n,$$

where

$$\Pi^{[2]} = hA(t_n + \frac{h}{2}),$$

in which $h = t_{n+1} - t_n$ and $x_{n+1} = x(t_{n+1})$.

By taking different values of $t \in [0, 1]$, such that $t_n = t_0 + nh$, the tables 3.5 and 3.6 are genrated.

TABLE 3.5: For h= 0.0625, iteration of Example 3.2 by MG2 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} 0 & 0.0625 \\ -0.062493 & -6.25 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.0625 \\ -0.062493 & -6.25 \end{bmatrix}$
$\exp(\Pi^{[2]})$	$\begin{bmatrix} 0.99947 & 0.00997 \\ -0.00997 & 0.00183 \end{bmatrix}$	$\begin{bmatrix} 0.99947 & 0.00997 \\ -0.00997 & 0.00183 \end{bmatrix}$
x_{n+1} $= \exp(\Pi^{[2]})x_n$	$x_1 = \exp(\Pi^{[2]})x_0$	$x_2 = \exp(\Pi^{[2]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.99947 \\ -0.00997 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.99885 \\ -0.00997 \end{bmatrix}$

TABLE 3.6: For h= 0.125, iteration of Example 3.2 by MG2 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} 0 & 0.125 \\ -0.1249875 & -12.5 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.125 \\ -0.1249875 & -12.5 \end{bmatrix}$
$\exp(\Pi^{[2]})$	$\begin{bmatrix} 0.99885 & 0.00998 \\ -0.00998 & -0.00009 \end{bmatrix}$	$\begin{bmatrix} 0.99885 & 0.00998 \\ -0.00998 & -0.00009 \end{bmatrix}$
x_{n+1} $= \exp(\Pi^{[2]})x_n$	$x_1 = \exp(\Pi^{[2]})x_0$	$x_2 = \exp(\Pi^{[2]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.99885 \\ -0.00998 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.99760 \\ -0.00997 \end{bmatrix}$

Now by using these $t_1, t_2, \cdots, t_5$ in exact solution. The comparison is given in the tables 3.7 and 3.8.

TABLE 3.7: For $h = 0.0625$, the approximate and real solutions and absolute error values of Example 3.2 obtained from the MG2 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	Error
0.0625	0.99947	0.99947	0
0.125	0.99885	0.99885	0
0.1875	0.99882	0.99882	0
0.25	0.99760	0.99760	0
0.3125	0.99697	0.99697	0

TABLE 3.8: For $h = 0.125$, the approximate and real solutions and absolute error values of example 3.2 obtained from the MG2 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	Error
0.125	0.99885	0.99885	0
0.25	0.99760	0.99760	0
0.375	0.99635	0.99635	0
0.5	0.99511	0.99510	0
0.625	0.99386	0.99385	0

Example 3.3 Consider a system of equations with initial conditions as discussed in [64]

$$
\begin{aligned}
x_1'(t) &= 9x_1(t) + 24x_2(t) + 5\cos(t) - \frac{1}{3}\sin(t), \\
x_2'(t) &= -24x_1(t) - 51x_2(t) - 9\cos(t) + \frac{1}{3}\sin(t), \\
x_1(0) &= \frac{4}{3}, x_2(0) = \frac{2}{3}.
\end{aligned} \tag{3.59}
$$

We solve this system using Magnus expansion technique and then compare the solution with exact solution. The exact solution of this system is

$$
\begin{aligned}
x_1(t) &= 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos(t), \\
x_2(t) &= -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos(t).
\end{aligned}
$$

To apply Magnus expansion method, first we rewrite the above system as

$$
\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}' = \begin{bmatrix} 9 & 24 & 5\cos(t) - \frac{1}{3}\sin(t) \\ -24 & -51 & -9\cos(t) + \frac{1}{3}\sin(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, \tag{3.60}
$$

subject to conditions

$$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix},$$

which imply

$$X' = AX,$$

where

$$A = \begin{bmatrix} 9 & 24 & 5\cos(t) - \frac{1}{3}\sin(t) \\ -24 & -51 & -9\cos(t) + \frac{1}{3}\sin(t) \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}.$$

Now applying the MG4, we have

$$x_{n+1} = \exp(\Pi^{[4]})x_n,$$

where

$$\Pi^{[4]} = \frac{1}{2}h(A_1 + A_2) + \frac{\sqrt{3}}{12}h^2[A_2, A_1],$$

and

$$A_1 = A\left(t_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h\right),$$

$$A_2 = A\left(t_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h\right),$$

in which $h = t_{n+1} - t_n$ and $x_{n+1} = x(t_{n+1})$.

By taking different values of $t \in [0, 1]$, such that $t_n = t_0 + nh$, the table 3.9 and 3.11 are genrated.

TABLE 3.9: For h=0.0625, iteration of Example 3.3 by MG4 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} 9 & 24 & 4.96046 \\ -24 & -51 & -8.94900 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 9 & 24 & 4.90639 \\ -24 & -51 & -8.86825 \\ 0 & 0 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} 9 & 24 & 4.93161 \\ -24 & -51 & -8.90664 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 9 & 24 & 4.86645 \\ -24 & -51 & -8.80586 \\ 0 & 0 & 0 \end{bmatrix}$
$\Pi^{[4]}$	$\begin{bmatrix} 0.5625 & 1.5 & 0.30870 \\ -1.5 & -3.1875 & -0.55716 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.5625 & 1.5 & 0.30476 \\ -1.5 & -3.1875 & -0.55107 \\ 0 & 0 & 0 \end{bmatrix}$
$x_{n+1} = \exp(\Pi^{[4]})x_n$	$x_1 = \exp(\Pi^{[4]})x_0$	$x_2 = \exp(\Pi^{[4]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 1.90178 \\ -0.98520 \\ 1 \end{bmatrix}$	$x_2 = \begin{bmatrix} 1.69471 \\ -0.99962 \\ 1 \end{bmatrix}$

Now by using these $t_1, t_2, \cdots, t_5$ in exact solution. The comparison is given in the table 3.10.

TABLE 3.10: For h= 0.0625, the approximate and real solutions and absolute error values of Example 3.3 obtained from the MG4 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	$x_2(t)$ Real	$x_2(t)$ Approximate	$x_1(t)$ Error	$x_2(t)$ Error
0.0625	1.9033	1.9017	-0.9869	-0.9852	0.0158	-0.0175
0.125	1.6976	1.6947	-1.0027	-0.9996	0.0296	-0.0313
0.1875	1.4663	1.4620	-0.8959	-0.8914	0.0429	-0.0444
0.25	1.2676	1.2620	-0.7952	-0.7894	0.0561	-0.0574
0.3125	1.1003	1.0935	-0.7087	-0.7019	0.0682	-0.0686

TABLE 3.11: For h=0.125, iteration of Example 3.3 by MG4 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} 9 & 24 & 4.89250 \\ -24 & -51 & -8.84673 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 9 & 24 & 4.71921 \\ -24 & -51 & -8.56736 \\ 0 & 0 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} 9 & 24 & 4.80164 \\ -24 & -51 & -8.70209 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 9 & 24 & 4.58544 \\ -24 & -51 & -8.34489 \\ 0 & 0 & 0 \end{bmatrix}$
$\Pi^{[4]}$	$\begin{bmatrix} 1.125 & 3 & 0.59989 \\ -3 & -6.375 & -1.08508 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1.125 & 3 & 0.57222 \\ -3 & -6.375 & -1.03867 \\ 0 & 0 & 0 \end{bmatrix}$
$x_{n+1} = \exp(\Pi^{[4]})x_n$	$x_1 = \exp(\Pi^{[4]})x_0$	$x_2 = \exp(\Pi^{[4]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 1.69084 \\ -0.99506 \\ 1 \end{bmatrix}$	$x_2 = \begin{bmatrix} 1.25522 \\ -0.78194 \\ 1 \end{bmatrix}$

Similarly by using these $t_1, t_2, \cdots, t_5$ in exact solution. The comparison is given in the table 3.12.

TABLE 3.12: For h= 0.125, the approximate and real solutions and absolute error values of Example 3.3 obtained from the MG4 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	$x_2(t)$ Real	$x_2(t)$ Approximate	$x_1(t)$ Error	$x_2(t)$ Error
0.125	1.6976	1.6908	-1.0027	-0.9950	0.0683	−0.0769
0.25	1.2676	1.2552	-0.7952	-0.7819	0.1242	−0.1328
0.375	0.9594	0.9290	-0.6348	-0.6093	0.3043	−0.2547
0.5	0.7387	0.7114	-0.5156	-0.4896	0.2729	−0.2597
0.625	0.5770	0.5467	-0.4236	-0.3939	0.3028	−0.2972

Example 3.4 Consider a system of equations with initial conditions as discussed in [65]

$$\begin{aligned} x_1'(t) &= -3x_1(t) + 3x_2(t) + 3\cos(t) - 3\sin(t), \\ x_2'(t) &= 2x_1(t) - 3x_2(t) - \cos(t) + 3\sin(t), \\ x_1(0) &= 1, x_2(0) = 0. \end{aligned} \tag{3.61}$$

We solve this system using Magnus expansion technique and then compare the solution with exact solution. The exact solution of this system is

$$x_1(t) = \cos(t), \quad x_2(t) = \sin(t). \tag{3.62}$$

To apply Magnus expansion method, first we rewrite the above system as

$$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}' = \begin{bmatrix} -3 & 3 & 3\cos(t) - 3\sin(t) \\ 2 & -3 & -\cos(t) + 3\sin(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, \tag{3.63}$$

subject to conditions

$$\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

which imply

$$X' = AX,$$

where

$$A = \begin{bmatrix} -3 & 3 & 3\cos(t) - 3\sin(t) \\ 2 & -3 & -\cos(t) + 3\sin(t) \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}.$$

Now applying the MG4, we have

$$x_{n+1} = \exp(\Pi^{[4]})x_n,$$

where

$$\Pi^{[4]} = \frac{1}{2}h(A_1 + A_2) + \frac{\sqrt{3}}{12}h^2[A_2, A_1],$$

and

$$A_1 = A\left(t_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h\right),$$

$$A_2 = A\left(t_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h\right),$$

in which $h = t_{n+1} - t_n$ and $x_{n+1} = x(t_{n+1})$.

By taking different values of $t \in [0, 1]$, such that $t_n = t_0 + nh$, the table 3.13 and 3.15 are genrated.

TABLE 3.13: For h=0.0625, iteration of Example 3.4 by MG4 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} -3 & 3 & 2.80676 \\ 2 & -3 & -0.81066 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -3 & 3 & 2.60256 \\ 2 & -3 & -0.61817 \\ 0 & 0 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} -3 & 3 & 2.64663 \\ 2 & -3 & -0.65911 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -3 & 3 & 2.43435 \\ 2 & -3 & -0.46115 \\ 0 & 0 & 0 \end{bmatrix}$
$\Pi^{[4]}$	$\begin{bmatrix} -0.1875 & 0.1875 & 0.1698 \\ 0.125 & -0.1875 & -0.0454 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.1875 & 0.1875 & 0.1568 \\ 0.125 & -0.1875 & -0.0332 \\ 0 & 0 & 0 \end{bmatrix}$
$x_{n+1} = \exp(\Pi^{[4]})x_n$	$x_1 = \exp(\Pi^{[4]})x_0$	$x_2 = \exp(\Pi^{[4]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.99048 \\ -0.07179 \\ 1 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.98227 \\ -0.14147 \\ 1 \end{bmatrix}$

Now by using these $t_1, t_2, \cdots, t_4$ in exact solution. The comparison is given in the table 3.14.

TABLE 3.14: For h= 0.0625, the approximate and real solutions and absolute error values of Example 3.4 obtained from the MG4 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	$x_2(t)$ Real	$x_2(t)$ Approximate	$x_1(t)$ Error	$x_2(t)$ Error
0.0625	0.9980	0.9904	0.0624	0.0717	0.0756	−0.0934
0.125	0.9921	0.9827	0.1246	0.1414	0.0941	−0.168
0.1875	0.9824	0.9746	0.1864	0.2104	0.0785	−0.2406
0.25	0.9689	0.3366	0.2474	0.7946	0.6322	-0.5472

TABLE 3.15: For h=0.125, iteration of Example 3.4 by MG4 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} -3 & 3 & 2.50346 \\ 2 & -3 & -0.52720 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -3 & 3 & 2.06735 \\ 2 & -3 & -0.14328 \\ 0 & 0 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} -3 & 3 & 2.26018 \\ 2 & -3 & -0.30995 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -3 & 3 & 1.79490 \\ 2 & -3 & 0.08481 \\ 0 & 0 & 0 \end{bmatrix}$
$\Pi^{[4]}$	$\begin{bmatrix} -0.1875 & 0.1875 & 0.1698 \\ 0.125 & -0.1875 & -0.0454 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.1875 & 0.1875 & 0.1568 \\ 0.125 & -0.1875 & -0.0332 \\ 0 & 0 & 0 \end{bmatrix}$
$x_{n+1} = \exp(\Pi^{[4]})x_n$	$x_1 = \exp(\Pi^{[4]})x_0$	$x_2 = \exp(\Pi^{[4]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.96158 \\ 0.16144 \\ 1 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.93553 \\ 0.30669 \\ 1 \end{bmatrix}$

Similarly by using these $t_1, t_2, \cdots, t_4$ in exact solution. The comparison is given in the table 3.16.

TABLE 3.16: For h= 0.125, the approximate and real solutions and absolute error values of Example 3.4 obtained from the MG4 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	$x_2(t)$ Real	$x_2(t)$ Approximate	$x_1(t)$ Error	$x_2(t)$ Error
0.125	0.9921	0.9615	0.1246	0.1614	0.3061	0.3677
0.25	0.9689	0.9355	0.2474	0.3066	0.3338	−0.5929
0.375	0.9305	0.9106	0.3662	0.4424	0.1984	0.0616
0.5	0.8775	0.8817	0.4794	0.5706	−0.414	−0.912

Example 3.5 Consider a system of equations with initial conditions as discussed in [66]

$$
\begin{aligned}
&x_1'(t) = -x_1(t) - 15x_2(t) + 15e^{-t}, \\
&x_2'(t) = 15x_1(t) - x_2(t) - 15e^{-t}, \\
&x_1(0) = 1, x_2(0) = 1.
\end{aligned}
\tag{3.64}
$$

We solve this system using Magnus expansion technique and then compare the solution with exact solution. The exact solution of this system is

$$
x_1(t) = x_2(t) = e^{-t}. \tag{3.65}
$$

To apply Magnus expansion method, first we rewrite the above system as

$$
\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}' = \begin{bmatrix} -1 & -15 & 15e^{-t} \\ 15 & -1 & -15e^{-t} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}, \tag{3.66}
$$

subject to conditions

$$
\begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},
$$

which imply

$$
X' = AX,
$$

where

$$A = \begin{bmatrix} -1 & -15 & 15e^{-t} \\ 15 & -1 & -15e^{-t} \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$X = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}.$$

Now applying the MG4, we have

$$x_{n+1} = \exp(\Pi^{[4]})x_n,$$

where

$$\Pi^{[4]} = \frac{1}{2}h(A_1 + A_2) + \frac{\sqrt{3}}{12}h^2[A_2, A_1],$$

and

$$A_1 = A\left(t_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)h\right),$$

$$A_2 = A\left(t_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)h\right),$$

in which $h = t_{n+1} - t_n$ and $x_{n+1} = x(t_{n+1})$.

By taking different values of $t \in [0, 1]$, such that $t_n = t_0 + nh$, the table 3.17 and 3.19 are genrated.

TABLE 3.17: For h=0.0625, iteration of Example 3.5 by MG4 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} -1 & -15 & 14.0911 \\ 15 & -1 & -14.0911 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -15 & 10.9117 \\ 15 & -1 & -10.9117 \\ 0 & 0 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} -1 & -15 & 13.4136 \\ 15 & -1 & -13.4136 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -15 & 12.6009 \\ 15 & -1 & -12.6009 \\ 0 & 0 & 0 \end{bmatrix}$
$\Pi^{[4]}$	$\begin{bmatrix} -0.062 & -0.937 & 0.864 \\ 0.937 & -0.062 & -0.853 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0625 & -0.937 & 0.721 \\ 0.937 & -0.062 & -0.749 \\ 0 & 0 & 0 \end{bmatrix}$
$x_{n+1} = \exp(\Pi^{[4]})x_n$	$x_1 = \exp(\Pi^{[4]})x_0$	$x_2 = \exp(\Pi^{[4]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.99998 \\ 0.99998 \\ 1 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.99996 \\ 0.99997 \\ 1 \end{bmatrix}$

Now by using these $t_1, t_2, \cdots, t_4$ in exact solution. The comparison is given in the table 3.18.

TABLE 3.18: For h= 0.0625, the approximate and real solutions and absolute error values of Example 3.5 obtained from the MG4 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	$x_2(t)$ Real	$x_2(t)$ Approximate	$x_1(t)$ Error	$x_2(t)$ Error
0.0625	0.93941	0.99998	0.93941	0.99998	−0.6057	−0.6057
0.125	0.8824	0.9999	0.8824	0.9999	-0.1174	-0.1174
0.1875	0.8290	0.9999	0.8290	0.9999	-0.1709	-0.1709
0.25	0.7788	0.9999	0.7788	0.9999	-0.2211	-0.2211

TABLE 3.19: For h=0.125, iteration of Example 3.5 by MG4 method

	$t_1 = t_0 + h$	$t_2 = t_1 + h$
A_1	$\begin{bmatrix} -1 & -15 & 12.89223 \\ 15 & -1 & -12.89223 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -15 & 11.37735 \\ 15 & -1 & -11.37735 \\ 0 & 0 & 0 \end{bmatrix}$
A_2	$\begin{bmatrix} -1 & -15 & 11.99482 \\ 15 & -1 & -11.99482 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -1 & -15 & 10.58539 \\ 15 & -1 & -10.58539 \\ 0 & 0 & 0 \end{bmatrix}$
$\Pi^{[4]}$	$\begin{bmatrix} -0.125 & -1.875 & 1.58377 \\ 1.875 & -0.125 & -1.52306 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.125 & -1.875 & 1.39767 \\ 1.875 & -0.125 & -1.34409 \\ 0 & 0 & 0 \end{bmatrix}$
x_{n+1} $= \exp(\Pi^{[4]})x_n$	$x_1 = \exp(\Pi^{[4]})x_0$	$x_2 = \exp(\Pi^{[4]})x_1$
x_{n+1}	$x_1 = \begin{bmatrix} 0.99998 \\ 0.99998 \\ 1 \end{bmatrix}$	$x_2 = \begin{bmatrix} 0.99997 \\ 0.99996 \\ 1 \end{bmatrix}$

Similarly by using these $t_1, t_2, \cdots, t_4$ in exact solution. The comparison is given in the table 3.20.

TABLE 3.20: For h= 0.125, the approximate and real solutions and absolute error values of Example 3.5 obtained from the MG4 method

t	$x_1(t)$ Real	$x_1(t)$ Approximate	$x_2(t)$ Real	$x_2(t)$ Approximate	$x_1(t)$ Error	$x_2(t)$ Error
0.125	0.8824	0.9999	0.8824	0.9999	-0.1174	-0.1174
0.25	0.7788	0.9999	0.7788	0.9999	-0.2211	-0.2211
0.375	0.6872	0.9999	0.6872	0.9999	-0.3126	-0.3126
0.5	0.6065	0.9999	0.6065	0.9999	-0.3934	-0.3934

Chapter 4

Acoustic Propagation with Arbitrary Expansion Chamber

In this chapter, we consider the acoustic wave propagation and scattering in a waveguide having arbitrary configuration of expansion chamber. The governing boundary value problem is solved by using Multimodal admittance method. The boundary value problem involved Helmholtz equation in accompany with rigid, soft and rigid-soft boundary conditions. Thus the use of Multimodal procedure leads to the accurate solution problem. The section wise detail is given as follows: Section 4.2 contains the mathematical formulation and solution of soft-soft, rigid-rigid and rigid-soft problem. The computational results and discussion is provided in Section 4.3.

4.1 Problem Formulation

Consider a two dimensional waveguide extended infinitely along x-direction and contains finite height along y-direction. The inside of the waveguide is filled with compressible fluid of density ρ and sound speed c. The physical configuration of the duct is as shown Figuer 4.1. The walls of waveguide are contained arbitrary geometrical configuration.

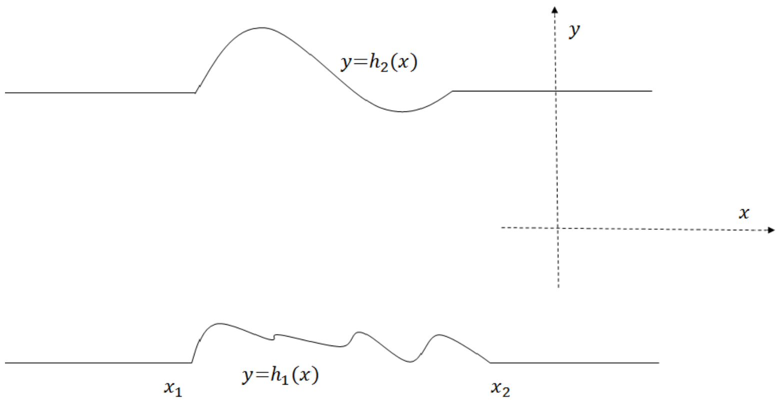

FIGURE 4.1: Geometry of the 2D waveguide.

The acoustic propagation in waveguide can be expressed using wave equation

$$\nabla^2\Phi(x,y,t) = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}, \tag{4.1}$$

where $\Phi(x,y,t)$ is the time dependent field potential. The acoustic pressure and normal velocity vectors are related to the field potential by the relations

$$p = -\rho\frac{\partial\Phi}{\partial t}, \tag{4.2}$$

and

$$v = \nabla\Phi, \tag{4.3}$$

respectively. Assuming the harmonic time dependence $e^{-i\omega t}$ in which ω is radiant frequency, we write

$$\Phi(x,y,t) = \phi(x,y)e^{-i\omega t}, \tag{4.4}$$

where $\phi(x,y)$ is the time independent field potential. By using (4.4) into (4.1), we find

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right\}\phi(x,y) = 0, \tag{4.5}$$

where $k = \dfrac{\omega}{c}$ is the wave number. The boundary conditions of the duct are assumed to contain three categories:

1) Soft-Soft.

2) Rigid-Rigid.

3) Soft-Rigid.

The solution of the governing boundary value problems with aforementioned condition is discussed in next section.

4.2 Multimodal Solution

In this section we formulate multimodal solution subject to soft-soft, rigid-rigid and soft-rigid settings.

4.2.1 Soft-Soft Case

The acoustically soft boundary condition of waveguide gives

$$\begin{aligned} \Phi(x, y) &= 0, \quad \text{at} \quad y = 0, \\ \Phi(x, y) &= 0, \quad \text{at} \quad y = b. \end{aligned} \tag{4.6}$$

Note that (4.6) is known as Dirichlet bounday condition.

Eigenvalue Problem

First, we determine the transverse mode through the respective transverse eigenvalue problem, that is

$$\frac{d^2 g}{dy^2} + \gamma^2 g = 0, \tag{4.7}$$

$$g(x, h_1) = 0, \tag{4.8}$$

$$g(x, h_2) = 0. \tag{4.9}$$

By using (4.7), we obtain

$$g(x,y) = A\cos\gamma y + B\sin\gamma y, \tag{4.10}$$

where A and B are arbitrary constants. By using boundary conditions (4.8), we find

$$A = -B\frac{\sin(\gamma h_1)}{\cos(\gamma h_1)}, \tag{4.11}$$

which on substituting into (4.10) leads to

$$g(x,y) = \frac{B}{\cos(\gamma h_1)}\sin(\gamma(y-h_1)). \tag{4.12}$$

Now by invoking (4.12), for non trival solution (4.9), reveals

$$\sin(\gamma(h_2 - h_1)) = 0,$$

which gives $\gamma \equiv \gamma_n = \dfrac{n\pi}{h_2 - h_1}; n = 1, 2\cdots$ and the eigenfunctions

$$g(x,y) \equiv g_n(x,y) = \sin\left(\frac{n\pi}{h(x)}(y-h_1)\right), \tag{4.13}$$

where $h(x) = h_2(x) - h_1(x)$. These eigenfunctions are orthogonal in nature and satisfy orthogonality relation

$$\int_{h_1}^{h_2} g_m g_n \, dy = \frac{h}{2}\delta_{mn}\epsilon_n, \tag{4.14}$$

where

$$\epsilon_n = \begin{cases} 2, & n = 0, \\ 0. \end{cases}$$

The corresponding orthonormal functions are found by dividing the respective weights. Thus, the orthonormal functions are

$$\phi_n = \sqrt{\frac{2}{h(x)\epsilon_n}}g_n(x,y); \quad n = 0, 1, 2..., \tag{4.15}$$

which satisfy

$$\int_{h_1}^{h_2} \phi_n \phi_m \, dy = \delta_{mn}. \tag{4.16}$$

4.2.2 Rigid-Rigid Case

The acoustically rigid boundary condition of waveguide gives

$$\begin{aligned} \frac{\partial \Phi}{\partial y}(x,y) &= 0, \quad \text{at} \quad y = 0, \\ \frac{\partial \Phi}{\partial y}(x,y) &= 0, \quad \text{at} \quad y = b. \end{aligned} \tag{4.17}$$

Note that (4.17) is known as Neumann bounday condition.

Eigenvalue Problem

First, we determine the transverse mode through the respective transverse eigenvalue problem, that is

$$\frac{d^2 g}{dy^2} + \gamma^2 g = 0, \tag{4.18}$$

$$\frac{\partial g}{\partial y}(x, h_1) = 0, \tag{4.19}$$

$$\frac{\partial g}{\partial y}(x, h_2) = 0. \tag{4.20}$$

By using (4.18), we obtain

$$g(x,y) = A\cos\gamma y + B\sin\gamma y, \tag{4.21}$$

where A and B are arbitrary constants. By using boundary conditions (4.19), we find

$$A = B\frac{\cos(\gamma h_1)}{\sin(\gamma h_1)}, \tag{4.22}$$

which on substituting into (4.21) leads to

$$g(x,y) = \frac{B}{\sin(\gamma h_1)}\cos(\gamma(y - h_1)). \tag{4.23}$$

Now by invoking (4.23), for non trival solution (4.20), reveals

$$\sin(\gamma(h_2 - h_1)) = 0,$$

which gives $\gamma \equiv \gamma_n = \dfrac{n\pi}{h_2 - h_1}; n = 0, 1, 2\cdots$ and the eigenfunctions

$$g(x, y) \equiv g_n(x, y) = \cos\left(\frac{n\pi}{h(x)}(y - h_1)\right), \tag{4.24}$$

where $h(x) = h_2(x) - h_1(x)$. These eigenfunctions are orthogonal in nature and satisfy orthogonality relation

$$\int_{h_1}^{h_2} g_m g_n \, dy = \frac{h}{2}\delta_{mn}\epsilon_n, \tag{4.25}$$

where

$$\epsilon_n = \begin{cases} 2, & n = 0, \\ 0. & \end{cases}$$

The corresponding orthonormal functions are found by dividing the respective weights. Thus, the orthonormal functions are

$$\phi_n = \sqrt{\frac{2}{h(x)\epsilon_n}} g_n(x, y); \quad n = 1, 2..., \tag{4.26}$$

which satisfy

$$\int_{h_1}^{h_2} \phi_n \phi_m \, dy = \delta_{mn}. \tag{4.27}$$

4.2.3 Soft-Rigid Case

The acoustically soft-rigid boundary condition of waveguide gives

$$\begin{aligned} \Phi(x, y) &= 0, \quad \text{at} \quad y = 0, \\ \frac{\partial \Phi}{\partial y}(x, y) &= 0, \quad \text{at} \quad y = b. \end{aligned} \tag{4.28}$$

Note that (4.28) is known as Dirichlet-Neumann bounday condition.

Eigenvalue Problem

In this technique, first we determine the transverse mode through the respective transverse eigen value problem, that is

$$\frac{d^2 g}{dy^2} + \gamma^2 g = 0, \tag{4.29}$$

$$g(x, h_1) = 0, \tag{4.30}$$

$$\frac{\partial g}{\partial y}(x, h_2) = 0. \tag{4.31}$$

By using (4.29), we obtain

$$g(x, y) = A \cos \gamma y + B \sin \gamma y, \tag{4.32}$$

where A and B are arbitrary constants. By using boundary conditions (4.30), we find

$$A = -B \frac{\sin(\gamma h_1)}{\cos(\gamma h_1)}, \tag{4.33}$$

which on substituting into (4.32) leads to

$$g(x, y) = \frac{B}{\cos(\gamma h_1)} \sin(\gamma(y - h_1)). \tag{4.34}$$

Now by invoking (4.34), for non trival solution (4.31), reveals

$$\cos(\gamma(h_2 - h_1)) = 0,$$

which gives $\gamma \equiv \gamma_n = \dfrac{(n+1)\pi}{2(h_2 - h_1)}; n = 1, 2 \cdots$ and the eigenfunctions

$$g(x, y) \equiv g_n(x, y) = \sin\left(\frac{(n+1)\pi}{2h(x)}(y - h_1)\right), \tag{4.35}$$

where $h(x) = h_2(x) - h_1(x)$. These eigenfunctions are orthogonal in nature and satisfy orthogonality relation

$$\int_{h_1}^{h_2} g_m g_n \, dy = \frac{h}{2}\delta_{mn}. \tag{4.36}$$

The corresponding orthonormal functions are found by dividing the respective weights. Thus, the orthonormal functions are

$$\phi_n = \sqrt{\frac{2}{h(x)}} g_n(x,y); \quad n = 1, 2..., \tag{4.37}$$

which satisfy

$$\int_{h_1}^{h_2} \phi_n \phi_m \, dy = \delta_{mn}. \tag{4.38}$$

To project solution, we consider first order evaluation Helmholtz equation along x-direction by assuming

$$\frac{\partial \phi}{\partial x} = \Psi. \tag{4.39}$$

On using (4.39) into (4.5), it is straightforward to write

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial^2 \phi}{\partial y^2} - k^2. \tag{4.40}$$

By combining (4.39) and (4.40)

$$\frac{\partial}{\partial x}\begin{pmatrix}\Phi \\ \Psi\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2}{\partial y^2} - k^2 & 0\end{pmatrix}\begin{pmatrix}\Phi \\ \Psi\end{pmatrix}. \tag{4.41}$$

Now the transverse modes of the waveguide are used to project Φ and Ψ as

$$\Phi = \sum_{n=1}^{\infty} a_n(x)\phi_n(x,y), \tag{4.42}$$

$$\Psi = \sum_{n=1}^{\infty} b_n(x)\phi_n(x,y), \tag{4.43}$$

where $a_n(x)$ and $b_n(x)$ are unknowns. These can be found through the coupled mode equations. To formulate these equations, we multiply (4.39) with $\phi_m(x,y)$ and integrate over $h_1 \leq y \leq h_2$ to get

$$\int_{h_1}^{h_2} \frac{\partial \Phi}{\partial x}\phi_m \, dy = \int_{h_1}^{h_2} \Psi \phi_m \, dy. \tag{4.44}$$

On substituting (4.42) and (4.43), (4.44) gives

$$\sum_{n=1}^{\infty}\int_{h_1}^{h_2}\left\{a_n'(x)\phi_n(x,y)+a_n(x)\phi_n'(x,y)\right\}\phi_m(x,y)\,dy=\sum_{n=1}^{\infty}\int_{h_1}^{h_2}b_n(x)\phi_n(x,y)\phi_m(x,y)\,dy, \tag{4.45}$$

or

$$\sum_{n=1}^{\infty}a_n'(x)\int_{h_1}^{h_2}\phi_n\phi_m\,dy+\sum_{n=1}^{\infty}a_n(x)\int_{h_1}^{h_2}\phi_n'\phi_m\,dy=\sum_{n=1}^{\infty}b_n(x)\int_{h_1}^{h_2}\phi_n\phi_m\,dy, \tag{4.46}$$

which on using (4.16) leads to

$$\sum_{n=1}^{\infty}a_n'(x)\delta_{mn}+\sum_{n=1}^{\infty}a_n(x)F_{mn}=\sum_{n=1}^{\infty}b_n(x)\delta_{mn}, \tag{4.47}$$

where F is a different for different boundary conditions.

- **For Soft-Soft Case**

$$F_{mn}=\begin{cases}-\frac{mn}{m^2-n^2}\frac{2}{h}[(-1)^{m+n}h_2'(x)-h_1'(x)], & m\neq n,\\ 0, & m=n.\end{cases}$$

- **For Rigid-Rigid Case**

$$F_{mn}=\begin{cases}-\frac{m^2}{m^2-n^2}\frac{2}{h}[(-1)^{m+n}h_2'(x)-h_1'(x)], & m\neq n,\\ \frac{1}{h(x)}[h_1'(x)-(-1)^{n+m}h_2'(x)], & m=n.\end{cases}$$

- **For Soft-Rigid Case**

$$F_{mn}=\begin{cases}-\frac{1}{h^2}\frac{(1+m)}{(n+m)(2+m+n)}[(1+m)(1)^{1+m}h_2'(x)-(1+n)h_1'(x)], & m\neq n,\\ 0, & m=n.\end{cases}$$

Accordingly, we multiply (4.40) with ϕ_m and integrate over $h_1\leq y\leq h_2$ to obtain

$$\int_{h_1}^{h_2}\frac{\partial\Psi}{\partial x}\phi_m\,dy=-\int_{h_1}^{h_2}\frac{\partial^2\Phi}{\partial y^2}\phi_m\,dy-k^2\int_{h_1}^{h2}\Phi\phi_m\,dy. \tag{4.48}$$

Applying integration by parts on the first term on right hand side of (4.48) and then simplifying the resulting leads to

$$\int_{h_1}^{h_2} \frac{\partial \Psi}{\partial x}\phi_m \, dy = -\gamma_m^2 \int_{h_1}^{h_2} \Phi\phi_m \, dy - k^2 \int_{h_1}^{h2} \Phi\phi_m \, dy. \tag{4.49}$$

By invoking (4.42)−(4.43) into (4.49), we find

$$\begin{aligned}&\sum_{n=1}^{\infty} b_n'(x) \int_{h_1}^{h_2} \phi_n\phi_m \, dy = (-\gamma_m^2 - k^2) \sum_{n=1}^{\infty} a_n(x) \int_{h_1}^{h_2} \phi_n\phi_m \, dy \\ &- \sum_{n=1}^{\infty} b_n(x) \int_{h_1}^{h2} \phi_n'\phi_m \, dy.\end{aligned} \tag{4.50}$$

On using (4.16), we get

$$\sum_{n=1}^{\infty} b_n'(x)\delta_{mn} - \sum_{n=1}^{\infty} b_n(x)F_{nm} = -k^2 \sum_{n=1}^{\infty} a_n\delta_{mn}. \tag{4.51}$$

Now from (4.16) and (4.51), we can write the coupled mode system

$$\begin{pmatrix} a'(x) \\ b'(x) \end{pmatrix} = \begin{pmatrix} -F & I \\ -K^2 & F^T \end{pmatrix} \begin{pmatrix} a(x) \\ b(x) \end{pmatrix} \tag{4.52}$$

where K is the diagonal matrix such that $K_{mn} = k_n\delta_{mn}$ with $k_n = \sqrt{k^2 - \frac{n^2\pi^2}{h^2}}$, F is matrix having entries F_{mn} and I is identity matrix.

Note that a and b represent vectors with entries a_n and b_n for $n = 1, 2...$, respectively. In this way, we get a first order differential system. We cannot treat it as a simple inital value problems system because of following two reason: first, the problem is not well posed because of given radiation condition at anechoic interface and source term at inlet interface and second reason is that the problem is unstable because of evanescent modes.

Therfore, we apply the admittance matrix method as proposed in [55, 67]. In this method we defined admittance matrix Y through the relation

$$b = Ya. \tag{4.53}$$

To compute the admittance matrix, we apply Magnus-Möbius scheme [55]. In this method, we apply numerical integration after descritizing the path between inlet and anechoic interface into some M points, such that $L = \tilde{x}_1 > \tilde{x}_2 > \cdots > \tilde{x}_M = 0$. The radiation condition is applied at $\tilde{x}_1 = L$. For the given problem, the radiation condition as defined in [68] is

$$Y(x = +\infty) = Y_c = ik. \tag{4.54}$$

Now to apply the scheme on (4.52), we defined

$$H(x) = \begin{pmatrix} -F(x) & I \\ -K^2(x) & F^T(x) \end{pmatrix}. \tag{4.55}$$

On using the Magnus-Möbius scheme, we can write

$$\begin{pmatrix} a(\tilde{x}_{n+1}) \\ b(\tilde{x}_{n+1}) \end{pmatrix} = e^{\Pi(\tilde{x}_n)} \begin{pmatrix} a(\tilde{x}_n) \\ b(\tilde{x}_n) \end{pmatrix}. \tag{4.56}$$

Here the expression for $H(\tilde{x}_n)$ depends on the scheme defined in Chapter 3. For scheme of second order

$$\Pi(\tilde{x}_n) = \delta_n H\left(\frac{\tilde{x}_n + \tilde{x}_{n+1}}{2}\right), \tag{4.57}$$

where $\delta_n = \tilde{x}_{n+1} - \tilde{x}_n$.

For scheme of fourth order

$$\Pi(\tilde{x}_n) = \frac{1}{2}\delta_n(H_1 + H_2) + \frac{\sqrt{3}}{12}\delta_n^2[H_2, H_1], \tag{4.58}$$

where

$$H_1 = H\left(\tilde{x}_n + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\delta_n\right),$$

and

$$H_2 = H\left(\tilde{x}_n + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\delta_n\right).$$

For $\tilde{x}_n$, the matrix exponential $e^{\Pi(\tilde{x}_n)}$ can be found as

$$e^{\Pi(\tilde{x}_n)} = \begin{pmatrix} E_1(\tilde{x}_n) & E_2(\tilde{x}_n) \\ E_3(\tilde{x}_n) & E_4(\tilde{x}_n) \end{pmatrix}, \tag{4.59}$$

by using (4.59) into (4.56), we get

$$\begin{pmatrix} a(\tilde{x}_{n+1}) \\ b(\tilde{x}_{n+1}) \end{pmatrix} = \begin{pmatrix} E_1(\tilde{x}_n) & E_2(\tilde{x}_n) \\ E_3(\tilde{x}_n) & E_4(\tilde{x}_n) \end{pmatrix} \begin{pmatrix} a(\tilde{x}_n) \\ b(\tilde{x}_n) \end{pmatrix}. \tag{4.60}$$

Now to get the admittance formulation from (4.53), we can write

$$\begin{aligned} b(\tilde{x}_n) &= Y(\tilde{x}_n)a(\tilde{x}_n), \\ b(\tilde{x}_{n+1}) &= Y(\tilde{x}_{n+1})a(\tilde{x}_{n+1}). \end{aligned} \tag{4.61}$$

By using (4.61) into (4.60)

$$\begin{pmatrix} a(\tilde{x}_{n+1}) \\ Y(\tilde{x}_{n+1})a(\tilde{x}_{n+1}) \end{pmatrix} = \begin{pmatrix} E_1(\tilde{x}_n) & E_2(\tilde{x}_n) \\ E_3(\tilde{x}_n) & E_4(\tilde{x}_n) \end{pmatrix} \begin{pmatrix} a(\tilde{x}_n) \\ Y(\tilde{x}_n)a(\tilde{x}_n) \end{pmatrix}, \tag{4.62}$$

which yield the following equaion

$$a(\tilde{x}_{n+1}) = E_1(\tilde{x}_n)a(\tilde{x}_n) + E_2(\tilde{x}_n)Y(\tilde{x}_n)a(\tilde{x}_n), \tag{4.63}$$

and

$$Y(\tilde{x}_{n+1})a(\tilde{x}_{n+1}) = E_3(\tilde{x}_n)a(\tilde{x}_n) + E_4(\tilde{x}_n)Y(\tilde{x}_n)a(\tilde{x}_n). \tag{4.64}$$

Using (4.63) into (4.64), we get

$$Y(\tilde{x}_{n+1}) = [E_3(\tilde{x}_n) + E_4(\tilde{x}_n)Y(\tilde{x}_n)][E_1(\tilde{x}_n) + E_2(\tilde{x}_n)Y(\tilde{x}_n)]^{-1}. \tag{4.65}$$

4.2.4 Reflection and Transmission Matrix

The wave components are divided into right and left going parts as stated in [69, 70], to obtain the reflection matrix R from the estimated admittance matrix

Y, we can write

$$a = a^+ + a^-,$$

and

$$b = Y_c(a^+ - a^-).$$

From the definition of reflection matrix $a^- = Ra^+$, we get

$$R = (Y_c + Y)^{-1}(Y_c - Y). \tag{4.66}$$

However, by defining the propagator matrix G in the same way as the Y matrix, it is possible to derive the transmission matrix at the same time as the Y matrix,

$$a(x_2) = G(x_2, x)a(x), \tag{4.67}$$

$G(x_2, x_2) = I$, where I is the identity matrix, and $x_2 \geq x$. The equation governing G is then found to be

$$G' = -G(-F + Y), \tag{4.68}$$

with the initial value $G(x_2, x = x_1) = I$, and the transmission matrix is given by

$$T = G(x_2, x = x_1)(I + R). \tag{4.69}$$

4.3 Numerical Results

In this section, the numerical result are presented. The reflection and transmission are plotted against frequency. For numerical computation, the dimensional height is defined by $h_1 = 0$, and $h_2 = 1 + 0.15(1 + \cos(\pi x/b))$ and frequency is along $2\pi \leq k \leq 6\pi$. The total number of modes that gives the dimension is 4 and the number of discretization along $-1.2 < x \leq 1.2$ is only 200. Figures 4.2-4.19, display the scatering against k with different material properties of the bounding walls. In Figures 4.2-4.7, the absloute value of reflecting modes R_{11}, R_{22}, and R_{33} and transmitting modes T_{11}, T_{22}, and T_{33} are portrayed for the acoustically soft wall conditions. From these figures it can seen that the fundamental mode reflection is almost 0 while the second and third modes contain maximum value in start that decreases by increasing k.

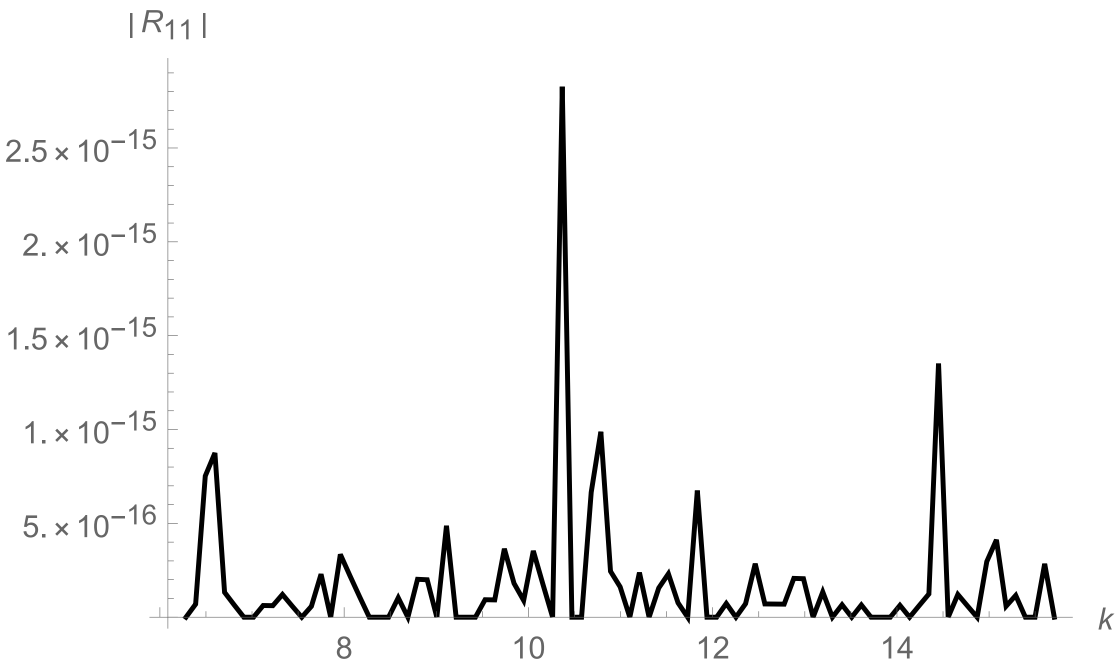

FIGURE 4.2: The absolute value of R_{11} against frequency k with soft condition.

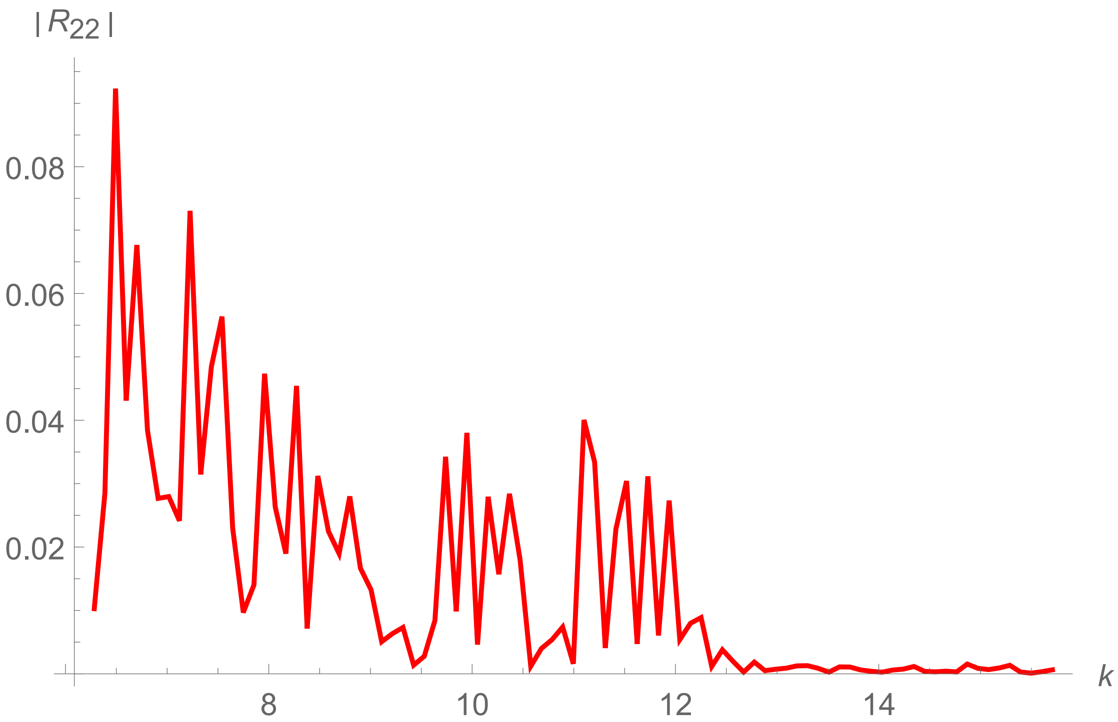

FIGURE 4.3: The absolute value of R_{22} against frequency k with soft condition.

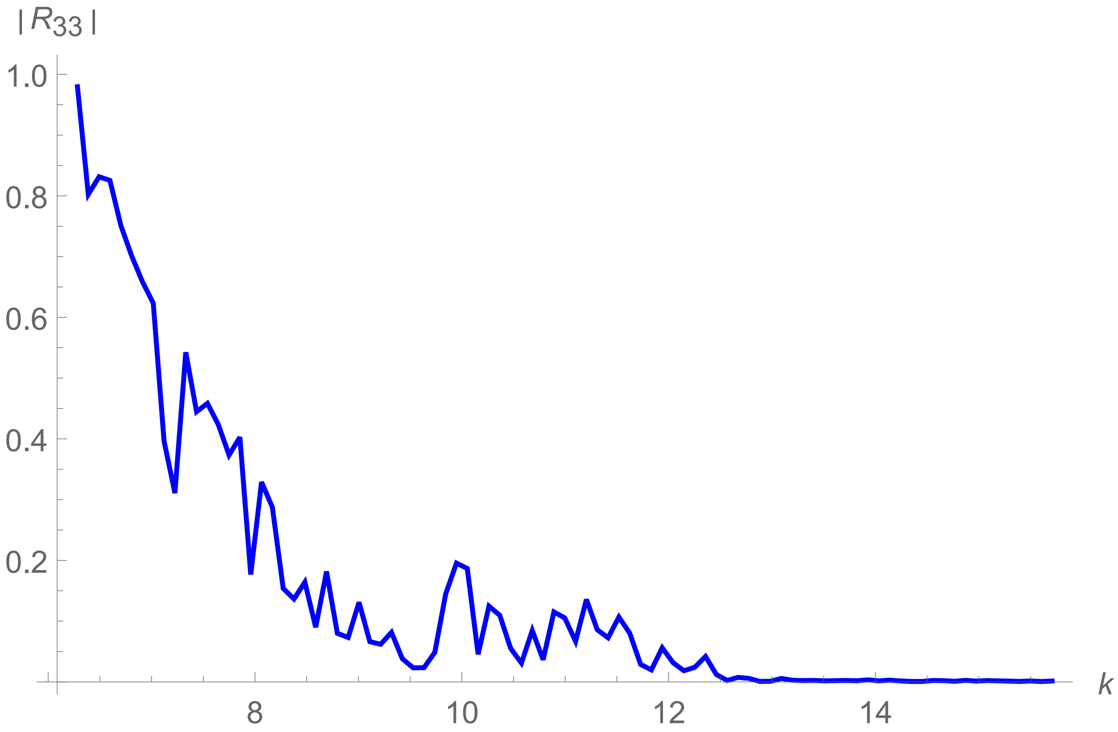

FIGURE 4.4: The absolute value of R_{33} against frequency k with soft condition.

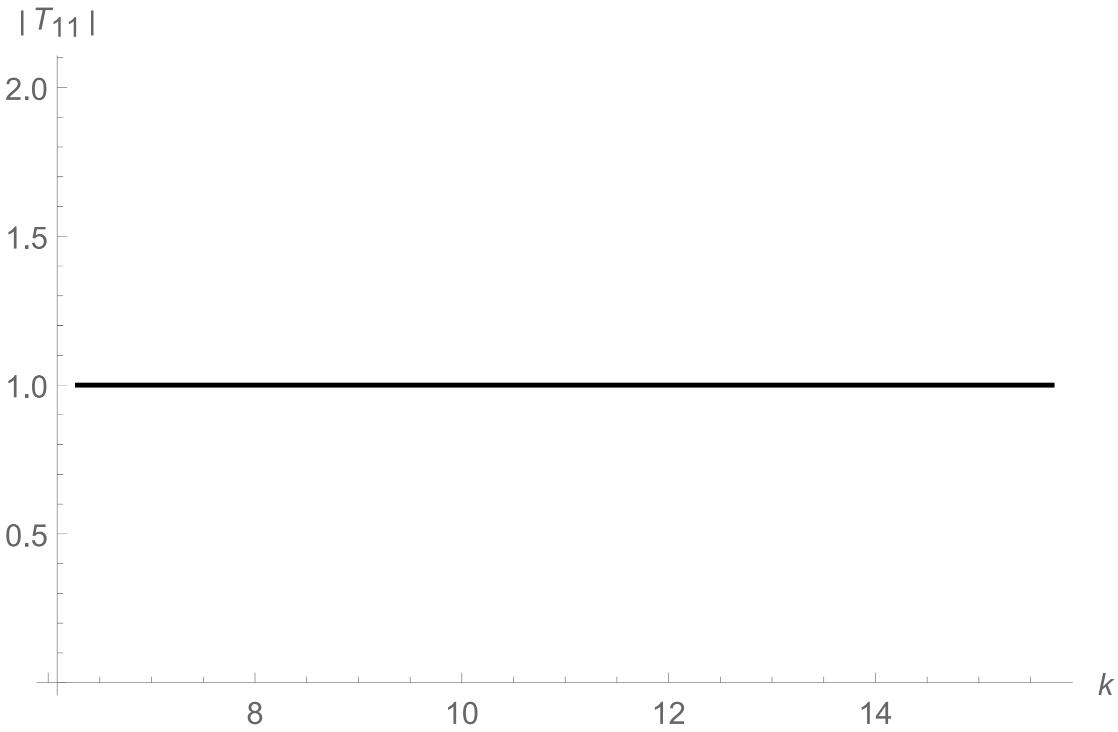

FIGURE 4.5: The absolute value of T_{11} against frequency k with soft condition.

On the other hand, the transmission is unity for all values k for the fundamental transmitting wave, see for instance Fig. 4.5. However, the second mode transitting modes contain a huge spike over $6 < k < 8$ for acoustically soft setting of boundaries, see Figs. 4.6-4.7.

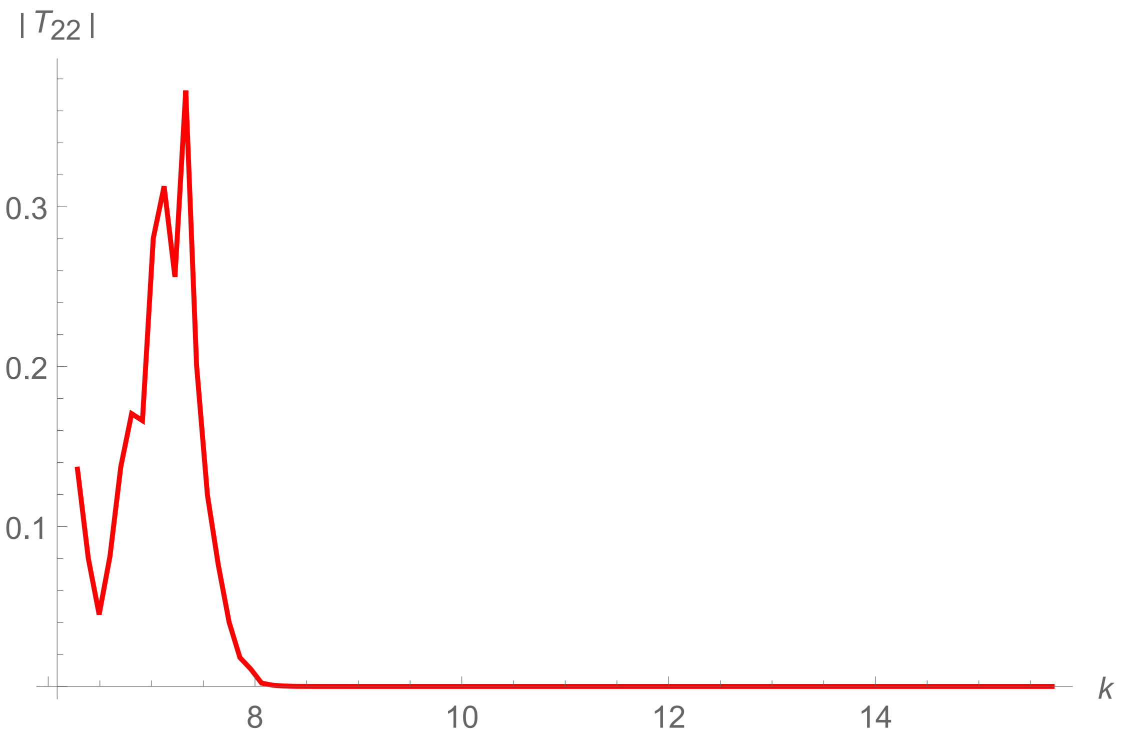

FIGURE 4.6: The absolute value of T_{22} against frequency k with soft condition.

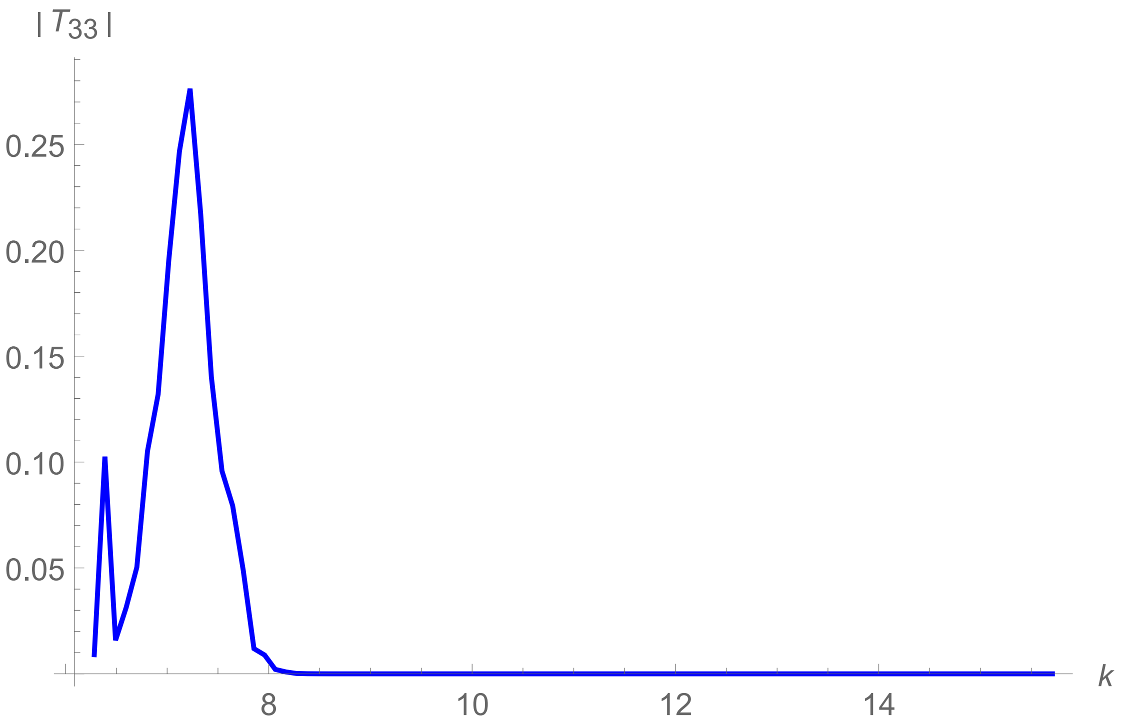

FIGURE 4.7: The absolute value of T_{33} against frequency k with soft condition.

For rigid-rigid wall conditions, the absloute values of reflecting modes R_{11}, R_{22} and R_{33} and transmitting modes T_{11}, T_{22} and T_{33} are portrayed for the acoustically soft wall condition in Figures 4.8-4.13.

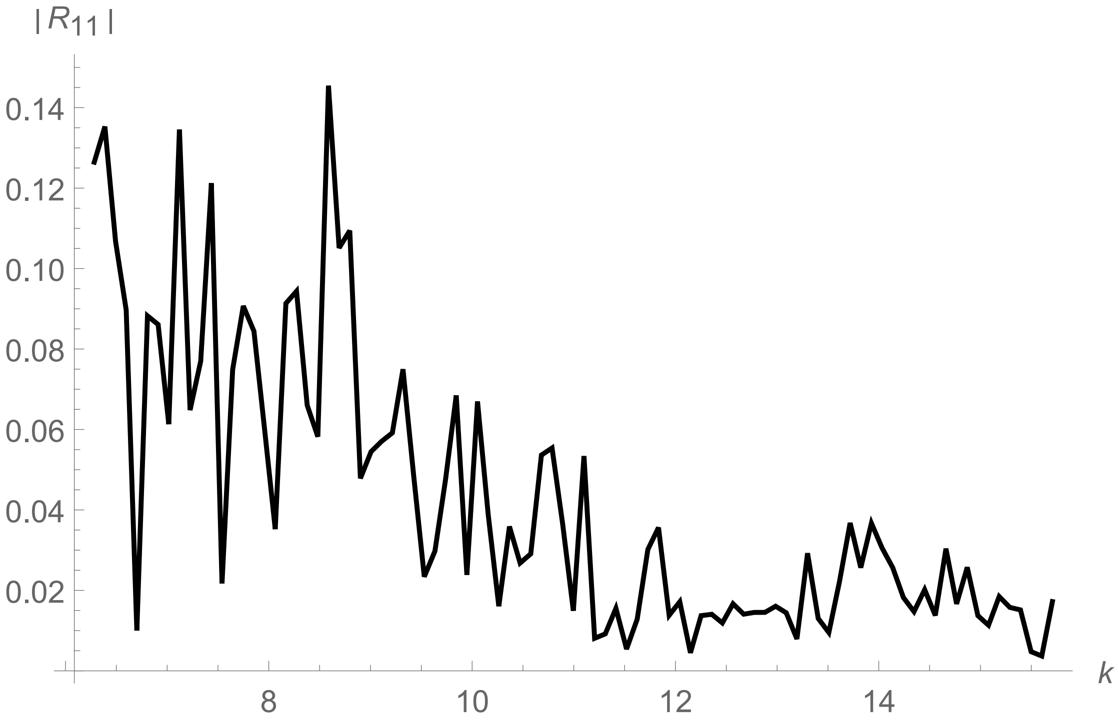

FIGURE 4.8: The absolute value of R_{11} against frequency k with rigid condition.

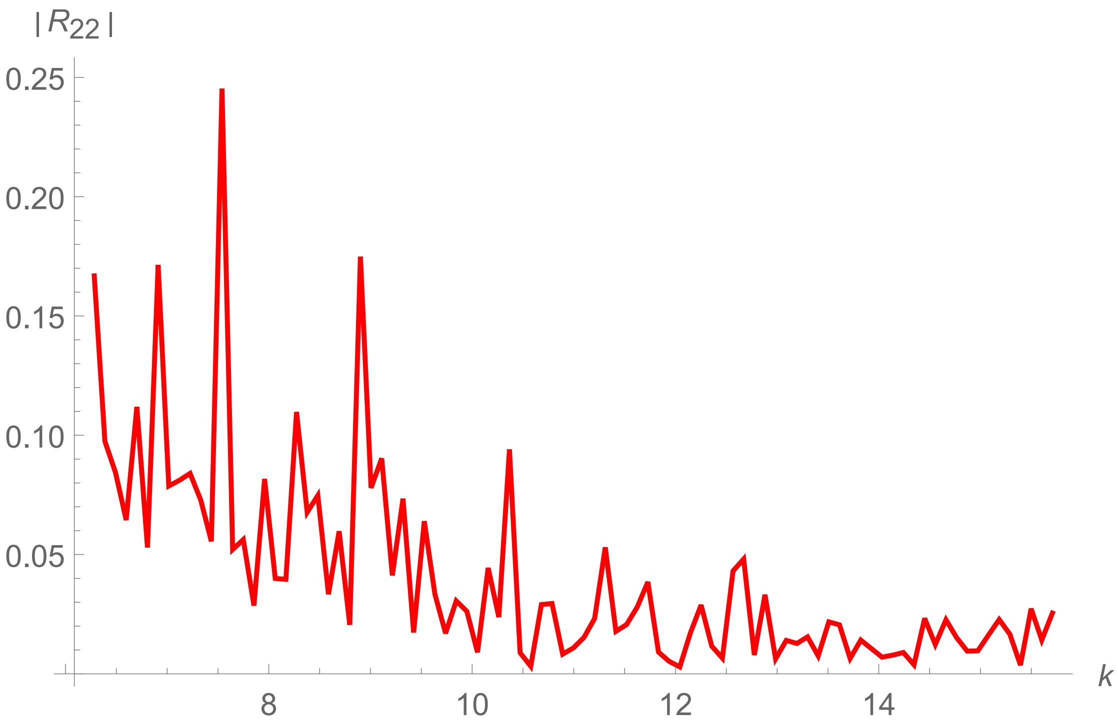

FIGURE 4.9: The absolute value of R_{22} against frequency k with rigid condition.

The fundamental reflecting mode is shown in Figure 4.8. Unlike to the soft conditions, the fundamental mode for rigid case is not zero and comparises the fluctuating behavior.

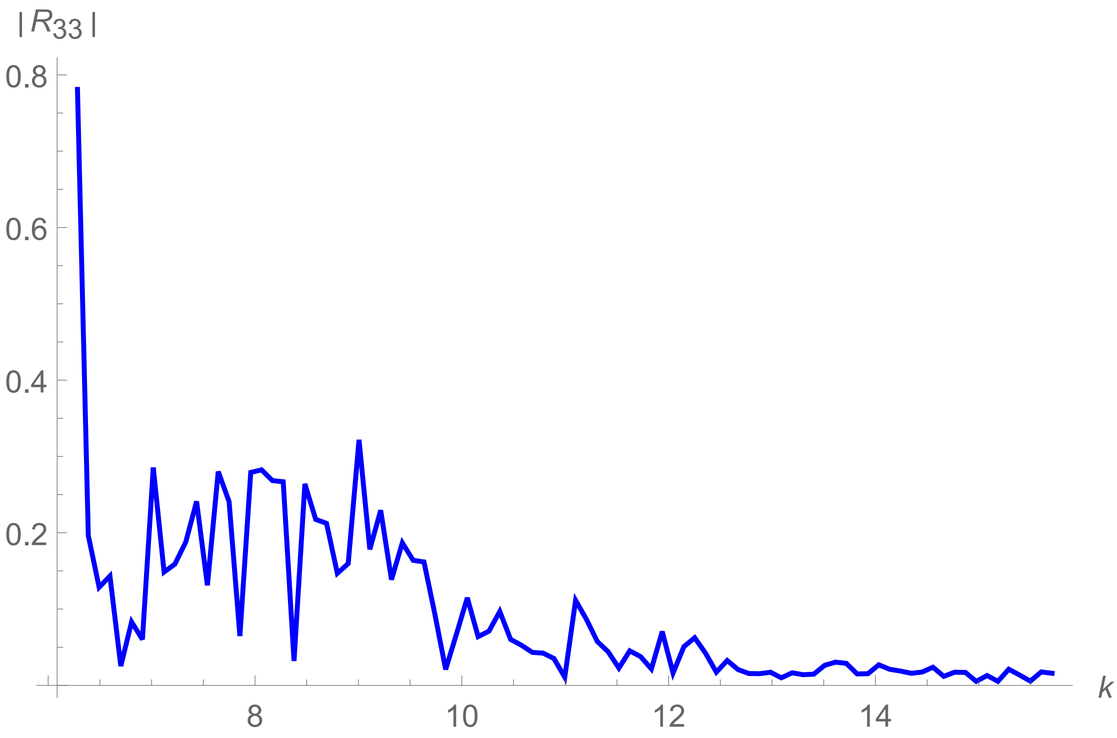

FIGURE 4.10: The absolute value of R_{33} against frequency k with rigid condition.

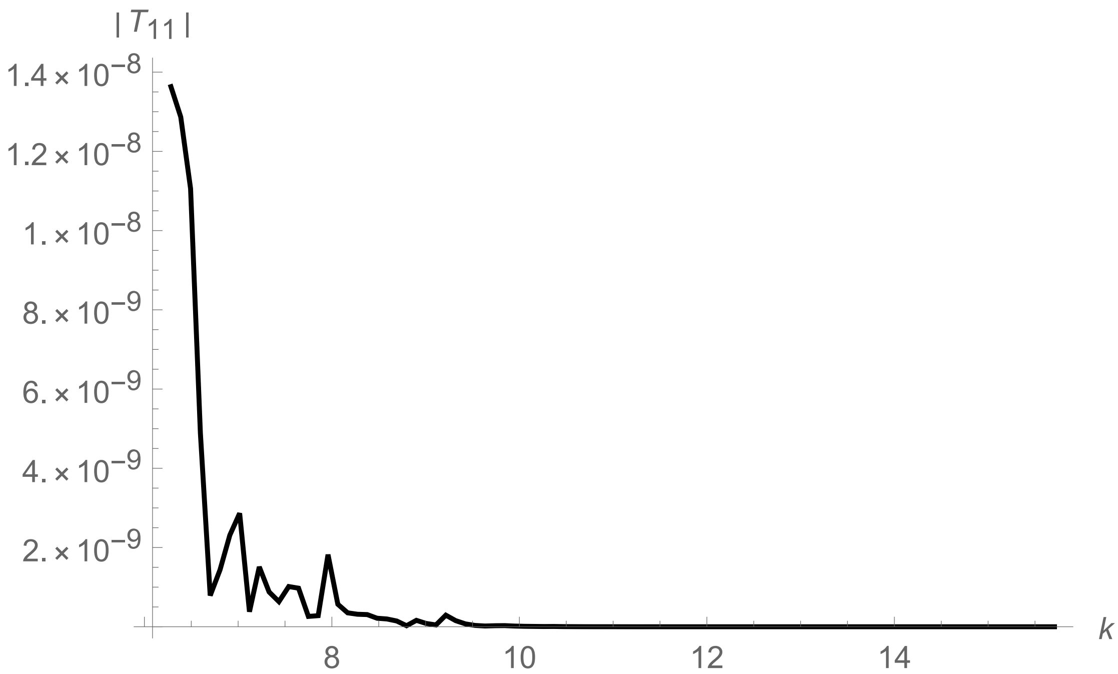

FIGURE 4.11: The absolute value of T_{11} against frequency k with rigid condition.

Accordingly, the fluctuating behavior is seen in Figure 4.9 and Figure 4.10. The transmitting mode against rigid setting is shown in Figures 4.11-4.13. It is observed that first two transmitting modes shown in Figures 4.11 and 4.12

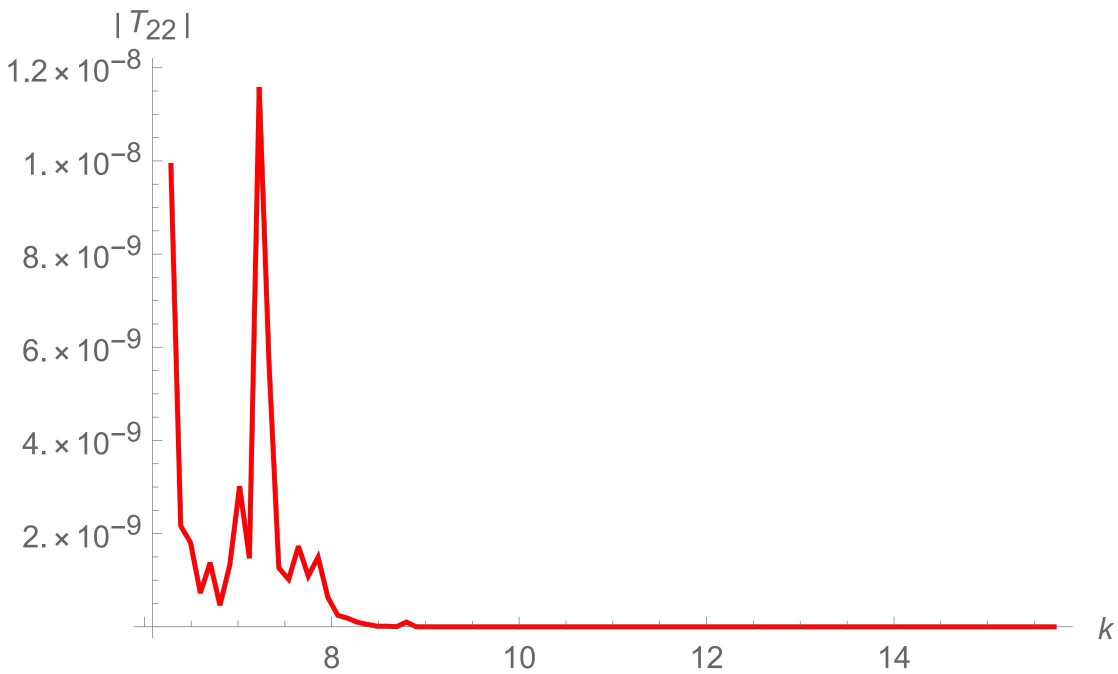

FIGURE 4.12: The absolute value of T_{22} against frequency k with rigid condition.

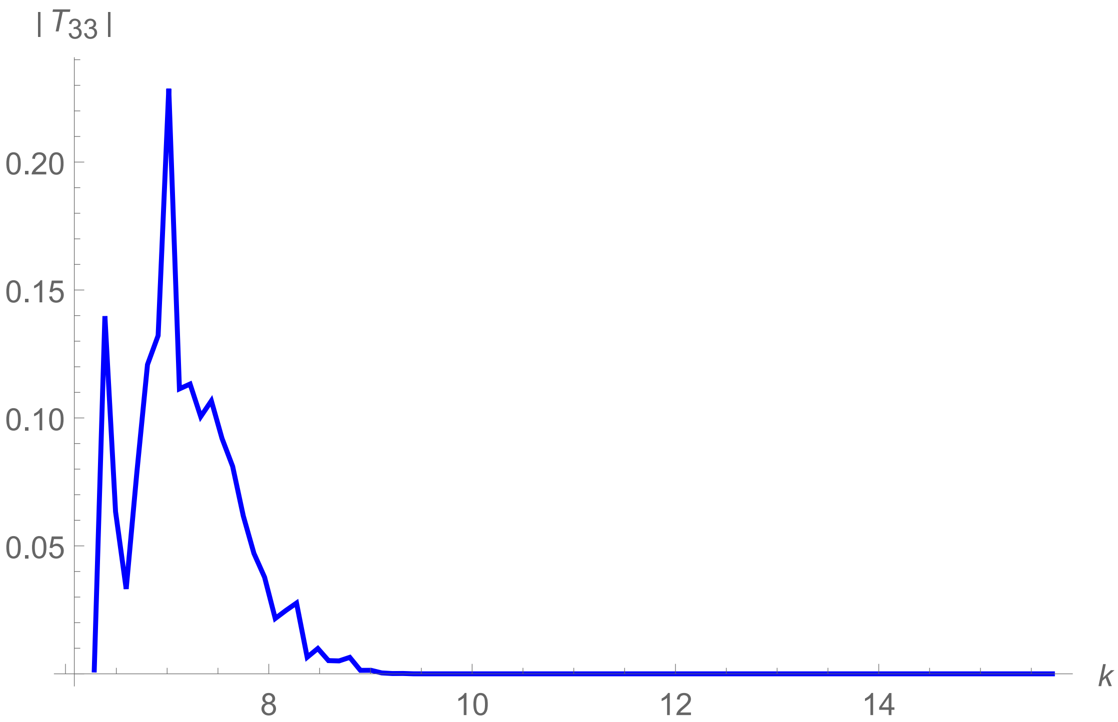

FIGURE 4.13: The absolute value of T_{33} against frequency k with rigid condition.

are almost zero while third mode is depicted in Figure 4.13 contain a huge spike over $6 < k < 8$.

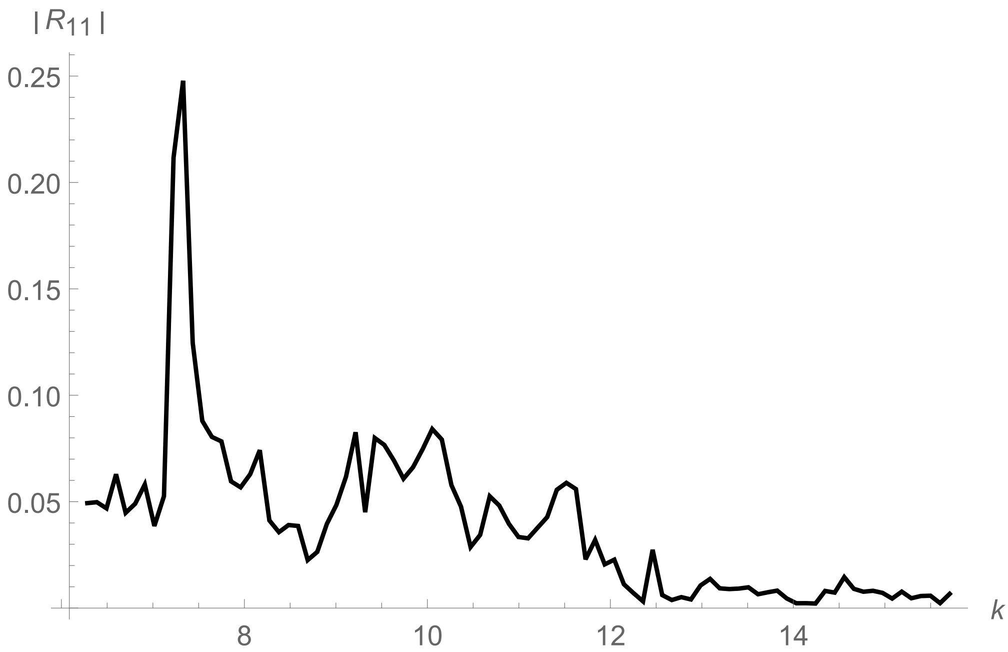

FIGURE 4.14: The absolute value of R_{11} against frequency k with soft-rigid condition.

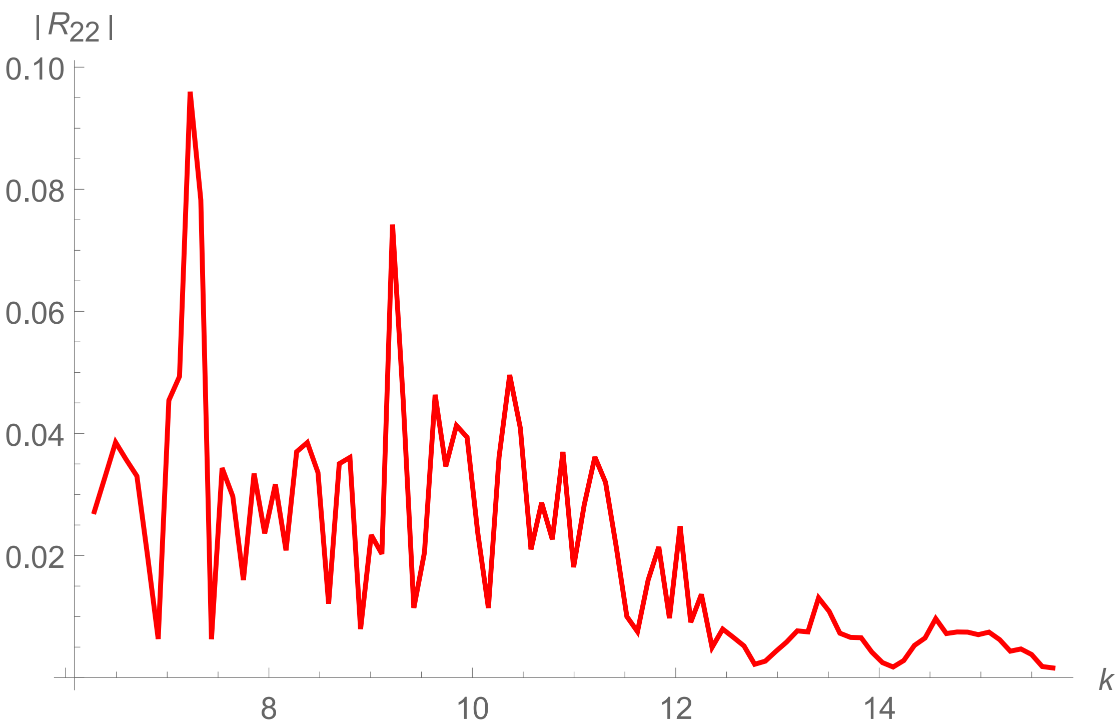

FIGURE 4.15: The absolute value of R_{22} against frequency k with soft-rigid condition.

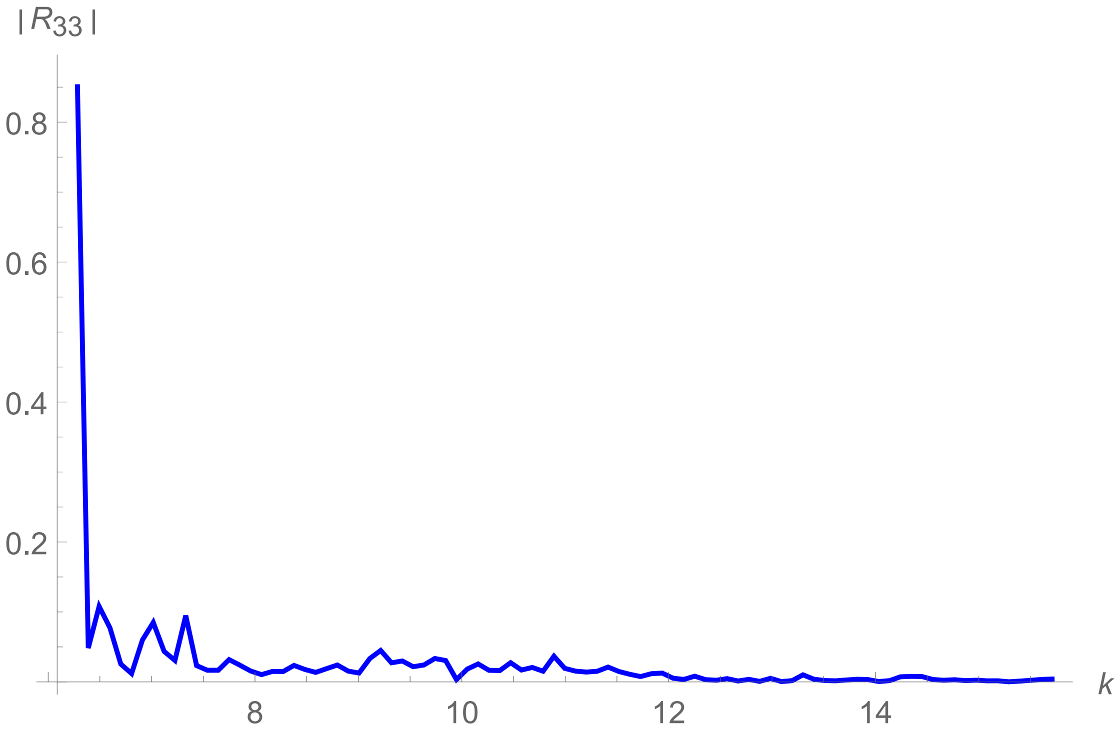

FIGURE 4.16: The absolute value of R_{33} against frequency k with soft-rigid condition.

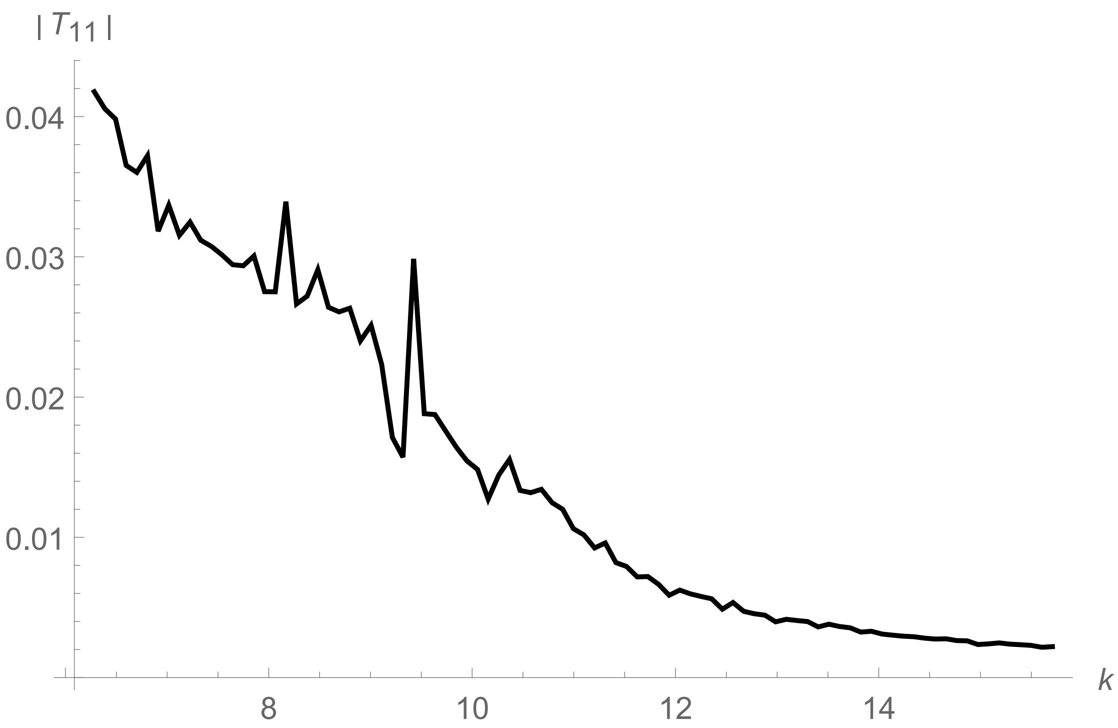

FIGURE 4.17: The absolute value of T_{11} against frequency k with soft-rigid condition.

Figs. 4.15-4.17, the reflection and transmission graph are shown for rigid-soft setting.

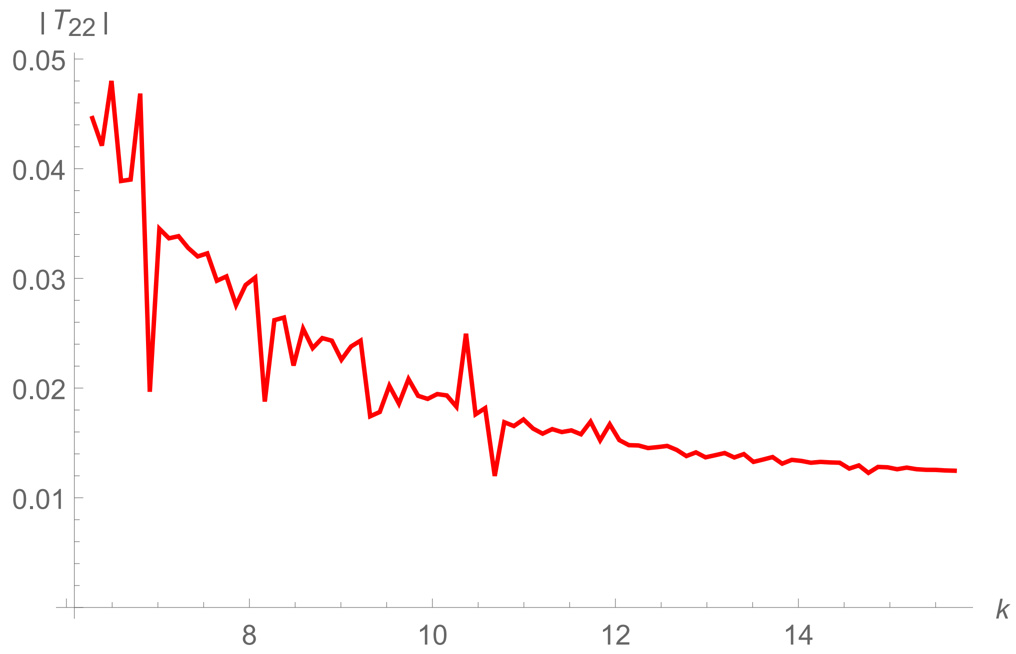

FIGURE 4.18: The absolute value of T_{22} against frequency k with soft-rigid condition.

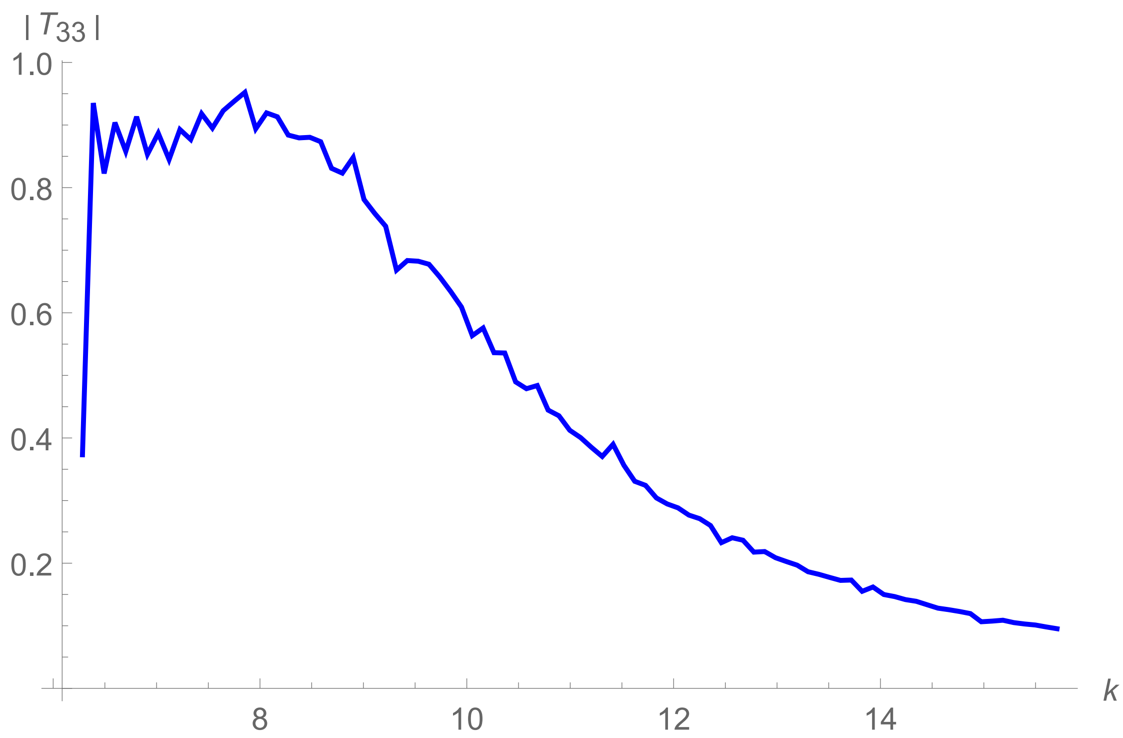

FIGURE 4.19: The absolute value of T_{33} against frequency k with soft-rigid condition.

Figures 4.14-4.19 show the absloute values of reflecting modes R_{11}, R_{22} and R_{33} and transmitting modes T_{11}, T_{22} and T_{33} are portrayed for the acoustically soft-rigid wall conditions. The behaviour of scattering modes with soft-rigid conditions is more different as compared with rigid-rigid with soft-soft conditions. The fundamental mode contains a spike over $7 < k < 8$ which decreases by increasing k with fluctuations, see Figure 4.14. Likewise the behaviour with different fluctuations is seen for second reflecting mode, see Figure 4.15. However, Figure 4.16 depicts a different behaviour. The transmitting modes for soft-rigid case are displayed in Figures 4.17-4.19. The behaviour of curves seen in Figures 4.17-4.19 comprise a maximum value in start that decreases by increasing frequency.

Chapter 5

Summary and Conclusion

The chapter wise summary and conclusion of the present study are enclosed in this chapter. Chapter 1 contain back ground and literature review to the current study alongwith thesis structure. The details of acoustic scattering are included in this thesis and a brief overview of the Magnus series expansion have been discussed. In chapter 2, we have discussed some basic definitions which are useful in understanding the mathematical modeling and associated physical characteristics of the work presented in rest of the chapters.

Chapter 3, presents a discussion on the basic development of the Magnus series expansion on the linear matrix differential equation. The method involves the derivation of matrix exponential by using Picard iteration. The Magnus method for order 2 and 4 is derived. The method is then applied to some initial value problems whose results are compared with analytical solution.

In chapter 4, the investigation of the wave propagation through a expansion chamber with arbitrary geometrical configuration is exploited. The physical problem are governed by Helmholtz's equation and contain boundary wall condition to be acoustically soft, rigd and soft-rigid. The Multimodal solution is found through projecting the local transverse modes to determine the coupled mode equation. By introducing the admittance matrix the Riccati equation is achieved, which is then integrated by the application of Magnus-Möbius method and radiation conditions. For a fixed shape of expansion chamber, the scattering modes are shown. It is observed that by changing the wall conditions the scttering is varied.

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