CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Best Proximity Points for Suzuki-Edelstein Proximal Contraction in b-Metric Space

by

Asad Ali

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2020

I dedicate my dissertation work to

my beloved ***family*** *specially*

my mother*,*

my wife*,*

and

my loving sisters*.*

A special feeling of gratitude is for

my father*,*

the most unswerving man,

I ever know in this world.

CERTIFICATE OF APPROVAL

Best Proximity Points for Suzuki-Edelstein Proximal Contraction in b-Metric Space

by

Asad Ali

(MMT181001)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Tayyab Kamran	QAU, Islamabad
(b)	Internal Examiner	Dr. Abdul Rehman Kashif	CUST, Islamabad
(c)	Supervisor	Dr. Rashid Ali	CUST, Islamaabad

Dr. Rashid Ali
Thesis Supervisor
July, 2020

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
July, 2020

Dr. Muhammad Abdul Qadir
Dean
Faculty of Computing
July, 2020

Author's Declaration

I, **Asad Ali** hereby state that my M.Phil thesis titled "**Best Proximity Points for Suzuki-Edelstein Proximal Contraction in b-Metric Space**" is my own work and has not been submitted previously by me for taking any degree from Capital University of Science and Technology, Islamabad or anywhere else in the country/abroad.

At any time if my statement is found to be incorrect even after my graduation, the University has the right to withdraw my M.Phil Degree.

(Asad Ali)

Registration No: MMT181001

Plagiarism Undertaking

I solemnly declare that research work presented in this thesis titled "**Best Proximity Point for Suzuki-Edelstein Proximal Contraction in b-metric space**" is solely my research work with no significant contribution from any other person. Small contribution/help wherever taken has been duly acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and Capital University of Science and Technology towards plagiarism. Therefore, I as an author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred/cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of M.Phil Degree, the University reserves the right to withdraw/revoke my M.Phil degree and that HEC and the University have the right to publish my name on the HEC/University website on which names of students are placed who submitted plagiarized work.

(Asad Ali)

Registration No: MMT181001

Acknowledgements

First and foremost, all praise and appreciation to **ALLAH** who is the most beneficent, merciful and the most kind. Countless respect and endurance for **Prophet Muhammad** (Peace Be Upon Him), the fortune of knowledge, who took the humanity out of ignorance and showed the right path.

I would like to express my warmest thanks to my kind nature supervisor **Dr. Rashid Ali** for the supervision and constant support. His invaluable help of constructive comments and suggestions throughout this work have contributed to the success of this research. This work would not have been possible without his guidance, support and encouragement.

In addition, I am thankful to the Mathematics department Capital University of Science and Technology **(CUST)** Islamabad specially the Head of the Department, **Dr. Muhammad Sagheer** for providing us a learning and creative environment.

I should be thankful to my respected teachers **Dr. Samina Rashid**, **Dr. Dur-E-Shehwar Sagheer**, for the encouragement and suggestions. May God keep blessing you all.

A special word of gratitude to my friends **Mr. Umair Ahmed Butt** and **Baber Sultan** who were always here to support and accomplish my research work. Most importantly, none of this would have been possible with out the love and patience of my family. I am extremely grateful to parents for their love, care and sacrifices for educating and preparing me for my future. My father **Muhammad Zarait Qureshi** sacrifices for me more than any other in life. My brother and sisters have also a great part of my work. Finally, this thesis might not exist at all without the love and support of **my wife**. I am thankful to her for support and encouragement.

(Asad Ali)

Registration No: MMT181001

Abstract

The concept of best proximity point in metric space under specific contraction mappings is demonstrated by many researchers. In the present dissertation, we discus the notion of modified Suzuki-Edelstein proximal contraction and acquire some best proximity points results for such contractions in the setting of b-metric spaces. The Suzuki-Edelstein proximal contraction performed the significant role in the extension and generalization of Banach contraction principle. As application, we formulate the fixed point results for modified Suzuki-Edelstein proximal contraction mappings in the setting of b-metric spaces. Our result will be valuable in solving particular best proximity points and fixed point results in the setting of b-metric spaces.

Contents

List of Figures

Symbols

(X, d)	Metric space
(X, d_ρ)	b-Metric space
d	distance function
$\mathbb{R}$	Real number
$\mathbb{N}$	Natural number
$\Rightarrow$	implies that
$\in$	Belongs to
$\forall$	for all
$\sum$	Sigma
$\Rightarrow$	Implies that
∞	Infinity
$\in$	Belongs to
$\lim_{x\to\infty}$	Limit

Chapter 1

Introduction

1.1 Background

Mathematics is one of the important branch of scientific knowledge having many applications in every sphere of life. Mathematics is further classified into multiple branches. One of the most important branch of mathematics is known as functional analysis. In functional analysis, fixed point theory is a valuable and dominant concept. The concept of fixed point theory has lot applications in various fields of science, such as optimisation theory, mathematical economics, variational inequalities and approximation theory etc.

Poincare [1] was the first mathematician to serve in the area of fixed point theory in 1886 and substantiate various fixed point results. Afterwards, the fixed point problem was taken into consideration by Brouwer [2] and established fixed point results for the solution of equation $T(\zeta) = \zeta$ in 1912. He also established fixed point results in various dimensions [3].

In 1922, a notable mathematician Banach [4] demonstrated a significant fixed point result in the area of functional analysis acknowledge as Banach Contraction principle (BCP). This result is declared to be the most fundamental consequence in the field of fixed point theory. BCP is stated as: “A contraction mapping in a complete metric space has a unique fixed point.” The two remarkable applications come from this principle. The first one is that it guarantees the existence and uniqueness of fixed point. The second and the very emotive one is that it devolved an approach to determine the fixed point of mapping. Due to its extensive application potential, this celebrated principle has been generalized in various ways over the year [5–9].

Afterwards, it was investigated by Kannan [10]. A lot of researchers in the field of mathematics [11–14] are fascinated to Banach contraction principle due to its generalization associated with fixed point theory in different spaces. Bakhtin [15] commenced the analysis of a popularize metric space known as b-metric space and accomplished the BCP [4] in b-metric space. Numerous former researchers investigated fixed point theory in distinguish mappings like mixed single as well as set-valued in b-metric space [16–18].

The answer of equation $T\zeta = \zeta$ are the fixed points of the mappings $T : X \to X$. If U and V are non-empty subsets of a metric space (X, d) and $T : U \to V$, then for the existence of a fixed point it is necessary that $T(U) \cap U \neq \phi$. If this does not hold, $d(\zeta, T\zeta) > 0$ for each $x \in U$ that is $d(\zeta, T\zeta)$ cannot be zero. In this situation our aim is to minimize the term $d(\zeta, T\zeta)$. The best approximation theory has been devolved in this sense. Thus, In 1969, Fan [19] suggested the idea of the best proximity point result for non-self continuous mappings. A lot of extensions of Fan's theorem were accustomed in publications like Reich [20], Sehgak and Singh [21] and Prolla [22]. The existence and convergence of best proximity points is attractive feature of optimization theory and it has pulled the consideration of lot of mathemacians [23–30].

In 2005, Elderd et al. [31] provided the existence and convergence of the best proximity points in the set of uniformly continuous Banach space. The best proximity point theorems for respective kinds of contractions were introduced in [32–35]. In 2008, Suzuki [9] presented a new kind of mappings and provided generalization of BCP. In 2010, Basha [36] introduced necessary and sufficient axoms to claim the existence of a best proximity point for proximal contraction of first and second kind (see Definition 3.2.4 and 3.2.5), which are non-self mappings and extended

Banach contraction principle for the best proximity point results for these contractions. Jleli et al.[37] established the existence of the best proximity point for generalized α-ψ-proximal contraction in complete metric space.

Recently, Hussain et al. [38, 39] acquire the Banach contraction principle for "Modified Suzuki α-proximal contractions in complete metric space". In my thesis, several artefact are reviewed which are indicated in previous debate but our main focus is on the work of Hussain et al.[39]: "The Best Proximity Point Results for Suzuki-Edelstein Proximal Contractions in metric space". We present the detailed study of result presented in [39]. This study leads to the extension of the best proximity, fixed point results in b-metric spaces. The results are then illustrated by suitable examples.

The rest of dissertation is orginized as below:

- **Chapter 2** includes the basic concepts, definitions and examples regarding metric space, b-metric space and fixed point.

- **Chapter 3** is about the literature review and the study of best proximity point results for modified Suzuki-Edelstein α-proximal contraction enclosed by metric space comprehensively.

- **Chapter 4** emphasizes the idea of modified Suzuki-Edelstein α-proximal contaction in b-metric space. Use this concept to acquire the best proximity points for "modified Suzuki-Edelstein α-proximal contaction in b-metric spaces".

Chapter 2

Preliminaries

In the present chapter, we discus about the fundamental definitions, results and examples which are use in subsequent chapters. The first section of this chapter covers some basics of metric space with few examples. The second section concerns with the b-metric space and related examples. The third section consists of fixed points in metric space and few significant fixed point results in metric space. The next section contains best proximity point in the setting of metric space and some basic definitions regarding best proximity points.

2.1 Metric Space

In mathematics, the ordinary distance or Euclidean distance is a straight line distance between two points. However, this distance may be other than the straight line like taxi cab distance. In literature the word "metric" is used to generalize the notion of distance and the space equipped with metric satisfying few properties known as metric space. The notion of metric space was initially prescribed by Frachet [40] in 1906.

Definition 2.1.1. [41](**Metric Space**)
"Let X be a non-empty set. Let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(M1) $d(x, y) \geq 0$

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$.

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ for al $x, y, z \in X$.

Then the mapping d is called a metric on X and the pair (X, d) is metric space."

Example 2.1.1. Consider $X = \mathbb{R}$, (the set of real numbers) and define $d : X \times X \to \mathbb{R}$ as;

$$d(\zeta, \mu) = |\zeta - \mu| \text{ for all } \zeta, \mu \in X$$

Then d is a metric on $\mathbb{R}$ and $(\mathbb{R}, d)$ is metric space.

To show that d is metric on X, It is easy to check that condition $(M_1), (M_2)$ and (M_3) hold. We just prove that (M_4) holds, that is

$$\begin{aligned} d(\zeta, \omega) &= d(\zeta, \omega), \\ &= |\zeta - \omega|, \\ &= |\zeta - \mu + \mu - \omega|, \\ &\leq |\zeta - \mu| + |\mu - \omega|, \\ &\leq d(\zeta - \mu) + d(\mu - \omega). \end{aligned}$$

Hence, d is metric on X.

Example 2.1.2. Consider $X = \mathbb{R}^2$, define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(\zeta, \mu) = \sqrt{(\zeta_1 - \mu_1)^2 + (\zeta_2 - \mu_2)^2}$$

Then d is a metric on $\mathbb{R}^2$ and $(\mathbb{R}^2, d)$ is a metric space.

Definition 2.1.2. [42](**Continuous mapping**)

"Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be continuous at a point x_0 if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$d(Tx, Tx_0) \leq \epsilon \quad \text{whenever} \quad d(x, x_0) \leq \delta.$$

for all $\zeta \in X$."

Example 2.1.3. Consider $X = \mathbb{R}$ equipped with usual metric d as stated in Example (2.1.1). A mapping $T : X \to X$ by

$$T(\zeta) = \zeta^3 \text{ where } \zeta \in X.$$

Then T is a continuous mapping.

Example 2.1.4. The space $C[a, b]$ is set of all read-valued continuous functions defined on $[a, b]$. The function $d : X \times X \to \mathbb{R}$ defined by

$$d(\zeta, \mu) = \max_{t \in [a,b]} |\zeta(t) - \mu(t)| \text{ for all } \zeta, \mu \in C[a, b]$$

is a metric on X and (X, d) is metric space.

Definition 2.1.3. [42](**Convergence of Sequence**)
"A sequence $\{x_n\}$ from the points in a metric space (X, d) is said to be converge to $x \in X$, if the sequence of non negative real numbers $d(x_n, x) \to 0$ when $n \to \infty$. In other words for every $\epsilon > 0$ there exist $N(\epsilon) \in \mathbb{N}$ so that $d(x_n, x) < \epsilon$, for each $n \geq N(\epsilon)$."

Example 2.1.5. "Consider again the set $\mathbb{R}$ (the set of real numbers) along with usual metric $d(\zeta, \mu) = |\zeta - \mu|$ then the sequence $\zeta_n = \frac{1}{n}$ in X is a convergent sequence."

Definition 2.1.4. [42](**Cauchy Sequence**)
"A sequence $\{x_n\}$ in a metric space (X, d) is said to be a Cauchy sequence if for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon$$

for all $m, n > \mathbb{N}$."

Definition 2.1.5. [42](**Complete Metric Space**)
"If every Cauchy sequence in a metric space (X, d) converges to a point $x \in X$ then X is called a complete metric space."

Example 2.1.6. "The closed interval $[0, 1]$ in $\mathbb{R}$ is a complete metric space the with usual metric on $\mathbb{R}$."

Definition 2.1.6. [42](**Compact Metric Space**)
"A metric space X is called compact if every sequence in X has a converget subsequence."

2.2 b-Metric Space

In current section, we present few fundamental concept, definitions and examples to understand the idea of b-metric spaces. The notion of b-metric was initiated by Bakhtin [43] and Czerwik [18] in 1989 which generalises the notion of metric space.

Definition 2.2.1. [43](b-**Metric Space**)
"Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is called a b-metric if it satisfies the following properties for each $x, y, z \in X$,
$(b1)$ $d(x, y) = 0 \Leftrightarrow x = y$,
$(b2)$ $d(x, y) = d(y, x)$,
$(b3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$, where $s \geq 1$.
The pair (X, d_b) is called a b-metric space with coefficient b."

Example 2.2.1. [43] "Let $Y = \{\zeta, \mu, \omega\}$ and $X = Y \cup \mathbb{N}$. Define $d_b : X \times X \to [0, \infty)$ as

$$d_b(\zeta, \mu) = d_b(\mu, \zeta) = d_b(\zeta, \omega) = d_b(\omega, \zeta) = 1,$$

$$d_b(\mu, \omega) = d_b(\omega, \mu) = \xi,$$

$$d_b(\zeta, \zeta) = d_b(\mu, \mu) = d_b(\omega, \omega) = 0, \qquad d_b(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right|,$$

where $\xi \in [2, \infty)$. Then we find that

$$d_b(\zeta, \mu) \leq \frac{\xi}{2}[d_b(\zeta, \omega) + d_b(\omega, \mu)] \text{ forall } \zeta, \mu, \omega \in X.$$

Hence (X, d_b) is b-metric space with coefficient $b = \frac{\xi}{2}$."

Note: "The class of a b-metric space is larger than the class of metric space. When $b = 1$ then the concept of b-metric space coincides with concept of metric space. Hance every metric space is a b-metric space. However, converse is not necessarily true."

Example 2.2.2. [43] "Consider $X = \mathbb{R}$ and $d_b(\zeta, \mu) = (\zeta - \mu)^2$. Then d_b is b-metric on $\mathbb{R}$ with $b = 2$, but (X, d_b) is not a metric space."

Example 2.2.3. [43] "Consider $X = \ell_\beta(\mathbb{R})$ with $0 < \beta < 1$, where

$$\ell_\beta(\mathbb{R}) = \left\{ \{\zeta_n\} \subseteq \mathbb{R} \,\middle|\, \sum_{n=1}^{\infty} |\zeta_n|^\beta < \infty \right\},$$

a function $d_b : X \times X \to \mathbb{R}^+$;

$$d_b(\zeta, \mu) = \left(\sum_{n=1}^{\infty} |\zeta_n - \mu_n|^\beta \right)^{\frac{1}{\beta}},$$

where $\zeta = \{\zeta_n\}$ and $\mu = \{\mu_n\}$ then d_b is a b-metric space having coefficient $b = 2^{\frac{1}{\beta}}$."

Definition 2.2.2. [44] (**Convergent Sequence**)
"Let (X, d_b) be a b-metric space. A sequence $\{x_n\}$ in X is said to be convergent and converges to x, if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $d_b(x_n, x) < \epsilon$ for all $n > n_0$ and this fact is represented bt $\lim_{n\to\infty} x_n = x$ of $n \to x$ as $n \to \infty$."

Definition 2.2.3. [44] (**Cauchy Sequence**)
"Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence, if for every $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$d_b(x_n, x_m) < \epsilon$$

for all $n, m > n_0$."

Definition 2.2.4. [44](**Complete Space**)
"The b-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent."

Remark 2.2.1. "A b-metric space need not be a continues function."

The following example illustrates.

Example 2.2.4. [44] "Let $X = \mathbb{N} \cup \{\infty\}$. A function $d_b : X \times X \to \mathbb{R}$ by:

$$d_b(m,n) = \begin{cases} 0 & \text{if} \quad m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if } m, n \text{ are even or } m, n = \infty, \\ 5 & \text{if } m \text{ and } n \text{ are odd and } m \neq n \text{ or } \infty \\ 2 & \text{otherwise} \end{cases}$$

It can be checked that for all $m, n, p \in X$, we have

$$d_b(m,p) \leq \frac{5}{2}\,(d_b(m,n) + d_b(n,p)).$$

Thus (X, d_b) is a b-metric space with $(b = \frac{5}{2})$. Let $x_n = 2n$ for each $n \in \mathbb{N}$, then

$$d_b(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as} \quad n \to \infty,$$

that is, $x_n \to \infty$, but $d_b(x_n, 1) = 2 \nrightarrow 5 = d_b(\infty, 1)\ as\ n \to \infty$."

2.3 Fixed Point and Contractions

A wide diversity of problems appearing in various fields of mathematics like differential equations, discrete and continuous system of dynamics can be demonstrated as fixed point problem. Therefore fixed point theory is quite convenient in finding solution of problem in structural optimization in science [45]. In this portion, we discuss the definition of fixed point as well as various types of contractions.

Definition 2.3.1. [46](**Fixed Point**)
"Let $T : X \to X$ be a mapping on a set X. A point $x \in X$ is said to be a fixed point of T if

$$Tx = x,$$

i.e. a point is mapped onto itself.
Geometrically,
if $y = fx$ is a real valued function on $\mathbb{R}$, then at fixed point of f, the geometry of f coincides with bar $y = x$. Thus a function may or may not have fixed point. Furthermore, fixed point may or may not be unique.

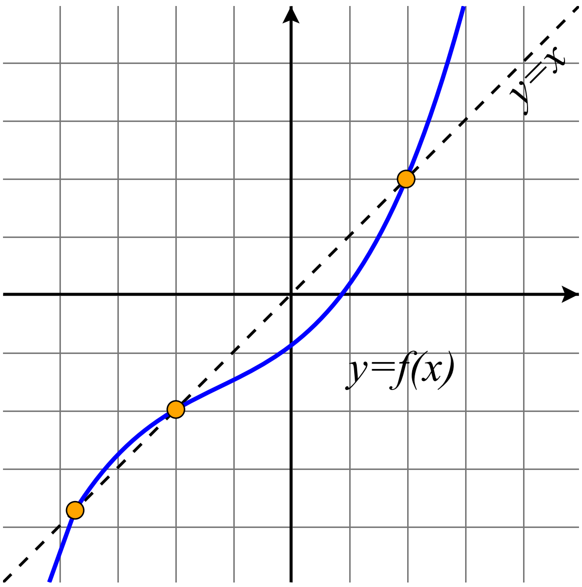

FIGURE 2.1: Three Fixed points

The graph mention above represents a function having three fixed points."
The following example illustrate having no fixed point.

Example 2.3.1. "Consider $X = \mathbb{R}$ with the usual metric d. Suppose mapping $T : X \to X$ by

$$T(x) = x + 1 \quad \forall \ \ x \in X$$

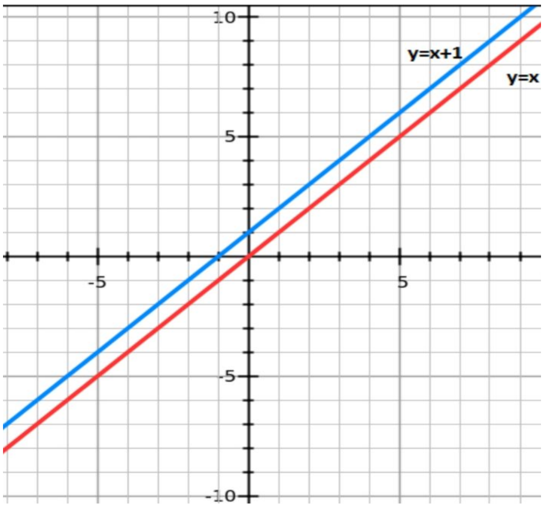

FIGURE 2.2: No Fixed Point

then T has no fixed point."

Example 2.3.2. "Consider $X = \mathbb{R}$ along with the usual metric d. Suppose mapping $T : X \to X$ by

$$T(\zeta) = 2\zeta + 1 \quad \forall \ \zeta \in X$$

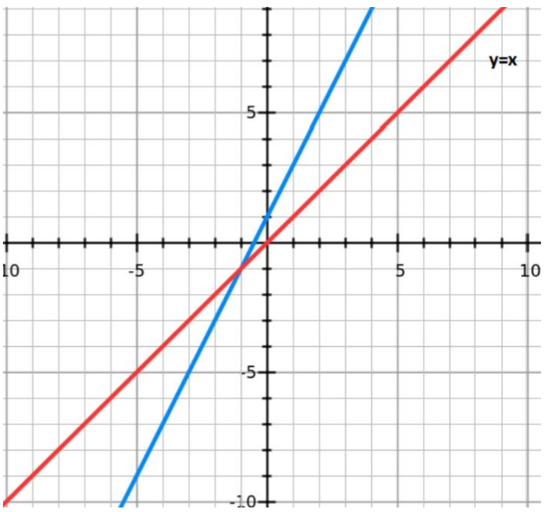

FIGURE 2.3: Unique Fixed Point

then T has a unique fixed point $\zeta = -1$."

Definition 2.3.2. [30] (**Contraction**)

"Let (X, d) be a metric space. A mapping $T :\to X$ is said to be a contraction if there is a positive real number $0 \leq \alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y).$$

This contraction is also known as Banach contraction."

Example 2.3.3. Suppose that $X = [0, 1]$ with usual metric $d(\zeta, \mu) = |\zeta - \mu|$.

A mapping $: X \to X$ as

$$T(\zeta) = \frac{1}{2 + \zeta}$$

is a contraction mapping.

Proof. Suppose that a mapping $T : X \to X$ as

$$T(\zeta) = \frac{1}{2 + \zeta}$$

$$\begin{aligned}
d(T\zeta, T\mu) \leq \quad & d\left(\frac{1}{2+\zeta}, \frac{1}{2+\mu}\right), \\
\leq & \left|\frac{1}{2+\zeta} - \frac{1}{2+\mu}\right|, \\
\leq & \left|\frac{2+\mu-2-\zeta}{(2+\zeta)(2+\mu)}\right|, \\
\leq & \left|\frac{-(\zeta-\mu)}{(2+\zeta)(2+\mu)}\right|, \\
\leq & \frac{|\zeta-\mu|}{(2)(2)}, \\
\leq & \frac{1}{4} d(\zeta, \mu),
\end{aligned}$$

is contraction with $\alpha = \frac{1}{4}$. □

Example 2.3.4. Suppose that a metric space (X, d) where $X = \mathbb{R}$ and $d(\zeta, \mu) = |\zeta - \mu|$. Define $T \colon X \to X$ by

$$T(\zeta) = \frac{\zeta}{5} + 4$$

is a contraction mapping.

Proof. Suppose that a mapping $T : X \to X$ as

$$T(\zeta) = \frac{\zeta}{5} + 4$$

$$\begin{aligned} d(T\zeta, T\mu) \leq \ & d\left(\frac{\zeta}{5} + 4, \frac{\mu}{5} + 4\right), \\ & \leq \left|\frac{\zeta}{5} + 4 - (\frac{\mu}{5} + 4)\right|, \\ & \leq \left|\frac{\zeta}{5} + 4 - \frac{\mu}{5} - 4\right|, \\ & \leq \left|\frac{\zeta}{5} - \frac{\mu}{5}\right|, \\ & \leq \frac{1}{5}|\zeta - \mu|, \\ & \leq \frac{1}{5}d(\zeta, \mu) \end{aligned}$$

is contraction with $\alpha = \frac{1}{5}$. □

In 1922, Banach [42] established fixed point theorem, popularly named as Banach contraction theorem.

Theorem 2.3.1. [47]
"Let (X, d) be a complete metric space. A mapping $T : X \to X$ such that

$$d(Tx, Ty) < \lambda d(x, y) \text{ for all } x, y \in X.$$

Where $0 \leq \lambda < 1$, then T has a unique fixed point."

In 1962, Edelstein [47] presented the following well known theorem.

Theorem 2.3.2. [47]
"Let (X, d) be a compact metric space, and let T be a mapping on X. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$.Then T has a unique fixed point."

Samet et al. [48] initiated the idea of α-admissible mappings in 2012. This idea was orilonged in various dimensions.

Definition 2.3.3. [38](**α-admissible mapping**)
"Let (X, d) be a metric space, T be a self mapping on X and $\alpha : X \times X \to [0, \infty)$ be a function. The mapping T is α-admissible if

$$\alpha(x, y) \geq 1 \text{ implies that } \alpha(Tx, Ty) \geq 1,$$

for all $x, y \in X$,"

Example 2.3.5. [48] Consider $X = (0, 1)$. Define a mapping $T : X \to X$ and also $\alpha : X \times X \to [0, \infty)$ as

$$T\zeta = \ln\ \zeta$$

for all $\zeta \in X$ and

$$\alpha(\zeta, \mu) = \begin{cases} 2 \text{ if } \zeta \geq \mu, \\ \\ 0 \text{ if } \zeta < \mu. \end{cases}$$

Then T is α-admissible.

Example 2.3.6. [48] "Consider $X = [0, \infty)$. Define a mapping $T : X \to X$ and also $\alpha : X \times X \to [0, \infty)$ as

$$T\zeta = \sqrt{\zeta}$$

for all $\zeta \in X$ and

$$\alpha(\zeta, \mu) = \begin{cases} e^{\zeta - \mu} & \text{if } \zeta \geq \mu, \\ 0 & \text{if } \zeta < \mu. \end{cases}$$

Then T is α-admissible."

Definition 2.3.4. [38](**α-admissible w.r.t.** η)
"Let (X, d) be a metric space, T be a self mapping on X and $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. Then mapping T is an α-admissible with respect to η if,

$$\alpha(x, y) \geq \eta(x, y) \text{ implies that } \alpha(Tx, Ty) \geq \eta(Tx, Ty)$$

for all $x, y \in X$."

Remark 2.3.1. "If $\eta(x,y) \geq 1$, then the above definition reduces to the definition of α-admissible mapping."

Theorem 2.3.3. [9]
"Let (X,d) be a complete metric space, and let T be a mapping on X. Define a non-increasing function θ from $[0,1)$ onto $(\frac{1}{2},1]$ by

$$\theta(r) = \begin{cases} 1 & \text{if} \quad 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & \text{if} \quad \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if} \quad \frac{1}{\sqrt{2}} \leq r \leq 1 \end{cases} .$$

We assume that there exists $r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \leq d(x,y)$$

which implies that

$$d(Tx,Ty) \leq rd(x,y) \text{ for all } x,y \in X.$$

Then there exist a unique fixed point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$."

Stimulated by previous results, Suzuki established an abstraction of Edelstein's fixed point result as below:

Theorem 2.3.4. [9]
"Let (X,d) be a compact metric space, and let T be a mapping on X. Assume that

$$\frac{1}{2}d(x,Tx) < d(x,y)$$

implies that

$$d(Tx,Ty) \leq d(x,y) \text{ for all } x \in X.$$

Then T has a unique fixed point."

Chapter 3

Suzuki-Edlstein α-proximal Contraction in Metric Space

In current chapter, we present few best proximity point results for Suzuki-Edlstein α-proximal contraction in metric space. Al-Thagfi et al [49] analyse the proximity point propositions for proximal contractions. In 2011, Basha [36] explained various best proximity point theorems for proximal contractions. Hussain et al. [38] acquired the best proximity point theorems for such contractions in metric space.

3.1 Best Proximity Point in Metric Space

Many problems can be expressed as an equation of the form $T\zeta = \zeta$, where T is self-mapping in appropriate domains. Yet, in the case that T is non-self mapping, the previous equation does not definitely have fixed point. In this situation, it is worthy to govern the estimated solution ζ such that the error $d(\zeta, T\zeta)$ is minimal. This study initiated the concept of best proximity point [50].

3.2 Some Basic Tools

In this section, we will introduced some major concepts and results that will be used in the rest of the chapter. We commence by using important notations.
Let (X, d) be metric space, U and V be non-empty subsets of (X, d). Describe

$$d(U, V) = \inf\{d(\zeta, \mu) : \zeta \in U, \mu \in V\}$$

$$U_0 = \{\zeta \in U : d(\zeta, \mu) = d(U, V)\} \text{ for some } \mu \in V\}$$

$$V_0 = \{\mu \in V : d(\zeta, \mu) = d(U, V) \text{ for some } \zeta \in U\}.$$

Definition 3.2.1. [39](**Best Proximity Point**)
"Let (X, d) be a metric space, A and B be two non-empty subsets of X. An element $x \in A$ is said to be best proximity point of mapping $T : A \to B$ if $d(x, Tx) = d(A, B)$."

Remark 3.2.1. "It is clear from the above definition that a counter notion of fixed point in the context of non-self mappings is so-called best proximity point."

Definition 3.2.2. [46](P**-property**)
"Let (A, B) be a pair of non-empty subsets of metric space X with $A_0 \neq \phi$. Then the pair (A, B) is said to have P-property iff

$$\begin{cases} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B), \end{cases} \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$."

Remark 3.2.2. "It is easy to see that, for any non-empty subset A of X, the pair (A, A) has the P-property."

Example 3.2.1. [46]
"Consider $X = \mathbb{R}^2$ with metric defined by

$$d((\zeta_1, \mu_1), (\zeta_2, \mu_2)) = \max\{|\zeta_1 - \mu_1|, |\zeta_2 - \mu_2|\}$$

Let $U = \{(\zeta, 0) : -1 \leq \zeta \leq 1\}$ and $V = \{(0, \mu) : -1 \leq \mu \leq 1\}$. Then (U, V) satisfy P-property."

Definition 3.2.3. [39] (**Weak P-property**)
"Let (A, B) be a pair of non-empty subsets of X with $A_0 \neq \phi$. Then the pair (A, B) is said to have the weak P-property if and only if

$$\begin{cases} d(x_1, y_1) = d(A, B), \\ d(x_2, y_2) = d(A, B), \end{cases} \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2),$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$."

Example 3.2.2. [24]
"Consider $X = \{(0, 1), (1, 0), (0, 3), (3, 0)\}$, endowed with the usual metric d. We suppose that

$$U = \{(0, 1), (1, 0)\},$$

$$V = \{(0, 3), (3, 0)\}.$$

Then for

$$d((0, 1), (0, 3)) = d(U, V),$$

$$d((1, 0), (3, 0)) = d(U, V),$$

we have

$$d((1, 0), (1, 0)) < d(0, 3), (3, 0)).$$

Also $U_0 \neq \phi$. Thus, the pair (U, V) satisfies weak P-property."

Example 3.2.3. Consider $X = [0, \infty) \times [0, \infty)$ with metric d defined by

$$d((\zeta_1, \zeta_2), (\mu_1, \mu_2)) = |\zeta_1 - \mu_1| + |\zeta_2 - \mu_2|$$

Let $U = \{1\} \times [0, \infty)$ and $V = \{0\} \times [0, \infty)$. Then

$$d(U, V) = d((1, 0), (0, 0)) = 1 \text{ and } U_0 = V \ , \ V_0 = V$$

then the pair (U, V) has weak P-property.

Example 3.2.4. "Consider $(\mathbb{R}^2, d)$, where $d = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and subset

$$U = \{(0,0)\} \textit{ and } V = \{y, 1 + \sqrt{1 \ - \ x^2}\}.$$

It is obvious that

$$U_0 = \{(0,0)\}, \quad V_0 \ = \ \{(-1,1),(1,1)\} \text{ and } d(U,V) = \sqrt{2},$$

insinuate we get,

$$d((0,0),(-1,1)) = \sqrt{2},$$

and also

$$d((0,0),(1,1)) = \sqrt{2}.$$

Although,

$$d((0,0),(0,0)) = 0.$$

and

$$d((-1,1),(1,1)) = 2.$$

$$\Rightarrow d((0,0),(0,0)) < d((-1,1),(1,1)).$$

We can see from above discussion that (U, V) has weak P-property but not P-property."

Remark 3.2.3. [51]

"It is easy to notice that if (A, B) has the P-property, then (A, B) has the weak $P-$ property."

Definition 3.2.4. [51](**Proximal Contraction of first kind**)

"Let (X, d) be a metric space, A and B be two non-empty subsets of X. A non-self mapping $T : A \to B$ is called Proximal contraction of first kind if there exists non-negative integer $\alpha > 1$, such that

$$\begin{cases} d(u,Tx) &= d(A,B), \\ d(v,Ty) &= d(A,B), \end{cases} \quad \Rightarrow \quad d(u,v) \leq \alpha\, d(x,y),$$

where $u, v, x, y \in A$."

Example 3.2.5. Let $U = [0,1]$, $V = [2,3]$ be subsets of $\mathbb{R}$ aling with Euclidean metric space. A mapping $T : U \to V$ defined by;

$$T(\zeta) = \begin{cases} 3-\zeta & \text{if} \quad \zeta \text{ is a rational,} \\ 2+\zeta & \quad \text{otherwise,} \end{cases}$$

T is Proximal contraction of first kind.

Definition 3.2.5. [51](**Proximal Contraction of second kind**)
"Let (X,d) be a metric space, A and B be two non-empty subsets of a X. A non-self mapping $T : U \to V$ is called Proximal contraction of second kind if there exists non-negative integer $\alpha > 1$, such that

$$\begin{cases} d(u,Tx) &= d(A,B), \\ d(v,Ty) &= d(A,B), \end{cases} \quad \Rightarrow \quad d(Tu,Tv) \leq \alpha\, d(Tx,Ty),$$

where $u, v, x, y \in A$."

Remark 3.2.4. [51]
"Notice that unlike the other definitions, these mappings may not be even continuous when we restrict to the self case."

Definition 3.2.6. [37](α**-proximal admissible mapping**)
"Let (X,d) be a metric space, A and B be two nonempty subsets of X, a non-self

mapping $T : A \to B$ is called α-proximal admissible if

$$\begin{cases} \alpha(x_1, x_2) \ \geq \ 1, \\ d(u_1, Tx_1) \ = \ d(A, B), \qquad \Rightarrow \alpha(u_1, u_2) \ \geq \ 1 \\ d(u_2, Tx_2) \ = \ d(A, B), \end{cases}$$

for all $x_1, x_2, u_1, u_2 \in A$ where $\alpha : A \times A \to [0, \infty)$."

Remark 3.2.5. "Clearly, if $A = B$, T is α-proximal admissible suggests that T is α-admissible."

Definition 3.2.7. [37] (α**-proximal admissible w.r.t.** η)
"Let $T : A \to B$ and $\alpha, \eta : A \times A \to [0, \infty)$ be the two functions. Then T is called α-proximal admissible with respect to η if

$$\begin{cases} \alpha(x_1, x_2) \ \geq \ \eta(x_1, x_2), \\ d(u_1, Tx_1) = d(A, B), \qquad \Rightarrow \alpha(u_1, u_2) \ \geq \ \eta(u_1, u_2) \\ d(u_2, Tx_2) = d(A, B), \end{cases}$$

$\forall \ x_1, x_2, u_1, u_2 \in A$."

Remark 3.2.6. By adopting $\eta(\zeta, \mu) = 1$ for all $\zeta, \mu \in U$, then above definition curtail to the idea of α-proximal admissible (definition 3.2.7).

Definition 3.2.8. [39]
"In consistence with [52], we denote Φ_φ the set of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following condition:

$$\varphi(t) \ \leq \ \frac{1}{2}t \quad \forall \ t \ \geq \ 0$$

We denote by Φ the set of nondecreasing functions $\phi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{n\to\infty} \phi^n(t) = 0$$

for all $t > 0$."

Lemma 3.2.1. [52]
"If $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$."

3.3 Best Proximity Point Theorems in Metric Space

We commence this portion from hypothesis of modified Suzuki-Edelstein α-proximal contraction.

Definition 3.3.1. [39](**Modified Suzuki-Edelstein α-proximal Contraction**)
"Let A and B be the two non-empty subset of a metric space (X, d), a non-self mapping $T : A \to B$ is said to be a modified Suzuki-Edelstein α-proximal contraction if

$$\varphi(d(x, Tx)) - 2d(A, B) \leq \alpha(x, y)d(x, y) \tag{3.1}$$

$$\Rightarrow \alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y)) \tag{3.2}$$

for all $x, y \in A$, where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\alpha : A \times A \to [0, \infty]$."

Theorem 3.3.1. [39] Suppose that U and V are the two non-empty closed subsets of a complete metric space (X, d) along with $U_0 \neq \phi$. Let $\alpha : U \times U \to [0, \infty]$ and $\phi \in \Phi$. Let $T : U \to V$ be a non-self mapping with $T(U_0) \subseteq V_0$ be continuous modified Suzuki-Edelstein proximal and α-proximal admissible mapping with respect to $\eta(\zeta, \mu) = 2$ and also pair (U, V) convince the weak P-property. Moreover, the elements ζ_0 and ζ_1 in U_0 with $d(\zeta_1, T\zeta_0) = d(U, V)$ satisfies $\alpha(\zeta_0, \zeta_1) \geq 2$. Then T has a unique best proximity point.

Proof. Let $\zeta_0 \in U_0$, since $T(U_0) \subseteq V_0$, then there exists component $\zeta_1 \in U_0$ such that

$$d(\zeta_1, T\zeta_0) = d(U, V)$$

then by assumption, $\alpha(\zeta_0, \zeta_1) \geq 2$. Since ζ_1 in U_0 and $T(U_0) \subseteq V_0$, then there exists a component ζ_2 in U_0 such that

$$d(\zeta_2, T\zeta_1) = d(U, V).$$

Since, T is α-proximal admissible mapping w.r.t. $\eta(\zeta, \mu) = 2$, and $\alpha(\zeta_0, \zeta_1) \geq 2$. Continuing this process to get ζ_{n+1}, ζ_n such that for all $n \in \mathbb{N}$,

$$d(\zeta_{n+1}, \zeta_n) = d(U, V) \text{ satisfies } \alpha(\zeta_n, \zeta_{n+1}) \geq \ 2 \tag{3.3}$$

As $\varphi \in \Phi$, by using Definition (3.2.9)

$$\varphi(d(\zeta_{n-1}, T\zeta_{n-1}) \leq \frac{1}{2}\, d(\zeta_{n-1}, T\zeta_{n-1}),$$

$$\leq 2d(\zeta_{n-1}, T\zeta_{n-1}),$$

by triangular inequality,

$$\varphi(d(\zeta_{n-1}, T\zeta_{n-1}) \leq 2(d(\zeta_{n-1}, \zeta_n) + d(\zeta_n, T\zeta_{n-1})),$$

$$= 2(d(\zeta_{n-1}, \zeta_n) + d(U, V)),$$

$$= 2d(\zeta_{n-1}, \zeta_n) + 2d(U, V),$$

From the above inequality,

$$\varphi(d(\zeta_{n-1}, T\zeta_{n-1}) - 2d(U,\ V) \ \leq 2d(\zeta_{n-1}, \zeta_n),$$

$$\leq \alpha(\zeta_{n-1}, \zeta_n) d(\zeta_{n-1}, \zeta_n),$$

Then by using the definition of modified Suzuki-Edelstein α-proximal contraction (definition 3.3.1)

$$\alpha(\zeta_{n-1}, \zeta_n) d(T\zeta_{n-1}, T\zeta_n) \leq \ \phi(d(\zeta_{n-1}, \zeta_n)$$

Now

$$d(T\zeta_{n-1}, T\zeta_n) \leq \ \alpha(\zeta_{n-1}, \zeta_n) d(T\zeta_{n-1}, T\zeta_n)$$

this implies that

$$d(T\zeta_{n-1}, T\zeta_n) \leq \ \phi(d(\zeta_{n-1}, \zeta_n)) \tag{3.4}$$

Now suppose that $\zeta_{n_0} = \zeta_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then we have

$$d(\zeta_{n_0}, T\zeta_{n_0}) = d(\zeta_{n_0+1}, T\zeta_{n_0}) \tag{3.5}$$

Using equation (3.2) in (3.4),

$$d(\zeta_{n_0}, T\zeta_{n_0}) = d(U, V) \tag{3.6}$$

Thus from the (3.5), we conclude that ζ_{n_0} is the best proximity. Therefore, we assume that $\zeta_{n_0} \neq \zeta_{n_0+1}$, that is $d(\zeta_n, \zeta_{n+1}) > 0 \ \forall \ n \in \mathbb{N} \cup \{0\}$. Seeing that ϕ is nondecreasing so from Inequality (3.3) and weak P-property of (U, V),

$$\begin{aligned} d(\zeta_{n+1}, \zeta_n) &\leq d(T\zeta_n, T\zeta_{n-1}), \\ &\leq \phi(d(\zeta_n, \zeta_{n-1})), \end{aligned} \tag{3.7}$$

So

$$\begin{aligned} d(\zeta_{n+1}, \zeta_n) &\leq \phi(d(\zeta_n, \zeta_{n-1})), \\ &\leq \phi(d(T\zeta_{n-1}, T\zeta_{n-2})), \\ &\leq \phi(\phi(d(\zeta_{n-1}, \zeta_{n-2}))), \\ d(\zeta_{n+1}, \zeta_n)) &\leq \phi^2(d(\zeta_{n-1}, \zeta_{n-2}))... \leq \ \phi^n(d(\zeta_0, \zeta_1)). \end{aligned}$$

Hence,

$$d(\zeta_{n+1}, \zeta_n) \ \leq \phi^n(d(\zeta_0, \zeta_1)) \tag{3.8}$$

Now by applying limit $n \to \infty$, the above inequality deduced to,

$$\lim_{n\to\infty} d(\zeta_{n+1}, \zeta_n) = 0$$

Now for a fixed $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(\zeta_{n+1}, \zeta_n) < \epsilon - \phi(\epsilon) \; \forall \; n \geq N.$$

As ϕ is nondecreasing, for all $n \in \mathbb{N}$

$$\phi(d(\zeta_{n+1}, \zeta_n)) \leq \phi(\epsilon - \phi(\epsilon))$$

$$\phi(d(\zeta_{n+1}, \zeta_n)) \leq \phi(\epsilon). \tag{3.9}$$

Then again by triangular inequality

$$\begin{aligned} d(\zeta_n, \zeta_{n+2}) &\leq d(\zeta_n, \zeta_{n+1}) + d(\zeta_{n+1}, \zeta_{n+2}), \\ &< \epsilon - \phi(\epsilon) + \phi(d(\zeta_n, \zeta_{n+1})), \\ &< \epsilon - \phi(\epsilon) + \phi(\epsilon), \\ &\leq \epsilon. \end{aligned}$$

Continuing the process in same the mechanism, we conclude

$$d(\zeta_n, \zeta_{n+j}) < \epsilon \; \forall \; n \geq N \text{ and } j \in \mathbb{N}.$$

Then there exist $j > 0$ such that $\forall \; m, n \in \mathbb{N}$, $\exists \; m < n$. By triangular inequality and (3.7),

$$\begin{aligned} d_b(\zeta_m, \zeta_n) \leq & d(\zeta_m, \zeta_{m+1}) + d(\zeta_{m+1}, \zeta_{m+2}) + d(\zeta_{m+2}, \zeta_{m+3}) + ... + d(\zeta_{n-1}, \zeta_n), \\ \leq & \sum_{j=m}^{n-2} \phi^j(d(\zeta_1, \zeta_0)), \end{aligned}$$

$$d_b(\zeta_m, \zeta_n) \leq \sum_{j=m}^{\infty} \phi^j(d(\zeta_1, \zeta_0)) \to \ 0 \text{ as } n \to \infty.$$

Consequently, we have

$$\lim_{m,n\to\infty} d_b(\zeta_m, \zeta_n) = 0$$

Hence this implies that $\{\zeta_n\}$ is Cauchy in U. Since X is complete and U is closed, therefore there exists $\omega \in U$ such that $\zeta_n \to \omega$ and the continuity condition of T implies $T\zeta_n \to T\omega$ as $n \to \infty$. So, (3.2) gives us

$$d(U, V) = \lim_{n\to\infty} d(\zeta_{n+1}, T\zeta_n)$$

$$\Rightarrow d(U, V) = d(\omega, T\omega)$$

Thus ω is best proximity point. Now for uniqueness, on contrary that $\mu, \omega \in U_0$ be best proximity points of T with $\mu \neq \omega$,

$$d(\mu, T\mu) = d(\omega, T\omega) = d(U, V),$$

now by using weak P-property,

$$d(\mu, \omega) \leq \ d(T\mu, T\omega) \tag{3.10}$$

Now,

$$d(\mu, T\omega) = 2d(U, V) - d(U, V) \tag{3.11}$$

Since we already know that Using (3.11) in (3.12),

$$\begin{aligned}\varphi(d(\mu, T\mu)) &\leq \frac{1}{2}d(\mu, T\mu),\\ &\leq \frac{1}{2}(2d(U, V) - d(U, V)),\\ &\leq \frac{1}{2}d(U, V),\\ &\leq 2d(U, V).\end{aligned}$$

So from inequality,

$$\varphi(d(\mu, T\mu)) \ - \ 2d(U, V) \ \leq \ 0 \ \leq \ \alpha(\mu, \omega)d(\mu, \omega) \tag{3.12}$$

Hence from the concept of modified Suzuki-Edelstein α-proximal contraction,

$$\alpha(\mu, \omega)d(T\mu, T\omega) \ \leq \ \phi(d(\mu, \omega)) \tag{3.13}$$

Since $\alpha(\mu, \omega) \geq 2$,

$$d(T\mu, T\omega) \leq \ \phi(d(\mu, \omega)) \tag{3.14}$$

By using (3.10),

$$d(\mu, \omega) \leq \phi(d(\mu, \omega)) < d(\mu, \omega) \tag{3.15}$$

contradiction, so $\mu = \omega$. Thus T has a unique best proximity point. □

The example below illustrate result (3.3.1)

Example 3.3.1. Consider $X = [0, \infty) \times [0, \infty)$ with metric d defined by

$$d((\zeta_1, \zeta_2), (\mu_1, \mu_2)) = |\zeta_1 - \mu_1| + |\zeta_2 - \mu_2|$$

Consider $U = \{1\} \times [0, \infty)$ and $V = \{0\} \times [0, \infty)$. Then

$$d(U, V) = d((1, 0), (0, 0)) = 1 \text{ and } U_0 = V, \ V_0 = V$$

Define $T : U \to V$ by

$$T(1, \zeta) \ = \ \begin{cases} (0, \frac{\zeta}{3}) \text{ if } \zeta \in [0, 1], \\ \\ (0, \zeta - \frac{2}{3}) \text{ if } \zeta > 1, \end{cases}$$

$\alpha : U \times U \to [0,\infty)$ by

$$\alpha((\zeta,\mu),(s,t)) = \begin{cases} 2 \text{ if } (\zeta,\mu),(s,t) \in [0,1]\times[0,1], \\ 0 \text{ otherwise,} \end{cases}$$

$\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \dfrac{999}{1000}t \ \forall\ t \geq 0$. Then

(a) The pair (U,V) has weak P-property.

(b) T is α-proximal admissible mapping w.r.t. η

(c) T has unique best proximity point.

Proof. It is understandable from mapping that $T(U_0) \subseteq V_0$. Let $(1,\zeta_1),(1,\zeta_2) \in U$ and $(0,u_1),(0,u_2) \in B$, such that

$$\begin{cases} d((1,\ \zeta_1),(0,\ \zeta_1)) = d(U,\ V) = 1, \\ d((1,\ \zeta_2),(0,\ u_2)) = d(U,\ V) = 1, \end{cases}$$

So

$$d((1,\zeta_1),(1,\mu_2)) \leq d((0,u_1),(0,u_2))$$

Hence, pair (U,V) has weak P-property.

To demonstrate T is α-proximal admissible w.r.t. η,

$$\begin{cases} \alpha((1,\zeta_1),(1,\zeta_2)) \geq\ 1, \\ d((1,u_1),T(1,\zeta_1)) = d(U,V) = 1, \\ d((1,u_2),T(1,\zeta_2)) = d(U,V) = 1. \end{cases}$$

Then we have

$$\begin{cases} \alpha((1,\zeta_1),(1,\zeta_2)) \in [0,1], \\ \\ d((1,u_1),T(1,\zeta_1)) = 1, \\ \\ d((1,u_2),T(1,\zeta_2)) = 1. \end{cases}$$

Then, $(\zeta_1, \zeta_2) \in [0,1] \times [0,1]$. We also have given that $u_1 = \frac{\zeta_1}{3}$ and $u_2 = \frac{\zeta_2}{3}$, that is $(1, u_1 = \frac{\zeta_1}{3}) \in [0,1] \times [0,1]$ and $(1, u_2 = \frac{\zeta_2}{3}) \in [0,1] \times [0,1]$ i.e. $\zeta_1, \zeta_2 \in [0,1]$.So, $\alpha((1,u_1),(1,u_2) \geq 2$.

Which means that T is α-proximal admissible w.r.t. $\eta(\zeta,\mu) = 2$.

Now to demonstrate that T has unique best proximity point. If

$$(1,\zeta),(1,\mu) \in [0,1] \times [0,1]$$

then $\alpha((1,\zeta),(1,\mu)) = 2$, then by using the definition of modified Suzuki-Edelstein α-proximal contraction, insinuate

$$\alpha((1,\zeta),(1,\mu))d(T((1,\zeta),T(1,\mu)) \leq \ \phi(d((1,\zeta),(1,\mu)))$$

Putting $\alpha((1,\zeta),(1,\mu)) = 2$, we get

$$2d(T((1,\zeta),T(1,\mu)) \leq \ \phi(d((1,\zeta),(1,\mu)))$$

$$2d(T((0,\frac{\zeta}{3}),(0,\frac{\mu}{3})) \leq \ \phi(d(\mid 1-1 \mid \ + \ \mid \zeta - \mu \mid)$$

$$2(|0-0| + |\frac{\zeta}{3} - \frac{\mu}{3}|) \leq \ \phi(\mid \zeta - \mu \mid)$$

As $\psi(t) = \dfrac{999}{1000} t \ \forall \ t \geq 0$. Using this inputs in inequality, we have

$$2(|\frac{\zeta}{3} - \frac{\mu}{3}|) \leq \ \frac{999}{1000} \ (\mid \zeta - \mu \mid)$$

$$2(\frac{1}{3}|\zeta - \mu|) \leq \ 0.99 \ (\mid \zeta - \mu \mid)$$

$$\Rightarrow 0.66 \ |\zeta - \mu| \leq \ 0.99 \ (\mid \zeta - \mu \mid)$$

Otherwise, $\alpha((1,\zeta),(1,\mu)) = 0$. Then above definition will reduced to

$$\varphi(d(1,\zeta),T(1,\zeta)) - 2d(U,V) \leq \alpha((1,\zeta),(1,\mu))d((1,\zeta),(1,\mu) = 0$$

implies

$$0 = \alpha((1,\zeta),(1,\mu))d(T(1,\zeta),T(1,\mu)) \leq \phi(d((1,\zeta),(1,\mu))$$

Hence,

$$\varphi(d(1,\zeta),T(1,\zeta))) - 2d(U,V) \leq \alpha((1,\zeta),(1,\mu))d((1,\zeta),(1,\mu))$$

implies

$$\alpha((1,\zeta),(1,\mu))d(T(1,\zeta),T(1,\mu)) \leq \phi(d((1,\zeta),(1,\mu)))$$

Thus axioms of result (3.3.1)are complacent. So, there exist best proximity point $\omega = (0,1)$ of T which is unique. □

Remark 3.3.1. *[39].*

"The best proximity point can also be obtained if we replace the condition of continuity of T in above theorem by the following $\mathfrak{H}$ property:

If $\{\zeta_n\}$ is a sequence in U such that $\alpha(\zeta_n,\zeta_{n+1}) \geq 2$ and $\zeta_n \to \omega \in U$ as $n \to \infty$ then $\alpha(\zeta_n,\omega) \geq 2$ $\forall n \in N$."

Theorem 3.3.2. Consider U and V are the two non-empty closed subsets of complete metric space (X,d), along with $U_0 \neq \phi$. Let a mapping $T : U \to V$ with $T(U_0) \subseteq V_0$ is modified Suzuki-Edelstein α-proximal admissible mapping w.r.t. to $\eta(\zeta,\mu) \geq 2$ and pair (U,V) gratifies weak P-property and the elements ζ_0 and ζ_1 in U_0 with $d(\zeta_1,T\zeta_0) = d(U,V)$ satisfies $\alpha(\zeta_0,\zeta_1) \geq 2$. Moreover, If $\{\zeta_n\}$ is sequence in U such that

$$\alpha(\zeta_n,\zeta_{n+1}) \geq 2, \zeta_n \to \omega \in U \text{ as } n \to \infty.$$

Then T has unique best proximity point.

Proof. Let ζ_0 and ζ_1 in U_0, since we have $T(U_0) \subseteq V_0$, such that

$$d(\zeta_1, T\zeta_0) = d(U, V) \text{ satisfies } \alpha(\zeta_0, \zeta_1) \geq 2.$$

By proceeding as previous theorem (3.3.1), we have Cauchy sequence $\zeta_n \to \omega$ as $n \to \infty$. Now by using the above property, we have $\alpha(\zeta_n, \omega) \geq 2 \; \forall \; n \in \mathbb{N}$. Now following (3.6), we gain

$$\begin{aligned} d(\zeta_{n+2}, \zeta_{n+1}) &\leq \phi(d(\zeta_{n+1}, \zeta_n), \\ &< d(\zeta_{n+1}, \zeta_n), \end{aligned}$$

which implies

$$d(\zeta_{n+2}, \zeta_{n+1}) < d(\zeta_{n+1}, \zeta_n) \; \forall \; n \in \mathbb{N}.$$

As $\varphi \in \Phi$, by using Definition (3.2.9)

$$\begin{aligned} \frac{1}{2} d(\zeta_n, \zeta_{n+1}) &\geq \varphi(d(\zeta_n, \zeta_{n+1})), \\ &> \alpha(\zeta_n, \omega) d(\zeta_n, \omega), \\ &\geq d(\zeta_n, \omega). \end{aligned} \tag{3.16}$$

$$\frac{1}{2} d(\zeta_n, \zeta_{n+1}) > d(\zeta_n, \omega) \text{ for some } n \; \in \mathbb{N}. \tag{3.17}$$

Similarly

$$\begin{aligned} \frac{1}{2} \, d(\zeta_{n+1}, \zeta_{n+2}) &\geq \varphi(d(\zeta_{n+1}, \zeta_{n+2})), \\ &> \alpha(\zeta_{n+1}, \omega) d(\zeta_{n+1}, \omega), \\ &\geq d(\zeta_{n+1}, \omega). \end{aligned} \tag{3.18}$$

i.e

$$\frac{1}{2} d(\zeta_{n+1}, \zeta_{n+2}) > d(\zeta_{n+1}, \omega) \text{ for some } n \; \in \mathbb{N}. \tag{3.19}$$

Since, by using triangular inequality

$$d(\zeta_n, \zeta_{n+1}) \leq d(\zeta_n, \omega) + d(\omega, \zeta_{n+1}),$$

$$d(\zeta_n, \zeta_{n+1}) \leq d(\zeta_n, \omega) + d(\zeta_{n+1}, \omega),$$

Using inequalities (3.16) and (3.18) in above inequality, we get

$$\begin{aligned} d(\zeta_n, \zeta_{n+1}) \leq & \ \frac{1}{2} d(\zeta_n, \zeta_{n+1}) + \frac{1}{2} \ d(\zeta_{n+1}, \zeta_{n+2}), \\ < & \ \frac{1}{2} d(\zeta_n, \zeta_{n+1}) + \frac{1}{2} \ d(\zeta_{n+1}, \zeta_n), \\ < & \ \frac{1}{2} d(\zeta_n, \zeta_{n+1}) + \frac{1}{2} \ d(\zeta_n, \zeta_{n+1}), \\ = & \ d(\zeta_n, \zeta_{n+1}), \end{aligned} \tag{3.20}$$

contradiction. Thus for all $n \in \mathbb{N}$, either

$$\varphi(\zeta_n, \zeta_{n+1}) \ \leq \ \alpha(\zeta_n, \omega) \ d(\zeta_n, \omega)$$

or

$$\varphi(\zeta_{n+1}, \zeta_{n+2}) \leq \ \alpha(\zeta_{n+1}, \omega) \ d(\zeta_{n+1}, \omega)$$

holds. Hence from the definition of modified Suzuki-Edelstein α-proximal admissible mapping (see Definition 3.3.1)

$$\begin{aligned} d(T\zeta_n, T\omega) \ & \leq \alpha\zeta_n, \omega) d(T\zeta_n, T\omega), \\ & \leq \phi(d(\zeta_n, \omega)). \end{aligned} \tag{3.21}$$

or

$$\begin{aligned} d(T\zeta_{n+1}, T\omega) \ & \leq \alpha(\zeta_{n+1}, \omega) d(T\zeta_{n+1}, T\omega), \\ & \leq \phi(d(\zeta_{n+1}, \omega)). \end{aligned} \tag{3.22}$$

Now taking limit as $n \to \infty$ in inequalities (3.20) and (3.21),

$$\lim_{n\to\infty} d(T\zeta_n, T\omega) = 0$$

or

$$d(T\zeta_{n+1}, T\omega) = 0.$$

$$T\zeta_n \to T_\omega$$

or

$$T\zeta_{n+1} \to T_\omega$$

Consequently, there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $T\zeta_{n_k} \to T_\omega$ as $x_{n_k} \to z$. *i.e*

$$\lim_{n\to\infty} d(\zeta_{n_{k+1}}, T\zeta_{n_k}) = d(\omega, T\omega)$$

and also

$$d(U, V) = \lim_{n\to\infty} d(\zeta_{n_{k+1}}, T\zeta_{n_k})$$

which further implies

$$d(U, V) \;=\; d(\omega, T\omega). \tag{3.23}$$

So, T has the best proximity point.

Now we have to spectacle that T has the unique best proximity point. For this, on contrary suppose $\mu, \omega \in U_0$ are the two best proximity points of T. Where $\zeta \neq \mu$

$$d(\mu, T\mu) = d(\omega, T\omega) = d(U, V)$$

By using weak P-property,

$$d(\mu, \omega) \leq \; d(T\mu, T\omega) \tag{3.24}$$

Now,

$$d(\mu, T\mu) = 2d(U, V) - d(U, V) \tag{3.25}$$

Since

$$\varphi(d(\mu, T\mu)) \leq \frac{1}{2}d(\mu, T\mu)$$

Using (3.24) in (3.25),

$$\begin{aligned}\varphi(d(\mu, T\mu)) &\leq \frac{1}{2}(2d(U,V) - d(U,V)),\\ &\leq \frac{1}{2}d(U,V),\\ &\leq 2d(U,V).\end{aligned}$$

So from the inequality

$$\varphi(d(\mu, T\mu)) - 2d(U,V) \leq 0 \leq \alpha(\mu,\omega)d(\mu,\omega) \tag{3.26}$$

Hence from the definition of modified Suzuki-Edelstein α-proximal contraction

$$\alpha(\mu,\omega)d(T\mu, T\omega) \leq \phi(d(\mu,\omega)) \tag{3.27}$$

Since $\alpha(\mu,\omega) \geq 2$,

$$d(T\mu, T\omega) \leq \phi(d(\mu,\omega)) \tag{3.28}$$

By using (3.10)

$$d(\mu,\omega) \leq \phi(d(\mu,\omega)) < d(\mu,\omega) \tag{3.29}$$

contradiction, so $\mu = \omega$. T has unique best proximity point. □

Theorem 3.3.3. Consider U and V are non-empty subsets of complete metric space (X,d) with $U_0 \neq \phi$. A function $\delta : [0,1) \to (0, \frac{1}{2}]$ and for all $\zeta, \mu \in U$ a mapping $T : U \to V$ be such that

$$\delta(\gamma)(\varphi(d(\zeta, T\zeta)) - 2d(U,V)) \leq d(\zeta,\mu) \tag{3.30}$$

$$\Rightarrow \delta(\gamma)d(T\zeta, T\mu) \leq \phi(d(\zeta,\mu)) \tag{3.31}$$

where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\gamma \in [0,1)$. Moreover, $T(U_0) \subseteq V_0$ and pair (U,V) satisfies weak P-property. Then T has unique best proximity point.

Proof. For fixed $\gamma \in [0,1)$, define $\alpha_\gamma(\zeta,\mu) = \dfrac{1}{\delta(\gamma)}$ for all $\zeta,\mu \in U$. Since $\dfrac{1}{\delta(\gamma)} \geq 2$ for all $\gamma \in [0,1)$, then $\alpha_r(x,y) \geq 2$ for all $\zeta,\mu \in U$. Now $\alpha_r(\zeta,\mu)$ is constant and $\alpha_r(x,y) \geq 2$ for all $\zeta,\mu \in U$. So from the above conditions it is clear that T is α_r-proximal admissible w.r.t. $\eta(\zeta,\mu) = 2$ and also property $\mathfrak{H}$ holds.
Now from the definition of modified Suzuki-Edelstein α-proximal contraction,

$$\varphi(d(\zeta,T\zeta) - 2d(U,V)) \leq \alpha_\gamma(\zeta,\mu)d(\zeta,\mu) \tag{3.32}$$

$$\Rightarrow \ \alpha(\zeta,\mu)d(T\zeta,T\mu) \leq \phi(d(\zeta,\mu))$$

then for some function $\delta(\gamma)$, the above inequality (3.30) become:

$$\delta(\gamma)(\varphi(d(\zeta,T\zeta)) - 2d(U,V) \leq d(\zeta,\mu)$$

by inequality (3.29),

$$\delta(\gamma)d(T\zeta,T\mu) \leq \phi(d(\zeta,\mu)).$$

Hence it is clear from the above results that all axioms of result (3.3.2) holds. So T has best proximity point which is unique. □

By using $\phi(t) = \xi t$, $\xi \in [0,1)$ in result (3.3.3), we gain corollary as follows:

Corollary 1. "Consider U and V are non-empty subsets of complete metric space (X,d) with $U_0 \neq \phi$. A function $\delta : [0,1) \to (0,\frac{1}{2}]$ and for all $\zeta,\mu \in U$, a mapping $T : U \to V$ be such that;

$$\delta(\gamma)(\varphi(d(\zeta,T\zeta)) - 2d(U,V)) \leq d(\zeta,\mu) \tag{3.33}$$

$$\Rightarrow \ \delta(\gamma)d(T\zeta,T\mu) \leq \xi d(\zeta,\mu) \tag{3.34}$$

where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\gamma \in [0,1)$. Moreover, $T(U_0) \subseteq V_0$ and pair (U,V) satisfies weak P-property. So, T has unique best proximity point."

Corollary 2. Consider U and V are non-empty subsets of the complete metric space (X, d) with $U_0 \neq \phi$, a non-increasing function $\theta : [0, 1) \to (0, \frac{1}{2}]$ defined as

$$\theta(\gamma) = \begin{cases} 1 \quad \text{if} \quad 0 \leq \gamma \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-\gamma}{\gamma^2} \quad \text{if} \quad \frac{\sqrt{5}-1}{2} \leq \gamma \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+\gamma} \quad \text{if} \quad \frac{1}{\sqrt{2}} \leq \gamma \leq 1 \end{cases} .$$

and for all $\zeta, \mu \in U$, a mapping $T : U \to V$ be such that

$$\frac{1}{2}\theta(\gamma)(\varphi(d(\zeta, T\zeta)) - 2d(U, V)) \leq d(\zeta, \mu) \tag{3.35}$$

$$\Rightarrow \frac{1}{2}\theta(\gamma)d(T\zeta, T\mu) \leq \xi d(\zeta, \mu) \tag{3.36}$$

where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\gamma \in [0, 1)$. Moreover, $T(U_0) \subseteq V_0$ and pair (U, V) convince weak P-property. So, T has unique best proximity point.

Proof. By putting $\delta(\gamma) = \frac{1}{2}\theta(r)$ in Corollary (1), we obtained the desired result. □

Corollary 3. "Consider U and V are non-empty subsets of complete metric space (X, d) along with $U_0 \neq \phi$. A non-increasing function $\delta : [0, 1) \to (\frac{1}{2}, 1]$ defined as

$$\delta(\gamma) = \frac{1}{2(1+\gamma)} \tag{3.37}$$

and for all $\zeta, \mu \in U$, a mapping $T : U \to V$ be such that;

$$\delta(\gamma)(\varphi(d(\zeta, T\zeta)) - 2d(U, V) \leq d(\zeta, \mu)) \tag{3.38}$$

$$\Rightarrow \delta(\gamma)d(T\zeta, T\mu) \leq \xi d(\zeta, \mu) \tag{3.39}$$

where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\gamma \in [0, 1)$. Moreover, $T(U_0) \subseteq V_0$ and pair (U, V) convince weak P-property. Then T has unique best proximity point."

Proof. By taking $\delta(\gamma) = \frac{1}{2(1+\rho)}$ in above result, we get our desired result. □

Corollary 4. "Consider U and V two non-empty subsets of complete metric space (X, d) where $U_0 \neq \phi$ and for all $\zeta, \mu \in U$, a mapping $T : U \to V$ be such that;

$$\frac{1}{2}(\varphi(d(\zeta, T\zeta)) - 2d(U, V) \leq d(\zeta, \mu)) \tag{3.40}$$

$$\Rightarrow \frac{1}{2}\gamma d(T\zeta, T\mu) \leq \xi d(\zeta, \mu) \tag{3.41}$$

where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\gamma \in [0, 1)$. Moreover, $T(U_0) \subseteq V_0$ and pair (U, V) convince weak P-property. Then T has unique best proximity point."

Proof. By putting $\delta(\gamma) = \frac{1}{2}$ in above Corollary, we will get our desired result. □

3.4 Applications

As an application of our important best proximity point theorems, deduce the new fixed point problems for Suzuki-Edelstain contraction in the structure of metric space. We convert best proximity points into fixed point by taking $U = V = X$. Let see few results as examples:

Theorem 3.4.1. "Consider (X, d) be complete metric space. A self-mapping $T : X \to X$ be continuous α-admissible mapping w.r.t. $n(\zeta, \mu) = 2$, where $\varphi \in \Phi_\varphi$ and also $\phi \in \Phi$ such that

$$(\varphi(d(\zeta,\ T\zeta)) \leq\ \alpha(\zeta,\ \mu)d(\zeta,\ \mu)) \tag{3.42}$$

$$\Rightarrow\ \alpha(\zeta,\ \mu)d(T\zeta,\ T\mu) \leq\ \phi(d(\zeta,\ \mu)) \tag{3.43}$$

for all $\zeta, \mu \in X$. Moreover, there exists element $\zeta_0 \in X$ such that $\alpha(\zeta_0,\ T\zeta_0) \geq 2$. Then T has fixed point which is unique."

Proof. If we take $U = V = X$ in Theorem (3.3.1) and (3.3.2), we obtained the desired results. □

The bellow example is taken from [51] to validate the above theorem.

Example 3.4.1. Consider $X = [0, \infty) \times [0, \infty)$ with metric described on X as

$$d(\zeta, \mu) = |\zeta - \mu|.$$

Consider $T : X \to X$ by

$$T(\zeta) = \begin{cases} \frac{\zeta}{2} - \frac{\zeta^2}{4} & \text{if } \ \zeta \in [0,1], \\ \\ \ln \zeta + \frac{1}{2} & \text{if } \ \zeta \in (1, \infty), \end{cases}$$

and $\alpha : X^2 \to \mathbb{R}^+$ by

$$\alpha(\zeta, \mu) = \begin{cases} 1 & \text{if } \ (\zeta, \mu) \in [0,1], \\ 0 & \text{otherwise}, \end{cases}$$

Consider $\psi(t) = \dfrac{1}{2}t \ \ \forall \ \ t \geq 0$ and $\zeta \leq \mu$. Then T has fixed point.

Proof. Consider $\zeta, \mu \in X$. First suppose $\alpha(\zeta, \mu) \geq 1$, then $\zeta, \mu \in [0, 1]$. On the other hand, if for all $\zeta, \mu \in [0, 1]$ then $T\zeta \leq \frac{1}{2}$. Hance $\alpha(T\zeta, T\mu) \geq 1$ which gives T is α-admissible. Clearly $\alpha(0, T_0) \geq 1$ and $0 \leq T_0$.
Now $\zeta \geq \mu$ and $\zeta, \mu \in [0, 1)$. Then

$$\alpha(\zeta, \mu) \ d(T\zeta, T\mu) \ = \ \frac{\zeta - \mu}{2} \ - \ (\frac{\zeta^2 - \mu^2}{4}),$$

$$\alpha(\zeta, \mu) \ d(T\zeta, T\mu) \ \leq \ \frac{\zeta - \mu}{2} \Rightarrow \phi(d(\zeta, \mu))$$

Now assume that $\alpha(\zeta, \mu) = 0$

$$0 \ = \ \alpha(\zeta, \mu) \ d(T\zeta, T\mu) \ \leq \ \phi(d(\zeta, \mu)).$$

From this result it is clear that all axioms of result (3.4.1) are fulfill. So, T has fixed point. □

Theorem 3.4.2. Consider (X, d) be complete metric space. A self-mapping $T : X \to X$ be continuous α-admissible w.r.t. $n(\zeta, \mu) = 2$, where $\varphi \in \Phi_{\varphi}$ and also

$\phi \in \Phi$ such that

$$(\varphi(d(\zeta,\ T\zeta)) \leq \alpha(\zeta,\ \mu)d(\zeta,\ \mu)) \tag{3.44}$$

$$\Rightarrow\ \alpha(\zeta, \mu)d(T\zeta, T\mu) \leq \phi d(\zeta, \mu) \tag{3.45}$$

for all $\zeta, \mu \in X$. Moreover, select an element $\zeta_0 \in X$ such that $\alpha(\zeta_0, Tx_0 \geq 2$ and either T is constant or the property $\mathfrak{H}$ given as a remark (3.3.1) satisfies. Then T has the unique fixed point.

Adopting $\phi(t) = \xi t$ in result (3.4.1) and (3.4.2) where $0\ \leq \xi < 1$, we get result as below:

Theorem 3.4.3. Consider (X, d) be complete metric space. A self-mapping $T : X \rightarrow X$ is continuous α-admissible w.r.t. $n(\zeta, \mu) = 2$, where $\varphi \in \Phi_\varphi$ and also $\phi \in \Phi$, such that

$$(\varphi(d(\zeta, T\zeta)) \leq \alpha(\zeta, \mu)d(\zeta, \mu)) \tag{3.46}$$

$$\Rightarrow\ \alpha(\zeta, \mu)d(T\zeta, T\mu) \leq \xi d(\zeta, \mu) \tag{3.47}$$

for all $\zeta, \mu \in X$. Moreover, select an element $\zeta_0 \in X$ such that $\alpha(\zeta_0, Tx_0) \geq 2$ and either T is constant or the property $\mathfrak{H}$ given as a remark (3.3.1) satisfies. Then T has the unique fixed point.

Chapter 4

Suzuki-Edlstein α-proximal Contraction in b-Metric Space

In current chapter, we extend the theory of Suzuki-Edlstein α-proximal contraction in b-metric space and also established few best proximity point theorems for such contraction in b-metric spaces.

4.1 Best Proximity Point in b-Metric Space

Let $(X, d\rho)$ is b-metric space with coefficient $\rho \geq 1$, U and V be non-empty subsets of $(X, d\rho)$. Define

$$d_\rho(U, V) = \inf\{d_\rho(\zeta, \mu) : \zeta \in U, \mu \in V\}$$

$$U_0 = \{\zeta \in U : d_\rho(\zeta, \mu) = d_\rho(U, V) \text{ for some } \mu \in V\}$$

$$V_0 = \{\mu \in V : d_\rho(\zeta, \mu) = d_\rho(U, V) \text{ for some } \zeta \in U\}.$$

Definition 4.1.1. (**Best Proximity Point**)
Let (X, d_ρ) be b-metric space along with parameter $\rho \geq 1$. Suppose that U and V be non-empty subsets of (X, d_ρ). A component $\zeta \in U$ is called best proximity

point of $T : U \to V$ if

$$d_\rho(\zeta, T\zeta) = d_\rho(U, V).$$

Definition 4.1.2. (**Weak P-property**)

"Let (U, V) be pair of non-empty subsets of b-metric space (X, d_ρ) with $U_0 \neq \phi$. The pair (U, V) satisfies weak P-property iff

$$\begin{cases} d_\rho(\zeta_1, \mu_1) \;=\; d_\rho(U, V), \\ \\ d_\rho(\zeta_2, \mu_2) \;=\; d_\rho(U, V), \end{cases} \quad \Rightarrow \quad d_\rho(\zeta_1, \zeta_2) \;\leq\; d_\rho(\mu_1, \mu_2),$$

where $\zeta_1, \zeta_2 \in U$ and $\mu_1, \mu_2 \in U$."

Definition 4.1.3. (α-**proximal admissible mapping**)

"Let (X, d_ρ) be a b-metric space with coefficient $\rho \geq 1$, U and V be two non-empty subsets of X, a mapping $T : U \to V$ is called as α-proximal admissible if

$$\begin{cases} \alpha(\zeta_1, \zeta_2) \;\geq\; 1, \\ \\ d_\rho(u_1, T\zeta_1) \;=\; d_\rho(U, V), \\ \\ d_\rho(u_2, T\zeta_2) \;=\; d_\rho(U, V), \end{cases} \quad \Rightarrow \alpha(u_1, u_2) \;\geq\; 1$$

for all $\zeta_1, \zeta_2, u_1, u_2 \in U$ where $\alpha : U \times U \to [0, \infty)$."

4.2 Best Proximity Point Theorems in b-Metric Space

Definition 4.2.1. (**Modified Suzuki-Edlstein α-proximal contraction in b-metric space**)

Let U and V be two non-empty closed subsets of complete b-metric space (X, d_ρ)

with coefficient $\rho \geq 1$, a mapping $T : U \to V$ is said to be modified Suzuki-Edelstein α-proximal contraction in b-metric space if

$$\varphi(d_\rho(\zeta, T\zeta)) - 2\ \rho\ d_\rho(U, V) \leq\ \rho\ \alpha(\zeta, \mu) d_\rho(\zeta, \mu) \tag{4.1}$$

$$\Rightarrow\ \alpha(\zeta, \mu) d_\rho(T\zeta, T\mu) \leq \phi(d_\rho(\zeta, \mu)) \tag{4.2}$$

for all $\zeta, \mu \in U$, where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $\alpha : U \times U \to [0, \infty]$.

Theorem 4.2.1. Suppose that (X, d_ρ) be a complete b-metric space having coefficient $\rho \geq 1$. Suppose that U and V be non-empty closed subsets of (X, d_ρ) where U_0 in non-empty. Let $\alpha : U \times U \to [0, \infty]$ and $\phi \in \Phi$. Let $T : U \to V$ is a non-self mapping fulfilling the following axioms:

(i) T is α-proximal admissible w.r.t. $\eta(\zeta, \mu) = 2$,

(ii) T is continuous modified Suzuki-Edelstein α-proximal admissible,

(iii) $T(U_0) \subseteq V_0$ and (U, V) satisfies weak P-property,

(iv) ζ_0 and ζ_1 in U with $d_\rho(\zeta_1, T\zeta_0) = d_\rho(U, V)$ satisfies $\alpha(\zeta_0, \zeta_1) \geq 2$.

Then T has unique best proximity point.

Proof. Consider $\zeta_0 \in U_0$. As $T(U_0) \subseteq V_0$, then there exists element $\zeta_1 \in U_0$ such that

$$d_\rho(\zeta_1, T\zeta_0) = d_\rho(U, V) \quad \text{satisfies} \quad \alpha(\zeta_0, \zeta_1) \geq 2.$$

Since $\zeta_1 \in U_0$ and $T(U_0) \subseteq V_0$, then there exists a component $\zeta_2 \in U_0$ such that

$$d_\rho(\zeta_2, T\zeta_1) = d_\rho(U, V)$$

Since, T is α-proximal admissible w.r.t. $\eta(\zeta, \mu) = 2$, then $\alpha(\zeta_0, \zeta_1) \geq 2$. Continuing this to get ζ_{n+1}, ζ_n such that for all $n \in \mathbb{N}$.

$$d_\rho(\zeta_{n+1}, \zeta_n) = d_\rho(U, V) \text{ satisfies } \alpha(\zeta_n, \zeta_{n+1}) \geq\ 2$$

As $\phi \in \Phi$, by using Definition (3.2.9),

$$\varphi(d_b(\zeta_{n-1}, T\zeta_{n-1}) \leq \frac{1}{2\rho}\, d_b(\zeta_{n-1}, T\zeta_{n-1}),$$

$$\leq 2\rho\ d_b(\zeta_{n-1}, T\zeta_{n-1}),$$

by using the triangular inequality of b-metric space,

$$\varphi(d_\rho(\zeta_{n-1}, T\zeta_{n-1}) \leq 2[\rho(d_\rho(\zeta_{n-1}, \zeta_n) + d_\rho(\zeta_n, T\zeta_{n-1}))],$$

$$= 2\ [\rho\ (d_\rho(\zeta_{n-1}, \zeta_n) + d_\rho(U, V))],$$

$$= 2\rho d_\rho(\zeta_{n-1}, \zeta_n) + 2\rho d_\rho(U, V),$$

From the above inequality,

$$\varphi(d_\rho(\zeta_{n-1}, T\zeta_{n-1}) - 2\rho d_\rho(U,\ V) \leq 2\rho d_\rho(\zeta_{n-1}, \zeta_n),$$

$$\leq \rho\alpha(\zeta_{n-1}, \zeta_n) d_\rho(\zeta_{n-1}, \zeta_n),$$

then by the definition of modified Suzuki-Edelstein α-proximal contraction in b-metric space

$$\rho\alpha(\zeta_{n-1}, \zeta_n) d_\rho(T\zeta_{n-1}, T\zeta_n) \leq \rho\phi(d_\rho(\zeta_{n-1}, \zeta_n)$$

$$\alpha(\zeta_{n-1}, \zeta_n) d_\rho(T\zeta_{n-1}, T\zeta_n) \leq \phi(d_\rho(\zeta_{n-1}, \zeta_n)$$

Now

$$d_b(T\zeta_{n-1}, T\zeta_n) \leq\ \alpha(\zeta_{n-1}, \zeta_n) d_b(T\zeta_{n-1}, T\zeta_n)$$

this implies that

$$d_\rho(T\zeta_{n-1}, T\zeta_n)\ \leq\ \rho\ \phi(d_\rho(\zeta_{n-1}, \zeta_n)) \tag{4.3}$$

Now suppose that $\zeta_n = \zeta_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then we must have

$$d_\rho(\zeta_{n_0}, T\zeta_{n_0}) = d_\rho(\zeta_{n_0+1}, T\zeta_{n_0}) \tag{4.4}$$

Using equation (4.2) in (4.4),

$$d_\rho(\zeta_{n_0}, T\zeta_{n_0}) = d_\rho(U, V) \tag{4.5}$$

Thus from (4.5), we conclude that ζ_{n_0} is the best proximity.Therefore, we assume that $\zeta_{n_0} \neq \zeta_{n+1}$, that is $d(\zeta_n, \zeta_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since by (3.3) ϕ is nondecreasing and weak P-property of (U, V),

$$\begin{aligned} d_\rho(\zeta_{n+1},\ \zeta_n) \leq& \rho d_\rho(T\zeta_n, T\zeta_{n-1}), \\ \leq& \rho\phi(d_\rho(\zeta_n, \zeta_{n-1})), \end{aligned}$$

So

$$\begin{aligned} d_\rho(\zeta_{n+1}, \zeta_n) \leq& \rho\phi(d_\rho(\zeta_n, \zeta_{n-1})), \\ \leq& \rho\ \phi(d_\rho(T\zeta_{n-1}, T\zeta_{n-2})), \\ \leq& \rho\ \phi(\phi(d_\rho(\zeta_{n-1}, \zeta_{n-2}))), \\ d_\rho(\zeta_{n+1}, \zeta_n)) \leq& \rho^2\phi^2(d_\rho(\zeta_{n-1}, \zeta_{n-2}))... \leq\ \rho^n\phi^n(d_\rho(\zeta_0, \zeta_1)). \end{aligned}$$

Hence,

$$d_\rho(\zeta_{n+1}, \zeta_n)\ \leq\ \rho^n\ \phi^n(d_\rho(\zeta_0, \zeta_1)) \tag{4.6}$$

by using limit as $n \to \infty$ and also continuity condition of d_ρ,

$$\lim_{n\to\infty} d_\rho(\zeta_{n+1}, \zeta_n) = 0$$

Now for fixed $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that

$$\begin{aligned} d_\rho(\zeta_{n+1}, \zeta_n) &< \frac{\epsilon}{\rho} - \phi(\epsilon), \\ &\leq \frac{\epsilon}{\rho}, \\ &\leq \epsilon, \end{aligned}$$

$\forall\ n \geq N$. As ϕ is nondecreasing,

$$\phi(d_\rho(\zeta_{n+1}, \zeta_n)) \leq \ \phi(\epsilon)\ \forall\ n \geq \mathbb{N} \tag{4.7}$$

Then again by triangular inequality

$$\begin{aligned} d_\rho(\zeta_n, \zeta_{n+2}) &\leq \rho[d_\rho(\zeta_n, \zeta_{n+1}) + d_\rho(\zeta_{n+1}, \zeta_{n+2})], \\ &< \ \rho[\frac{\epsilon}{\rho} - \phi(\epsilon) + \phi(\epsilon)], \\ &\leq \ \rho[\frac{\epsilon}{\rho}], \\ &< \ \epsilon. \end{aligned}$$

Continuing the process in this scheme, we conclude

$$d_\rho(\zeta_n, \zeta_{n+j}) < \epsilon\ \forall\ \ n \geq \ N \text{ and } j\ \in \mathbb{N}.$$

Then there exist $j > 0$ such that $\forall\ \ m, n \in \mathbb{N}$, there exist $m < n$. Ensuing triangular inequality and (4.7),

$$\begin{aligned} d_\rho(\zeta_m, \zeta_n) &\leq \rho d_\rho(\zeta_m, \zeta_{m+1}) + \rho^2 d_\rho(\zeta_{m+1}, \zeta_{m+2}) + ... + \rho^{n-m-1} d_\rho(\zeta_{n-1}, \zeta_n), \\ &\leq \sum_{j=m}^{n-2}\ \rho^{j-m+1}\ \phi^j\ (d_\rho(\zeta_1, \zeta_0)), \\ &\leq \sum_{j=m}^{\infty} \rho^j\ \phi^j\ (d_\rho(\zeta_1, \zeta_0))\ \rightarrow\ 0 \text{ as } n \rightarrow\ \infty. \end{aligned}$$

Consequently, we have

$$\lim_{m,n \to \infty}\ d_\rho(\zeta_m, \zeta_n)\ =\ 0$$

Hence this impels that $\{\zeta_n\}$ is Cauchy sequence in U. X is complete and U is closed, therefore there exists $\omega \in U$ such that $\zeta_n \to \omega$, by continuity of T implies

$T\zeta_n \to T\omega$ as $n \to \infty$. So inequality (4.2) gives

$$d_\rho(U,V) = \lim_{n\to\infty} d_\rho(\zeta_{n+1}, T\zeta_n) = d_\rho(\omega, T\omega).$$

Thus ω is best proximity point. Now, for uniqueness, assume that on contrary that $\mu, \omega \in U_0$ are best proximity points of T having $\mu \neq \omega$,

$$d_\rho(\mu, T\mu) = d_\rho(\omega, T\omega) = d_\rho(U,V).$$

Now by using weak P-property,

$$d_\rho(\mu,\omega) \leq \; d_\rho(T\mu, T\omega) \tag{4.8}$$

Now,

$$d_\rho(\mu, T\mu) = 2\ \rho d_\rho(U,V) - \rho d_\rho(U,V) \tag{4.9}$$

Since

$$\varphi(d_\rho(\mu, T\mu)) \leq \; \frac{1}{2\rho} d_\rho(\mu, T\mu)$$

Using (4.9) in (4.10),

$$\begin{aligned}
\varphi(d_\rho(\mu, T\mu)) &\leq \frac{1}{2\rho} d_\rho(\mu, T\mu), \\
&\leq \frac{1}{2\rho}\ (2\rho d_\rho(U,\ V) - \rho\ d_\rho(U,\ V)), \\
&\leq \frac{1}{2\rho}\ \rho\ d_\rho(U,\ V), \\
&\leq \frac{1}{2}\ d_\rho(U,\ V), \\
&\leq 2 d_\rho(U,\ V).
\end{aligned}$$

So from the inequality,

$$\varphi(d_\rho(\mu, T\mu)) - 2d_\rho(U,V) \leq \; 0 \; \leq \; \rho\alpha(\mu,\omega) d_\rho(\mu,\omega) \tag{4.10}$$

Hence from the definition of modified Suzuki-Edelstein α-proximal contraction in b-metric space,

$$\rho\ \alpha(\mu,\omega)d_\rho(T\mu,T\omega) \leq\ \rho\phi(d_\rho(\mu,\omega)) \tag{4.11}$$

Since $\alpha(\mu,\omega) \geq 2$, from the above inequality,

$$\rho\ d_\rho(T\mu,T\omega) \leq\ \rho\phi(d_\rho(\mu,\omega)) \tag{4.12}$$

$$d_\rho(T\mu,T\omega) \leq\ \phi(d_\rho(\mu,\omega)) \tag{4.13}$$

By using (4.10),

$$d_\rho(\mu,\omega) \leq\ \phi(d_\rho(\mu,\omega)) < d_\rho(\mu,\omega) \tag{4.14}$$

is contradiction, so $\mu = \omega$.

Hence T has unique best proximity point. □

Example 4.2.1. Suppose that $X = \{(0,2),(1,2),(1,3),(2,2),(2,3),(3,3),(4,6),(5,6)\}$ be the complete b-metric space (X,d_ρ) with the co-efficient $\rho = 2$, Define

$$d_\rho(\zeta,\mu) = \max\{|\zeta_1-\mu_1|^2, |\zeta_2-\mu_2|^2\} \text{ for all } \zeta,\mu \in X.$$

Let

$$U = \{(0,2),(2,2),(2,3)\}$$

$$V = \{(1,2),(1,3),(3,3)\}$$

be the two non-empty subsets of X. Now,

$$\begin{aligned} d_\rho(U,V) =&\ \inf\{d_\rho(\zeta,\mu) : \zeta \in U, \mu \in V\},\\ =&\ \inf\{d_\rho((0,2),(1,2)), d_\rho((0,2),(1,3)), d_\rho((0,2),(3,3)), d_\rho((2,2),(1,2)),\\ &+ d_\rho((2,2),(1,3)), d_\rho((2,2),(3,3)), d_\rho((2,3),(1,2)), d_\rho((2,3),(1,3)),\\ &+ d_\rho((2,3),(3,3))\}, \end{aligned}$$

$$d_\rho(U,V) = \inf\{\max(|0-1|^2,|2-2|^2),\max(|0-1|^2,|2-3|^2),\max(|0-3|^2,|2-3|^2),$$
$$+\ \max(|2-1|^2,|2-2|^2),\max(|2-1|^2,|2-3|^2),\max(|2-3|^2,|2-3|^2),$$
$$+\ \max(|2-1|^2,|3-2|^2),\max(|2-1|^2,|3-3|^2),\max(|2-3|^2,|3-3|^2)\},$$

$$d_\rho(U,V) = \inf\{\max(1,0),\ \max(1,1),\ \max(9,1),\ \max(1,0),\ \max(1,1),\max(1,1),$$
$$+\ \max(1,1),\max(1,1),\max(1,1),\max(1,0),\max(1,0)\}$$
$$d_\rho(U,V) = \inf\{1,1,9,1,1,1,1,1,1\},$$
$$= 1,$$

Hence

$$\Rightarrow d_\rho(U,V) = 1$$

and also

$$U_0 = \{(0,2),(2,2),(2,3)\} = U \quad and$$
$$V_0 = \{(1,2),(1,3),(3,3)\} = V.$$

For all $\zeta_1,\zeta_2 \in U_0 \subseteq U and\ \mu_1,\mu_2 \in V_0 \subseteq V$. Moreover, pair (U,V) fulfils weak P-property, as (X,d_ρ) is b-metric space with $\rho=2$. Now, define $T:U\to V$, as

$$T(\zeta) = \begin{cases} \{(1,2)\} \text{ if } \zeta \in \{0,2\}, \\ \{(1,3),(3,3)\} \text{ if } \zeta \in \{(2,2),(2,3)\}. \end{cases}$$

$\alpha: U\times U\to X$ by

$$\alpha((\zeta,\mu),(s,t)) = \begin{cases} 2 \text{ if } (\zeta,\mu),(s,t)\in\{(1,2),(1,3),(3,3)\}, \\ 0 \text{ otherwise,} \end{cases}$$

Clearly $T(U_0) \subseteq V_0$. Now, to prove that T satisfy the Modified Suzuki-Edlstein α-proximal contraction in b-metric space. The following portion of modified Suzuki-Edelstein α-proximal contraction holds for all $\zeta, \mu \in U_0$, that is

$$\varphi(d_\rho(\zeta, T\zeta)) - 2\ \rho\ d_\rho(U, V) \leq\ \rho\ \alpha(\zeta, \mu) d_\rho(\zeta, \mu) \tag{4.15}$$

Now, we must prove that second part of modified Suzuki-Edelstein α-proximal contraction valids for all $\zeta, \mu \in U_0$, that is

$$\Rightarrow\ \alpha(\zeta, \mu) d_\rho(T\zeta, T\mu) \leq\ \phi(d_\rho(\zeta, \mu)). \tag{4.16}$$

Now, Suppose if $\zeta \in (0, 2)$ and $\mu \in \{(2, t), t \in \{2, 3\}\}$, where $\zeta \neq \mu$. Then we have

$$d_\rho(\zeta, \mu) = 4, d_\rho(T\zeta, T\mu) = 4$$

Further, if $\zeta \in \{(2, t), t \in \{2, 3\}\}$ and $\zeta \in \{(0, 2), (2, 2), (2, 3)\}$, then

$$d_\rho(\zeta, \mu) = 1, d_\rho(T\zeta, T\mu) = 0.$$

So, the inequality (4.16) holds for all $\zeta \neq \mu \in U_0$. By considering $\rho \geq 2$, $\alpha(\zeta, \mu) = 1$ foe all $\zeta, \mu \in U$ and $\psi(t) = \dfrac{999}{1000} t\ \forall\ t \geq 0,\ \forall\ u, v \in U$, which represents that T is modified Suzuki-Edelstein α-proximal contraction condition in b-metric space. Further, all the axioms of above Theorem holds. Therefore T has best proximity point.

These results can also be established by replacing the axiom of continuity of T in above result by the $\mathfrak{H}$ property:
$\mathfrak{H}$: "If $\{\zeta_n\}$ is a sequence in U such that $\alpha(\zeta_n, \zeta_{n+1}) \geq 2$ and $\zeta_n \to \omega \in U$ as $n \to \infty$ then $\alpha(\zeta_n, \omega) \geq 2\ \forall\ n \in N$."

Theorem 4.2.2. Let (X, d_ρ) be a complete b-metric space along with coefficient $\rho \geq 1$. Suppose that U and V be the non-empty closed subsets of (X, d_ρ) with $U_0 \neq \phi$. Let $\alpha : U \times U \to [0, \infty]$ and $\phi \in \Phi$. Let $T : U \to V$ is a non-self mapping satisfying following axioms:

(i) T is modified Suzuki-Edelstein α-proximal admissible w.r.t. $\eta(\zeta,\mu)=2$,

(ii) $T(U_0)\subseteq V_0$, (U,V) satisfies weak P-property,

(iii) ζ_0, ζ_1 in U with $d_\rho(\zeta_1,T\zeta_0)=d_\rho(U,V)$ satisfies $\alpha(\zeta_0,\zeta_1)\geq 2$,

(iv) Moreover, If $\{\zeta_n\}$ is a sequence in U such that $\alpha(\zeta_n,\zeta_{n+1})\geq 2$ $\zeta_n\to\omega\in U$ as $n\to\infty$..

Then T has best proximity point which is unique.

Proof. Consider sequence ζ_n. Suppose ζ_0 , $\zeta_1\in U_0$, since $T(U_0)\subseteq V_0$, such that

$$d_\rho(\zeta_1,T\zeta_0)=d_\rho(U,V) \text{ satisfies } \alpha(\zeta_0,\zeta_1)\ \geq\ 2.$$

By progressing the substantiation of result (4.2.1), we have Cauchy sequence $\zeta_n\to\omega$ as $n\to\infty$. Now by using the above property, $\alpha(\zeta_n,\omega)\geq 2$ for all $n\in\mathbb{N}$. While, from the succeeding inequality (4.6),

$$\begin{aligned} d_\rho(\zeta_{n+2},\zeta_{n+1}) &\leq \phi(d_\rho(\zeta_{n+1},\zeta_n), \\ &< d_\rho(\zeta_{n+1},\zeta_n), \end{aligned}$$

implies

$$d_\rho(\zeta_{n+2},\ \zeta_{n+1})<d_\rho(\zeta_{n+1},\zeta_n)\ \forall\ n\in\mathbb{N}.$$

Also

$$\begin{aligned} \frac{1}{2\rho}d_\rho(\zeta_n,\zeta_{n+1}) &\geq \varphi(d_\rho(\zeta_n,\zeta_{n+1})), \\ &> \alpha(\zeta_n,\omega)d_\rho(\zeta_n,\omega), \\ &\geq d_\rho(\zeta_n,\omega). \end{aligned} \tag{4.17}$$

that is

$$\frac{1}{2\rho}d_\rho(\zeta_n,\zeta_{n+1})>d_\rho(\zeta_n,\omega)\ for\ some\ n\in\mathbb{N}. \tag{4.18}$$

Similarly

$$\frac{1}{2\rho}d_\rho(\zeta_{n+1},\zeta_{n+2}) \geq \varphi(d_\rho(\zeta_{n+1},\zeta_{n+2})),$$

$$> \alpha(\zeta_{n+1},\omega)\ d_\rho(\zeta_{n+1},\omega), \tag{4.19}$$

$$\geq d_\rho(\zeta_{n+1},\omega).$$

that is

$$\frac{1}{2\rho}d_\rho(\zeta_{n+1},\ \zeta_{n+2}) > d_\rho(\zeta_{n+1},\ \omega) \text{ for some } n \in \mathbb{N}. \tag{4.20}$$

Triangular inequality of b-metric space,

$$d_\rho(\zeta_n,\zeta_{n+1}) \leq\ \rho[d_\rho(\zeta_n,\omega) + d_\rho(\omega,\zeta_{n+1})],$$

Using inequalities (4.18) and (4.20) in above inequality,

$$d_\rho(\zeta_n,\zeta_{n+1}) \leq \rho[\frac{1}{2\rho}d_\rho(\zeta_n,\zeta_{n+1}) + \frac{1}{2\rho}d_\rho(\zeta_{n+1},\zeta_{n+2})],$$

$$< \rho[\frac{1}{\rho}(\frac{1}{2}d_\rho(\zeta_n,\zeta_{n+1}) + \frac{1}{2}d_\rho(\zeta_n,\zeta_{n+1}))],$$

$$< \frac{1}{2}\ d_\rho(\zeta_n,\zeta_{n+1}) + \frac{1}{2}\ d_\rho(\zeta_n,\zeta_{n+1}),$$

$$d_\rho(\zeta_n,\zeta_{n+1})\ \leq\ d_\rho(\zeta_n,\zeta_{n+1}).$$

contradiction. Thus $\forall\ n \in \mathbb{N}$, either

$$\varphi(\zeta_n,\zeta_{n+1}) \leq\ \alpha(\zeta_n,\omega)d_\rho(\zeta_n,\omega)$$

or

$$\varphi(\zeta_{n+1},\zeta_{n+2}) \leq \alpha(\zeta_{n+1},\omega)d_\rho(\zeta_{n+1},\omega) \text{ holds.}$$

Hence from the concept of modified Suzuki-Edelstein α-proximal admissible mapping,

$$\begin{aligned} d_\rho(T\zeta_n, T\omega) \leq & \alpha(\zeta_n, \omega) d_\rho(T\zeta_n, T\omega), \\ \leq & \phi(d_\rho(\zeta_n, \omega)). \end{aligned} \tag{4.21}$$

or

$$\begin{aligned} d_\rho(T\zeta_{n+1}, T\omega) \leq & \ \alpha(\zeta_{n+1}, \omega) d_\rho(T\zeta_{n+1}, T\omega), \\ \leq & \ \phi(d_\rho(\zeta_{n+1}, \omega)). \end{aligned} \tag{4.22}$$

Now applying limit as $n \to \infty$ in inequalities (4.21) and (4.22),

$$\lim_{n\to\infty} d_\rho(T\zeta_n, T\omega) \ = \ 0$$

or

$$d_\rho(T\zeta_{n+1}, T\omega) \ = \ 0$$

implies

$$T\zeta_n \to T_\omega$$

or

$$T\zeta_{n+1} \to T_\omega$$

Consequently, there exists subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that $T\zeta_{n_k} \to T_\omega$ as $\zeta_{n_k} \to \omega$.

$$\lim_{n\to\infty} d_\rho(\zeta_{n_{k+1}}, T\zeta_{n_k}) \ = \ d_\rho(\omega, T\omega)$$

and

$$d_\rho(U, V) = \lim_{n\to\infty} d_\rho(\zeta_{n_{k+1}}, T\zeta_{n_k})$$

Consequently implies that

$$d_\rho(U, V) = d_\rho(\omega, T\omega). \tag{4.23}$$

This shows that T has best proximity point.

For the uniqueness, assume on contrary $\mu, \omega \in U_0$ are best proximity points, where $\zeta \neq \mu$

$$d_\rho(\mu, T\mu) = d_\rho(\omega, T\omega) = d_\rho(U, V).$$

Now weak P-property,

$$d_\rho(\mu, \omega) \leq d_\rho(T\mu, T\omega) \tag{4.24}$$

Now,

$$d_\rho(\mu, T\mu) = 2\rho d_\rho(U, V) - \rho d_\rho(U, V) \tag{4.25}$$

Since

$$\varphi(d_\rho(\mu, T\mu)) \leq \frac{1}{2\rho} d_\rho(\mu, T\mu)$$

Using (4.11) in (4.12),

$$\begin{aligned}
\varphi(d_\rho(\mu, T\mu)) &\leq \frac{1}{2\rho} d_\rho(\mu, T\mu), \\
&\leq \frac{1}{2\rho}(2\rho d_\rho(U, V) - \rho d_\rho(U, V)), \\
&\leq \frac{1}{2\rho} \rho d_\rho(U, V), \\
&\leq \frac{1}{2} d_\rho(U, V), \\
&\leq 2 d_\rho(U, V).
\end{aligned}$$

So from the inequality,

$$\varphi(d_\rho(\mu, T\mu)) - 2d_\rho(U, V) \leq 0 \leq \rho\alpha(\mu, \omega) d_\rho(\mu, \omega) \tag{4.26}$$

Hence from the concept of modified Suzuki-Edelstein α-proximal contraction in b-metric space,

$$\rho\ \alpha(\mu, \omega) d_\rho(T\mu, T\omega) \leq \rho\phi(d_\rho(\mu, \omega))$$

Since $\alpha(\mu, \omega) \geq 2$,

$$\rho\ d_\rho(T\mu, T\omega) \leq \rho\phi(d_\rho(\mu, \omega))$$

$$d_\rho(T\mu, T\omega) \leq \phi(d_\rho(\mu, \omega)) \tag{4.27}$$

By using (4.26),

$$d_\rho(\mu, \omega) \leq \phi(d_\rho(\mu, \omega)) < d_\rho(\mu, \omega)$$

contradiction, so $\mu = \omega$.

T has a best proximity point which is unique. □

Corollary 5. Consider U and V are non-empty subsets of the complete b-metric space (X, d_ρ) where $\rho \geq 1$ with $U_0 \neq \phi$. A function $\delta : [0, 1) \to (0, \frac{1}{2}]$ and for all $\zeta, \mu \in U$, a mapping $T : U \to V$ be such that;

$$\delta(r)(\varphi(d_\rho(\zeta, T\zeta)) - 2\rho d_\rho(U, V)) \leq \rho d(\zeta, \mu) \tag{4.28}$$

$$\Rightarrow \delta(r) d_\rho(T\zeta, T\mu) \leq \xi d_\rho(\zeta, \mu) \tag{4.29}$$

where $\varphi \in \Phi_\varphi$, $\phi \in \Phi$ and $r \in [0, 1)$. Moreover, $T(U_0) \subseteq V_0$ and pair (U, V) satisfies weak P-property. So, T has best proximity point that is unique.

Theorem 4.2.3. Consider (X, d_ρ) be the b-complete metric space along with parameter $\rho \geq 1$. A mapping $T : X \to X$ be continuous α-admissible w.r.t. $n(\zeta, \mu) = 2$, where $\varphi \in \Phi_\varphi$ and also $\phi \in \Phi$ such that;

$$\varphi(d_\rho(\zeta, T\zeta)) \leq \rho\, \alpha(\zeta, \mu)\, d_\rho(\zeta, \mu)) \tag{4.30}$$

$$\Rightarrow \alpha(\zeta, \mu)\, d_\rho(T\zeta, T\mu) \leq \phi(d_\rho(\zeta, \mu)) \tag{4.31}$$

for all $\zeta, \mu \in X$. Moreover, there exists element $\zeta_0 \in X$ such that $\alpha(\zeta_0, T\zeta_0) \geq 2$. Then T has unique fixed point.

Proof. By selecting $U = V = X$ in results (4.2.1) and (4.2.2), we obtained the desired results. □

Chapter 5

Conclusion

The work of Hussain et al on "Best proximity point results for Suzuki-Edlstein α-proximal contraction" is investigated in this thesis and also the brief description of their work and achievement.

The aim of this research was to discus the results established by Hussain et al [39] in b-metric space. For this, the definition of modified Suzuki-Edlstein α-proximal contraction is formulated in the setting of b-metric space.

As application, the fixed point theorems for then established for modified Suzuki-Edelstein proximal contraction in the setting of b-metric space.
These results might be valuable in solving particular best proximity points in addition to fixed point theory in perception of b-metric spaces.

Bibliography

[1] H. Poincare, "Surless courbes define barles equations differentiate less," *J. de Math*, vol. 2, pp. 54–65, 1886.

[2] L. E. J. Brouwer, "Über abbildung von mannigfaltigkeiten," *Mathematische annalen*, vol. 71, no. 1, pp. 97–115, 1911.

[3] L. Brouwer, "Collected works vol. 1 north-holland," 1975.

[4] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fund. math*, vol. 3, no. 1, pp. 133–181, 1922.

[5] A. Arvanitakis, "A proof of the generalized banach contraction conjecture," *Proceedings of the American Mathematical Society*, vol. 131, no. 12, pp. 3647–3656, 2003.

[6] D. W. Boyd and J. S. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, no. 2, pp. 458–464, 1969.

[7] B. S. Choudhury and K. Das, "A new contraction principle in menger spaces," *Acta Mathematica Sinica, English Series*, vol. 24, no. 8, p. 1379, 2008.

[8] J. Merryfield, B. Rothschild, and J. Stein Jr, "An application of ramsey's theorem to the banach contraction principle," *Proceedings of the American Mathematical Society*, vol. 130, no. 4, pp. 927–933, 2002.

[9] T. Suzuki, "A generalized banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[10] R. Kannan, "Some results on fixed points," *Bull. Cal. Math. Soc.*, vol. 60, pp. 71–76, 1968.

[11] B. Samet, "Coupled fixed point theorems for a generalized meir–keeler contraction in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 12, pp. 4508–4517, 2010.

[12] B. S. Choudhury, "Unique fixed point theorem for weakly c-contractive mappings," *Kathmandu university journal of science, engineering and technology*, vol. 5, no. 1, pp. 6–13, 2009.

[13] P. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces," *Fixed Point Theory and Applications*, vol. 2008, no. 1, p. 406368, 2008.

[14] P. Z. Daffer and H. Kaneko, "Fixed points of generalized contractive multivalued mappings," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, 1995.

[15] I. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Func. An., Gos. Ped. Inst. Unianowsk*, vol. 30, pp. 26–37, 1989.

[16] M. Boriceanu, "Fixed point theory for multivalued generalized contraction on a set with two b-metrics." *Studia Universitatis Babes-Bolyai, Mathematica*, no. 3, 2009.

[17] M. Boriceanu, A. Petrusel, and I. Rus, "Fixed point theorems for some multivalued generalized contractions in b-metric spaces," *International Journal of Mathematics and Statistics*, vol. 6, no. S10, pp. 65–76, 2010.

[18] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[19] K. Fan, "Extensions of two fixed point theorems of fe browder," *Mathematische zeitschrift*, vol. 112, no. 3, pp. 234–240, 1969.

[20] S. Reich, "Approximate selections, best approximations, fixed points, and invariant sets," *Journal of Mathematical Analysis and Applications*, vol. 62, no. 1, pp. 104–113, 1978.

[21] V. Sehgal and S. Singh, "A generalization to multifunctions of fan's best approximation theorem," *Proceedings of the American Mathematical Society*, vol. 102, no. 3, pp. 534–537, 1988.

[22] J. B. Prolla, "Fixed-point theorems for set-valued mappings and existence of best approximants," *Numerical Functional Analysis and Optimization*, vol. 5, no. 4, pp. 449–455, 1983.

[23] A. Abkar and M. Gabeleh, "The existence of best proximity points for multivalued non-self-mappings," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 107, no. 2, pp. 319–325, 2013.

[24] M. U. Ali, T. Kamran, and N. Shahzad, "Best proximity point for–proximal contractive multimaps," in *Abstract and Applied Analysis*, vol. 2014. Hindawi, 2014.

[25] A. Amini-Harandi, "Best proximity points for proximal generalized contractions in metric spaces," *Optimization letters*, vol. 7, no. 5, pp. 913–921, 2013.

[26] S. S. Basha, "Best proximity point theorems on partially ordered sets," *Optimization Letters*, vol. 7, no. 5, pp. 1035–1043, 2013.

[27] M. Gabeleh, "Best proximity points: global minimization of multivalued nonself mappings," *Optimization Letters*, vol. 8, no. 3, pp. 1101–1112, 2014.

[28] N. Hussain, A. Latif, and P. Salimi, "Best proximity point results for modified suzuki α-ψ-proximal contractions," *Fixed Point Theory and Applications*, vol. 2014, no. 1, p. 10, 2014.

[29] E. Karapınar, "Best proximity points of cyclic mappings," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1761–1766, 2012.

[30] A. Latif, "Banach contraction principle and its generalizations," in *Topics in fixed point theory*. Springer, 2014, pp. 33–64.

[31] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[32] M. A. Al-Thagafi and N. Shahzad, "Best proximity sets and equilibrium pairs for a finite family of multimaps," *Fixed Point Theory and Applications*, vol. 2008, no. 1, p. 457069, 2008.

[33] W. K. Kim, S. Kum, and K. H. Lee, "On general best proximity pairs and equilibrium pairs in free abstract economies," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 68, no. 8, pp. 2216–2227, 2008.

[34] C. Mongkolkeha and P. Kumam, "Best proximity point theorems for generalized cyclic contractions in ordered metric spaces," *Journal of Optimization Theory and Applications*, vol. 155, no. 1, pp. 215–226, 2012.

[35] K. Włodarczyk, R. Plebaniak, and A. Banach, "Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3332–3341, 2009.

[36] S. Sadiq Basha, "Full length article: Best proximity point theorems," *Journal of Approximation Theory*, vol. 163, no. 11, pp. 1772–1781, 2011.

[37] M. Jleli and B. Samet, "Best proximity points for α–ψ-proximal contractive type mappings and applications," *Bulletin des Sciences Mathematiques*, vol. 137, no. 8, pp. 977–995, 2013.

[38] N. Hussain, M. Kutbi, and P. Salimi, "Best proximity point results for modified–proximal rational contractions," in *Abstract and Applied Analysis*, vol. 2013. Hindawi, 2013.

[39] A. Hussain, M. Q. Iqbal, and N. Hussain, "Best proximity point results for suzuki-edelstein proximal contractions via auxiliary functions," *Filomat*, vol. 33, no. 2, 2019.

[40] M. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, vol. 22, no. 1, pp. 1–72, 1906.

[41] B. K. Singh and P. K. Pathak, "Some fixed point theorems in g–metric spaces," *Journal of Computer and Mathematical Sciences*, vol. 10, no. 1, pp. 151–158, 2019.

[42] E. Kreyszig, *Introductory functional analysis with applications.* wiley New York, 1978, vol. 1.

[43] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.

[44] A. Latif, V. Parvaneh, P. Salimi, and A. Al-Mazrooei, "Various suzuki type theorems in b-metric spaces," *J. Nonlinear Sci. Appl*, vol. 8, no. 4, pp. 363–377, 2015.

[45] R. LEVY, "Fixed point theory and structural optimization," *Engineering optimization*, vol. 17, no. 4, pp. 251–261, 1991.

[46] C. Mongkolkeha, C. Kongban, and P. Kumam, "Existence and uniqueness of best proximity points for generalized almost contractions," in *Abstract and Applied Analysis*, vol. 2014. Hindawi, 2014.

[47] M. Edelstein, "On fixed and periodic points under contractive mappings," *Journal of the London Mathematical Society*, vol. 1, no. 1, pp. 74–79, 1962.

[48] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α–ψ-contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[49] M. Al-Thagafi and N. Shahzad, "Convergence and existence results for best proximity points," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 10, pp. 3665–3671, 2009.

[50] A. Eldered and P. Veeramani, "Convergence and existence for best proximity points," *J. Math. Anal. Appl*, vol. 323, no. 2, 2006.

[51] A. Fernández-León, "Best proximity points for proximal contractions," *arXiv preprint arXiv:1207.4349*, 2012.

[52] P. Salimi and E. Karapınar, "Suzuki-edelstein type contractions via auxiliary functions," *Mathematical Problems in Engineering*, vol. 2013, 2013.