CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Prešić Form of Nonself Operators and Best Proximity Point Results in b-Metric Spaces

by

Khola Shuaib

A thesis submitted in partial fulfillment for the degree of Master of Philosophy

in the

Faculty of Computing
Department of Mathematics

2021

I dedicate my dissertation work to my beloved ***family***

CERTIFICATE OF APPROVAL

Prešić Form of Nonself Operators and Best Proximity Point Results in b-Metric Spaces

by

Khola Shuaib

(MMT183022)

THESIS EXAMINING COMMITTEE

S. No.	Examiner	Name	Organization
(a)	External Examiner	Dr. Muhammad Usman Ali	COMSATS, Attock
(b)	Internal Examiner	Dr. Abdul Rehman Kashif	CUST Islamabad
(c)	Supervisor	Dr. Samina Batul	CUST Islamabad

Supervisor Name
Dr. Samina Batul
April, 2021

Dr. Muhammad Sagheer
Head
Dept. of Mathematics
April, 2021

Dr. M. Abdul Qadir
Dean
Faculty of Computing
April, 2021

Author's Declaration

I, **Khola Shuaib** hereby state that my M.Phil thesis titled "**Prešić Form of Nonself Operators and Best Proximity Point Results in b-Metric Spaces**" is my own work and has not been submitted previously by me for taking any degree from Capital University of Science and Technology, Islamabad or anywhere else in the country/abroad.

At any time if my statement is found to be incorrect even after my graduation, the University has the right to withdraw my M.Phil Degree.

(Khola Shuaib)

Registration No: MMT183022

Plagiarism Undertaking

I solemnly declare that research work presented in this thesis titled "**Prešić Form of Nonself Operators and Best Proximity Point Results in b-Metric Spaces**" is solely my research work with no significant contribution from any other person. Small contribution/help wherever taken has been duly acknowledged and that complete thesis has been written by me.

I understand the zero tolerance policy of the HEC and Capital University of Science and Technology towards plagiarism. Therefore, I as an author of the above titled thesis declare that no portion of my thesis has been plagiarized and any material used as reference is properly referred/cited.

I undertake that if I am found guilty of any formal plagiarism in the above titled thesis even after award of M.Phil Degree, the University reserves the right to withdraw/revoke my M.phil degree and that HEC and the University have the right to publish my name on the HEC/University website on which names of students are placed who submitted plagiarized work.

(Khola Shuaib)

Registration No: MMT183022

Acknowledgement

My first and foremost gratitude and thanks goes to **Almighty ALLAH** for countless things He gave without asking.
I also express my sincere gratitude to Holy Prophet **Hazrat Muhammad** (Peace Be Upon Him) for guiding us on right path by His teaching of patience, motivations and immense knowledge.
I owe my profound gratitude to my honourable, kind nature supervisor, and my mentor **Dr. Samina Batul** for her superb guidance and inexhaustible inspiration throughout this work and which has contributed to the success of this research. In addition, I am thankful to the Mathematics department Capital University of Science and Technology **(CUST)** Islamabad specially the Head of the Department, **Dr. Muhammad Sagheer** for providing us a learning and creative environment. I should be thankful to my respected teachers **Dr. Rashid Ali**, **Dr. Dur-e-Shehwar Sagheer**, and **Dr. Abdul Rahman Kashif** for the encouragement and suggestions. May Allah keep blessing you all.
I would like to thank my parents for all their love, encouragement and support. My greatest thanks, however, must go to my husband **Kashif Zubair** for his help, support, motivation and all contribution of his time, ideas and suggestion to make my dissertation productive and stimulating. My brothers **Awais kiani** and **Sohaib Kiani** have also a great part of my work. I dedicate my work to my beloved son **Muhammad Abdul Rahman**. It would not be justice if I don't mention the support of my mother in law and my loving sisters **Fatima**, **Jawariya** and **Aisha**, because this thesis might not exist at all without their support. I am thankful to all of them for support and encouragement.

(Khola Shuaib)

Abstract

The concept of best proximity point in metric spaces for Prešić type nonself operators is demonstrated by many researchers. In the present dissertation, we discussed the notion of Prešić type nonself operators and acquire some best proximity points results for such operators in the setting of $b-$metric spaces. The Prešić type nonself operators played an important role in the extension and generalization of Banach contraction principle. Our result will be valuable in solving particular best proximity points and fixed point results in the setting of $b-$metric spaces.

Contents

Author's Declaration iv

Plagiarism Undertaking v

Acknowledgement vi

Abstract vii

List of Figures x

Abbreviations xi

Symbols xii

1 Introduction 1
- 1.1 Background 1

2 Preliminaries 4
- 2.1 Graph Theory 4
- 2.2 Metric Spaces and b-Metric Spaces 7
- 2.3 BCP and Fixed Point Theory 11

3 Prešić Form of Nonself Operators and Related Best Proximity Results 14
- 3.1 BBP in Metric Spaces 14
- 3.2 Some Basic Definitions 15
- 3.3 BBP Theorems in Metric Spaces 16

4 Prešić Form of Nonself Operators and Best Proximity Point Results in b-Metric Spaces 44
- 4.1 BBP in b-Metric Spaces 44
- 4.2 BBP Results in b-Metric Spaces 45

5 Conclusion 62

Bibliography 63

List of Figures

2.1 A graph having a loop and multiple edges 5
2.2 Path diagram 6
2.3 A directed graph 6
2.4 Three fixed points 11
2.5 No fixed point 12

Abbreviations

BCP	Banach Contraction Principle
BPP	Best Proximity Point

Symbols

$(\mathbb{X}, \mathtt{d})$	Metric space
$(\mathbb{X}, \mathtt{d}_b)$	$b-$ Metric space
$\mathtt{d}$	Distance function
$\mathbb{R}$	The set of real numbers
$\mathbb{N}$	The set of natural numbers
$\Rightarrow$	Implies that
$\in$	Belongs to
$\forall$	For all
$\sum$	Sigma
$\rightarrow$	Approaches to
∞	Infinity
max	Maximum
inf	Infimum
lim	Limit
ϕ	Non empty set

Chapter 1

Introduction

1.1 Background

Mathematics has numerous uses in every field of life and it is one of the main branches of scientific knowledge. In scientific knowledge, mathematics has an important role, so it is called the mother of sciences. It is further classified into several divisions and functional analysis is considered as one of the foremost important branches of mathematics. Fixed point theory is a significant notion in functional analysis. Fixed point theory provides suitable conditions for the existence of a problem solution. In various fields of science, the idea of fixed point theory has many applications, such as optimization theory, mathematical economics and approximation theory etc. From last five to six decades, fixed point theory has become the fascinating and greatest growing research area for mathematicians.

In 1886, Poincare [1] was the first mathematician who study the field of fixed point theory. Afterward Brouwer [2] considered the fixed point problem and established fixed point theorems for the solution of the equation $\mathbb{T}(s) = s$. In different dimensions, he also proved several fixed point results.

Metric space is pivotal point of fixed-point theory. The definition of metric space was initiated for the first time by a French mathematician Frechet [3]. He is regarded as a founder of modern topology and he contributed significantly in the field of set theory and functional theory. He introduced the general concept of entire metric space soley.

Stefan Banach [4] proved a significant result known as Banach Contraction Principle (BCP) in 1922. The analysis of BCP is considered to be the most fundamental consequence in the field of fixed point theory. The two main points come from this principle. The first is that it guarantees the presence and uniqueness of fixed point. The second and most important one is that the fixed point of mapping can be determined by an approach.

Substantially, BCP was also invesstigated by Kannan [5] and Chatterjea [6] by replacing contraction conditions. In literature of fixed point theory, a lot of researchers [7–10] used various methods for extension and generalization of BCP, either it is using the different spaces or replacing the contraction conditions. Bakhtin [11] initiated the analysis of one of the most interesting generalization of metric spaces known as b-metric spaces and he extended the BCP [4] in setting of b-metric spaces. Many researchers investigated fixed point theory in distinguishing mappings like mixed single as well as set-valued in b-metric spaces [12–15].

Prešić [16] gave a contractive condition on the finite product of metric spaces and proved a fixed point theorem. For the operators defined on product spaces, the analysis of Prešić is considered foremost important extensions of BCP. A lot of other researchers worked on the other forms of Prešić results like Berinde et al. [17], Khan et al. [18], Păcurar [19], and Shukla et al. [20, 21].

In addition to differential equations, several problems that arise in various domain of mathematics, such as optimization theory, can also be formulated as a fixed-point equation of the form $Ts = s$. If T is self mapping and the other conditions are fulfilled, then the above equation has a solution. While, if T is non-self-mapping, then there is no solution to the equation given above. Therefore the value of the element s must be determined which is closest to Ts in this situation. So, Fan [22] suggested the idea of best proximity point(BPP) result for non-self continuous mappings in 1969. Numerous former researchers works on extensions of Fan's theorem like Reich [23], Sehgak and Singh [24] and Prolla [25].

In the literature, many researchers have examined the presence of best proximity points by using various approaches. In 2010, Basha [26] presented the concept and extended BCP for the existance of BBP and stated some results for proximal contraction. By using the generalized $\alpha - \psi$-proximal contractions in complete metric spaces, Jleli and Samet [27] discussed the nature of BPP. Markin and Shahzad[28] took relatively u-continuous mappings to acquire the best proximity points. Nawab et al. [29] established the BCP for modified Suzuki α-proximal contractions in the setting of complete metric spaces. For further details of BBP we can see for example [30–34].

In this thesis, the main focus on discussion is on "Presic type nonself operators and related best proximity results" by Usman et al. [35]. After the comprehensive analysis of the paper, results have been extended in setting of b-matric spaces.

The rest of dissertation is organized as follows:

- **Chapter 2**

This chapter includes the basic concepts, definitions and examples regarding graph theory, metric spaces, $b-$metric spaces and fixed point theory.

- **Chapter 3**

This chapter is about the literature review and study of BBP results for Presic type nonself operators enclosed by metric spaces comprehensively.

- **Chapter 4**

This chapter emphasizes on the idea of Presic type nonself operators in $b-$metric spaces.

- **Chapter 5**

In this chapter, the conclusion is presented.

Chapter 2

Preliminaries

In this chapter few basic definitions, results and examples are presented which are used in subsequent chapters. The first section covers some basics of graph theory with examples. The second section concerns with the metric spaces and b-metric spaces with related examples. The last section concerns with the BCP and fixed points in metric spaces.

2.1 Graph Theory

Graph theory deals with investigation as well as analysis of graphs. Whereas, graph is the class of mathematical structures that involves physical representation of problems related to daily life which contains piecewise relations that is developed between points. Graphs are designed by vertices which is also known as nodes or point. Furthermore, they are fixed by edges or links or lines. Indeed, prior text related to concept of graph theory appeared in 1936.

Definition 2.1.1.

Graph is the combination of vertices and edges where vertex means points shown on the graph and the lines that joins those points(vertices) are called edges of graph.

Definition 2.1.2.

"An edge with identical ends is called a **loop**." [36]

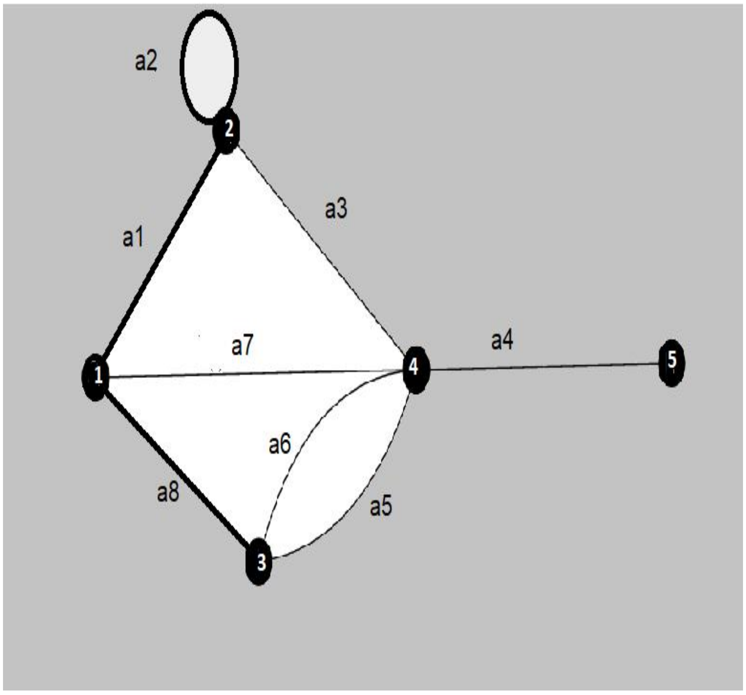

FIGURE 2.1: A graph having a loop and multiple edges

In the above graph, vertices = $\{1, 2, 3, 4, 5\}$ and edges = $\{a1, a2, a3, a4, a5, a6, a7, a8\}$.

Definition 2.1.3.

Parallel edges are those edges in which the edges of two point having the same end.

Definition 2.1.4.

"A **Simple graph** is a graph that has no self-loops and edges." [37]

Definition 2.1.5.

A walk in a graph $G = (V, E)$ is a sequence of vertices

$$v_0, v_1, v_2, ..., v_k \in V$$

such that for every $i = 1, ..., k$, $(v_{i-1}, v_i) \in E$. In this case, we say that the walk is from v_0 to v_k. Furthermore, if all the vertices are distinct, then the walk is called a path.

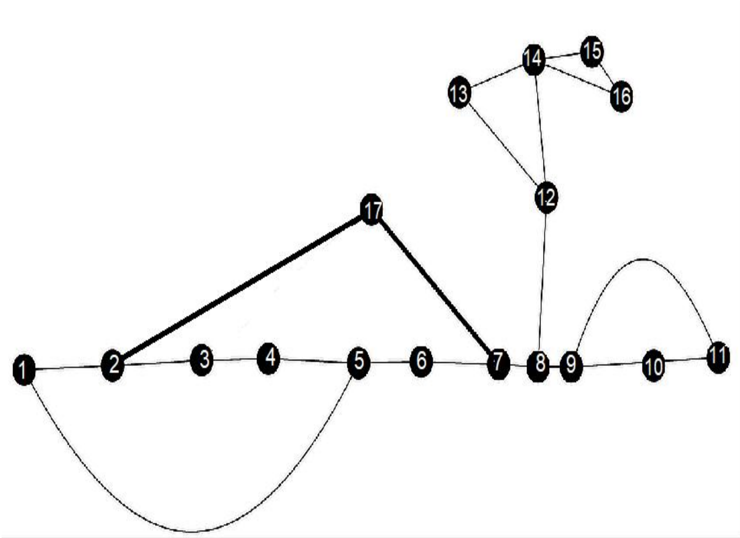

FIGURE 2.2: Path diagram

In figure 2.2, $v_0 = 1; v_i = 11$ and $\{v_i\} : i = 0, 1, 2, ..., 10$ is a path.

Definition 2.1.6.

A graph in which the path is defined from each edge from one vertex to the other is called **directed** graph.

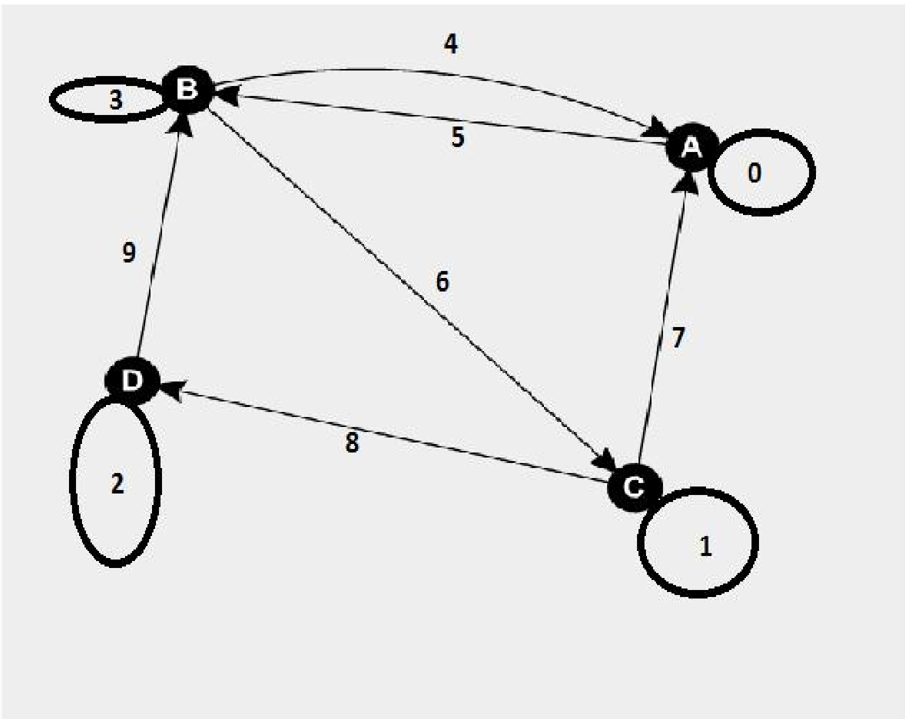

FIGURE 2.3: A directed graph

2.2 Metric Spaces and b-Metric Spaces

Frachet [3] developed the idea of metric spaces in 1906.

Definition 2.2.1.

"A metric space is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

$(M1)$ d is real-valued, finite and non negative;

$(M2)$ $d(x, y) = 0$ if and only if $x = y$;

$(M3)$ $d(x, y) = d(y, x)$ **(Symmetry)**;

$(M4)$ $d(x, y) \leq d(x, z) + d(z, y)$ **(Triangle inequality)**." [38]

Example 2.2.1. Consider $\mathbb{X} = \mathbb{R}$, define a metric $\mathtt{d} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ as

$$\mathtt{d}(s, c) = \sqrt{|s - c|} \qquad \text{for all} \quad s, c \in \mathbb{X},$$

then (R, d) is a metric space.

Example 2.2.2. Consider $\mathbb{X} = \mathbb{R}^2$, define $\mathtt{d} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\mathtt{d}(s, c) = \max\{|s_1 - c_1|, |s_2 - c_2|\}.$$

Then $(\mathbb{R}^2, \mathtt{d})$ is a metric space.

Definition 2.2.2.

"Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be metric spaces. A mapping $T : X \to Y$ is said to be **continuous** at a point $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\tilde{d}(Tx, Tx_0) < \epsilon \quad \text{for all } x \text{ satisfying} \quad d(x, x_0) < \delta.$$

T is said to be continuous if it is continuous at every point of X." [38]

Example 2.2.3. Let us define a self mapping $\mathbb{T}$ on set of real numbers $\mathbb{R}$ endowed with usual metric, such that

$$\mathbb{T}(s) \;=\; s^7 \quad \text{where} \quad s \;\in\; \mathbb{X}.$$

Then $\mathbb{T}$ a is continuous mapping.

Definition 2.2.3.

"A sequence (x_n) in a metric space $X = (X, d)$ is said to **converge** or to be **convergent** if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the limit of (x_n) and we write

$$\lim x_n = x,$$

or, simply,

$$x_n \to x.$$

We say that (x_n) converges to x or has the limit x. If (x_n) is not convergent, it is said to be divergent." [38]

Example 2.2.4.

The sequence $s_n = \frac{1}{n}$ where $n \in \mathbb{R}$ is a convergent sequence in $(\mathbb{R}, \mathtt{d})$, where $\mathtt{d}$ is usual metric.

Definition 2.2.4.

"A metric space X is called **compact** if every sequence in X has a convergent subsequence." [38]

Definition 2.2.5.

"A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be **Cauchy sequence** (or fundamental) if for each $\epsilon > 0$ there exist a positive number $\mathbb{N} = \mathbb{N}(\epsilon)$ such that

$$d(x_n, x_m) < \epsilon \quad \text{for all} \quad m, n > \mathbb{N}.\text{" [38]}$$

Definition 2.2.6.

"If every Cauchy sequence in a metric space (X, d) converges to a point $x \in X$ then X is called **complete metric space**." [38]

Example 2.2.5.

Closed interval $[2, 4]$ in $\mathbb{R}$ is complete with usual metric on $\mathbb{R}$.

Example 2.2.6.

The real line $\mathbb{R}$ and the complex plane $\mathbb{C}$ are complete with usual metric on $\mathbb{R}$ and $\mathbb{C}$ respectively.

Concept of b-metric space is prior by Bakhtin [11].

Definition 2.2.7.

"Consider a non-empty set X with a real number $s \geq 1$. A function
$d : X \times X \to [0, \infty)$ is called ***b*-metric** if it satisfies the following properties for each $x, y, z \in X$,

$(b1)$ $d(x, y) = 0 \Leftrightarrow x = y$;

$(b2)$ $d(x, y) = d(y, x)$;

$(b3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$;

the pair (X, d) is called a b-metric space." [39]

Remark 2.1.

When $s = 1$ then the concept of b-metric space coincides with concept of metric space.

Example 2.2.7.

Consider $\mathbb{X} = \mathbb{R}$ be a function defined by

$$\mathtt{d}(s, c) = (s - c)^2 \quad \forall \quad s, c \in \mathbb{X}.$$

Then a pair $(\mathbb{R}, \mathtt{d})$ is b-metric space with $b = 2$.

Remark 2.2.

Any metric space is a b-metric space. The converse, however, is not necessarily valid.

Definition 2.2.8.

"Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is called **convergent** if and only if there exist $x \in X$ such that $d_b(x_n, x) \to 0$ as $n \to 0$." [40]

Definition 2.2.9.

"Let (X, d) be a b-metric space. A sequence $\{x_n\}$ in X is called **Cauchy** if and only if $d_b(x_n, x_m) \to 0$ as $m, n \to \infty$." [40]

Definition 2.2.10.

"The b-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent." [40]

Remark 2.3.

A b-metric space is not a continues function.

Example 2.2.8. Consider $\mathbb{X} = \mathbb{N} \cup \{\infty\}$. A function $\mathtt{d}_b : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ by:

$$\mathtt{d}_b(s_1, s_2) = \begin{cases} 0 & \text{if} \quad s_1 = s_2, \\ \left|\frac{1}{s_1} - \frac{1}{s_2}\right| & \text{if one of } s_1, s_2 \text{ is even and other is even or } \infty, \\ 5 & \text{if one of } s_1, s_2 \text{ is odd and other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

It can be verified that $\forall\ s_1, s_2, s_3 \in \mathbb{X}$, we have

$$\mathtt{d}_b(s_1, s_3) \leq \frac{5}{2}\left(\mathtt{d}_b(s_1, s_2)\ +\ \mathtt{d}_b(s_2, s_3)\right).$$

Thus $(\mathbb{X}, \mathtt{d}_b)$ is a b-metric space with $b = \frac{5}{2}$.

Consider a sequence $s_m = 2m$ for each $m \in \mathbb{N}$, then

$$\mathtt{d}_b(2m, \infty) = \frac{1}{2m} \to 0 \quad \text{as} \quad m \to \infty,$$

further

$$\lim_{m\to\infty} \mathtt{d}_b(2m, \infty) = 0\ ,$$

but

$$\lim_{m\to\infty} \mathtt{d}_b(s_m, 1) = 2 \nrightarrow 5$$

$$= d_b(\infty, 1).$$

$\Rightarrow$ It is not a continues.

2.3 BCP and Fixed Point Theory

A wide diversity of problems appearing in various areas of pure and applied mathematics like differential equations, discrete and continuous system of dynamics can be modeled as fixed point equation.

Definition 2.3.1.
"Let $T : X \to X$ be a mapping on a set X. A point $x \in X$ is said to be a **fixed point** of T if

$$Tx = x$$

that is, a point is mapped onto itself.
Geometrically, if $y = \mathbb{T}x$ is real valued function, then by a fixed point of $\mathbb{T}$ means where the line $y = x$ intersect the graph of $\mathbb{T}$. A function may therefore have a fixed point or not have one. In addition, the fixed point might or may not be unique."[41]

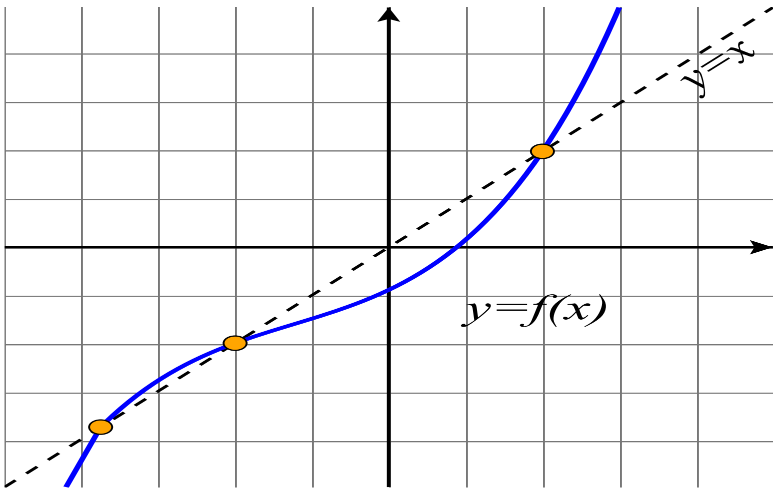

FIGURE 2.4: Three fixed points

The graph mention above represents a function having three fixed points.

Example 2.3.1. Let $\mathbb{X} = \mathbb{R}$ be endowed with metric $d(s, c) =\mid s - c \mid$ and $\mathbb{T} : \mathbb{X} \to \mathbb{X}$, define by

$$\mathbb{T}s = (s + 2) \quad \text{for each} \quad s \in \mathbb{X}$$

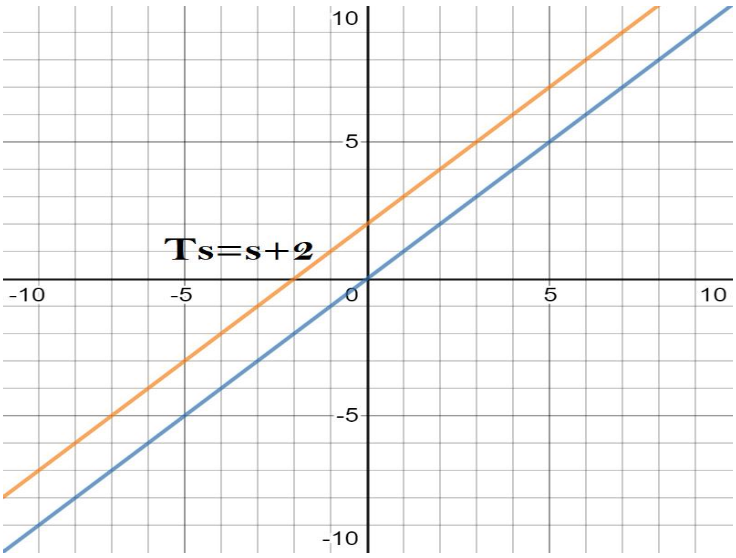

FIGURE 2.5: No fixed point

Then, there is no fixed point for $\mathbb{T}$.

Definition 2.3.2.

"Let $X = (X, d)$ be a metric space. A mapping $T : X \to X$ is called a **contraction** on X if there is a positive real number $\alpha < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq \alpha d(x, y) \quad (\alpha < 1).$$

Geometrically this means that any points x and y have images that are closer together than those points x and y more precisely, the ratio $d(Tx, Ty)/d(x, y)$ does not exceeds a constant α which is strictly less than 1." [38]

Example 2.3.2. Consider $\mathbb{X} = [0, 3]$ with the usual metric space $\mathtt{d}(s, c) = |s - c|$. Then $\mathbb{T} : \mathbb{X} \to \mathbb{X}$, define by

$$\mathbb{T}(s) = \frac{1}{4+s}$$

is a contraction mapping. Since

$$\begin{aligned} \mathtt{d}(\mathbb{T}s, \mathbb{T}c) \leq\ & \mathtt{d}(\frac{1}{4+s}, \frac{1}{4+c}), \\ \leq\ & |\frac{1}{4+s} - \frac{1}{4+c}|, \\ \leq\ & |\frac{4+c-4-s}{(4+s)(4+c)}|, \end{aligned}$$

$$
\begin{aligned}
&\leq \; \Big| \frac{-(s-c)}{(4+s)(4+c)} \Big|, \\
&\leq \frac{| s-c |}{(4)(4)}, \\
&\leq \frac{1}{16}\mathtt{d}(s,c).
\end{aligned}
$$

Here $\alpha = \frac{1}{16}$.

In 1922, Banach [38] established the following fixed point result, popularly named as Banach contraction theorem.

Theorem 2.3.1.
"Consider a metric space $X = (X, d)$, where $X \neq \phi$. Suppose that X is complete and let $T : X \to X$ be a contraction on X. Then T has precisely one fixed point." [38]

Example 2.3.3. Consider $\mathbb{X} = \mathbb{R}$ endowed with usual metric $\mathtt{d}(\mathbb{T}(s_1), \mathbb{T}(s_2)) =| s_1 - s_2 |$. Define a mappig $\mathbb{T} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ by

$$
\mathbb{T}s \;=\; 1 + \frac{s}{3},
$$

here $s = \frac{3}{2}$ is a unique fixed point.

Chapter 3

Prešić Form of Nonself Operators and Related Best Proximity Results

In this chapter, we will discuss few best proximity results in metric spaces. Basha [26] explained certain BBP results for proximal contractions in 2011. Ali et al. [35] acquired the best proximity theorms for Prešić type non self operators in metric spaces.

3.1 BBP in Metric Spaces

Consider a mapping $\mathbb{T} : \mathbb{X} \to \mathbb{Y}$, define by

$$\mathbb{T}(s) = s, \tag{3.1}$$

here $\mathbb{T}$ is non-self mapping, therefore (3.1) does not definitely have a fixed point. In this situation, it is worthy to find the estimated solution s such that the error $\mathtt{d}(s, Ts)$ is minimal. This is the concept behind the best approximation thoery. The existence of BBP in Banach space was introduced by Elderd et al. [42] in 2005.

3.2 Some Basic Definitions

This section presents few major concepts and notions that will be used during the following chapter. We commence by using important notations. Throughout $(\mathbb{X}, \mathtt{d})$ is a metric space and a directed graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is defined on a $\mathbb{X}$.

Definition 3.2.1. Consider $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$, then define

$\mathtt{d}(\mathbb{H}, \mathbb{K}) = \inf\{\mathtt{d}(h, k) : h \in \mathbb{H}, k \in \mathbb{K}\}$,

$\mathtt{d}(x_0, \mathbb{K}) = \inf \{\mathtt{d}(x_0, k) : k \in \mathbb{K}\}$,

$\mathbb{H}_0 = \{h \in \mathbb{H} : \mathtt{d}(h, k) = \mathtt{d}(\mathbb{H}, \mathbb{K}) \text{ for some } k \in \mathbb{K}\}$,

$\mathbb{K}_0 = \{k \in \mathbb{K} : \mathtt{d}(h, k) = \mathtt{d}(\mathbb{H}, \mathbb{K}) \text{ for some } h \in \mathbb{H}\}$.

Definition 3.2.2.

"Consider a metric space (X, d). Suppose A and B be two non-empty subsets of X. An element $x \in A$ is said to be a BPP of the mapping $T : A \to B$ if

$$d(x, Tx) = d(A, B)." \text{ [43]}$$

Remark 3.1. From the above definition, it is obvious that the BPP deduces to fixed point with self mapping.

Basha and Shahzad [44] have presented the following definition.

Definition 3.2.3.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$ where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$. If each sequence $\{k_n\}$ in $\mathbb{K}$ with $\mathtt{d}(h, k_n) \to \mathtt{d}(h, \mathbb{K})$, for some $h \in \mathbb{H}$, has a convergent subsequence. Then, $\mathbb{K}$ is called approximately compact with respect to $\mathbb{H}$.

Definition 3.2.4.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$. Then $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ is called to be path admissible, when:

$$\begin{cases} \mathtt{d}(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}), \\ \mathtt{d}(w_2, \mathbb{T}(h_2, h_3)) = \mathtt{d}(\mathbb{H}, \mathbb{K}), \\ h_1 P h_3, \end{cases} \qquad \Rightarrow (w_1, w_2) \in \mathbb{E},$$

where $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$.

Here $h_1 P h_3, \Rightarrow \quad h_1, h_2, h_3 \in \mathbb{V}$ and $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$.

3.3 BBP Theorems in Metric Spaces

Definition 3.3.1.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$. An element $h_* \in \mathbb{H}$ is said to be BPP of $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ if

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}), \tag{3.2}$$

where

$$\mathtt{d}(\mathbb{H}, \mathbb{K}) \;=\; \inf\{\mathtt{d}(h, k) : h \in \mathbb{H}, k \in \mathbb{K}\}.$$

Theorem 3.3.1.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and $\mathtt{d}(w_1, \mathbb{T}(h_1, h_2)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}(w_1, w_2) \leq \Gamma \max\{\mathtt{d}(h_1, h_2), \mathtt{d}(h_2, h_3)\}, \tag{3.3}$$

where $\Gamma \in [0, 1)$.

Further, suppose all the following asumptions are true:

(i) $\mathbb{T}$ is path admissible;

(ii) $\exists\, h_0, h_1, h_2 \in \mathbb{H}$ which satisfy $\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{K}$ is approximately compact with respect to $\mathbb{H}$;

(vi) When $\{h_j\} \subseteq \mathbb{X}$ such that $h_j P h_{j+2}$ for all $j \in \mathbb{N}$ and $h_j \to a$ as $j \to \infty$, then $(h_j, a) \in \mathbb{E}\ \forall\ j \in \mathbb{N}$ and $(a, a) \in \mathbb{E}$,

then, $\exists$ a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From assumption (ii), $h_0, h_1, h_2 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

and $h_0 P h_2$, that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From assumption (iv), we have $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

Since from assumption (i) , we get $(h_2, h_3) \in \mathbb{E}$. Thus, $h_1 P h_3$.
Continuing this procedure, we build $\{h_{j\geq 2}\}$ in $\mathbb{H}$ which satisfy:

$$\mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}) \ \ \forall \ \ j \in \mathbb{N}, \tag{3.4}$$

and $h_{j-1} P h_{j+1}$, that is, $(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E}$ for all $j \in \mathbb{N}$.

From (3.3), we have:

$$\mathtt{d}(h_j, h_{j+1}) \;\leq\; \Gamma\ \max\{\mathtt{d}(h_{j-2}, h_{j-1}), \mathtt{d}(h_{j-1}, h_j)\} \text{ for } j = 2, 3, 4, \cdots . \tag{3.5}$$

Consider $\mathtt{d}_j = \mathtt{d}(h_j, h_{j+1})$ for all $j \in \mathbb{N} \cup \{0\}$.

Then we can rewrite (3.5) as

$$\mathtt{d}_j \ \leq \Gamma\ \max\{\mathtt{d}_{j-2}, \mathtt{d}_{j-1}\} \ \ \forall \ \ j = 2, 3, 4, \cdots . \tag{3.6}$$

It is obviously true for $j = 0, 1$ because if we consider

$$Z = \max\{\mathtt{d}_0/\psi, \mathtt{d}_1/\psi^2\}, \quad \text{where} \quad \psi = \Gamma^{1/2}.$$

Since Z is $\max\{\mathtt{d}_0/\psi, \mathtt{d}_1/\psi^2\}$, then

$$\mathtt{d}_0 \leq Z\psi \quad \text{and} \quad \mathtt{d}_1 \leq Z\psi^2.$$

Take $j = 2, 3, 4, \cdots$ in (3.6),

$$\begin{aligned}
\mathtt{d}_2 &\leq \Gamma \max\,\{\mathtt{d}_0, \mathtt{d}_1\} \leq \Gamma \max\{Z\psi, Z\psi^2\} \leq \Gamma Z\psi = Z\psi^3,\\
\mathtt{d}_3 &\leq \Gamma \max\,\{\mathtt{d}_1, \mathtt{d}_2\} \leq \Gamma \max\{Z\psi^2, Z\psi^3\} \leq \Gamma Z\psi^2 = Z\psi^4,\\
\mathtt{d}_4 &\leq \Gamma \max\,\{\mathtt{d}_2, \mathtt{d}_3\} \leq \Gamma \max\{Z\psi^3, Z\psi^4\} \leq \Gamma Z\psi^3 = Z\psi^5,
\end{aligned}$$

continuing the same process, we obtain

$$\mathtt{d}_p \leq \Gamma \max\{\mathtt{d}_{p-1}, \mathtt{d}_{p-2}\} \leq \Gamma \max\{Z\psi^p, Z\psi^{p-1}\} \leq \Gamma Z\psi^{p-1}, = Z\psi^{p+1}.$$

Thus, by using induction, we get:

$$\mathtt{d}_{p-1} \leq Z\psi^p \quad \text{for each} \quad p \in \mathbb{N}. \tag{3.7}$$

By triangle inequality,

$$\begin{aligned}
\mathtt{d}(h_p, h_{p+q}) &\leq \mathtt{d}(h_p, h_{p+1}) + \mathtt{d}(h_{p+1}, h_{p+q}),\\
&\leq \mathtt{d}(h_p, h_{p+1}) + \mathtt{d}(h_{p+1}, h_{p+2}) + \mathtt{d}(h_{p+2}, h_{p+q}),\\
&\leq \mathtt{d}(h_p, h_{p+1}) + \mathtt{d}(h_{p+1}, h_{p+2}) + \mathtt{d}(h_{p+2}, h_{p+3}) + \cdots + \mathtt{d}(h_{p+q-1}, h_{p+q}),\\
&\leq Z\psi^{p+1} + Z\psi^{p+2} + Z\psi^{p+3} + \cdots + Z\psi^{p+q},\\
&\leq Z\psi^{p+1}\{1 + \psi + \psi^2 + \cdots + \psi^{q-1}\},\\
&\leq \frac{1-\psi^q}{1-\psi}\, Z\psi^{p+1},\\
&\leq \frac{\psi^{p+1}}{1-\psi}\, Z,
\end{aligned}$$

where $\psi = \Gamma^{1/2} < 1$.

Letting $p \to \infty$,

$$\lim_{p\to\infty} \mathtt{d}(h_p, h_{p+q}) \le \lim_{p\to\infty} \frac{\psi^{p+1}}{1-\psi} Z,$$

$$\Rightarrow \lim_{p\to\infty} \mathtt{d}(h_p, h_{p+q}) \le 0 .$$

$$\Rightarrow \lim_{p\to\infty} \mathtt{d}(h_p, h_{p+q}) = 0.$$

Therefore, we get a Cauchy sequence $\{h_j\}$ in $\mathbb{H}$. Therefore, $\{h_j\}$ converges to some point $h_* \in \mathbb{H}$ and $h_j \in \mathbb{H}_0$ which satisfy

$$\mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)),$$

that is, $(h_j, h_*) \in \mathbb{E}$.

Furthermore, we have to prove that $\mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)) \to \mathtt{d}(h_*, \mathbb{K}) \quad as \quad j \to \infty$. Since we know that

$$\mathtt{d}(\mathbb{H}, \mathbb{K}) \le \mathtt{d}(h_*, \mathbb{K}).$$

$$\begin{aligned} \mathtt{d}(h_*, \mathbb{K}) &\le \mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)), \\ &= \mathtt{d}(h_*, h_{j+1}) + \mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)), \\ &= \mathtt{d}(h_*, h_{j+1}) + \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{by} \quad (3.4) \\ &\le \mathtt{d}(h_*, h_{j+1}) + \mathtt{d}(h_*, \mathbb{K}). \end{aligned}$$

Therefore, $\mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)) \to \mathtt{d}(h_*, \mathbb{K}) \quad as \quad j \to \infty$.

Since assumption (v) hold, therefore $\{\mathbb{T}(h_{j-1}, h_j)\}$ has a subsequence $\{\mathbb{T}(h_{j_{m-1}}, h_{j_m})\}$, that converges to $k_* \in \mathbb{K}$ such that

$$\mathtt{d}(h_*, k_*) = \lim_{m\to\infty} \mathtt{d}(h_{j_{m+1}}, \mathbb{T}(h_{j_{m-1}}, h_{j_m}) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

We also have

$$\mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $(h_{j-1}, h_j), (h_j, h_*) \in \mathbb{E}$.

Hence, $h_* \in \mathbb{H}_0$. As we know $\mathbb{T}(h_j, h_*) \in \mathbb{K}_0$, and by definition of $\mathbb{K}_0$, we have

$g \in \mathbb{H}$ which satisfy

$$\mathtt{d}(g, \mathbb{T}(h_j, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}). \tag{3.8}$$

From assumption (vi), we know that $(h_j, h_*) \in \mathbb{E} \;\; \forall \;\; j \in \mathbb{N}$. Hence, we have

$$\mathtt{d}(g, \mathbb{T}(h_j, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}) \quad \text{and} \quad \mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}(\mathbb{H}, \mathbb{K}) \;\; \text{for each} \;\; j \in \mathbb{N}.$$

Thus, we get $h_{j-1}Ph_*$, that is, $(h_{j-1}, h_j), (h_j, h_*) \in \mathbb{E}, \;\; \forall \;\; j \in \mathbb{N}$.

Hence, from (3.3), we get:

$$\mathtt{d}(h_{j+1}, g) \;\leq\; \Gamma \max\{\mathtt{d}(h_{j-1}, h_j), \mathtt{d}(h_j, h_*)\} \text{ for each } j = 2, 3, 4, \cdots .$$

Letting $j \to \infty$,

$$\mathtt{d}(h_{j+1}, g) \;\leq\; \Gamma \max\{\mathtt{d}(h_{j-1}, h_j), \mathtt{d}(h_j, h_*)\} \to 0 \quad \text{as} \quad j \to \infty,$$

$$\Rightarrow \lim_{j\to\infty} \mathtt{d}(h_{j+1}, g) \;\leq\; 0,$$

$$\Rightarrow \mathtt{d}(h_*, g) = 0, \quad \text{that is} \quad g = h_*.$$

Put $g = h_*$ in (3.8),

$$\mathtt{d}(h_*, \mathbb{T}(h_j, h_*)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $(h_*, h_*) \in \mathbb{H}$.

Furthermore, note that $\mathbb{T}(h_*, h_*) \in \mathbb{K}_0$, and by definition of $\mathbb{K}_0$, we have $t \in \mathbb{H}$ which satisfy

$$\mathtt{d}(t, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}). \tag{3.9}$$

From assumption (vi), we know that $(h_*, h_*) \in \mathbb{E}$, and we have,

$$\mathtt{d}(t, \mathbb{T}(h_*, h_*)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}) \quad \text{and} \quad \mathtt{d}(h_*, \mathbb{T}(h_j, h_*)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}) \;\; \forall \;\; j \in \mathbb{N}.$$

Thus we get $h_jPh_* \;\; \forall \;\; j \in \mathbb{N}$, that is, $(h_j, h_*), (h_*, h_*) \in \mathbb{E} \;\; \forall \;\; j \in \mathbb{N}$.

Thus, by (3.3),

$$\begin{aligned} \mathtt{d}(h_*, t) &\leq \Gamma \ \max\{\mathtt{d}(h_j, h_*), \mathtt{d}(h_*, h_*)\} \ \ \forall \ \ j \in \mathbb{N}, \\ &= \Gamma \ \max\{\mathtt{d}(h_j, h_*), \mathtt{d}(h_*, h_*)\} \ \to 0 \ \text{ as } \ j \to \infty, \\ &\Rightarrow \ \mathtt{d}(h_*, t) = 0, \qquad \text{that is,} \qquad t = h_*. \end{aligned}$$

By using $t = h_*$ in (3.9),

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

□

Example 3.3.1. Consider $\mathbb{X} = \mathbb{R}^2$ endowed with usual metric

$$\mathtt{d}((s_1, s_2), (c_1, c_2)) = |s_1 - c_1| + |s_2 - c_2| \ \text{ for each } \ s, c \in \mathbb{R}^2.$$

Define a graph $\mathbb{G}$ as $\mathbb{V} = \mathbb{R}^2$ and

$$\mathbb{E} = \{(s_1, s_2), (c_1, c_2) : s_1, s_2, c_1, c_2 \in [0, 1]\} \cup \{(s, s) : s \in \mathbb{R}^2\}.$$

Take

$$\mathbb{H} = \{(0, s) : s \in [-2, 2]\}, \quad \text{and} \quad \mathbb{K} = \{(1, s) : s \in [-2, 2]\}.$$

Define:

$$\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}, \ \ \mathbb{T}((0, s), (0, c)) = \begin{cases} (1, \frac{s+c+2}{4}) & \text{when } s, c \ \geq 0, \\ (1, |s + c| - 2) & \text{otherwise.} \end{cases}$$

Let $\overline{h_1}, \ \overline{h_2}, \ \overline{h_3} \geq 0$, and $\overline{h_1} = (0, h_1), \ \overline{h_2} = (0, h_2), \ \overline{h_3} = (0, h_3) \in [-2, 2]$,

To find w_1 and w_2, we have:

$$\mathtt{d}(\overline{w_1}, \mathbb{T}(\overline{h_1}, \overline{h_2})) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(\overline{w_2}, \mathbb{T}(\overline{h_2}, \overline{h_3})). \tag{3.10}$$

For this consider

$$\begin{aligned}
\mathtt{d}(\mathbb{H},\mathbb{K}) &= \inf\{\mathtt{d}(\overline{h},\overline{k}) : \overline{h}\in\mathbb{H}, \overline{k}\in\mathbb{K}\},\\
&= \inf\{\mathtt{d}((0,s),(1,s)) : \quad \text{where} \quad s\in[-2,2]\},\\
&= \inf\{\mid 0-1\mid + \mid s-s\mid \quad \text{where} \quad s\in[-2,2]\},\\
&= 1.
\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\mathbb{H},\mathbb{K}) \;=\; 1. \tag{3.11}$$

$$\begin{aligned}
\mathtt{d}(\overline{w_1},\mathbb{T}(\overline{h_1},\overline{h_2})) &= \mathtt{d}((0,w_1),\mathbb{T}((0,h_1),(0,h_2))),\\
&= \mathtt{d}((0,w_1),(1,\frac{h_1+h_2+2}{4})),\\
&= \mid 0-1\mid + \mid w_1 - \frac{h_1+h_2+2}{4}\mid,\\
&= 1 + \mid w_1 - \frac{h_1+h_2+2}{4}\mid .
\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\overline{w_1},\mathbb{T}(\overline{h_1},\overline{h_2})) \;=\; 1 + w_1 - \frac{h_1+h_2+2}{4}. \tag{3.12}$$

Using (3.11) and (3.12) in (3.10), we obtain

$$1 = 1 + w_1 - \frac{h_1+h_2+2}{4}.$$

$$\Rightarrow \quad w_1 = \frac{h_1+h_2+2}{4}.$$

Similarly

$$\begin{aligned}
\mathtt{d}(\overline{w_2},\mathbb{T}(\overline{h_2},\overline{h_3})) &= \mathtt{d}((0,w_2),\mathbb{T}((0,h_2),(0,h_3))),\\
&= \mathtt{d}((0,w_2),(1,\frac{h_2+h_3+2}{4})),\\
&= \mid 0-1\mid + \mid w_2 - \frac{h_2+h_3+2}{4}\mid,\\
&= 1 + w_2 - \frac{h_2+h_3+2}{4}.
\end{aligned}$$

From (3.10), we obtain

$$\Rightarrow \quad w_2 = \frac{h_2+h_3+2}{4}.$$

$$\overline{w_1} = (0, \tfrac{h_1+h_2+2}{4}), \quad \overline{w_2} = (0, \tfrac{h_2+h_3+2}{4}),$$

$\Rightarrow \overline{h_1}, \overline{h_2}, \overline{h_3}, \overline{w_1}, \overline{w_2} \in \mathbb{H}, \quad$ with $\quad \overline{h_1} P \overline{h_3}.$

Also we have

$$\mathtt{d}(\overline{w_1}, \overline{w_2}) \leq \Gamma \max\{\mathtt{d}(\overline{h}_1, \overline{h}_2), \mathtt{d}(\overline{h}_2, \overline{h}_3)\},$$

where

$$\begin{aligned}\mathtt{d}(\overline{w_1}, \overline{w_2}) &= \mathtt{d}((0, w_1), (0, w_2)),\\ &= \mid 0 - 0 \mid + \mid w_1 - w_2 \mid,\\ &= \mid \frac{h_1 + h_2 + 2}{4} - \frac{h_2 + h_3 + 2}{4} \mid,\\ &= \frac{\mid h_1 - h_3 \mid}{4}.\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\overline{h_3}, \overline{w_2}) = \frac{\mid h_1 - h_3 \mid}{4}.$$

Using above equation in (3.3),

$$\Rightarrow \mathtt{d}(\overline{w_1}, \overline{w_2}) = \frac{\mid h_1 - h_3 \mid}{4} = \psi \max\{\mathtt{d}(\overline{h}_1, \overline{h}_2), \mathtt{d}(\overline{h}_2, \overline{w}_1)\}.$$

Here $\psi = \Gamma^{\frac{1}{2}} = \frac{1}{2} \in [0, 1)$.

Consider

$$\overline{h_1} = (0, h_1), \quad \overline{h_2} = (0, h_2), \quad \overline{h_3} = (0, h_3) \in \mathbb{H} \quad \text{such that} \quad \overline{h_1} P \overline{h_3},$$

Since

$$(0, w_1) = (0, \tfrac{h_1 + h_2 + 2}{4}), \quad (0, w_2) = (0, \tfrac{h_2 + h_3 + 2}{4}).$$

We have

$$\mathtt{d}((0, w_1), \mathbb{T}((0, h_1), (0, h_2))) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}((0, w_2), \mathbb{T}((0, h_2), (0, h_3))),$$

then $((0, w_1), (0, w_2)) \in \mathbb{E}$.

Thus, $\mathbb{T}$ is path admissible.

Now we will prove that

$$\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2.$$

To do so consider

$$\overline{h_1} = (0,0), \ \ \overline{h_2} = (0, \frac{1}{2}), \ \ \overline{h_3} = (0, \frac{5}{8}) \in \mathbb{H},$$

such that

$$\begin{aligned}\mathtt{d}((0,\frac{5}{8}), \mathbb{T}((0,0),(0,\frac{1}{2}))) &= \mid (0,\frac{5}{8}) - (1, \frac{0+\frac{1}{2}+2}{4}) \mid, \\ &= \mid (0-1 \mid + \mid \frac{5}{8} - \frac{5}{8} \mid, \\ &=1, \\ &=\mathtt{d}(\mathbb{H}, \mathbb{K}).\end{aligned}$$

and $(0,0)P(0,\frac{5}{8})$.

Moreover, assumption (v) holds such that $h_j P h_{j+2}\ \forall\ j \in \mathbb{N}$, and $h_j \to a$ as $j \to \infty$, then $(h_j, a) \in \mathbb{E}$ for each $j \in \mathbb{N}$ and $(a,a) \in \mathbb{E}$.
Therefore, all axioms are true. Hence $\mathbb{T}$ has a BPP.

Theorem 3.3.2.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ such that for $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and $\mathtt{d}(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}(h_3, w_2) \leq \Gamma \max\{\mathtt{d}(h_1, h_2), \mathtt{d}(h_2, w_1)\}, \tag{3.13}$$

where $\Gamma \in [0,1)$.
Moreover, suppose the following asumptions are true:

(i) $\mathbb{T}$ is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ satisfying $\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{K}$ is approximately compact with respect to $\mathbb{H}$;

(vi) When $\{h_j\} \subseteq \mathbb{X}$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to a$ as $j \to \infty$, then $(h_j, a) \in \mathbb{E}$ for each $j \in \mathbb{N}$ and $(a, a) \in \mathbb{E}$,

then, $\exists$ a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From assumption (ii), we have $h_0, h_1, h_2 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

and $h_0 P h_2$, that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.
From assumption (iv), we have $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

Since from assumption (i) , we get $(h_2, h_3) \in \mathbb{E}$. Thus, $h_1 P h_3$.
Continuing the same procedure, we build a sequence $\{h_{j\geq 2}\}$ in $\mathbb{H}$ which satisfy:

$$\mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) \ = \ \mathtt{d}(\mathbb{H}, \mathbb{K}) \ \ \forall \ \ j \in \mathbb{N},$$

and $h_{j-1} P h_{j+1}$, that is, $(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E}$ for all $j \in \mathbb{N}$.

From (3.13), we have:

$$\mathtt{d}(h_j, h_{j+1}) \ \leq \ \Gamma \ \max\{\mathtt{d}(h_{j-2}, h_{j-1}), \mathtt{d}(h_{j-1}, h_j)\} \text{ for every } j = 2, 3, 4, \cdots . \quad (3.14)$$

Let $\mathtt{d}_j = \mathtt{d}(h_j, h_{j+1})$ for all $j \in \mathbb{N} \cup \{0\}$.

Then we can rewrite (3.14) as

$$\mathtt{d}_j \leq \Gamma \max\{\mathtt{d}_{j-2}, \mathtt{d}_{j-1}\} \;\; \forall \;\; j = 2, 3, 4, \cdots . \tag{3.15}$$

From (3.15) and Theorem 3.3.1, we get a Cauchy sequence $\{h_j\} \subseteq \mathbb{H}$ such that $h_j \to h_*$, $h_* \in \mathbb{H}_0$. As $\mathbb{T}(h_j, h_*) \in \mathbb{K}_0$, we have $w \in \mathbb{H}$ satisfying

$$\mathtt{d}(w, \mathbb{T}(h_j, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}). \tag{3.16}$$

From assumption (vi), $(h_j, h_*) \in \mathbb{E} \;\; \forall \;\; j \in \mathbb{N}$ and we also have

$$\mathtt{d}(h_*, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

Thus, we get $h_{j-1} P h_*$, that is, $(h_{j-1}, h_j), (h_j, h_*) \in \mathbb{E}$, for each $j \in \mathbb{N}$.

Hence, from (3.13), we get:

$$\mathtt{d}(h_*, w) \leq \Gamma \max\{\mathtt{d}(h_{j-1}, h_j), \mathtt{d}(h_j, h_{j+1})\} \;\; \forall \;\; j \in \mathbb{N}.$$

Letting $j \to \infty$,

$$\lim_{j\to\infty} \mathtt{d}(h_*, w) \;\leq\; \Gamma \max \lim_{j\to\infty} \{\mathtt{d}(h_{j-1}, h_j), \mathtt{d}(h_j, h_{j+1})\} \;\; \forall \;\; j \in \mathbb{N}.$$

$$\Rightarrow \lim_{j\to\infty} \mathtt{d}(h_*, w) \;\leq\; 0,$$

$$\Rightarrow \;\; \mathtt{d}(h_*, w) = 0, \quad \text{that is,} \quad w = h_*.$$

Using $w = h_*$ in (3.16), we get

$$\mathtt{d}(h_*, \mathbb{T}(h_j, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

That is, $(h_*, h_*) \in \mathbb{E}$. Furthermore, note that $\mathbb{T}(h_*, h_*) \in \mathbb{K}_0$, and there is $q \in \mathbb{H}$ which satisfy

$$\mathtt{d}(q, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}). \tag{3.17}$$

From assumption (vi), $(h_*, h_*) \in \mathbb{E}$. Thus,

$$\mathtt{d}(h_*, \mathbb{T}(h_j, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad \mathtt{d}(q, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

$$\text{and } h_j P h_*, \quad \text{that is,} \quad (h_j, h_*) \in \mathbb{E} \text{ and } (h_*, h_*) \in \mathbb{E}.$$

Hence, from (3.13),

$$\mathtt{d}(h_*, q) \ \leq \Gamma \max\left\{\mathtt{d}(h_j, h_*), \mathtt{d}(h_*, h_*)\right\} \ \forall \ j \ \in \ \mathbb{N}.$$

$$\Rightarrow \ \mathtt{d}(h_*, q) \ \leq \Gamma \max\left\{\mathtt{d}(h_j, h_*), \mathtt{d}(h_*, h_*)\right\} \ \rightarrow \ 0 \text{ as } j \rightarrow \infty.$$

$$\Rightarrow \ \mathtt{d}(h_*, q) = 0, \quad \text{that is} \quad q = h_*.$$

Thus, by putting $q = h_*$ in (3.17) we have

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

□

Example 3.3.2. Consider $\mathbb{X} = \mathbb{R}^2$ endowed with usual metric

$$\mathtt{d}((s_1, s_2), (c_1, c_2)) \ = \ |s_1 - c_1| + |s_2 - c_2| \ \text{ for each } \ s, c \in \mathbb{R}^2.$$

Define a graph $\mathbb{G}$ as $\mathbb{V} = \mathbb{R}^2$ and

$$\mathbb{E} = \{(s_1, s_2), (c_1, c_2) : s_1, s_2, c_1, c_2 \in [0, 1]\} \cup \{(s, s) : s \in \mathbb{R}^2\}.$$

Take

$$\mathbb{H} = \{(0, s) : s \in [-2, 2]\}, \quad \text{and} \quad \mathbb{K} = \{(1, s) : s \in [-2, 2]\}.$$

Define:

$$\mathbb{T} : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{K}, \ \ \mathbb{T}((0, s), (0, c)) = \ (1, c) \ \ \forall \ \ (0, s), (0, c) \in \mathbb{H}.$$

Let $\overline{h_1} = (0, h_1), \ \overline{h_2} = (0, h_2), \ \overline{h_3} = (0, h_3) \in [-2, 2].$

To find w_1 and w_2, we have:

$$\mathtt{d}(\overline{w_1}, \mathbb{T}(\overline{h_1}, \overline{h_2})) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(\overline{w_2}, \mathbb{T}(\overline{h_2}, \overline{h_3})). \tag{3.18}$$

For this consider

$$\begin{aligned}
\mathtt{d}(\mathbb{H}, \mathbb{K}) &= \inf\{\mathtt{d}(\overline{h}, \overline{k}) : \overline{h} \in \mathbb{H}, \overline{k} \in \mathbb{K}\},\\
&= \inf\{\mathtt{d}((0, s), (1, s)) : \quad \text{where} \quad s \in [-2, 2]\},\\
&= \inf\{\mid 0 - 1 \mid + \mid s - s \mid: \quad \text{where} \quad s \in [-2, 2]\},\\
&= 1.
\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\mathbb{H}, \mathbb{K}) \ = \ 1. \tag{3.19}$$

$$\begin{aligned}
\mathtt{d}(\overline{w_1}, \mathbb{T}(\overline{h_1}, \overline{h_2})) &= \mathtt{d}((0, w_1), \mathbb{T}((0, h_1), (0, h_2))),\\
&= \mathtt{d}((0, w_1), (1, h_2)),\\
&= \mid 0 - 1 \mid + \mid w_1 - h_2 \mid,\\
&= 1 + w_1 - h_2.
\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\overline{w_1}, \mathbb{T}(\overline{h_1}, \overline{h_2})) \ = \ 1 + w_1 - h_2. \tag{3.20}$$

Using (3.19) and (3.20) in (3.18), we obtain

$$1 = 1 + w_1 - h_2.$$

$$\Rightarrow \quad w_1 = h_2.$$

Similarly

$$\begin{aligned}
\mathtt{d}(\overline{w_2}, \mathbb{T}(\overline{h_2}, \overline{h_3})) &= \mathtt{d}((0, w_2), \mathbb{T}((0, h_2), (0, h_3))),\\
&= \mathtt{d}((0, w_2), (1, h_3)),\\
&= \mid 0 - 1 \mid + \mid w_2 - h_3 \mid,\\
&= 1 + w_2 - h_3.
\end{aligned}$$

From (3.18), we obtain

$$\Rightarrow \quad w_2 = h_2.$$

$$\overline{w_1} = (0, w_1) = (0, h_2), \quad \overline{w_2} = (0, w_2) = (0, h_3),$$

$$\Rightarrow \overline{h_1}, \overline{h_2}, \overline{h_3}, \overline{w_1}, \overline{w_2} \in \mathbb{H}, \qquad \text{with} \quad \overline{h_1} P \overline{h_3}.$$

Also we have

$$\mathtt{d}(\overline{h_3}, \overline{w_2}) \leq \Gamma \max\{\mathtt{d}(\overline{h}_1, \overline{h}_2), \mathtt{d}(\overline{h}_2, \overline{w}_1)\}, \tag{3.21}$$

where

$$\begin{aligned}\mathtt{d}(\overline{h}_3, \overline{w}_2) &= \mathtt{d}((0, h_3), (0, w_2)),\\ &=\mid 0 - 0 \mid + \mid h_3 - w_2 \mid,\\ &=\mid h_3 - h_3 \mid,\\ &= 0.\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\overline{h}_3, \overline{w}_2) = 0.$$

Using above equation in (3.21),we get

$$\Rightarrow \mathtt{d}(\overline{h_3}, \overline{w_2}) = \ 0 \ = \psi \max\{\mathtt{d}(\overline{h}_1, \overline{h}_2), \mathtt{d}(\overline{h}_2, \overline{w}_1)\}.$$

Here we say $\psi = \Gamma^{\frac{1}{2}} = \frac{1}{2} \in [0, 1)$.

Now we will prove condition (i) of Theorem 3.3.2. To do so we consider

$$\overline{h_1} = (0, h_1), \quad \overline{h_2} = (0, h_2), \quad \overline{h_3} = (0, h_3) \in \mathbb{H} \quad \text{such that} \quad \overline{h_1} P \overline{h_3}.$$

Since

$\overline{w}_1 = (0, w_1) = (0, h_2), \quad \overline{w}_2 = (0, w_2) = (0, h_3),$

and now we prove

$$\mathtt{d}((0, w_1), \mathbb{T}((0, h_1), (0, h_2))) = \mathtt{d}(\mathbb{H}, \mathbb{K}) \text{ and } \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}((0, w_2), \mathbb{T}((0, h_2), (0, h_3))),$$

$$
\begin{aligned}
\mathtt{d}(\overline{w_1}, \mathbb{T}(\overline{h_1}, \overline{h_2})) &= \mathtt{d}((0, w_1), \mathbb{T}((0, h_1), (0, h_2))),\\
&= \mathtt{d}((0, h_2), (1, h_2)),
\end{aligned}
$$

$$
\begin{aligned}
&=\mid 0 - 1 \mid + \mid h_2 - h_2 \mid,\\
&= 1\\
&= \mathtt{d}(\mathbb{H}, \mathbb{K}).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\mathtt{d}(\overline{w_2}, \mathbb{T}(\overline{h_2}, \overline{h_3})) &= \mathtt{d}((0, w_2), \mathbb{T}((0, h_2), (0, h_3))),\\
&= \mathtt{d}((0, h_2), (1, h_3)),\\
&=\mid 0 - 1 \mid + \mid h_3 - h_3 \mid,\\
&= 1 = \mathtt{d}(\mathbb{H}, \mathbb{K}).
\end{aligned}
$$

$\Rightarrow ((0, w_1), (0, w_2)) \in \mathbb{E}$.

Thus, $\mathbb{T}$ is path admissible.

Now we will prove condition (ii).

$$
\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2.
$$

To do so consider

$$
\overline{h_1} = (0, 0), \quad \overline{h_2} = (0, \frac{1}{2}), \quad \overline{h_3} = (0, \frac{5}{8}) \in \mathbb{H},
$$

such that

$$
\begin{aligned}
\mathtt{d}((0, \frac{5}{8}), \mathbb{T}((0, 0), (0, \frac{1}{2}))) &= \mid (0, \frac{5}{8}) - (1, \frac{0 + \frac{1}{2} + 2}{4}) \mid,\\
&= \mid (0 - 1 \mid + \mid \frac{5}{8} - \frac{5}{8} \mid,\\
&=1 = \mathtt{d}(\mathbb{H}, \mathbb{K}).
\end{aligned}
$$

and $(0, 0) P (0, \frac{5}{8})$.

Moreover, assumption (v) holds such that $h_j P h_{j+2} \ \forall \ j \in \mathbb{N}$, and $h_j \to a$ as $j \to \infty$

, then $(h_j, a) \in \mathbb{E}$ for each $j \in \mathbb{N}$ and $(a, a) \in \mathbb{E}$.

Therefore, all axioms are true. Hence $\mathbb{T}$ has a BPP.

Remark 3.2. Notice that, in the above example, Theorem 3.3.1 is not applicable.

Counter Example: Let $\overline{h_1} = (0, \frac{5}{8})$, $\overline{h_2} = (0, \frac{1}{2})$, $\overline{h_3} = (0, 0) \in \mathbb{H}$, and

$\overline{w}_1 = (0, w_1) = (0, h_2) = (0, \frac{1}{2})$, $\overline{w}_2 = (0, w_2) = (0, h_3) = (0, 0)$.

$$\begin{aligned} \mathtt{d}(\overline{w}_1, \overline{w}_2) &= \mathtt{d}((0, w_1), (0, w_2)), \\ &= \mid 0 - 0 \mid + \mid w_1 - w_2 \mid, \\ &= \mid \frac{1}{2} - 0 \mid, \\ &= \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} \mathtt{d}(\overline{h}_1, \overline{h}_2) &= \mathtt{d}((0, \frac{5}{8}), (0, \frac{1}{2})), \\ &= \mid 0 - 0 \mid + \mid \frac{5}{8} - \frac{1}{2} \mid, \\ &= \frac{1}{8}. \end{aligned}$$

$$\begin{aligned} \mathtt{d}(\overline{h}_2, \overline{h}_3) &= \mathtt{d}((0, \frac{1}{2}), (0, 0)), \\ &= \mid 0 - 0 \mid + \mid 0 - \frac{1}{2} \mid, \\ &= \frac{1}{2}. \end{aligned}$$

Use the above values in (3.3),we get

$$\frac{1}{2} \leq \Gamma \max\{\frac{1}{8}, \frac{1}{2}\} \quad \text{where} \quad \Gamma \in [0, 1).$$

Let $\Gamma = \frac{1}{2} \in [0, 1)$,then

$$\frac{1}{2} \leq \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}.$$

Which is contradiction.

Theorem 3.3.3.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Let $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ be a mapping such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and $\mathtt{d}(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}(\mathbb{T}(h_2, w_1), \mathbb{T}(h_3, w_2)) \leq \Gamma \mathtt{d}(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3)), \tag{3.22}$$

where $\Gamma \in [0, 1)$.

Moreover, suppose that the following asumptions are true:

(i) $\mathbb{T}$ is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ satisfying $\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{H}$ is approximately compact with respect to $\mathbb{K}$;

(vi) When $\{h_j\}, \{\overline{h_j}\} \subseteq \mathbb{X}$ such that $h_j \to h$ and $\overline{h_j} \to \overline{h}$, then $\mathbb{T}(h_j, \overline{h_j}) \to \mathbb{T}(h, \overline{h})$,

then, $\exists$ a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From assumption (ii), we get $h_0, h_1, h_2 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

and $h_0 P h_2$, that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From assumption (iv), we have $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

Since from assumption (i), we get $(h_2, h_3) \in \mathbb{E}$. Thus, $h_1 P h_3$.

By continuing the same method further, we construct a sequence $\{h_j : j \in \mathbb{N} \setminus \{1\}\} \in \mathbb{H}$ which satisfy:

$$\mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}) \;\;\forall\;\; j \in \mathbb{N}, \tag{3.23}$$

and $\;h_{j-1} P h_{j+1}$, that is, $\;(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E} \;\;\forall\;\; j \in \mathbb{N}.$

From (3.22), we have:

$$\mathtt{d}(\mathbb{T}(h_{j-1}, h_j), \mathbb{T}(h_j, h_{j+1})) \;\leq\; \Gamma \mathtt{d}(\mathbb{T}(h_{j-2}, h_{j-1}), \mathbb{T}(h_{j-1}, h_j)) \;\;\forall\;\; j = 2, 3, 4, \cdots. \tag{3.24}$$

For convenience, we take $\mathbb{T}_{j-1} = \mathbb{T}(h_{j-1}, h_j)$ for each $j = 2, 3, 4, \cdots.$

Then rewrite (3.24) as

$$\mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j) \;\leq\; \Gamma \mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}) \;\text{ for each }\; j = 2, 3, 4, \cdots. \tag{3.25}$$

By using induction,

$$\begin{aligned}
\mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j) &\leq \Gamma\ \mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \\
&\leq \Gamma(\Gamma\ \mathtt{d}(\mathbb{T}_{j-3}, \mathbb{T}_{j-2})), \\
&= \Gamma^2 \mathtt{d}(\mathbb{T}_{j-3}, \mathbb{T}_{j-2}), \\
&\leq \Gamma^2 \Gamma \mathtt{d}(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \\
&= \Gamma^3 \mathtt{d}(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \\
&\;\;\vdots \\
&\leq \Gamma^{j-1} \mathtt{d}(\mathbb{T}_0, \mathbb{T}_1) \;\text{ for }\; j = 2, 3, 4, \cdots.
\end{aligned}$$

Hence,

$$\mathtt{d}(\mathbb{T}_p, \mathbb{T}_{p+1}) \leq \Gamma^p \mathtt{d}(\mathbb{T}_1, \mathbb{T}_0) \quad \text{for} \quad p = 1, 2, 3, \cdots.$$

$$\mathtt{d}(\mathbb{T}(h_p, h_{p+1}), \mathbb{T}(h_{p+1}, h_{p+2})) \leq \Gamma^p \mathtt{d}(\mathbb{T}(h_0, h_1), \mathbb{T}(h_1, h_2)) \quad \text{for} \quad p = 1, 2, 3, \cdots. \tag{3.26}$$

Now using triangle inequality,

$$\begin{aligned}
\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1})) \leq &\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))+\mathtt{d}(\mathbb{T}(h_{p+1},h_{p+2}),\\
&\mathbb{T}(h_{p+s},h_{p+s+1})),\\
\leq &\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))+\mathtt{d}(\mathbb{T}(h_{p+1},h_{p+2}),\\
&\mathbb{T}(h_{p+2},h_{p+3}))+\mathtt{d}(\mathbb{T}(h_{p+2},h_{p+3}),\mathbb{T}(h_{p+s},h_{p+s+1})),\\
\leq &\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))+\mathtt{d}(\mathbb{T}(h_{p+1},h_{p+2}),\\
&\mathbb{T}(h_{p+2},h_{p+3}))+\cdots+\mathtt{d}(\mathbb{T}(h_{p+s-1},h_{p+s}),\\
&\mathbb{T}(h_{p+s},h_{p+s+1})),
\end{aligned}$$

We can write the above equation as

$$\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1})) \leq \sum_{k=p}^{p+s-1}\mathtt{d}(\mathbb{T}(h_k,h_{k+1}),\mathbb{T}(h_{k+1},h_{k+2}). \tag{3.27}$$

By using (3.26) in (3.27), we get

$$\begin{aligned}
\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1})) \leq &\sum_{k=p}^{p+s-1}\Gamma^k\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2)),\\
\leq &\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2))\Gamma^p\sum_{k=0}^{s-1}\Gamma^k,\\
\leq &\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(_1,h_2))\Gamma^p\{1+\Gamma^1+\Gamma^2+\cdots+\Gamma^{s-1}\},\\
\leq &\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2))\Gamma^p\frac{1-\Gamma^s}{1-\Gamma},\\
\leq &\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2))\Gamma^p\{\frac{1}{1-\Gamma}\}.
\end{aligned}$$

Now by picking $\lim_{p\to\infty}$ in above inequality, the inequality deduced to,

$$\lim_{p\to\infty}\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1})) \leq 0.$$

Further we have

$$\Rightarrow \lim_{p\to\infty}\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1})) = 0.$$

This proves that we get a Cauchy sequence $\{\mathbb{T}(h_{j-1},h_j)\}$ in $\mathbb{K}$. Since $\mathbb{X}$ is complete metric space, therefore $\mathbb{T}(h_{j-1}, h_j) \to k_* \in \mathbb{K}$ and $\mathbb{T}(h_{j-1}, h_j) \in \mathbb{K}_0$.

Moreover,

$$\begin{aligned} \mathtt{d}(k_*, \mathbb{H}) &\leq \mathtt{d}(k_*, h_{j+1}), \\ &\leq \mathtt{d}(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)), \\ &= \mathtt{d}(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{by} \quad (3.23) \\ &\leq \mathtt{d}(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}(k_*, \mathbb{H}). \end{aligned}$$

Therefore, $\mathtt{d}(k_*, h_{j+1}) \quad \to \quad \mathtt{d}(k_*, \mathbb{H}) \quad \text{as} \quad j \to \infty.$

Since from assumption (v), $\{h_j\}$ has a subsequence $\{h_{j_l}\}$ which converges to an element $h_* \in \mathbb{H}$. Thus, we have:

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \lim_{l \to \infty} \mathtt{d}(h_{j_{l+1}}, \mathbb{T}(h_{j_{l-1}}, h_{j_l})) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

$$\Rightarrow \mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

□

Example 3.3.3. Consider $\mathbb{X} = \mathbb{R}^2$ endowed with usual metric

$$\mathtt{d}((s_1, s_2), (c_1, c_2)) \;=\; |s_1 - c_1| + |s_2 - c_2| \;\text{ for each }\; s, c \in \mathbb{R}^2.$$

Define a graph $\mathbb{G}$ as $\mathbb{V} = \mathbb{R}^2$ and $\mathbb{E} = \mathbb{R}^4$

Take

$$\mathbb{H} = \{(0, s) : s \in [-2, 2]\}, \quad \text{and} \quad \mathbb{K} = \{(1, s) : s \in [-2, 2]\}.$$

Define:

$$\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}, \;\; \mathbb{T}((0, s), (0, c)) = \;(1, \frac{c}{2}) \;\; \forall \;\; (0, s), (0, c) \in \mathbb{H}.$$

Let $\overline{h_1} = (0, h_1)$, $\overline{h_2} = (0, h_2)$, $\overline{h_3} = (0, h_3) \in [-2, 2]$.

To find w_1 and w_2, we have:

$$\mathtt{d}(\overline{w_1}, \mathbb{T}(\overline{h_1}, \overline{h_2})) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(\overline{w_2}, \mathbb{T}(\overline{h_2}, \overline{h_3})). \tag{3.28}$$

For this consider

$$\begin{aligned}
\mathtt{d}(\mathbb{H},\mathbb{K}) =& \inf\{\mathtt{d}(\overline{h},\overline{k}) : \overline{h}\in\mathbb{H},\overline{k}\in\mathbb{K}\},\\
=& \inf\{\mathtt{d}((0,s)(1,s)) : \quad \text{where} \quad s\in[-2,2]\},\\
=& \inf\{\mid 0-1\mid + \mid s-s\mid \quad \text{where} \quad s\in[-2,2]\},\\
=& 1.
\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\mathbb{H},\mathbb{K}) \;=\; 1. \tag{3.29}$$

$$\begin{aligned}
\mathtt{d}(\overline{w_1},\mathbb{T}(\overline{h_1},\overline{h_2})) &= \mathtt{d}((0,w_1),\mathbb{T}((0,h_1),(0,h_2))),\\
&= \mathtt{d}((0,\frac{h_2}{2}),(1,\frac{h_2}{2})),\\
&= \mathtt{d}((0,w_1),(1,\frac{h_2}{2})),\\
&=\mid 0-1\mid + \mid w_1-\frac{h_2}{2}\mid,\\
&= 1+w_1-\frac{h_2}{2}.
\end{aligned}$$

$$\Rightarrow \quad \mathtt{d}(\overline{w_1},\mathbb{T}(\overline{h_1},\overline{h_2})) \;=\; 1+w_1-\frac{h_2}{2}. \tag{3.30}$$

Using (3.29) and (3.30) in (3.28), we obtain

$$1 = 1+w_1-\frac{h_2}{2}.$$

$$\Rightarrow \quad w_1 = \frac{h_2}{2}.$$

Similarly

$$\begin{aligned}
\mathtt{d}(\overline{w_2},\mathbb{T}(\overline{h_2},\overline{h_3})) &= \mathtt{d}((0,w_2),\mathbb{T}((0,h_2),(0,h_3))),\\
&= \mathtt{d}((0,w_2),(1,\frac{h_3}{2})),\\
&=\mid 0-1\mid + \mid w_2-\frac{h_3}{2}\mid,\\
&= 1+w_2-\frac{h_3}{2}.
\end{aligned}$$

From (3.28), we obtain

$$\Rightarrow \quad w_2 = \frac{h_3}{2}.$$

$\overline{w_1} = (0, w_1) = \frac{h_2}{2}, \quad \overline{w_2} = (0, w_2) = \frac{h_3}{2},$

$\Rightarrow \overline{h_1}, \overline{h_2}, \overline{h_3}, \overline{w_1}, \overline{w_2} \in \mathbb{H}, \qquad \text{with} \quad \overline{h_1} P \overline{h_3}.$

Also we have

$$\mathtt{d}(\mathbb{T}(\overline{h_2}, \overline{w}_1), \mathbb{T}(\overline{h_3, w_2})) \leq \Gamma \mathtt{d}(\mathbb{T}(\overline{h_1}, \overline{h_2}), \mathbb{T}(\overline{h_2}, \overline{h_3})),$$

where

$$\begin{aligned}
\mathtt{d}(\mathbb{T}(\overline{h}_2, \overline{w}_1), \mathbb{T}(\overline{h}_3, \overline{w}_2)) &= \mathtt{d}\mathbb{T}((0, h_2), (0, w_1)), (\mathbb{T}((0, h_3), (0, w_2))), \\
&= \mathtt{d}((1, \frac{h_2}{4}), (1, \frac{h_3}{4})), \\
&=| 1 - 1 | + | \frac{h_2}{4} - \frac{h_3}{4} |, \\
&= \frac{1}{4} | h_2 - h_3 |, \\
&= \frac{1}{2} | \frac{h_2}{2} - 1 + 1 - \frac{h_3}{2} |, \\
&= \frac{1}{2} \mathtt{d}((1, \frac{h_2}{2}), (1, \frac{h_3}{2})), \\
&= \Gamma \mathtt{d}(\mathbb{T}(\overline{h_1}, \overline{h_2}), \mathbb{T}(\overline{h_2}, \overline{h_3})).
\end{aligned}$$

$$\Rightarrow \mathtt{d}(\mathbb{T}(\overline{h}_2, \overline{w}_1), \mathbb{T}(h_3, w_2)) = \frac{1}{2} \mathtt{d}(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3)).$$

Here $\Gamma = \frac{1}{2} \in [0, 1)$.

Now we will prove condition (i).

To do so we consider

$$\overline{h_1} = (0, h_1), \quad \overline{h_2} = (0, h_2), \quad \overline{h_3} = (0, h_3) \in \mathbb{H} \quad \text{such that} \quad \overline{h_1} P \overline{h_3},$$

Since

$\overline{w}_1 = (0, w_1) = (0, h_2), \quad \overline{w}_2) = (0, w_2) = (0, h_3).$

and we already prove

$$\mathtt{d}((0, w_1), \mathbb{T}((0, h_1), (0, h_2))) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}((0, w_2), \mathbb{T}((0, h_2), (0, h_3))),$$

$\Rightarrow ((0, w_1), (0, w_2)) \in \mathbb{E}$.

Thus, $\mathbb{T}$ is path admissible.

Moreover,

Now we will prove condition (ii).

$$\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2.$$

To do so consider

$$\overline{h_1} = (0, 0), \quad \overline{h_2} = (0, \frac{1}{2}), \quad \overline{h_3} = (0, \frac{5}{8}) \in \mathbb{H},$$

such that

$$\begin{aligned}
\mathtt{d}((0, \frac{5}{8}), \mathbb{T}((0, 0), (0, \frac{1}{2}))) &= \mid (0, \frac{5}{8}) - (1, \frac{0 + \frac{1}{2} + 2}{4}) \mid, \\
&= \mid (0 - 1 \mid + \mid \frac{5}{8} - \frac{5}{8} \mid, \\
&= 1, \\
&= \mathtt{d}(\mathbb{H}, \mathbb{K}).
\end{aligned}$$

and $(0, 0) P (0, \frac{5}{8})$.

Moreover, assumption (v) holds such that $h_j P h_{j+2} \ \forall \ j \in \mathbb{N}$, and $h_j \to a$ as $j \to \infty$, then $(h_j, a) \in \mathbb{E}$ for each $j \in \mathbb{N}$ and $(a, a) \in \mathbb{E}$.

Therefore, all axioms are true. Hence $\mathbb{T}$ has a BPP.

Theorem 3.3.4.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ such that for $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$ and $\mathtt{d}(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}(\mathbb{T}(h_2, w_1), \mathbb{T}(h_3, w_2)) \leq \Gamma \max\{\mathtt{d}(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3)), \mathtt{d}(\mathbb{T}(h_2, h_3), \mathbb{T}(w_1, w_2))\}, \tag{3.31}$$

where $\Gamma \in [0, 1)$.

Moreover, suppose that the following asumptions are true:

(i) $\mathbb{T}$ is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ satisfying $\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{H}$ is approximately compact with respect to $\mathbb{K}$;

(vi) When $\{h_j\}$, $\{\overline{h_j}\} \subseteq \mathbb{X}$ such that $h_j \to h$ and $\overline{h_j} \to \overline{h}$, then $\mathbb{T}(h_j, \overline{h_j}) \to \mathbb{T}(h, \overline{h})$,

then, $\exists$ a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From assumption (ii), we have $h_0, h_1, h_2 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

and $h_0 P h_2$, that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From assumption (iv), we have $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

Since we have $(h_2, h_3) \in \mathbb{E}$ from assumption (i). Thus, $h_1 P h_3$.

Continuing the same procedure further, we build a sequence $\{h_{j \geq 2}\}$ in $\mathbb{H}$ which satisfy:

$$\mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) \ = \ \mathtt{d}(\mathbb{H}, \mathbb{K}) \ \ \forall \ \ j \in \mathbb{N}, \tag{3.32}$$

and $h_{j-1} P h_{j+1}$, that is, $(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E}$ for all $j \in \mathbb{N}$.

From (3.31), we have:

$$\begin{aligned} \mathtt{d}(\mathbb{T}(h_{j-1}, h_j), \mathbb{T}(h_j, h_{j+1})) \ &\leq \Gamma \max\{\mathtt{d}(\mathbb{T}(h_{j-2}, h_{j-1})), \mathbb{T}(h_{j-1}, h_j)), \\ &\qquad \mathtt{d}(\mathbb{T}(h_{j-1}, h_j), \mathbb{T}(h_j, h_{j+1}))\}. \end{aligned} \tag{3.33}$$

For convenience, we take $\mathbb{T}_{j-1} = \mathbb{T}(h_{j-1}, h_j)$ for each $j = 2, 3, 4, \cdots$.

Then rewrite (3.33) as

$$\mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j) \ \leq \ \Gamma \max\{\mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j)\}.$$

Either,

$$\max\{\mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j)\} = \mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}) \ \text{ for each } \ j = 2, 3, 4, \cdots.$$

Or, we take ,

$$\max\{\mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j)\} = \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j) \ \text{ for each } \ j = 2, 3, 4, \cdots.$$

If we take $\max\{\mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j)\} = \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j)$ for each $j = 2, 3, 4, \cdots$.

which is contradiction.

Thus we have:

$$\mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j) \ \leq \ \Gamma \mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}) \ \text{ for each } \ j = 2, 3, 4, \cdots. \tag{3.34}$$

By using induction,

$$\begin{aligned} \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j) &\leq \Gamma \ \mathtt{d}(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \\ &\leq \Gamma(\Gamma \ \mathtt{d}(\mathbb{T}_{j-3}, \mathbb{T}_{j-2})), \\ &= \Gamma^2 \mathtt{d}(\mathbb{T}_{j-3}, \mathbb{T}_{j-2}), \\ &\leq \Gamma^2 \Gamma \mathtt{d}(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \end{aligned}$$

$$=\Gamma^3\mathtt{d}(\mathbb{T}_{j-4},\mathbb{T}_{j-3}),$$
$$\vdots$$
$$\leq\Gamma^{j-1}\mathtt{d}(\mathbb{T}_0,\mathbb{T}_1) \text{ for } j=2,3,4,\cdots.$$

Hence,

$$\mathtt{d}(\mathbb{T}_p,\mathbb{T}_{p+1})\leq\Gamma^p\mathtt{d}(\mathbb{T}_1,\mathbb{T}_0) \quad \text{for} \quad p=1,2,3,\cdots.$$

$$\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))\leq\Gamma^p\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2)) \text{ for } p=1,2,3,\cdots. \tag{3.35}$$

Now using triangle inequality,

$$\begin{aligned}\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1}))\leq&\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))+\mathtt{d}(\mathbb{T}(h_{p+1},h_{p+2}),\\ &\mathbb{T}(h_{p+s},h_{p+s+1})),\\ \leq&\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))+\mathtt{d}(\mathbb{T}(h_{p+1},h_{p+2}),\\ &\mathbb{T}(h_{p+2},h_{p+3}))+\mathtt{d}(\mathbb{T}(h_{p+2},h_{p+3}),\mathbb{T}(h_{p+s},h_{p+s+1})),\\ \leq&\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+1},h_{p+2}))+\mathtt{d}(\mathbb{T}(h_{p+1},h_{p+2}),\\ &\mathbb{T}(h_{p+2},h_{p+3}))+\cdots+\mathtt{d}(\mathbb{T}(h_{p+s-1},h_{p+s}),\\ &\mathbb{T}(h_{p+s},h_{p+s+1})).\end{aligned}$$

We can write the above equation as

$$\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1}))\leq\sum_{k=p}^{p+s-1}\mathtt{d}(\mathbb{T}(h_k,h_{k+1}),\mathbb{T}(h_{k+1},h_{k+2})). \tag{3.36}$$

By using (3.35) in (3.36), we get

$$\begin{aligned}\mathtt{d}(\mathbb{T}(h_p,h_{p+1}),\mathbb{T}(h_{p+s},h_{p+s+1}))\leq&\sum_{k=p}^{p+s-1}\Gamma^k\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2)),\\ \leq&\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2))\Gamma^p\sum_{k=0}^{s-1}\Gamma^k,\\ \leq&\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(_1,h_2))\Gamma^p\{1+\Gamma^1+\Gamma^2+\cdots+\Gamma^{s-1}\},\\ \leq&\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2))\Gamma^p\frac{1-\Gamma^s}{1-\Gamma},\\ \leq&\mathtt{d}(\mathbb{T}(h_0,h_1),\mathbb{T}(h_1,h_2))\Gamma^p\{\frac{1}{1-\Gamma}\}.\end{aligned}$$

Now applying $\lim_{p\to\infty}$ in above inequality, the inequality deduced to,

$$\lim_{p\to\infty} \mathtt{d}(\mathbb{T}(h_p, h_{p+1}), \mathbb{T}(h_{p+s}, h_{p+s+1})) \leq 0.$$

$$\Rightarrow \lim_{p\to\infty} \mathtt{d}(\mathbb{T}(h_p, h_{p+1}), \mathbb{T}(h_{p+s}, h_{p+s+1})) = 0.$$

This proves that we get a Cauchy sequence $\{\mathbb{T}(h_{j-1}, h_j)\}$ in $\mathbb{K}$. Since $\mathbb{X}$ is complete metric space, therefore $\mathbb{T}(h_{j-1}, h_j) \to k_* \in \mathbb{K}$ and $\mathbb{T}(h_{j-1}, h_j) \in \mathbb{K}_0$.
Moreover,

$$\begin{aligned}
\mathtt{d}(k_*, \mathbb{H}) &\leq \mathtt{d}(k_*, h_{j+1}),\\
&\leq \mathtt{d}(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}(h_{j+1}, \mathbb{T}(h_{j-1}, h_j),\\
&= \mathtt{d}(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}(\mathbb{H}, \mathbb{K}), \quad \text{by} \quad (3.32)\\
&\leq \mathtt{d}(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}(k_*, \mathbb{H}).
\end{aligned}$$

Therefore, $\mathtt{d}(k_*, h_{j+1}) \quad \to \quad \mathtt{d}(k_*, \mathbb{H}) \quad \text{as} \quad j \to \infty$.

Since from assumption (v), the sequence $\{h_j\}$ has a subsequence $\{h_{j_l}\}$ which converges to an element $h_* \in \mathbb{H}$. Thus, we have:

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \lim_{l\to\infty} \mathtt{d}(h_{j_{l+1}}, \mathbb{T}(h_{j_{l-1}}, h_{j_l})) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

$$\Rightarrow \mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}).$$

□

Theorem 3.3.5.

Consider a complete metric space $(\mathbb{X}, \mathtt{d})$ endowed with graph $\mathbb{G}$. Suppose $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and $\mathtt{d}(w_1, \mathbb{T}(h_1, h_2)) \;=\; \mathtt{d}(\mathbb{H}, \mathbb{K}) = \mathtt{d}(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}(\mathbb{T}(h_2, h_3), \mathbb{T}(w_1, w_2)) \leq \Gamma \max\{\mathtt{d}(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3)), \mathtt{d}(\mathbb{T}(h_2, w_1), \mathbb{T}(h_3, w_2))\}, \tag{3.37}$$

where $\Gamma \in [0, 1)$.

Moreover, suppose that the following asumptions are true:

(i) $\mathbb{T}$ is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ satisfying $\mathtt{d}(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{H}$ is approximately compact with respect to $\mathbb{K}$;

(vi) When $\{h_j\}$, $\{\overline{h_j}\} \subseteq \mathbb{X}$ such that $h_j \to h$ and $\overline{h_j} \to \overline{h}$, then $\mathbb{T}(h_j, \overline{h_j}) \to \mathbb{T}(h, \overline{h})$,

then, $\exists$ a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. This theorem can be proved in a similar way to the proof of Theorem 3.3.4. □

Chapter 4

Prešić Form of Nonself Operators and Best Proximity Point Results in b-Metric Spaces

This chapter is about the extention of Prešić form of non self operators and BBP results in b-metric spaces.

4.1 BBP in b-Metric Spaces

Let $(\mathbb{X}, \mathtt{d}_b)$ is a b-metric space with coefficient $b \geq 1$ and $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ is a directed graph defined on $\mathbb{X}$. Consider $\mathbb{H}$ and $\mathbb{K}$ be two nonempty subsets of $\mathbb{X}$.
Define

$$\begin{gathered}
\mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \inf\{\mathtt{d}_b(h, k) : h \in \mathbb{H}, k \in \mathbb{K}\}, \\
\mathtt{d}_b(x_0, \mathbb{K}) = \inf\{\mathtt{d}_b(x_0, k) : k \in \mathbb{K}\}, \\
\mathbb{H}_0 = \{h \in \mathbb{H} : \mathtt{d}_b(h, k) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \text{ for some } k \in \mathbb{K}\}, \\
\mathbb{K}_0 = \{k \in \mathbb{K} : \mathtt{d}_b(h, k) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \text{ for some } h \in \mathbb{H}\}.
\end{gathered}$$

Definition 4.1.1.

Consider a b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$ endowed with the $\mathbb{G}$ graph.

Suppose $\mathbb{H}$ and $\mathbb{K}$ are nonempty subsets of $\mathbb{X}$. A mapping $\mathbb{T} : \mathbb{H} \to \mathbb{K}$ has the BBP $h_* \in \mathbb{H}$ if

$$\mathtt{d}_b(h_*, \mathbb{T}(h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \tag{4.1}$$

where

$$\mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \inf\{\mathtt{d}_b(h, k) : h \in \mathbb{H}, k \in \mathbb{K}\}.$$

Definition 4.1.2.

Consider a b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$, where b-metric is continuous, and consider two nonempty sets $\mathbb{H}$, $\mathbb{K} \subseteq \mathbb{X}$. $\mathbb{K}$ is called approximately compact with respect to $\mathbb{H}$, when each $\{k_n\} \subseteq \mathbb{K}$ with $\mathtt{d}(h, k_n) \to \mathtt{d}(h, \mathbb{K})$, for some $h \in \mathbb{H}$, has a convergent subsequence.

Definition 4.1.3.

Let $(\mathbb{X}, \mathtt{d}_b)$ be a b-metric space with coefficient $b \geq 1$ with the $\mathbb{G}$ graph, where b-metric is continuous, and $\mathbb{H}$, $\mathbb{K}$ are nonempty subsets of $\mathbb{X}$. A mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ is called path admissible, when:

$$\begin{cases} \mathtt{d}_b(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \\ \mathtt{d}_b(w_2, \mathbb{T}(h_2, h_3)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \\ h_1 P h_3, \end{cases} \quad \Rightarrow (w_1, w_2) \in \mathbb{E},$$

where $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ and $h_1 P h_3$.

$\Rightarrow \quad h_1, h_2, h_3 \in \mathbb{V}$ and $(h_1, h_2) \in \mathbb{E}$ and $(h_2, h_3) \in \mathbb{E}$.

4.2 BBP Results in b-Metric Spaces

Theorem 4.2.1.

Consider a complete b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$ endowed with $\mathbb{G}$ graph, where b-metric is continuous. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and

$\mathtt{d}_b(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \mathtt{d}_b(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}_b(w_1, w_2) \leq \Gamma \max\{\mathtt{d}_b(h_1, h_2), \mathtt{d}_b(h_2, h_3)\}, \tag{4.2}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true:

(i) Mapping is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ which satisfy $\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{K}$ is approximately compact with respect to $\mathbb{H}$;

(vi) When $\{h_j\} \subseteq \mathbb{X}$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to x_*$ as $j \to \infty$, then $(h_j, x_*) \in \mathbb{E}$ for all $j \in \mathbb{N}$ and $(x_*, x_*) \in \mathbb{E}$,

then, there exists a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From condition (ii), we have $h_0, h_1, h_2 \in \mathbb{H}$ satisfying

$$\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2,$$

that is $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From condition (iv), $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

Since from condition (i), we have $(h_2, h_3) \in \mathbb{E}$. Hence, $h_1 P h_3$.

By continuing same process, we build a sequence $\{h_{j\geq 2}\} \subseteq \mathbb{H}$ which satisfy

$$\mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \text{ for each } j \in \mathbb{N}, \tag{4.3}$$

and $h_{j-1} P h_{j+1}$, that is $(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E} \ \forall \ j \in \mathbb{N}$.

From (4.2), we have:

$$\mathtt{d}_b(h_j, h_{j+1}) \leq \Gamma \max\{\mathtt{d}_b(h_{j-2}, h_{j-1}), \mathtt{d}_b(h_{j-1}, h_j)\} \text{ for each } j = 2, 3, 4, \cdots . \tag{4.4}$$

For convenience, we take $\mathtt{d}_j = \mathtt{d}_b(h_j, h_{j+1})$ for each $j \in \mathbb{N} \cup \{0\}$.

Then we can rewrite (4.4) as

$$\mathtt{d}_j \leq \Gamma \max\{\mathtt{d}_{j-2}, \mathtt{d}_{j-1}\} \text{ for each } j = 2, 3, 4, \cdots . \tag{4.5}$$

It is obviously true for $j = 0, 1$ because if we consider

$$Z = \max\{\mathtt{d}_0/\psi, \mathtt{d}_1/\psi^2\}, \quad \text{where } \psi = \Gamma^{1/2}.$$

Since Z is $\max\{\mathtt{d}_0/\psi, \mathtt{d}_1/\psi^2\}$, then

$$\mathtt{d}_0 \leq Z\psi \quad \text{and} \quad \mathtt{d}_1 \leq Z\psi_2.$$

Take $j = 2, 3, 4, \cdots$ in (4.5), we get:

$$\begin{aligned}
\mathtt{d}_2 &\leq \Gamma \max\{\mathtt{d}_0, \mathtt{d}_1\} \leq \Gamma \max\{Z\psi, Z\psi^2\} \leq \Gamma Z\psi = Z\psi^3, \\
\mathtt{d}_3 &\leq \Gamma \max\{\mathtt{d}_1, \mathtt{d}_2\} \leq \Gamma \max\{Z\psi^2, Z\psi^3\} \leq \Gamma Z\psi^2 = Z\psi^4, \\
\mathtt{d}_4 &\leq \Gamma \max\{\mathtt{d}_2, \mathtt{d}_3\} \leq \Gamma \max\{Z\psi^3, Z\psi^4\} \leq \Gamma Z\psi^3 = Z\psi^5, \\
&\vdots \\
\mathtt{d}_j &\leq \Gamma \max\{\mathtt{d}_{j-1}, \mathtt{d}_{j-2}\} \leq \Gamma \max\{Z\psi^j, Z\psi^{j-1}\} \leq \Gamma Z\psi^{j-1} = Z\psi^{j+1}.
\end{aligned}$$

Thus, by using induction we have,

$$\mathtt{d}_{j-1} \leq Z\psi^j \quad \forall \quad j \in \mathbb{N}.$$

$$\Rightarrow \quad \mathtt{d}_b(h_{j-1}, h_j) \;\leq\; Z\psi^j \quad \forall \quad j \in \mathbb{N}. \tag{4.6}$$

By using triangle inequality we get

$$\begin{aligned}
\mathtt{d}_b(h_j, h_{j+q}) &\leq b\{\mathtt{d}_b(h_j, h_{j+1}) + \mathtt{d}_b(h_{j+1}, h_{j+q})\},\\
&= b\mathtt{d}_b(h_j, h_{j+1}) + b\mathtt{d}_b(h_{j+1}, h_{j+q}),\\
&\leq b\mathtt{d}_b(h_j, h_{j+1}) + bb\{\mathtt{d}_b(h_{j+1}, h_{j+2}) + \mathtt{d}_b(h_{j+2}, h_{j+q})\},\\
&= b\mathtt{d}_b(h_j, h_{j+1}) + b^2\mathtt{d}_b(h_{j+1}, h_{j+2}) + b^2\mathtt{d}_b(h_{j+2}, h_{j+q}),\\
&\leq b\mathtt{d}_b(h_j, h_{j+1}) + b^2\mathtt{d}_b(h_{j+1}, h_{j+2}) + \cdots + b^q\mathtt{d}_b(h_{j+q-1}, h_{j+q}),\\
&\leq bZ\psi^{j+1} + b^2Z\psi^{j+2} + b^3Z\psi^{j+3} + \cdots + b^qZ\psi^{j+q}, \quad \text{by (4.6)}\\
&\leq b\psi^{j+1}\{1 + b\psi + b^2\psi^2 \cdots + b^{q-1}\psi^{q-1}\}Z,\\
&\leq \frac{1-(b\psi)^q}{1-b\psi}\; Zb\psi^{j+1},\\
&< \frac{b\psi^{j+1}}{1-b\psi}\; Z.
\end{aligned}$$

Note that $\psi \;=\; \Gamma^{1/2} < 1$.

Letting $j \to \infty$,

$$\lim_{j\to\infty} \mathtt{d}_b(h_j, h_{j+q}) \;\leq\; \lim_{j\to\infty} \frac{b\psi^{j+1}}{1-b\psi}\; Z,$$

$$\Rightarrow \quad \lim_{j\to\infty} \mathtt{d}_b(h_j, h_{j+q}) = 0.$$

Therefore, we get b-Cauchy sequence $\{h_j\} \in \mathbb{H}$. Thus, $\exists$ an element $h_* \in \mathbb{H}$ such that $h_j \to h_*$ and $h_j \in \mathbb{H}_0$ which satisfy

$$\mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)),$$

that is, $(h_j, h_*) \in \mathbb{E}$.

Furthermore, we have to prove that $\mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)) \to \mathtt{d}_b(h_*, \mathbb{K}) \quad as \quad j \to \infty$.

$$\begin{aligned}
\mathtt{d}_b(h_*, \mathbb{K}) &\leq \mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)),\\
&\leq b\{\mathtt{d}_b(h_*, h_{j+1}) + \mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)\},\\
&= b\mathtt{d}_b(h_*, h_{j+1}) + b\mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{by (4.3)}\\
&\leq b\mathtt{d}(h_*, h_{j+1}) + b\mathtt{d}_b(h_*, \mathbb{K}).
\end{aligned}$$

Therefore, $\mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)) \to \mathtt{d}_b(h_*, \mathbb{K}) \quad as \quad j \to \infty.$

Since hyphothesis (v) hold, the sequence $\mathbb{T}(h_{j-1}, h_j)$ has a subsequence $\mathbb{T}(h_{j_{m-1}}, h_{j_m})$, which converges to a point $k_* \in \mathbb{K}$.

$$\Rightarrow \mathtt{d}_b(h_*, k_*) = \lim_{m\to\infty} \mathtt{d}_b(h_{j_{m+1}}, \mathbb{T}(h_{j_{m-1}}, h_{j_m}) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

We already have

$$\mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}),$$

that is, $(h_{j-1}, h_j), (h_j, h_*) \in \mathbb{E}$.

Hence, $h_* \in \mathbb{H}_0$. As we know $\mathbb{T}(h_j, h_*) \in \mathbb{K}_0$, we have $g \in \mathbb{H}$ satisfying

$$\mathtt{d}_b(g, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}). \tag{4.7}$$

Condition (vi) implies $(h_j, h_*) \in \mathbb{E} \;\; \forall \;\; j \in \mathbb{N}$. Thus, we have

$$\mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad \mathtt{d}_b(g, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \;\; \forall \;\; j \in \mathbb{N}.$$

Hence, we get $h_{j-1}Ph_*, \Rightarrow (h_{j-1}, h_j), (h_j, h_*) \in \mathbb{E}, \;\; \forall \;\; j \in \mathbb{N}$.

Hence, from (4.2),

$$\mathtt{d}_b(h_{j+1}, g) \leq \Gamma \max\{\mathtt{d}_b(h_{j-1}, h_j), \mathtt{d}_b(h_j, h_*)\} \text{ for each } j = 2, 3, 4, \cdots.$$

Letting $j \to \infty$,

$$\lim_{j\to\infty} \mathtt{d}_b(h_{j+1}, g) \leq \Gamma \lim_{j\to\infty} \max\{\mathtt{d}_b(h_{j-1}, h_j), \mathtt{d}_b(h_j, h_*)\},$$

$$\Rightarrow \mathtt{d}_b(h_*, g) = 0, \quad \text{that is} \quad g = h_*.$$

Put $g = h_*$ in (4.7),

$$\mathtt{d}_b(h_*, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}),$$

that is, $(h_*, h_*) \in \mathbb{H}$.

Furthermore, we know that $\mathbb{T}(h_*, h_*) \in \mathbb{K}_0$, and we have an element $t \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(t, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}). \tag{4.8}$$

Condition (vi) implies that, $(h_*, h_*) \in \mathbb{E}$. Hence, we have

$$\mathtt{d}_b(t, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad \mathtt{d}_b(h_*, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \quad \text{for each} \quad j \in \mathbb{N}.$$

Thus we get $h_j P h_*$ for each $j \in \mathbb{N}$, that is, $(h_j, h_*), (h_*, h_*) \in \mathbb{E}$ for each $j \in \mathbb{N}$
Thus, from (4.2), we get:

$$\begin{aligned} \mathtt{d}_b(h_*, t) &\leq \Gamma \ \max\{\mathtt{d}_b(h_j, h_*), \mathtt{d}_b(h_*, h_*)\} \text{ for each } \ j \in \mathbb{N}, \\ &= \Gamma \ \max\{\mathtt{d}_b(h_j, h_*), \mathtt{d}_b(h_*, h_*)\} \ \to 0 \ \text{ as } \ j \to \infty, \\ &\Rightarrow \ \mathtt{d}_b(h_*, t) = 0, \qquad \text{that is} \qquad t = h_*. \end{aligned}$$

Put $t = h_*$ (4.8), we get

$$\mathtt{d}_b(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

□

Theorem 4.2.2.

Consider a complete b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$ endowed with $\mathbb{G}$ graph, where b-metric is continuous. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \to \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and
$\mathtt{d}_b(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \mathtt{d}_b(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}_b(h_3, w_2) \leq \Gamma \max\left\{\mathtt{d}_b(h_1, h_2), \mathtt{d}_b(h_2, w_1)\right\}, \tag{4.9}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.
Furthermore, suppose that the subsequent conditions are true:

(i) Mapping is path admissible;

(ii) $\exists \ h_0, h_1, h_2 \in \mathbb{H}$ which satisfy $\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{K}$ is approximately compact with respect to $\mathbb{H}$;

(vi) When $\{h_j\} \subseteq \mathbb{X}$ such that $h_j P h_{j+2}$ for each $j \in \mathbb{N}$ and $h_j \to x_*$ as $n \to \infty$, then $(h_j, x_*) \in \mathbb{E}$ for all $j \in \mathbb{N}$ and $(x_*, x_*) \in \mathbb{E}$,

then, there exists a point $h_* \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From condition (ii), we have $h_0, h_1, h_2 \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2,$$

that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From condition (iv), we have $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

Since $\mathbb{T}$ is path admissible, so $(h_2, h_3) \in \mathbb{E}$. Hence, $h_1 P h_3$.

By continuing the same procedure, we build a sequence $\{h_{j \geq 2}\} \subseteq \mathbb{H}$ which satisfy:

$$\mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \ \text{ for each } \ j \in \mathbb{N}, \tag{4.10}$$

and $h_{j-1} P h_{j+1}$, that is $\quad (h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E} \ \forall \ j \in \mathbb{N}$.

From (4.9), we have:

$$\mathtt{d}_b(h_j, h_{j+1}) \ \leq \Gamma \ \max\{\mathtt{d}_b(h_{j-2}, h_{j-1}), \mathtt{d}_b(h_{j-1}, h_j)\} \text{ for each } j = 2, 3, 4, \cdots . \tag{4.11}$$

For convenience, we take $\mathtt{d}_j = \mathtt{d}_b(h_j, h_{j+1}) \;\; \forall \;\; j \in \mathbb{N} \cup \{0\}$.

Then we can rewrite (4.11) as

$$\mathtt{d}_j \;\leq\; \Gamma \max\{\mathtt{d}_{j-2}, \mathtt{d}_{j-1}\} \;\text{ for each }\; j = 2, 3, 4, \cdots . \tag{4.12}$$

From the (4.12) and Theorem 3.3.1, $\{h_j\}$ is a b-Cauchy sequence in $\mathbb{H}$ such that $h_j \to h_*$ and $h_* \in \mathbb{H}_0$. As $\mathbb{T}(h_j, h_*) \in \mathbb{K}_0$, we have $w \in \mathbb{H}$ satisfying

$$\mathtt{d}_b(w, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}). \tag{4.13}$$

From assumption (vi), we get $(h_j, h_*) \in \mathbb{E} \; \forall \; j \in \mathbb{N}$ and we already have

$$\mathtt{d}_b(h_*, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

Thus, we get $h_{j-1} P h_*$, that is $(h_{j-1}, h_j), (h_j, h_*) \in \mathbb{E}, \forall \; j \in \mathbb{N}$.
Hence, from (4.9), we get:

$$\mathtt{d}_b(h_*, w) \;\leq \Gamma \max\{\mathtt{d}_b(h_{j-1}, h_j), \mathtt{d}_b(h_j, h_{j+1})\} \;\text{ for each }\; j \in \mathbb{N}. \tag{4.14}$$

Taking $j \to \infty$ on (4.14) ,

$$\lim_{j\to\infty} \mathtt{d}_b(h_*, w) \;\leq\; \Gamma \max \lim_{j\to\infty} \{\mathtt{d}_b(h_{j-1}, h_j), \mathtt{d}_b(h_j, h_{j+1})\} \quad \forall \quad j \in \mathbb{N}.$$

$$\Rightarrow \;\; \lim_{j\to\infty} \mathtt{d}_b(h_*, w) \;\leq 0,$$

$$\Rightarrow \;\; \mathtt{d}_b(h_*, w) = 0, \quad \text{that is} \quad w = h_*.$$

Using $w = h_*$ in (4.13),

$$\mathtt{d}_b(h_*, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

That is, $(h_*, h_*) \in \mathbb{E}$. Furthermore, note that $\mathbb{T}(h_*, h_*) \in \mathbb{K}_0$, and there is $q \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(q, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}). \tag{4.15}$$

Condition (vi) implies, $(h_*, h_*) \in \mathbb{E}$. Hence, we have

$$\mathtt{d}_b(h_*, \mathbb{T}(h_j, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad \mathtt{d}_b(q, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}),$$

$$\text{and } h_j P h_*, \quad \text{that is} \quad (h_j, h_*) \in \mathbb{E} \text{ and } (h_*, h_*) \in \mathbb{E}.$$

Thus, from (4.9),

$$\mathtt{d}_b(h_*, q) \ \leq \Gamma \max\{\mathtt{d}_b(h_j, h_*), \mathtt{d}(h_*, h_*)\} \ \text{ for each } \ j \ \in \ \mathbb{N}.$$

$$\mathtt{d}_b(h_*, q) \ \leq \Gamma \max\{\mathtt{d}_b(h_j, h_*), \mathtt{d}_b(h_*, h_*)\} \ \rightarrow \ 0 \text{ as } j \rightarrow \infty.$$

$$\mathtt{d}_b(h_*, q) = 0, \quad \textit{that is} \quad q = h_*.$$

Thus, by putting $q = h_*$ in (4.15) we have

$$\mathtt{d}_b(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

□

Theorem 4.2.3.

Consider a complete b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$ endowed with $\mathbb{G}$ graph, where b-metric is continuous. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and
$\mathtt{d}_b(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \mathtt{d}_b(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}_b(\mathbb{T}(h_2, w_1), \mathbb{T}(h_3, w_2)) \leq \Gamma\{\mathtt{d}_b(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3))\}, \tag{4.16}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.
Furthermore, suppose that the subsequent conditions are true:

(i) Mapping is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ which satisfy $\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{H}$ is approximately compact with respect to $\mathbb{K}$;

(vi) When $\{h_j\}$ and $\{\overline{h_j}\}$ are in $\mathbb{X}$ such that $h_j \to h$ and $\overline{h_j} \to \overline{h}$, then $\mathbb{T}(h_j, \overline{h_j}) \to \mathbb{T}(h, \overline{h})$,

then, there exists a point $h_* \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From condition (ii), we have $h_0, h_1, h_2 \in \mathbb{H}$ satisfying

$$\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2,$$

that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From condition (iv), $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

Since $\mathbb{T}$ is path admissible, so we have $(h_2, h_3) \in \mathbb{E}$. Hence, $h_1 P h_3$.

By similar method, we build a sequence $\{h_{j \geq 2}\} \subseteq \mathbb{H}$ which satisfy:

$$\mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \ \ \forall \ \ j \in \mathbb{N}, \tag{4.17}$$

and $h_{j-1} P h_{j+1}$, that is, $(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E} \ \ \forall \ \ j \in \mathbb{N}$.

From (4.16), we have:

$$\mathtt{d}_b(h_j, h_{j+1}) \ \leq \Gamma \ \{\mathtt{d}_b(h_{j-2}, h_{j-1}), \mathtt{d}_b(h_{j-1}, h_j)\} \text{ for each } j = 2, 3, 4, \cdots . \tag{4.18}$$

For convenience, we take $\mathbb{T}_{j-1} = \mathbb{T}(h_{j-1}, h_j)$ for each $j = 2, 3, 4, \cdots$.

Then rewrite (4.18) as

$$\mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j) \leq \Gamma \mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}) \text{ for each } j = 2, 3, 4, \cdots. \tag{4.19}$$

By using induction, we get

$$\begin{aligned}
\mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j) \leq & \Gamma\ \mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \\
\leq & \Gamma(\Gamma\ \mathtt{d}_b(\mathbb{T}_{j-3}, \mathbb{T}_{j-2})), \\
= & \Gamma^2 \mathtt{d}_b(\mathbb{T}_{j-3}, \mathbb{T}_{j-2}), \\
\leq & \Gamma^2 \Gamma \mathtt{d}_b(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \\
= & \Gamma^3 \mathtt{d}_b(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \\
& \vdots \\
\leq & \Gamma^{j-1} \mathtt{d}_b(\mathbb{T}_1, \mathbb{T}_0) \text{ for } j = 2, 3, 4, \cdots.
\end{aligned}$$

Hence,

$$\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) \leq \Gamma^j \mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1) \quad \text{for} \quad j = 1, 2, 3, \cdots. \tag{4.20}$$

By using triangle inequality,

$$\begin{aligned}
\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+p}) &\leq b\{\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + \mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+p})\}, \\
&= b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + b\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+p}), \\
&\leq b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + bb\{\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+2}) + \mathtt{d}_b(\mathbb{T}_{j+2}\mathbb{T}_{j+p})\}, \\
&= b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + b^2\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+2}) + b^2\mathtt{d}_b(\mathbb{T}_{j+2}\mathbb{T}_{j+p}), \\
&\leq b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + b^2\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+2}) + \cdots + \mathtt{d}_b(\mathbb{T}_{j+p-1}\mathbb{T}_{j+p}).
\end{aligned} \tag{4.21}$$

By using (4.20) in (4.21), we get

$$\begin{aligned}
\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+p}) \leq & b\Gamma^j \mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1) + b^2\Gamma^{j+1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1) + \cdots + b^p\Gamma^{j+p-1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1), \\
= & b\Gamma^j \mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1)(1 + b\Gamma + b^2\Gamma^2 + \cdots + b^{p-1}\Gamma^{p-1}), \\
\leq & b\Gamma^{j+1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1)\frac{1-(b\Gamma)^p}{1-\Gamma}, \\
< & b\Gamma^{j+1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1)\frac{1}{1-\Gamma}.
\end{aligned}$$

Applying $\lim_{j\to\infty}$ on above inequality , the inequality deduced to,

$$\lim_{j\to\infty} \mathtt{d}_b(\mathbb{T}(h_j, \mathbb{T}_{j+1}), \mathbb{T}(h_{j+p}, h_{j+p+1}) \leq 0.$$

$$\Rightarrow \lim_{j\to\infty} \mathtt{d}_b(\mathbb{T}(h_j, h_{j+1}), \mathbb{T}(h_{j+p}, h_{j+p+1}) = 0.$$

This proves that, we get a b-Cauchy sequence $\mathbb{T}(h_{j-1}, h_j)$ in closed subset of $\mathbb{K}$. Since $\mathbb{X}$ is complete, consider $k_* \in \mathbb{K}$ such that $\mathbb{T}(h_{j-1}, h_j) \to k_*$.
Moreover,

$$\begin{aligned}
\mathtt{d}_b(k_*, \mathbb{H}) \leq \mathtt{d}_b(k_*, h_{j+1}), & \\
&\leq b\{\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)\}, \\
&= b\{\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}_b(\mathbb{H}, \mathbb{K})\}, \text{ by (4.17)} \\
&= b\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + b\mathtt{d}_b(\mathbb{H}, \mathbb{K}), \\
&\leq b\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + b\mathtt{d}_b(k_*, \mathbb{H}).
\end{aligned}$$

Therefore, $\mathtt{d}_b(k_*, h_{j+1}) \to \mathtt{d}_b(k_*, \mathbb{H})$ as $j \to \infty$.
Since from condition (v), $\{h_j\}$ has a subsequence $\{h_{j_l}\}$ that converges an element $h_* \subseteq \mathbb{H}$ such that:

$$\mathtt{d}_b(h_*, \mathbb{T}(h_*, h_*)) = \lim_{l\to\infty} \mathtt{d}_b(h_{j_{l+1}}, \mathbb{T}(h_{j_{l-1}}, h_{j_l})) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

$$\Rightarrow \mathtt{d}_b(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

□

Theorem 4.2.4.
Consider a complete b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$ endowed with $\mathbb{G}$ graph, where b-metric is continuous. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H}\times\mathbb{H} \to \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and
$\mathtt{d}_b(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \mathtt{d}_b(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}_b(\mathbb{T}(h_2, w_1), \mathbb{T}(h_3, w_2)) \leq \Gamma \max\{\mathtt{d}_b(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3)), \mathtt{d}_b(\mathbb{T}(h_2, h_3), \mathbb{T}(w_1, w_2))\}, \tag{4.22}$$

where $\Gamma \in [0,1)$ such that $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true:

(i) Mapping is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ which satisfy $\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{H}$ is approximately compact with respect to $\mathbb{K}$;

(vi) When $\{h_j\}$, $\{\overline{h_j}\}$ in $\mathbb{X}$ such that $h_j \to h$ and $\overline{h_j} \to \overline{h}$, then $\mathbb{T}(h_j, \overline{h_j}) \to \mathbb{T}(h, \overline{h})$,

then, there exists a point $h_* \in\ \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. From condition (ii), $h_0, h_1, h_2 \in \mathbb{H}$ satisfying

$$\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}), \quad \text{and} \quad h_0 P h_2,$$

that is, $(h_0, h_1), (h_1, h_2) \in \mathbb{E}$.

From condition (iv), $\mathbb{T}(h_1, h_2) \in \mathbb{K}_0$, and by the definition of $\mathbb{K}_0$, we have $h_3 \in \mathbb{H}$ which satisfy

$$\mathtt{d}_b(h_3, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

Since $\mathbb{T}$ is path admissible, so we get $(h_2, h_3) \in \mathbb{E}$. Hence, $h_1 P h_3$.

By similar method, we build a sequence $\{h_{j \geq 2}\}$ in $\mathbb{H}$ which satisfy:

$$\mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) \ \ \forall \ \ j \in \mathbb{N}, \tag{4.23}$$

and $h_{j-1}Ph_{j+1}$, that is $(h_{j-1}, h_j), (h_j, h_{j+1}) \in \mathbb{E} \;\forall\; j \in \mathbb{N}$.
From (4.22), we have:

$$\begin{aligned} \mathtt{d}_b(\mathbb{T}(h_{j-1}, h_j), \mathbb{T}(h_j, h_{j+1})) \;&\leq \Gamma \max\{\mathtt{d}_b(\mathbb{T}(h_{j-2}, h_{j-1})), \mathbb{T}(h_{j-1}, h_j)), \\ &\qquad \mathtt{d}_b(\mathbb{T}(h_{j-1}, h_j), \mathbb{T}(h_j, h_{j+1}))\}. \end{aligned} \tag{4.24}$$

For convenience, we take $\mathbb{T}_{j-1} = \mathbb{T}(h_{j-1}, h_j)$ for each $j = 2, 3, 4, \cdots$.
Then rewrite (4.24) as

$$\mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j) \;\leq\; \Gamma \max\{\mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}(\mathbb{T}_{j-1}, \mathbb{T}_j)\}.$$

Either

$$\max\{\mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j)\} = \mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}) \text{ for each } j = 2, 3, 4, \cdots,$$

or ,

$$\max\{\mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j)\} = \mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j) \text{ for each } j = 2, 3, 4, \cdots.$$

If we take $\max\{\mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j)\} = \mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j)$, which is contradiction.
Hence we have:

$$\mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}(h_j) \;\leq\; \Gamma \mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}) \text{ for each } j = 2, 3, 4, \cdots. \tag{4.25}$$

By using induction, we get

$$\begin{aligned} \mathtt{d}_b(\mathbb{T}_{j-1}, \mathbb{T}_j) &\leq \Gamma\; \mathtt{d}_b(\mathbb{T}_{j-2}, \mathbb{T}_{j-1}), \\ &\leq \Gamma(\Gamma\; \mathtt{d}_b(\mathbb{T}_{j-3}, \mathbb{T}_{j-2})), \\ &= \Gamma^2 \mathtt{d}_b(\mathbb{T}_{j-3}, \mathbb{T}_{j-2}), \\ &\leq \Gamma^2 \Gamma \mathtt{d}_b(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \\ &= \Gamma^3 \mathtt{d}_b(\mathbb{T}_{j-4}, \mathbb{T}_{j-3}), \\ &\;\vdots \\ &\leq \Gamma^{j-1} \mathtt{d}_b(\mathbb{T}_1, \mathbb{T}_0) \text{ for } j = 2, 3, 4, \cdots. \end{aligned}$$

Hence,

$$\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) \leq \Gamma^j \mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1) \quad \text{for} \quad j = 1, 2, 3, \cdots . \tag{4.26}$$

By using triangle inequality,

$$\begin{aligned}
\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+p}) &\leq b\{\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + \mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+p})\}, \\
&= b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + b\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+p}), \\
&\leq b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + bb\{\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+2}) + \mathtt{d}_b(\mathbb{T}_{j+2}\mathbb{T}_{j+p})\}, \\
&= b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + b^2\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+2}) + b^2\mathtt{d}_b(\mathbb{T}_{j+2}\mathbb{T}_{j+p}), \\
&\leq b\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+1}) + b^2\mathtt{d}_b(\mathbb{T}_{j+1}, \mathbb{T}_{j+2}) + \cdots + \mathtt{d}_b(\mathbb{T}_{j+p-1}\mathbb{T}_{j+p}).
\end{aligned} \tag{4.27}$$

By using (4.26) in (4.27), we get

$$\begin{aligned}
\mathtt{d}_b(\mathbb{T}_j, \mathbb{T}_{j+p}) &\leq b\Gamma^j\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1) + b^2\Gamma^{j+1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1) + \cdots + b^p\Gamma^{j+p-1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1), \\
&= b\Gamma^j\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1)(1 + b\Gamma + b^2\Gamma^2 + \cdots + b^{p-1}\Gamma^{p-1}), \\
&\leq b\Gamma^{j+1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1)\frac{1-(b\Gamma)^p}{1-\Gamma}, \\
&< b\Gamma^{j+1}\mathtt{d}_b(\mathbb{T}_0, \mathbb{T}_1)\frac{1}{1-\Gamma}.
\end{aligned}$$

Applying $\lim_{j\to\infty}$ on above inequality , the inequality deduced to,

$$\lim_{j\to\infty} \mathtt{d}_b(\mathbb{T}(h_j, \mathbb{T}_{j+1}), \mathbb{T}(h_{j+p}, h_{j+p+1}) \leq 0.$$

$$\Rightarrow \lim_{j\to\infty} \mathtt{d}_b(\mathbb{T}(h_j, h_{j+1}), \mathbb{T}(h_{j+p}, h_{j+p+1}) = 0.$$

This proves that, we get a b-Cauchy sequence $\mathbb{T}(h_{j-1}, h_j)$ in closed subset of $\mathbb{K}$. Since $\mathbb{X}$ is complete, consider $k_* \in \mathbb{K}$ such that $\mathbb{T}(h_{j-1}, h_j) \to k_*$.

Moreover,

$$\begin{aligned}
\mathtt{d}_b(k_*, \mathbb{H}) &\leq \mathtt{d}_b(k_*, h_{j+1}), \\
&\leq b\{\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}_b(h_{j+1}, \mathbb{T}(h_{j-1}, h_j))\}, \\
&= b\{\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + \mathtt{d}_b(\mathbb{H}, \mathbb{K})\}, \quad \text{by (4.23)} \\
&= b\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + b\mathtt{d}_b(\mathbb{H}, \mathbb{K}), \\
&\leq b\mathtt{d}_b(k_*, \mathbb{T}(h_{j-1}, h_j)) + b\mathtt{d}_b(k_*, \mathbb{H}).
\end{aligned}$$

Therefore, $\mathtt{d}_b(k_*, h_{j+1}) \quad \rightarrow \quad \mathtt{d}_b(k_*, \mathbb{H}) \qquad \text{as} \qquad j \rightarrow \infty$.

Condition (iv) implies, $\{h_j\}$ has a subsequence $\{h_{j_l}\}$ that converges an element $h_* \in \mathbb{H}$ such that:

$$\mathtt{d}_b(h_*, \mathbb{T}(h_*, h_*)) = \lim_{l\to\infty} \mathtt{d}_b(h_{j_{l+1}}, \mathbb{T}(h_{j_{l-1}}, h_{j_l})) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

$$\Rightarrow \mathtt{d}_b(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}).$$

□

Theorem 4.2.5.

Consider a complete b-metric space $(\mathbb{X}, \mathtt{d}_b)$ with coefficient $b \geq 1$ endowed with $\mathbb{G}$ graph, where b-metric is continuous. Suppose that $\mathbb{H}, \mathbb{K} \neq \phi$, where $\mathbb{H}, \mathbb{K} \subseteq \mathbb{X}$ are closed. Consider a mapping $\mathbb{T} : \mathbb{H}\times\mathbb{H} \rightarrow \mathbb{K}$ such that for each $h_1, h_2, h_3, w_1, w_2 \in \mathbb{H}$ with $h_1 P h_3$, that is, $(h_1, h_2), (h_2, h_3) \in \mathbb{E}$, and
$\mathtt{d}_b(w_1, \mathbb{T}(h_1, h_2)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K}) = \mathtt{d}_b(w_2, \mathbb{T}(h_2, h_3))$, we have:

$$\mathtt{d}_b(\mathbb{T}(h_2, h_3), \mathbb{T}(w_1, w_2)) \leq \Gamma \max\{\mathtt{d}_b(\mathbb{T}(h_1, h_2), \mathbb{T}(h_2, h_3)), \mathtt{d}_b(\mathbb{T}(h_2, w_1), \mathbb{T}(h_3, w_2))\}, \tag{4.28}$$

where $\Gamma \in [0, 1)$ such that $b\Gamma < 1$.

Furthermore, suppose that the subsequent conditions are true:

(i) Mapping is path admissible;

(ii) $\exists\ h_0, h_1, h_2 \in \mathbb{H}$ which satisfy $\mathtt{d}_b(h_2, \mathbb{T}(h_0, h_1)) = \mathtt{d}_b(\mathbb{H}, \mathbb{K})$ and $h_0 P h_2$;

(iii) $\mathbb{H}_0$ is nonempty;

(iv) $\mathbb{T}(\mathbb{H} \times \mathbb{H}_0) \subseteq \mathbb{K}_0$;

(v) $\mathbb{H}$ is approximately compact with respect to $\mathbb{K}$;

(vi) When $\{h_j\}$, $\{\overline{h_j}\} \subseteq \mathbb{X}$ such that $h_j \rightarrow h$ and $\overline{h_j} \rightarrow \overline{h}$, then $\mathbb{T}(h_j, \overline{h_j}) \rightarrow \mathbb{T}(h, \overline{h})$, then, there exists a point $h_* \in \mathbb{H}$ which satisfy

$$\mathtt{d}(h_*, \mathbb{T}(h_*, h_*)) = \mathtt{d}(\mathbb{H}, \mathbb{K}),$$

that is, $\mathbb{T}$ has a BPP.

Proof. This theorem can be proved in a similar way to the proof of Theorem 4.2.4. □

Chapter 5

Conclusion

The work of Ali et al. [35] on "Prešić type nonself operators and related best Proximity results" is discussed and elaborated to represent the complete analysis of the article [35] in this thesis.

The main purpose of this research was to discuss and extend the above results in b-metric spaces. For this, the definition of best proximity point is formulated in the setting of b-metric spaces. Then the fixed point theorems are established for Prešić form of non self operators and best proximity results in the setting of b-metric spaces. These results might be valuable in solving particular best proximity points in addition to fixed point theory in perception of b-metric spaces.

Bibliography

[1] H. Poincare, "Surless courbes define barles equations differentiate less," *J. de Math*, vol. 2, pp. 54–65, 1886.

[2] L. E. J. Brouwer, "Über abbildung von mannigfaltigkeiten," *Mathematische annalen*, vol. 71, no. 1, pp. 97–115, 1911.

[3] M. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, vol. 22, no. 1, pp. 1–72, 1906.

[4] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fund. math*, vol. 3, no. 1, pp. 133–181, 1922.

[5] R. Kannan, "Some results on fixed points," *Bull. Cal. Math. Soc.*, vol. 60, pp. 71–76, 1968.

[6] M. Khan, P. HK, and M. Khan, "Some fixed point theorems in metrically convex spaces," *Georgian Mathematical Journal*, vol. 7, no. 3, pp. 523–530, 2000.

[7] B. Samet, "Coupled fixed point theorems for a generalized meir–keeler contraction in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 12, pp. 4508–4517, 2010.

[8] B. S. Choudhury, "Unique fixed point theorem for weakly c-contractive mappings," *Kathmandu university journal of science, engineering and technology*, vol. 5, no. 1, pp. 6–13, 2009.

[9] P. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces," *Fixed Point Theory and Applications*, vol. 2008, no. 1, p. 406368, 2008.

[10] P. Z. Daffer and H. Kaneko, "Fixed points of generalized contractive multi-valued mappings," *Journal of Mathematical Analysis and Applications*, vol. 192, no. 2, 1995.

[11] I. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Func. An., Gos. Ped. Inst. Unianowsk*, vol. 30, pp. 26–37, 1989.

[12] M. Boriceanu, "Fixed point theory for multivalued generalized contraction on a set with two b-metrics.," *Studia Universitatis Babes-Bolyai, Mathematica*, no. 3, 2009.

[13] M. Boriceanu, A. Petrusel, and I. Rus, "Fixed point theorems for some multivalued generalized contractions in b-metric spaces," *International Journal of Mathematics and Statistics*, vol. 6, no. S10, pp. 65–76, 2010.

[14] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[15] R. Fagin and L. Stockmeyer, "Relaxing the triangle inequality in pattern matching," *International Journal of Computer Vision*, vol. 30, no. 3, pp. 219–231, 1998.

[16] S. Presic, "Sur la convergence des suites," *COMPTES RENDUS HEBDOMADAIRES DES SEANCES DE L ACADEMIE DES SCIENCES*, vol. 260, no. 14, p. 3828, 1965.

[17] V. Berinde and M. Păcurar, "Stability of k-step fixed point iterative methods for some prešić type contractive mappings," *Journal of Inequalities and Applications*, vol. 2014, no. 1, pp. 1–12, 2014.

[18] M. S. Khan, M. Berzig, and B. Samet, "Some convergence results for iterative sequences of prešić type and applications," *Advances in Difference Equations*, vol. 2012, no. 1, pp. 1–12, 2012.

[19] M. Păcurar, "Common fixed points for almost presić type operators," *Carpathian Journal of Mathematics*, pp. 117–126, 2012.

[20] S. Shukla and S. Radenović, "Some generalizations of prešić type mappings and applications," *Annals of the Alexandru Ioan Cuza University-Mathematics*, vol. 1, no. ahead-of-print, 2015.

[21] S. Shukla, R. Sen, and S. Radenović, "Set-valued prešić type contraction in metric spaces," *Annals of the Alexandru Ioan Cuza University-Mathematics*, vol. 1, no. ahead-of-print, 2014.

[22] K. Fan, "Extensions of two fixed point theorems of fe browder," *Mathematische zeitschrift*, vol. 112, no. 3, pp. 234–240, 1969.

[23] S. Reich, "Approximate selections, best approximations, fixed points, and invariant sets," *Journal of Mathematical Analysis and Applications*, vol. 62, no. 1, pp. 104–113, 1978.

[24] V. Sehgal and S. Singh, "A generalization to multifunctions of fan's best approximation theorem," *Proceedings of the American Mathematical Society*, vol. 102, no. 3, pp. 534–537, 1988.

[25] J. B. Prolla, "Fixed-point theorems for set-valued mappings and existence of best approximants," *Numerical Functional Analysis and Optimization*, vol. 5, no. 4, pp. 449–455, 1983.

[26] S. Sadiq Basha, "Full length article: Best proximity point theorems," *Journal of Approximation Theory*, vol. 163, no. 11, pp. 1772–1781, 2011.

[27] M. Jleli and B. Samet, "Best proximity points for α–ψ-proximal contractive type mappings and applications," *Bulletin des Sciences Mathematiques*, vol. 137, no. 8, pp. 977–995, 2013.

[28] J. Markin and N. Shahzad, "Best proximity points for relatively-continuous mappings in banach and hyperconvex spaces," in *Abstract and Applied Analysis*, vol. 2013, Hindawi, 2013.

[29] N. Hussain, M. Kutbi, and P. Salimi, "Best proximity point results for modified–proximal rational contractions," in *Abstract and Applied Analysis*, vol. 2013, Hindawi, 2013.

[30] C. Di Bari, T. Suzuki, and C. Vetro, "Best proximity points for cyclic meir–keeler contractions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 11, pp. 3790–3794, 2008.

[31] M. A. Alghamdi, M. A. Alghamdi, and N. Shahzad, "Best proximity point results in geodesic metric spaces," *Fixed Point Theory and Applications*, vol. 2012, no. 1, pp. 1–12, 2012.

[32] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[33] B. S. Choudhury, N. Metiya, M. Postolache, and P. Konar, "A discussion on best proximity point and coupled best proximity point in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2015, no. 1, pp. 1–17, 2015.

[34] G. K. Jacob, M. Postolache, M. Marudai, and V. Raja, "Norm convergence iterations for best proximity points of non-self non-expansive mappings," *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys*, vol. 79, no. 1, pp. 49–56, 2017.

[35] M. U. Ali, M. Farheen, T. Kamran, and G. Maniu, "Prešić type nonself operators and related best proximity results," *Mathematics*, vol. 7, no. 5, p. 394, 2019.

[36] J. A. Bondy and U. S. R. Murty, *Graph theory with applications*, vol. 290. Macmillan London, 1976.

[37] J. L. Gross and J. Yellen, *Handbook of graph theory.* CRC press, 2003.

[38] E. Kreyszig, *Introductory functional analysis with applications*, vol. 1. wiley New York, 1978.

[39] T. Kamran, M. Samreen, and Q. UL Ain, "A generalization of b-metric space and some fixed point theorems," *Mathematics*, vol. 5, no. 2, p. 19, 2017.

[40] A. Latif, V. Parvaneh, P. Salimi, and A. Al-Mazrooei, "Various suzuki type theorems in b-metric spaces," *J. Nonlinear Sci. Appl*, vol. 8, no. 4, pp. 363–377, 2015.

[41] C. Mongkolkeha, C. Kongban, and P. Kumam, "Existence and uniqueness of best proximity points for generalized almost contractions," in *Abstract and Applied Analysis*, vol. 2014, Hindawi, 2014.

[42] A. Eldered and P. Veeramani, "Convergence and existence for best proximity points," *J. Math. Anal. Appl*, vol. 323, no. 2, 2006.

[43] A. Hussain, M. Q. Iqbal, and N. Hussain, "Best proximity point results for suzuki-edelstein proximal contractions via auxiliary functions," *Filomat*, vol. 33, no. 2, 2019.

[44] S. S. Basha and N. Shahzad, "Best proximity point theorems for generalized proximal contractions," *Fixed Point Theory and Applications*, vol. 2012, no. 1, p. 42, 2012.