CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

Some Fixed Point Theorems in Fuzzy 2-Metric and Fuzzy 3-Metric spaces

by

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Declaration of Authorship

I, Haseeb Wali, declared that this thesis titled, 'Some Fixed Point Theorems in Fuzzy 2-Metric and Fuzzy 3-Metric spaces ' and the work presented in it are my own. I confirm that:

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“ Pure mathematics is, in its way, the poetry of logical ideas.”

Albert Einstein

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Abstract

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In this thesis we reviewed a Presic type contraction of $2k$-weakly compatible mapping by using φ-function in Fuzzy metric space. which is the generalization of Banach's contraction principle. We extend the notion of Presic type contraction of $2k$-weakly compatible mapping by using φ-function on Fuzzy 2-Metric, Fuzzy 3-Metric Spaces and obtain fixed point results. Finally, we prove fixed point theorems in Fuzzy 2-metric and Fuzzy 3-metric spaces.

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Haseeb Wali

Contents

Dedicated To My Parents.

Chapter 1

Introduction

The theory of fixed point is one of the most powerful tools of modern mathematics. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory is a beautiful mixture of analysis, topology and geometry. In particular fixed point theorem has been applied in such field as numerical methods like Newton-Raphson Method and establishing Picard's Existence Theorem regarding existence and uniqueness of solution of first order differential equation, existence of solution of integral equations and a system of linear equations. Fixed point theorems are powerful tools not only in mathematics but also economics. Fixed Point Theorems are used for the solution of the 1D wave equation. etc.

In 1886, Poincare [8] was the first to work in this field. Then Brouwer [15] in 1912, proved fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani [31]. Mean while Banach principle came in to existence which was considered as one of the fundamental principle in the field of functional analysis. In 1922, Banach [24] proved that a contraction mapping on a complete metric space possesses a unique fixed point. Later on it was developed by Kannan [23]. The fixed point theory (as well as Banach contraction principle) has been studied and generalized in different spaces and various fixed point theorem were developed. See for the instances the work presented in [5],[33].

In 1963 Gahler [28] introduced the notion of 2-metric spaces. Later, Dhage [4] presented the metric space generalization and called it D-metric space. Currently, some mathematicians worked on G-metric spaces and got some coincident point

theorems on G-metric spaces [35]. In 1989, Bakhtin [9] introduced the concept of a b-metric space as a generalization of metric spaces. In 1993, Czerwik [27] [26] extended many fixed point results in the setting of b-metric spaces. In 1994, Matthews [29] introduced the concept of partial metric space in which the self distance of any point of space may not be zero. In 1996, O'Neill generalized the concept of partial metric space by admitting negative distances. In 2013, Shukla [32] generalized both the concept of b-metric and partial metric spaces by introducing the partial b-metric spaces. In 1942, Manger [11] introduced the notion Probabilistic metric spaces.
In 1965, L.A.Zadeh introduced the innovate notion of fuzzy set [14]. In fuzzy topology there are so many opinions about the concept of the metric space. It can be divided into two groups: First group contains those results in which a fuzzy metric on a set X is considered as a map $d : X \times X \to \mathbb{R}^+$ where X shows the totality of every fuzzy point of a set and obeys some properties which are similar to the common metric properties. For such progress numerical distances are begin between the fuzzy objects. As far as the second group is concerned, it involves study of such results in which the distances between elements are fuzzy and the elements themselves may or may not be fuzzy. Erceg [17], Seikkala and Kaleva [19] and Kramosil and Michalek [10] described fuzzy metric spaces with explanation. A fixed point theorem in fuzzy metric space was proved by Grabiecs in [18] by universalizing the contraction mapping principle due to Banach. Subramanyam [21] generalized Grabiecs result for a pair of commuting maps in the lines of Jungck [6]. New definition of Fuzzy Metric spaces was first introduced by George and Veermani [2] who modified the concept of fuzzy metric spaces and defined a Hausdorff topology on this space and it has many applications in quantum mechanics specially in connection with both string and E-infinity theory. They have also proved that every metric induces a fuzzy metric in Hausdroff topology.
In the present work we reviewed the fixed point theorem in [20] under the title " Some Fixed Point Theorem in Fuzzy 2 and Fuzzy 3-metric Spaces" They established fixed point theorem for Presic types contraction in Fuzzy metric spaces. To prove the existence of a common fixed point of 3-maps in Fuzzy metric space, they have used the non-decreasing continuous function φ defined in $[0, 1]^{2k}$.
After detailed review of their results, We extend the notion of Presic type contraction of $2k$-weakly compatible mapping on Fuzzy 2-Metric, Fuzzy 3-Metric Spaces and obtain fixed point results.

Rest of the thesis is organized as follows.

- In Chapter 2, we recall the basic definition of Metric spaces, 2-Metric space and Fuzzy set , Fuzzy metric space, Fuzzy 2-metric space, and Fuzzy 3-metric space and presented few examples which hold the properties of Metric spaces and fuzzy metric spaces.
- In Chapter 3, The result present in [20] are studied and reviewed. Particularly we study the common fixed point theorem of Presic types for 3 maps in Fuzzy metric spaces.
- In Chapter 4, we extend the results reviewed in Chapter 3, in the sitting of fuzzy 2 and fuzzy 3-metric spaces.

Chapter 2

Preliminaries

In this chapter we will present the basic definitions, theorems, lemmas and examples of various abstract spaces. In Sections 2.1 and 2.2 we study the definitions , examples of metric space and 2-metric space. In Section 2.3 we study fuzzy set and their examples. In Sections 2.4, 2.5 and 2.6 we study the definition and examples of fuzzy metric space, fuzzy 2-metric and fuzzy 3-metric spaces, respectively.

2.1 Metric Space

In mathematics, the ordinary distance or Euclidean distance is a straight line distance between two points. However, distance may be other than straight line like taxicab distance. In literature the word "metric" is used to generalize the notion of distance and the space equipped with metric, satisfying some properties, is called metric space. The formal definition of metric is as follows

Definition 2.1.1. Let X be a non-empty set. A mapping $d : X \times X \to \mathbb{R}$ is said to be a metric if and only if d satisfies the following .

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ iff $x = y$,
3. $d(x, y) = d(y, x)$ for all $x, y \in X$,
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$

If d is a metric for X, then the order pair (X, d) is called a metric space and $d(x, y)$ is the distance between x *and* y.

Example 2.1.2. Let $d(x, y) = |x - y|$, (X, d) is a metric space. the first three condition are obvioursly satisfied, and the 4th follows from ordinary triangular inequality for real numbers:

$$\begin{aligned} d(x, y) &= |x - y| \\ &= |(x - z) + (y - z)| \\ &\leq |x - z| + |y - z| \\ &= d(x, z) + d(z, y) \end{aligned}$$

Definition 2.1.3. (**Sequence**)
A sequence $\{x_n\}$ in a metric space X is a collection $\{x_1, x_3, \ldots, x_n, \ldots\}$ of elements in X enumerated by natural numbers.

Definition 2.1.4. (**Convergent Sequence**)
Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to a point $a \in X$ if for every $\epsilon > 0$ there exist an $n \in \mathbb{N}$ such that

$$d(x_n, a) < \epsilon \quad \text{for all} \quad n \geq N.$$

We write

$$\lim_{n \to \infty} x_n = a \; or \; x_n \to a$$

Example 2.1.5. The following are the examples of convergent sequence

1. The sequence $x_n = \dfrac{1}{n}$ converges to 0.
2. The sequence $x_n = (1)^n$ does not converge.

Definition 2.1.6. (**Cauchy Sequence**)
A sequence $\{x_n\}$ in a metric space (X, d) is a Cauchy sequence if for each $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \epsilon \quad \text{whenever} \quad n, m \geq N.$$

Definition 2.1.7. (**Complete Metric Space**)
A metric space (X, d) is said to be complete if and only if every Cauchy sequence in X converges to a point in X.

Example 2.1.8. (i) The closed unit interval $[0, 1]$ is a complete metric space (under the absolute-value metric). This is easy to prove, using the fact that $\mathbb{R}$ is complete.
(ii) The open unit interval $(0, 1)$ in $\mathbb{R}$, with the usual metric, is an incomplete metric space.

Definition 2.1.9. (**Continuous mapping**)
Consider X and Y are two metric spaces and $F : X \to Y$. Then mapping F is continuous at point $p_0 \in X$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $p \in X$

$$d(Fp, Fp_0) < \varepsilon \quad \text{whenever} \quad d(p, p_0) < \delta$$

If mapping F is continuous at each point of X. Then F is continuous on X.
For instance see the following examples.

(i) Consider a mapping $F : \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = x^2 \quad \forall \quad x \in \mathbb{R}$$

Then mapping F is continuous on $\mathbb{R}$

(ii) Define $F : (0, 1) \to \mathbb{R}$, by

$$F(x) = \frac{3}{x} \quad \forall \quad x \in \mathbb{R}$$

Then mapping F is continuous on $(0, 1)$
Then F is not continuous at $x = 0 \in \mathbb{R}$, hence F is not continuous on $\mathbb{R}$.

Definition 2.1.10. (Monotonic Function)
Consider a function F which is defined on a subset of real numbers. Then F is said to be a monotonic function, if the increment $\Delta[F(x)]$ of F does not change sign when $\Delta x > 0$ *i.e.*, the increment is always positive or negative or we say that the function always varies in the same direction.
For example, a function $F(a_1, a_2, \ldots, a_n)$ defined on $\mathbb{R}^n$ is called monotonic function if for $a_1 \leq a_1^{'}, \ldots, a_n \leq a_n^{'}$

$$F(a_1, \ldots, a_n) \leq F(a_1^{'}, \ldots, a_n^{'})$$

or

$$F(a_1, \ldots, a_n) \geq F(a_1^{'}, \ldots, a_n^{'})$$

Definition 2.1.11. (Monotonic Increasing Function)
Consider a function F which is defined on a subset of real numbers. Then F is said to be a monotonic increasing function, if for each $a_1, a_2 \in \mathbb{R}$ with $a_1 \leq a_2$ we have $F(a_1) \leq F(a_2)$.
For example, the function defined on real numbers by $F(x) = e^x$, is monotonic increasing function.

Definition 2.1.12. (Monotonic Decreasing Function)
Consider a function F which is defined on a subset of real numbers. Then F is said to be a monotonic decreasing function, if for each$a_1, a_2 \in \mathbb{R}$ with $a_1 \leq a_2$ we have $F(a_1) \geq F(a_2)$.
For example, if we define a function $F : (0, 1) \to [0, 1)$ by

$$F(u) = \frac{1}{u+1} \quad \forall\, u \in (0, 1).$$

Then F is a monotonic decreasing function.

Definition 2.1.13. (Lipschitzian Mapping)
Let F be a self mapping on a metric space X, *i.e*, $F : X \to X$, then F is said to be Lipschitzian if there exists a constant $\alpha > 0$ such that

$$d(F(\xi_1), F(\xi_2)) \leq \alpha d(\xi_1, \xi_2), \quad \forall \quad \xi_1, \xi_2 \in X$$

The constant α is called Lipschitzian constant of F.

Example 2.1.14. Consider a metric space $X = \mathbb{R}$, endowed with the standard metric d. Define a mapping $F : X \to X$ by

$$F(z) = 3z \quad \forall \quad z \in X$$

$$\begin{aligned} d(F(z_1), F(z_2)) &= d(3z_1, 3z_2) \\ &= |3z_1 - 3z_2| \\ &= 3|\ z_1 - z_2| \\ &= 3d(z_1, z_2) \end{aligned}$$

Then F is Lipschitzian mapping with Lipschitzian constant $\alpha = 3$.

Definition 2.1.15. (Contraction Mapping)
Let (X, d) be a metric space. A self map $F : X \to X$ is said to be a contraction if its Lipschitzian constant $\alpha < 1$, *i.e.* there exists $0 \leq \alpha < 1$ such that

$$d(F(\xi_1), F(\xi_2)) \leq \alpha d(\xi_1, \xi_2), \quad \forall\, \xi_1, \xi_2 \in X \; and \quad \xi_1 \neq \xi_2$$

Geometrically a contraction means, any points $\xi_1, \xi_2 \in X$ have the images $F(\xi_1)$ and $F(\xi_2)$ under the mapping F, that are much closer than those points.

Example 2.1.16. Consider a metric space $X = (0, \frac{1}{4}) \subseteq \mathbb{R}$ endowed with the standard metric d. Define a mapping $F : X \to X$ by

$$F(X) = x^2 \quad \forall x \in X$$

Since $\sup(0, \frac{1}{4}) = \frac{1}{4}$, Which implies that for every $x \in X,\ |x| < \frac{1}{4}$.
Now for $\xi_1, \xi_2 \in X$ we have

$$\begin{aligned} d(\xi_1, \xi_2) &= |\xi_1 - \xi_2| \\ d(F(\xi_1), F(\xi_2)) &= |F(\xi_1) - F(\xi_2)| \\ &= |\xi_1^2 - \xi_2^2| \\ &= |\xi_1^2 - \xi_2^2| d(\xi_1, \xi_2) \\ &\leq \frac{1}{2} d(\xi_1, \xi_2) \\ &\leq \alpha d(\xi_1, \xi_2) \end{aligned}$$

Hence F is a contraction with $\alpha \in (0, \frac{1}{4})$

Definition 2.1.17. (Contractive Mapping or Strict Contraction)
Let X be a metric space. A self map $F : X \to X$ is said to be contractive if

$$d(F(\xi_1), F(\xi_2)) < d(\xi_1, \xi_2), \quad \forall \quad \xi_1, \xi_2 \in X, \quad \text{where} \quad \xi_1 \neq \xi_2$$

Every contraction is contractive mapping but the converse is not true in general, for instance see the following example.

Example 2.1.18. Consider a metric space X = [0,1) endowed with the standard metric d. Define a mapping $f : X \to X$ by
$f(x) = x + \frac{1}{x},\ \forall\, x \in X.$

Then we have

$$\begin{aligned} d(f(x_1), f(x_2)) &= d(x_1 + \frac{1}{x_1}, x_2 + \frac{1}{x_2}) \\ &= |\frac{x_1+1}{x_1} - \frac{x_2+1}{x_2}| \\ &= |(x_1 + x_2) + (\frac{1}{x_1} - \frac{1}{x_2})| \\ &= |(x_1 - x_2) + (\frac{x_2 - x_1}{x_1 x_2})| \\ &= |(x_1 - x_2) + (\frac{x_1 - x_2}{x_1 x_2})| \\ &= |x_1 - x_2||1 - \frac{1}{x_1 x_2}| \\ &< |x_1 - x_2| \\ &= d(x_1, x_2) \end{aligned}$$

Hence f is contractive but not a contraction.

Definition 2.1.19. (Non-expensive Mapping)
Let (X, d) be a metric space with metric d and $F : X \to X$ be a self map, then F is called a non-expensive mapping if for each $\xi_1, \xi_2 \in X$ and $\xi_1 \neq \xi_2$

$$d(F(\xi_1), F(\xi_2)) \leq d(\xi_1, \xi_2)$$

Every contractive mapping is a non-expensive mapping but every non-expensive mapping need not be contractive mapping and hence is not a contraction.

Example 2.1.20. Consider a metric space $X = \mathbb{R}$ endowed with a usual metric d. Define a mapping $I : X \to X$ by

$$I(\xi) = \xi \ ; \quad \forall \quad \xi \in X \qquad \text{(Identity map)}$$

Now

$$d((I\xi_1), (I\xi_2)) = d(\xi_1, \xi_2) \quad \forall \quad \xi_1, \xi_2 \in X$$

implies that I is non-expensive mapping but not contractive.

Definition 2.1.21. (Fixed Point)
Consider a metric space (X, d) and $T : X \to X$ be a self map. A point $z \in X$ is said to be a fixed point of T if $T(z) = z$.
Generally a point that does not move by a given transformation is called fixed point of that transformation.

Geometrically, if $y = T(x)$ is a real valued function, then by a fixed point of T we means the points where the graph of T intersect with line $y = x$.
Thus a mapping T may or may not have fixed point. Further fixed point may not be unique

Example 2.1.22. Let a metric space $X = \mathbb{R}$ be endowed with a usual metric d. Define $f : X \to X$ by

$$f(t) = 2(t) + 1; \quad \forall \quad t \in X$$

then f has a unique fixed point at $t = -1 \in \mathbb{R}$

Example 2.1.23. Let a metric space $X = \mathbb{R}$ be endowed with a usual metric d. Define a mapping $f : X \to X$ by

$$f(t) = t + 1, \quad \forall \quad t \in X,$$

then f has no fixed point, because $t = t + 1$ has no solution.

Example 2.1.24. Let a metric space $X = \mathbb{R}$ be endowed with a metric d and let I be the identity map on X *i.e.*

$$I(t) = t\ ; \quad \forall \quad t \in X$$

Then each point of X is a fixed point of I.
If we define a real valued function $g(x)$ on an interval I, then finding zeros of $g(x)$ is the same as finding the fixed point of $F(x)$ where

$$F(x) = x - g(x)$$

Since by zeros of $g(x)$ we means those point x for which $g(x) = 0$ implies that

$$x - g(x) = x \quad or \quad F(x) = x$$

i,e. x is then a fixed point of mapping F.

Example 2.1.25. Consider a quadratic polynomial $g(x) = x^2 + 5x + 4$
Clearly zeros of $g(x)$ are $x = -4$ and $x = -1$.
Rewrite the function $g(x) = 0$ as

$$x^2 + 5x + 4 = 0$$

$$x^2 + 4 = -5x$$
$$x = \frac{x^2 + 4}{-5} = F(x)$$

Clearly finding zeros of $g(x)$ is the same as the problem of finding fixed point of $F(x)$ such that $F(x) = x$.

In 1922 Banach proved the following theorem, popularly known as Banach contraction principle

Theorem 2.1.26. [24] Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping such that
$$d(Tx, Ty) \leq \lambda d(x, y)$$
for all $x, y \in X$, where $0 \leq \lambda < 1$, then there exists a unique $x \in X$ such that $T(x) = x$.

2.2 2-Metric Space

In 1962,the notion of 2-metric space was introduced by Ghler ([30], [28]) as a generalization of the usual notion of metric space (X, d). It has been developed extensively by Gahler and many other mathematicians ([3], [37], [36]).

Definition 2.2.1. [7] Let X be a non empty set. A real valued function d defined on $X \times X \times X$ is said to be a 2-metric space on X if

1. Given distinct elements x, y of X. There exists an element z of X such that $d(x, y, z) \neq 0$
2. $d(x, y, z) = 0$ when at least two of x, y, z are equal,
3. $d(x, y, z) = d(x, z, y) = d(z, x, y)$ for all x, y, z in X, and
4. $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.

When d is a 2-metric on X, then the order pair (X, d) is called 2-metric space.

Example 2.2.2. Let a mapping $d : \mathbb{R}^3 \to [0, +\infty)$ be defined by

$$d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}.$$

Then d is a 2-metric on $\mathbb{R}$, *i.e.*, the following inequality holds:
$d(x, y, z) \leq d(x, y, w) + d(y, z, w) + d(z, x, w)$
for arbitrary real numbers x, y, z, w.

Definition 2.2.3. (Convergent Sequence)
[7] A sequence $\{x_n\}$ in 2-metric space (X, d) is said to be convergent to an element $x \in X$ if

$$\lim_{n\to\infty} d(x_n, x, a) = 0 \ \forall\ a \in X$$

It follows that if the sequence $\{x_n\}$ converges to a point x then

$$\lim_{n\to\infty} d(x_n, a, b) = d(x, a, b) \quad \forall\ a, b \in X$$

Definition 2.2.4. (Cauchy Sequence)
[7] A sequence $\{x_n\}$ in a 2-metric space X is a Cauchy sequence if

$$d(x_m, x_n, a) = 0 \quad \text{as} \quad m, n \to \infty \quad \forall\ a \in X.$$

Definition 2.2.5. (Complete 2-Metric Space)
A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent

2.3 Fuzzy Set

A fuzzy set is a class of objects with a continuum of grades of membership. Such a set characterized by membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one.
Fuzzy sets were first proposed by Lofti A. Zadeh in his 1965 [14] paper entitled none other than: Fuzzy Sets. This paper laid the foundation for all fuzzy logic that followed by mathematically defining fuzzy sets and their properties. The mathematical definition of a fuzzy set is as follows.

Definition 2.3.1. A fuzzy set A in X is a function with domain X and values in $[0, 1]$.
A fuzzy set A in X is characterized by a membership function $f_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$, with the values of $f_A(x)$ at x representing the grade of membership of x in A. Thus, the nearer the

value of $f_A(x)$ to unity, the higher the grade of membership of x in A.
This definition of a fuzzy set is like a superset of the definition of a set in the ordinary sense of the term. The grades of membership of 0 and 1 correspond to the two possibilities of truth and false in an ordinary set. The ordinary boolean operators that are used to combine sets will no longer apply; we know that 1 AND 1 is 1, but what is 0.7 AND 0.3? This will be covered in the fuzzy operations section.

Example 2.3.2. Let X be the real line $\mathbb{R}$ and let A be a fuzzy set of numbers which are much greater than 1. Then, one can give a precise, albeit subjective, characterization of A by specifying $f_A(x)$ as a function on $\mathbb{R}$. Representative values of such a function might be: $f_A(0) = 0$; $f_A(1) = 0$; $f_A(5) = 0.01$; $f_A(10) = 0.2$; $f_A(100) = 0.95$; $f_A(500) = 1$, etc

Definition 2.3.3. (Triangular norm)
A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called triangular norm (t-norm) if satisfies the following condition.

1. $a * b = b * a, \forall a, b \in [0,1]$;
2. $a * 1 = a, \forall a \in [0,1]$;
3. $(a * b) * c = a * (b * c)$, for all $a, b, c \in [0,1]$;
4. if $a \leq c$ and $b \leq d$, with $a, b, c \in [0,1]$, then $a * b \leq c * d$

Example 2.3.4. Three basic examples of continuous t-norms are $\wedge, \cdot, *$, Which are defined by

$$a \wedge b = \min(a, b), \quad a \cdot b = ba \quad \text{and} \quad a * b = \max\{a + b - 1, 0\}$$

2.4 Fuzzy Metric Space

The notion of a Fuzzy metric space was introduced by Erceg [17] Kaleva and Seikkala [19] and Kramosil and Michalekin [10] in detail. Grabiecs [18] proved a fixed point theorem in fuzzy metric space by generalizing the Banach contraction mapping principle.

Definition 2.4.1. A 3-tuple$(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $t, s > 0$,

FM-1 $M(x, y, t) \geq 0$

FM-2 $M(x, y, t) = 1$ if and only if $x = y$

FM-3 $M(x, y, t) = M(y, x, t)$

FM-4 $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$

FM-5 $M(x, y, .) : (0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$

Example 2.4.2. Let (X, d) be a metric space. Define $a*b = ab$ or $a*b = \min\{a, b\}$ for all $x, y \in X$.

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + d(x, y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Remark 2.4.3. i . Since $*$ is continuous, it follows from **(FM-4)**, *i.e.*

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$$

that the limit of the sequence in fuzzy metric space is uniquely determined.

ii . Let $(X, M, *)$ be a fuzzy metric space with the following condition **(FM-6)**

$$\lim_{t\to\infty} (M, x, y, t) = 1 \quad \forall \quad x, y \in X$$

Example 2.4.4. Let $X = \left\{ \dfrac{1}{n} : n \in N \right\} \cup \{0\}$ and let $*$ be the continuous t-norm defined by $a * b = ab$ for all $a, b \in [0, 1]$. For each $t > 0$ and $x, y \in X$, define M, by

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + d(x, y)} & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Clearly, $(X, M, *)$ is a complete fuzzy metric space.

Lemma 2.4.5. [18] For all $x, y \in X$, $M(x, y, .)$ is non-decreasing.

Proof. Suppose

$$M(x, y, t) > M(x, y, s) \quad \text{for some} \quad 0 < t < s$$

Then

$$M(x, y, t) * M(y, y, st) \leq M(x, y, s) < M(x, y, t)$$

By property(ii),

$$M(y, y, s + t) = 1$$

thus

$$M(x, y, t) \leq M(x, y, s) < M(x, y, t)$$

Which is a contradiction. Hence M is non decreasing

□

Lemma 2.4.6. [34] Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with condition **(FM-6)**.
If there exists a number $q \in (0, 1)$ such that

$$M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t) \quad \forall \quad t > 0 \quad \text{and} \quad n = 1, 2, \ldots$$

then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.4.7. [13] Let $(X, M, *)$ be a fuzzy metric space. If there exists $q \in (0, 1)$ such that

$$M(x, y, qt) \geq M(x, y, t) \quad \forall \quad x, y \in X \text{and} \quad t > 0$$

then $\quad x = y$

Lemmas 2.4.5, 2.4.6, 2.4.7 and Remark (2.4.3) hold for fuzzy 2-metric spaces and fuzzy 3-metric spaces also.

Definition 2.4.8. (Continuous Function)
A function M is continuous in fuzzy metric space if and only if whenever $x_n \to x, \; y_n \to y$ then

$$\lim_{t \to \infty} M(x_n, y_n, t) = M(x, y, t) \quad \text{for each} \quad t > 0$$

Definition 2.4.9. (Compatible Mapping)
Let A and B map from a fuzzy metric space $(X, M, *)$ into itself. The maps A and B are said to be compatible (or asymptotically commuting), if for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$$

Where $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \quad \text{for some} \quad z \in X$$

From the above definition it is inferred that A and B are non-compatible maps from a fuzzy metric space $(X, M, *)$ into itself if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, but either $lim_{n\to\infty} M(ABx_n, BAx_n, t) \neq 1$. or the limit does not exist.

Definition 2.4.10. (Weakly Commuting)
Two mappings A and S on fuzzy metric space X are weakly commuting if and only if

$$M(ASu, SAu, t) \geq M(Au, Su, t) \quad \forall \quad u \in X \quad \text{and} \quad t > 0$$

2.5 Fuzzy 2-Metric Space

D. Singh[1] introduced the notion of fuzzy 2-metric space by using t-norm. He also proved a fixed point theorem in fuzzy 2-metric space

Definition 2.5.1. (t-norm)
A binary operation $* : [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that

$$a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$$

whenever

$$a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 \quad \text{for all} \quad a_1, a_2, b_1, b_2, c_1, c_2 \in [0, 1]$$

Definition 2.5.2. The 3-tuple $(X, M, *)$ is called a fuzzy 2-metric space if X is an anbitrary set, $*$ is a contineous t-norm and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions. For all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$

FM2-1 $M(x, y, z, 0) = 0$

FM2-2 $M(x, y, z, t) = 1, t > 0$ and when at least two of these three points are equal,

FM2-3 $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ (symmetry about three variables),

FM2-4 $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3)$

FM2-5 $M(x, y, z, .) : [0, \infty) \to [0, 1]$ is left continuous.

Example 2.5.3. Let (X, d) be a 2-metric space. Define $a_1 * b_1 * c_1 = a_1 b_1 c_1$ or $a_1 * b_1 * c_1 = \min\{a_1, b_1, c_1\}$ for all $x, y, z \in X$.

$$M(x, y, z, w, t) = \begin{cases} \dfrac{t}{t + d(x, y, z)}, & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Then $(X, M, *)$ is a fuzzy 2-metric space. We call this fuzzy 2-metric M induced by the metric d the standard fuzzy metric.

Definition 2.5.4. (Convergent Sequence)
Let $(X, M, *)$ be a fuzzy 2-metric space, then A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, a, t) = 1 \quad \forall\, a \in X \quad \text{and} \quad t > 0$$

Definition 2.5.5. (Cauchy Sequence)
Let $(X, M, *)$ be a fuzzy 2-metric space, then A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$$\lim_{n \to \infty} M(x_{n+p}, x_n, a, t) = 1 \quad \forall\, a \in X \text{ and } \quad t > 0,\ p > 0$$

Definition 2.5.6. (Complete Fuzzy 2-Metric Space)
A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5.7. (Continuous Mapping)
A function M is continuous in fuzzy 2-metric space if and only if whenever
$x_n \to x,\ y_n \to y$ then

$$\lim_{t \to \infty} M(x_n, y_n, a, t) = M(x, y, a, t) \quad \forall\, a \in X \text{ and } t > 0$$

Definition 2.5.8. (Weakly Commuting)
Two mapping A and S an fuzzy 2-metric space X are weakly commuting iff

$$M(ASu, SAu, a, t) \geq M(Au, Su, a, t) \quad \forall\, u, a \in X \text{ and } t > 0$$

2.6 Fuzzy 3-Metric Space

The notion of Fuzzy 2-metric spaces can be extended to more generalized notion of fuzzy 3-metric spaces as introduced by Zaheer K. Ansari [13]

Definition 2.6.1. (t-norm)
A binary operation $* : [0,1] \times [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit 1 such that

$$a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2 \quad \text{whenever} \quad a_1 \leq a_1, b_1 \leq b_2, c_1 \leq c_2 \text{ and } d_1 \leq d_2$$

for all $a_1, a_2, b_1, b_2, c_1, c_2\ d_1, d_2$ are in $[0,1]$.

Definition 2.6.2. The 3- tuple $(X, M, *)$ is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z, w, u \in X$ and $t_1, t_2, t_3, t_4 > 0$.

FM3-1 $M(x, y, z, w, 0) = 0$,

FM3-2 $M(x, y, z, w, t) = 1$ for all $t > 0$, [only when the three simplex (x, y, z, w) degenerate]

FM3-3 $M(x, y, z, w, t) = M(x, w, z, y, t) = M(y, z, w, x, t) = M(z, w, x, y, t) = \dots$

FM3-4 $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M(u, y, z, w, t_4)$

FM-5 $M(x, y, z, w) : [0, \infty) \to [0, 1]$ is left continuous.

Example 2.6.3. Let (X, d) be a 3-metric space. Define $a_1 * b_1 * c_1 * d_1 = a_1 b_1 c_1 d_1$ or $a_1 * b_1 * c_1 * d_1 = \min\{a_1, b_1, c_1, d_1\}$ for all $x, y, z, w \in X$.

$$M(x, y, z, w, t) = \begin{cases} \dfrac{t}{t + d(x, y, z, w)}, & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

Then $(X, M, *)$ is a fuzzy 3-metric space. We call this fuzzy 3-metric M induced by the 3-metric d the standard fuzzy 3-metric.

Definition 2.6.4. (Convergent Sequence)
Let $(X, M, *)$ be a fuzzy 3-metric space, then a sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$$

for all $a, b \in X$ *and* $t > 0$.

Definition 2.6.5. (Cauchy Sequence)
Let $(X, M, *)$ be a fuzzy 3-metric space, then a sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, b, t) = 1$$

for all $a, b \in X$ *and* $t > 0, p > 0$.

Definition 2.6.6. (Complete Fuzzy 3-Metric space)
A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.6.7. (Continuous Function)
A function M is continuous in fuzzy 3-metric space iff whenever
$x_n \to x,\ y_n \to y$, then

$$\lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$$

$\forall\ a, b \in X$ and $t > 0$.

Definition 2.6.8. (Weakly Commutable Mapping)
Two mappings A and S on fuzzy 3-metric space are weakly commuting iff

$$M(ASu, SAu, a, b, t) \geq M(Au, Su, a, b, t) \quad \forall\ u, a, b \in X \text{ and } t > 0$$

Chapter 3

Fixed point theorem in Fuzzy Metric Spaces

In this chapter we review common fixed point theorems of Presic type in Fuzzy metric spaces which extends the results of R. George [22].

3.1 Presic Type Contractions

In 1965 S.B Presic [25]. generalize the Banach Contraction(2.1.26) into Presic type Contraction
Let $T : X \to X$, where $k \geq 1$ is a positive integer. A point $x^* \in X$ is a fixed point of T if $x^* = T(x^*, x^*, \ldots, x^*)$.
Consider the $k-$order nonlinear difference equation

$$x_{n+1} = T(x_{n-k+1}, x_{n-k+2}, \ldots, x_n) \quad \text{for} \quad n = k-1, k, k+1, \ldots \tag{3.1}$$

with the initial values $x_0, x_1, x_2, \ldots, x_{k-1} \in X$.
Equation (3.1) can be considered by means of fixed point theory in view of the fact that $x \in X$ is a solution of (3.1) if and only if x is a fixed point of T. The following theorem is the generalization of Banach contraction theorem(2.1.26).

Theorem 3.1.1. [25] Let (X, d) be a complete metric space, k be a positive integer, and $T : X^k \to X$. Suppose that

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1})$$

for every $x_1, x_2, \ldots, x_k, x_{k+1} \in X$, where $q_i \geq 0$ and $\sum_{i=1}^{k} q_i \in [0, 1)$. Then T has a unique fixed point x^*. Moreover for any arbitrary points $x_1, x_2, \ldots, x_{k+1} \in X$, the sequence $\{x_n\}$ defined by $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, for all $n \in \mathbb{N}$ converges to x^*.

In 2007 Ciric and Presic generalized the above theorem as follows.

Theorem 3.1.2. [16] Let (X, d) be a complete metric space, k be a positive integer, and $T : X^k \to X$. Suppose that

$$d(T(x_1, x_2, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})) \leq \lambda\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}$$

for every $x_1, x_2, \ldots, x_k, x_{k+1} \in X$, where $\lambda \in [0, 1)$. Then T has a fixed point x^*. Moreover for any arbitrary points $x_1, x_2, \ldots, x_{k+1} \in X$, the sequence $\{x_n\}$ defined by $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, for all $n \in \mathbb{N}$ converges to x^*.
Moreover, if $d(T(u, u, \ldots, u), T(v, v, \ldots, v)) < d(u, v)$ holds for all $u, v \in X$ with $u \neq v$ then x^* is the unique fixed point of T.

Before presenting the main theorem we define φ-function as follows:
Let a function $\varphi : [0, 1]^{2k} \to [0, 1]$ such that:

1. φ is an increasing function, i.e. $x_1 \leq y_1, \ldots, x_{2k} \leq y_{2k}$, implies

$$\varphi(x_1, x_2, x_3, \ldots, x_{2k}) \leq \varphi(y_1, y_2, y_3, \ldots, y_{2k})$$

2. $\varphi(t, t, t, \ldots) \geq t$, for all $t \in X$.

3. φ is continuous in each coordinate variable.

Theorem 3.1.3. Let $(X, M, *)$ be a fuzzy metric space and $S, T : X^{2k} \to X$, $f : X \to X$ be mappings satisfying for each positive integer k,

$$M(S(x_1, x_2, \ldots, x_{2k}), T(x_2, x_3, \ldots, x_{2k+1}), qt) \geq \varphi\{M(fx_i, fx_{i+1}, t) : 1 \leq i \leq 2k\} \tag{3.2}$$

for all $x_1, x_2, x_3, \dots, x_{2k+1} \in X, 0 < q < \frac{1}{2}$, and $t \in [0, \infty)$

$$M(T(y_1, y_2, \dots, y_{2k}), S(y_2, y_3, \dots, y_{2k+1}), qt) \geq \varphi\{M(fy_i, fy_{i+1}, t) : 1 \leq i \leq 2k\} \tag{3.3}$$

for all $y_1, y_2, y_3, \dots, y_{2k+1} \in X$, and

$$M(S(u, u, \dots, u), T(v, v, \dots, v), qt) > M(fu, fv, t) \quad \forall\, u, v \in X \text{and } u \neq v \tag{3.4}$$

Suppose then $f(X)$ is complete and either (f, S) or (f, T) is $2k$-weakly compatible pair. Then there exist a unique point p in X such that

$$fp = p = S(p, p, p, \dots, p) = T(p, p, \dots, p)$$

Proof. Suppose $x_1, x_2, x_3, \dots, x_{2k}$ are arbitrary points in X and for $n \in \mathbb{N}$. Define

$$fx_{2n+2k-1} = S(x_{2n-1}, x_{2n}, x_{2n+1}, \dots, x_{2n+2k-2})$$

and

$$fx_{2k+2n} = T(x_{2n}, x_{2n+1}, \dots, x_{2n+2k-1})$$

Let $\quad \alpha_n = M(fx_n, fx_{n+1}, qt)$

Claim $\quad \alpha_n \geq \left(\frac{K - \beta^n}{K + \beta^n}\right)^2$ for all $n \in \mathbb{N}$. where $\beta = \frac{1}{2q}$

and

$$K = \min\left\{\frac{\beta(1 + \sqrt{\alpha_1})}{1 - \sqrt{\alpha_1}}, \frac{\beta^2(1 + \sqrt{\alpha_2})}{1 - \sqrt{\alpha_2}}, \frac{\beta^3(1 + \sqrt{\alpha_3})}{1 - \sqrt{\alpha_3}}, \dots, \frac{\beta^{2k}(1 + \sqrt{\alpha_k})}{1 - \sqrt{\alpha_k}}\right\}$$

So we have $\alpha_n \geq \left(\frac{K - \beta^n}{K + \beta^n}\right)^2$ for $n = 1, 2, 3, \dots, 2k$

Now

$$\begin{aligned}
\alpha_{2k+1} &= M(fx_{2k+1}, fx_{2k+2}, qt) \\
&= M(S(x_1, x_2, \dots, x_{2k}), T(x_2, x_3, \dots, x_{2k+1}), qt) \\
&\geq \varphi\{M(fx_i, fx_{i+1}, t)\},\ i = 1, 2, \dots, 2k\}
\end{aligned}$$

From (3.2),

$$\begin{aligned}
\alpha_{2k+1} &\geq \varphi\{\alpha_1, \alpha_2, \ldots, \alpha_{2k}\} \\
&\geq \varphi\left\{\left(\frac{k-\beta^1}{K+\beta^1}\right)^2, \left(\frac{K-\beta^2}{K+\beta^2}\right)^2, \ldots, \left(\frac{K-\beta^{2k-1}}{K+\beta^{2k-1}}\right)^2, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2\right\} \\
&\geq \varphi\left\{\left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \ldots, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2\right\} \\
&\geq \varphi\left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2 \\
&\geq \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2
\end{aligned}$$

Thus $\quad \alpha_{2k+1} \geq \left(\frac{k-\beta^{2k+1}}{k+\beta^{2k+1}}\right)^2$

Similarly we have

$$\begin{aligned}
\alpha_{2k+2} &= M(fx_{2k+2}, fx_{2k+3}, qt) \\
&= M(T(x_2, x_3, \ldots, x_{2k+1}), S(x_3, x_4, \ldots, x_{2k+2}), qt) \\
&\geq \varphi\{M(fx_i, fx_{i+1}, t),\ i = 2, 3, \ldots, 2k+1\}
\end{aligned}$$

$$\begin{aligned}
\alpha_{2k+2} &\geq \varphi\{\alpha_1, \alpha_2, \ldots, \alpha_{2k+1}\} \\
&\geq \varphi\left\{\left(\frac{k-\beta^2}{k+\beta^2}\right)^2, \left(\frac{K-\beta^3}{K+\beta^3}\right)^2, \ldots, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2\right\} \\
&\geq \varphi\left\{\left(\frac{K-\beta^{2k+1}}{K+\theta^{2k+1}}\right)^2, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2, \ldots, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2\right\} \\
&\geq \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2 \\
&\geq \left(\frac{K-\beta^{2k+2}}{K+\beta^{2k+2}}\right)^2
\end{aligned}$$

So we have, $\quad \alpha_{2k+2} \geq \left(\frac{K-\beta^{2k+2}}{K+\beta^{2k+2}}\right)^2$

Hence our claim is true.

Now we show that $\{x_n\}$ is a Cauchy sequence in X for any $n, p \in \mathbb{N}$, we have

$$\begin{aligned}
&M(fx_n, fx_{n+p}, t) \\
&\geq M(fx_n, fx_{n+1}, \frac{t}{2}) * M(fx_{n+1}, fx_{n+2}, \frac{t}{2^2}) * \cdots * M(fx_{n+p-1}, fx_{n+p}, \frac{t}{2^p}) \\
&\geq \alpha_n * \alpha_{n+1} * \ldots * \alpha_{n+p-1} \\
&\geq \left(\frac{K-2^n}{K+2^n}\right)^2 * \left(\frac{K-2^{2n}}{K+2^{2n}}\right)^2 * \ldots * \left(\frac{K-2^{np}}{K+2^{np}}\right)^2 \\
&\to\ 1 * 1 * \ldots * 1 \quad \text{as} \quad n \to \infty
\end{aligned}$$

Hence $\{fx_n\}$ is a Cauchy sequence.
Since $f(X)$ is a complete sub space of X, Then there exist z in $f(X)$ such that

$$\lim_{n\to\infty} fx_n = z$$

For $z \in f(X)$, there exist $p \in X$ such that $z = fp$. Then for any integer, using (3.2) and (3.3), We have

$$\begin{aligned}
&M(S(p, p, \ldots, p), fp, t) \\
&= \lim_{n\to\infty} M(S(p, p, p, \ldots, p), fx_{2n+2k-1}, t) \\
&= \lim_{n\to\infty} M(S(p, p, p, \ldots, p), S(x_{2n-1}, x_{2n}, \ldots, x_{2n+2k-2}), t) \\
&\geq \lim_{n\to\infty} M(S(p, p, \ldots, p), T(p, p, \ldots, x_{2n-1}), \frac{t}{2}) \\
&* M(T(p, p, \ldots, x_{2n-1}), S(p, p, \ldots, x_{2n-1}, x_{2n}), \frac{t}{2^3}) * \ldots \\
&* M(T(p, x_{2n-1}, \ldots, x_{2n+2k-3}), S(x_{2n-1}, x_{2n}, \ldots, x_{2n+2k-2}), \frac{t}{2^{k-1}}) \\
&\geq \lim_{n\to\infty} \varphi\{M(fp, fp, t), M(fp, fp, t), \ldots, M(fp, fx_{2n-1}, t)\} \\
&* \varphi\{M(fp, fp, t), M(fp, fp, t), \ldots, M(fx_{2n-1}, fx_{2n}, t)\} * \ldots \\
&* \varphi\{M(fp, fx_{2n-1}, t), M(fx_{2n-1}, fx_{2n}, t), \ldots, M(fx_{2n+2k-3}, fx_{2n+2k-2}, t)\} \\
&\longrightarrow 1.
\end{aligned}$$

i.e

$$M(S(p, p, p, \ldots, p), fp, t) = 1$$

So $c(f, T) \neq \Phi$, where $c(f, T)$ denote the set of all coincidence points of the mappings f and T, so that

$$S(p, p, ..., p) = fp \tag{3.5}$$

Consider

$$\begin{aligned} M(fp, T(p,p,\ldots,p),t) &= M(S(p,p,\ldots,p),T(p,p,\ldots,p),t) \\ &\geq \varphi\{M(fp,fp,t),M(fp,fp,t),\ldots,M(fp,fp,t)\} \\ &\geq M(fp,fp,t) = 1 \end{aligned}$$

Thus

$$T(p,p,p,.....,p) = fp \tag{3.6}$$

Now suppose that (f,S) is $2k$-weakly compatible pair. Then we have

$$f(S(p,p,\ldots,p)) = S(fp,fp,\ldots,fp)$$

$$f^2p = f(fp) = f(S(p,p,\ldots,p)) = S(fp,fp,\ldots,fp)$$

Suppose that $fp \neq p$, then from (3.3), we have

$$\begin{aligned} M(f^2p,fp,t) &= M(S(fp,fp,\ldots,fp),T(p,p,\ldots,p),t) \\ &\geq \{M(f^2p,fp,t),M(f^2p,fp,t),\ldots,M(f^2p,fp,t)\} \\ &\geq M(f^2p,fp,t) \end{aligned}$$

This is contradiction. So our supposition is wrong. Hence $fp = p$
Now from (3.5) and (3.6), we have

$$fp = p = S(p,p,\ldots,p) = T(p,p,\ldots,p)$$

To prove the uniqueness we proceed as follows;
Suppose there exist a point $q \neq p$ in X such that

$$fq = q = S(q,q,\ldots,q) = T(q,q,\ldots,q)$$

Consider

$$\begin{aligned} M(fp,fq,t) &= M(S(p,p,\ldots,p),T(q,q,\ldots,q),t) \\ &\geq \varphi\{M(fp,fq,t),M(fp,fq,t),\ldots,M(fp,fq,t)\} \\ &\geq M(fp,fq,t) \end{aligned}$$

It is contradiction. Therefore $q = p$ □

The Presic types contraction in fuzzy metric space for k-weakly compatible mapping we define φ-function as follows:
Let a function $\varphi : [0,1]^k \to [0,1]$ such that:

1. φ is an increasing function, i.e. $x_1 \leq y_1, \ldots, x_k \leq y_k$, implies

$$\varphi(x_1, x_2, x_3, \ldots, x_k) \leq \varphi(y_1, y_2, y_3, \ldots, y_k)$$

2. $\varphi(t, t, t, \ldots) \geq t$, for all $t \in X$.

3. φ is continuous in each coordinate variable.

Theorem 3.1.4. Let $(X, M, *)$ be a fuzzy metric space and $S, T : X^k \to X$, $f : X \to X$ be mappings satisfying for each positive integer k;

$$M(S(x_1, x_2, x_3, \ldots, x_k), T(x_2, x_3, \ldots, x_{k+1})q, t) \geq \varphi\{M(fx_i, fx_{i+1}, t) : 1 \leq i \leq k\} \tag{3.7}$$

for all $x_1, x_2, x_3, \ldots, x_k, x_{k+1} \in X,\ 0 < q < \dfrac{1}{2},$ and $t \in [0, \infty)$;

$$M(T(y_1, y_2, y_3, \ldots, y_k), S(y_2, y_3, \ldots, y_{k+1}), qt) \geq \varphi\{M(fy_i, fy_{i+1}, t) : 1 \leq i \leq k \tag{3.8}$$

for all $y_1, y_2, y_3, \ldots\ldots., y_k, y_{k+1} \in X,$ and

$$M(S(u, u, u, \ldots, u), T(v, v, v, \ldots, v), qt) > M(fu, fv, t) \tag{3.9}$$

for all $u, v \in X$, with $u \neq v$.
Suppose then $f(X)$ is complete and either (f, S) or (f, T) is k-weakly compatible pair. Then there exist a unique point p in X such that

$$fp = p = S(p, p, p, \ldots, p) = T(p, p, p \ldots, p)$$

Proof. Suppose $x_1, x_2, x_3, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$. Define

$$fx_{n+k-1} = S(x_{n-1}, x_n, x_{n+1}, \ldots, x_{n+k-1}) \quad \text{and} \quad fx_{k+n} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$$

Let $\quad \alpha_n = M(fx_n, fx_{n+1}, qt)$
Claim $\quad \alpha_n \geq \left(\dfrac{K - \beta^n}{K + \beta^n}\right)^2$ for all $n \in \mathbb{N}$. where $\beta = \dfrac{1}{q}$
and

$K = \min\left\{\frac{\beta(1+\sqrt{\alpha_1})}{1-\sqrt{\alpha_1}}, \frac{\beta^2(1+\sqrt{\alpha_2})}{1-\sqrt{\alpha_2}}, \frac{\beta^3(1+\sqrt{\alpha_3})}{1-\sqrt{\alpha_3}}, \dots, \frac{\beta^k(1+\sqrt{\alpha_k})}{1-\sqrt{\alpha_k}}\right\}$ So we have $a_n \geq \left(\frac{K-\beta^n}{K+\beta^n}\right)^2$ for $n = 1, 2, 3, \dots, 2k$
Now

$$\begin{aligned} a_{k+1} &= M(fx_{k+1}, fx_{k+2}, qt) \\ &= M(S(x_1, x_2, x_3, \dots, x_{k-1}, x_k), T(x_2, x_3, \dots, x_k, x_{k+1},)qt) \\ &\geq \varphi\{M(fx_i, fx_{i+1}, t)\},\ i = 1, 2, \dots, k\} \end{aligned}$$

From (3.7),

$$\begin{aligned} \alpha_{k+1} &\geq \varphi\{\alpha_1, \alpha_2, \dots, \alpha_k\} \\ &\geq \varphi\left\{\left(\frac{K-\beta^1}{K+\beta^1}\right)^2, \left(\frac{K-\beta^2}{K+\beta^2}\right)^2, \dots, \left(\frac{K-\beta^{k-1}}{K+\beta^{k-1}}\right)^2, \left(\frac{K-\beta^k}{K+\beta^k}\right)^2\right\} \\ &\geq \varphi\left\{\left(\frac{K-\beta^k}{K+\beta^k}\right)^2, \left(\frac{K-\beta^k}{K+\beta^k}\right)^2, \dots, \left(\frac{K-\beta^k}{K+\beta^k}\right)^2, \left(\frac{K-\beta^k}{K+\beta^k}\right)^2\right\} \\ &\geq \varphi\left(\frac{k-\beta^k}{K+\beta^k}\right)^2 \\ &\geq \left(\frac{K-\beta^{k+1}}{K+\beta^{k+1}}\right)^2 \end{aligned}$$

Thus $\alpha_{k+1} \geq \left(\frac{K-\beta^{k+1}}{K+\beta^{k+1}}\right)^2$
Similarly we have

$$\begin{aligned} a_{k+2} &= M(fx_{k+2}, fx_{k+3}, qt) \\ &= M(T(x_2, x_3, \dots, x_k, x_{k+1}), S(x_3, x_4, \dots, x_{k+1}, x_{k+2}), qt) \\ &\geq \varphi\{M(fx_i, fx_{i+1}, t),\ i = 2, 3, \dots, k+1\} \\ &\geq \varphi\{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{k+1}\} \\ &\geq \varphi\left\{\left(\frac{K-\beta^2}{K+\beta^2}\right)^2, \left(\frac{K-\beta^3}{K+\beta^3}\right)^2, \dots, \left(\frac{K-\beta^k}{K+\beta^k}\right)^2, \left(\frac{K-\beta^{k+1}}{K+\beta^{k+1}}\right)^2\right\} \\ &\geq \varphi\left\{\left(\frac{k-\beta^{k+1}}{K+\beta^{k+1}}\right)^2, \left(\frac{K-\beta^{k+1}}{K+\beta^{k+1}}\right)^2, \dots, \left(\frac{K-\beta^{k+1}}{k+\beta^{k+1}}\right)^2, \left(\frac{K-\beta^{k+1}}{K+\beta^{k+1}}\right)^2\right\} \\ &\geq \left(\frac{K-\beta^{k+1}}{K+\beta^{k+1}}\right)^2 \\ &\geq \left(\frac{K-\beta^{k+2}}{K+\beta^{k+2}}\right)^2 \end{aligned}$$

So we have, $\alpha_{k+2} \geq \left(\frac{K-\beta^{k+2}}{K+\beta^{k+2}}\right)^2$
Hence our claim is true.
Now we show that $\{x_n\}$ is a Cauchy sequence in X. For any $n, p \in \mathbb{N}$, we have

$$
\begin{aligned}
& M(fx_n, fx_{n+p}, t) \\
& \geq M(fx_n, fx_{n+1}, \frac{t}{2}) * M(fx_{n+1}, fx_{n+2}, \frac{t}{2^2}) * \ldots * M(fx_{n+p-1}, fx_{n+p}, \frac{t}{2^p}) \\
& \geq \alpha_n, \alpha_{n+1}, * \ldots * \alpha_{n+p-1} \\
& \geq \left(\frac{K-2^n}{K+2^n}\right)^2 * \left(\frac{K-2^{2n}}{K+2^{2n}}\right)^2 * \ldots * \left(\frac{K-2^{np}}{K+2^{np}}\right)^2 \\
& \rightarrow \ 1 * 1 * \ldots * 1 \quad \text{as} \quad n \rightarrow \infty
\end{aligned}
$$

Hence $\{fx_n\}$ is a Cauchy sequence.
Since $f(X)$ is a complete sub space of X. Then there exist z in $f(X)$ such that $\lim_{n\to\infty} fx_n = z$ For $z \in f(X)$, there exist $p \in X$ such that $z = fp$. Then for any integer, using (3.7) and (3.8), We have

$$
\begin{aligned}
& M(S(p,p,p,\ldots,p), fp, t) = \lim_{n\to\infty} M(S(p,p,p,\ldots,p), fx_{2n+k-1}, t) \\
& = \lim_{n\to\infty} M(S(p,p,p,\ldots,p), S(x_{2n-1}, x_{2n}, \ldots, x_{2n+k-2}), t) \\
& \geq \lim_{n\to\infty} M(S(p,p,\ldots,p), T(p,p,\ldots,x_{2n-1}), \frac{t}{2}) \\
& * M(T(p,p,\ldots,x_{2n-1}), S(p,p,\ldots,x_{2n-1},x_{2n}), \frac{t}{2^3}) * \ldots \\
& * M(T(p,x_{2n-1},\ldots,x_{2n+k-3}), S(x_{2n-1},x_{2n},\ldots,x_{2n+k-2}), \frac{t}{2^{k-1}}) \\
& \geq \lim_{n\to\infty} \varphi\{M(fp,fp,t), M(fp,fp,t), \ldots, M(fp, fx_{2n-1})\} \\
& * \varphi\{M(fp,fp,t), M(fp,fp,t), \ldots, M(fx_{2n-1}, fx_{2n})\} * \ldots \\
& * \varphi\{M(fp, fx_{2n-1}, t), M(fx_{2n-1}, fx_{2n}, t), \ldots, M(fx_{2n+k-3}, fx_{2n+k-2})\} \\
& \rightarrow 1
\end{aligned}
$$

i.e; $M(S(p,p,\ldots,p), fp, t) = 1$
So $c(f,T) \neq \Phi$, where $c(f,T)$ denote the set of all coincidence points of the mappings f and T, so that

$$S(p,p,\ldots,p) = fp \tag{3.10}$$

Consider

$$\begin{aligned} M(fp, T(p,p,\ldots,p),t) &= M(S(p,p,\ldots,p), T(p,p,\ldots,p),t) \\ &\geq \varphi\{M(fp,fp,t), M(fp,fp,t),\ldots,M(fp,fp,t)\} \\ &\geq M(fp,fp,t) = 1 \end{aligned}$$

Thus

$$T(p,p,p,.....,p) = fp \tag{3.11}$$

Now suppose that (f,S) is k-weakly compatible pair, then we have

$$f(S(p,p,\ldots,p)) = S(fp,fp,\ldots,fp)$$

$$f^2p = f(fp) = f(S(p,p,\ldots,p)) = S(fp,fp,\ldots,fp)$$

Suppose that $fp \neq p$, then from eq.(3.9), we have

$$\begin{aligned} M(f^2p, fp, t) &= M(S(fp,fp,\ldots,fp), T(p,p,\ldots,p),t) \\ &\geq \{M(f^2p,fp,t), M(f^2p,fp,t),\ldots,M(f^2p,fp,t)\} \\ &\geq M(f^2p,fp,t) \end{aligned}$$

This is contradiction. So our supposition is wrong. Hence $fp = p$
Now from eq.(3.10) and eq.(3.11), we have

$$fp = p = S(p,p,\ldots,p) = T(p,p,\ldots,p)$$

Uniqueness:
Suppose there exist a point $q \neq p$ in X such that

$$fp = q = S(q,q,\ldots,q) = T(q,q,\ldots,q)$$

Consider

$$\begin{aligned} M(fp, fq, t) &= M(S(p,p,\ldots,p), T(q,q,\ldots,q),t) \\ &\geq \varphi\{M(fp,fq,t), M(fp,fq,t),\ldots,M(fp,fq,t)\} \\ &\geq M(fp,fq,t) \end{aligned}$$

It is contradiction. Therefore $q = p$ □

Chapter 4

Fixed Point Theorems in Fuzzy 2-Metric and Fuzzy 3-Metric Space

In this chapter we introduce the notion of Presic type contraction in Fuzzy 2-metric space(2.5.2) and Fuzzy 3-metric space(2.6.2) and our aim to prove some fixed point theorems as an extension of the results presented in Chapter 3.

4.1 Presic Type Contractions

Before introducing the Presic type contraction in fuzzy 2-metric Space, and the related fixed point result, we first define the φ-function as follows.
Let a function $\varphi : [0,1]^{2k} \to [0,1]$ such that:

1. φ is an increasing function, i.e. $x_1 \leq y_1, \ldots, x_{2k} \leq y_{2k}, a \leq a'$ implies

$$\varphi(x_1, x_2, x_3, \ldots, x_{2k}, a) \leq \varphi(y_1, y_2, y_3, \ldots, y_{2k}, a')$$

2. $\varphi(t, t, t, \ldots) \geq t$, for all $t \in X$.

3. φ is continuous in each coordinate variable.

Using the φ-function, we now state the following fixed point Theorem.

Theorem 4.1.1. Let $(X, M, *,)$ be a fuzzy 2-metric space and $S, T : X^{2k} \to X$, $f : X \to X$ be mappings satisfying for each positive integer K,

$$M(S(x_1, x_2, \ldots, x_{2k}), T(x_2, x_3, \ldots, x_{2k+1}), a, qt) \geq \varphi\{M(fx_i, fx_{i+1}, a, t) : 1 \leq i \leq 2k\} \tag{4.1}$$

for all $x_1, x_2, x_3, \ldots, x_{2k+1}, a \in X, 0 < q < \dfrac{1}{2}$, and $t \in [0, \infty)$

$$M(T(y_1, y_2, \ldots, y_{2k}), S(y_2, y_3, \ldots, y_{2k+1}), a', qt) \geq \varphi\{M(fy_i, fy_{i+1}, a', t) : 1 \leq i \leq 2k\} \tag{4.2}$$

for all $y_1, y_2, y_3, \ldots, y_{2k+1}, a' \in X$, and

$$M(S(u, u, \ldots, u), T(v, v, \ldots, v), a, qt) > M(fu, fv, a, t) \quad \forall\, u, v, a \in X \text{and } u \neq v \tag{4.3}$$

Suppose then $f(X)$ is complete and either (f, S) or (f, T) is $2k$-weakly compatible pair. Then there exist a unique point p in X such that

$$fp = p = S(p, p, p, \ldots, p) = T(p, p, \ldots, p)$$

Proof. Choose $x_1, x_2, x_3, \ldots, x_{2k}$ are arbitrary points in X and for $n \in \mathbb{N}$.
Such that $fx_{2n+2k-1} = S(x_{2n-1}, x_{2n}, x_{2n+1}, \ldots, x_{2n+2k-2})$
and

$$fx_{2k+2n} = T(x_{2n}, x_{2n+1}, \ldots, x_{2n+2k-1})$$

For simplicity set $\alpha_n = M(fx_n, fx_{n+1}, a, qt)$
We claim that $\alpha_n \geq \left(\dfrac{K - \beta^n}{K + \beta^n}\right)^2$ for all $n \in \mathbb{N}$. where $\beta = \dfrac{1}{2q}$
and

$$K = \min\left\{\frac{\beta(1 + \sqrt{\alpha_1})}{1 - \sqrt{\alpha_1}}, \frac{\beta^2(1 + \sqrt{\alpha_2})}{1 - \sqrt{\alpha_2}}, \frac{\beta^3(1 + \sqrt{\alpha_3})}{1 - \sqrt{\alpha_3}}, \ldots, \frac{\beta^{2k}(1 + \sqrt{\alpha_k})}{1 - \sqrt{\alpha_k}}\right\}$$

By selection of K we have $\alpha_n \geq \left(\dfrac{K - \beta^n}{K + \beta^n}\right)^2$ for $n = 1, 2, 3, \ldots, 2k$
Now

$$\begin{aligned}
\alpha_{2k+1} &= M(fx_{2k+1}, fx_{2k+2}, a, qt) \\
&= M(S(x_1, x_2, \ldots, x_{2k}), T(x_2, x_3, \ldots, x_{2k+1}), a, qt) \\
&\geq \varphi\{M(fx_i, fx_{i+1}, a, t)\},\ i = 1, 2, \ldots, 2k\}
\end{aligned}$$

Using (4.1), we have

$$\begin{aligned}
\alpha_{2k+1} &\geq \varphi\{\alpha_1, \alpha_2, \ldots, \alpha_{2k}\} \\
&\geq \varphi\left\{\left(\frac{K-\beta^1}{K+\beta^1}\right)^2, \left(\frac{K-\beta^2}{K+\beta^2}\right)^2, \ldots, \left(\frac{K-\beta^{2k-1}}{K+\beta^{2k-1}}\right)^2, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2\right\} \\
&\geq \varphi\left\{\left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \ldots, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2\right\} \\
&\geq \varphi\left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2 \\
&\geq \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2
\end{aligned}$$

Thus $\quad \alpha_{2k+1} \geq \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2$

Similarly we have

$$\begin{aligned}
\alpha_{2k+2} &= M(fx_{2k+2}, fx_{2k+3}, a, qt) \\
&= M(T(x_2, x_3, \ldots, x_{2k+1}), S(x_3, x_4, \ldots, x_{2k+2}), a, qt) \\
&\geq \varphi\{M(fx_i, fx_{i+1}, a, t),\ i = 2, 3, \ldots, 2k+1\}
\end{aligned}$$

$$\begin{aligned}
\alpha_{2k+2} &\geq \varphi\{\alpha_1, \alpha_2, \ldots, \alpha_{2k+1}\} \\
&\geq \varphi\left\{\left(\frac{K-\beta^2}{K+\beta^2}\right)^2, \left(\frac{K-\beta^3}{K+\beta^3}\right)^2, \ldots, \left(\frac{K-\beta^{2k}}{K+\beta^{2k}}\right)^2, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2\right\} \\
&\geq \varphi\left\{\left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2, \ldots, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2, \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2\right\} \\
&\geq \left(\frac{K-\beta^{2k+1}}{K+\beta^{2k+1}}\right)^2 \\
&\geq \left(\frac{K-\beta^{2k+2}}{K+\beta^{2k+2}}\right)^2
\end{aligned}$$

Thus $\quad \alpha_{2k+2} \geq \left(\frac{K-\beta^{2k+2}}{K+\beta^{2k+2}}\right)^2$. Than our claim is true.

Now by our claim we show that $\{x_n\}$ is a Cauchy sequence in X for any $n, p \in \mathbb{N}$, we have

$$
\begin{aligned}
&M(fx_n, fx_{n+p}, a, t)\\
&\geq M(fx_n, fx_{n+1}, a, \frac{t}{2}) * M(fx_{n+1}, fx_{n+2}, a, \frac{t}{2^2}) * \cdots * M(fx_{n+p-1}, fx_{n+p}, a, \frac{t}{2^p})\\
&\geq \alpha_n * \alpha_{n+1} * \ldots * \alpha_{n+p-1}\\
&\geq \left(\frac{K-2^n}{K+2^n}\right)^2 * \left(\frac{K-2^{2n}}{K+2^{2n}}\right)^2 * \ldots * \left(\frac{K-2^{np}}{K+2^{np}}\right)^2\\
&\rightarrow \; 1 * 1 * \ldots * 1 \quad \text{as} \quad n \rightarrow \infty
\end{aligned}
$$

by which we conclude that $\{fx_n\}$ is a Cauchy sequence. Since $f(X)$ is a complete sub space of X, Then there exist z in $f(X)$ such that

$$\lim_{n\to\infty} fx_n = z$$

For $z \in f(X)$, there exist $p \in X$ such that $z = fp$. Then for any integer, using (4.1) and (4.2), We have

$$
\begin{aligned}
&M(S(p, p, \ldots, p), fp, a, t)\\
&= \lim_{n\to\infty} M(S(p, p, p, \ldots, p), fx_{2n+2k-1}, a, t)\\
&= \lim_{n\to\infty} M(S(p, p, p, \ldots, p), S(x_{2n-1}, x_{2n}, \ldots, x_{2n+2k-2}), a, t)\\
&\geq \lim_{n\to\infty} M(S(p, p, \ldots, p), T(p, p, \ldots, x_{2n-1}), a, \frac{t}{2})\\
&* M(T(p, p, \ldots, x_{2n-1}), S(p, p, \ldots, x_{2n-1}, x_{2n}), a, \frac{t}{2^3}) * \ldots\\
&* M(T(p, x_{2n-1}, \ldots, x_{2n+2k-3}), S(x_{2n-1}, x_{2n}, \ldots, x_{2n+2k-2}), a, \frac{t}{2^{k-1}})\\
&\geq \lim_{n\to\infty} \varphi\{M(fp, fp, a, t), M(fp, fp, a, t), \ldots, M(fp, fx_{2n-1}, a, t)\}\\
&* \varphi\{M(fp, fp, a, t), M(fp, fp, a, t), \ldots, M(fx_{2n-1}, fx_{2n}, a, t)\} * \ldots\\
&* \varphi\{M(fp, fx_{2n-1}, a, t), M(fx_{2n-1}, fx_{2n}, a, t), \ldots, M(fx_{2n+2k-3}, fx_{2n+2k-2}, a, t)\}\\
&\longrightarrow 1.
\end{aligned}
$$

i.e;

$$M(S(p, p, p, \ldots, p), fp, a, t) = 1$$

So $c(f, T) \neq \Phi$, where $c(f, T)$ denote the set of all coincidence points of all mappings f and T, so that

$$S(p, p, ..., p) = fp \tag{4.4}$$

Consider

$$\begin{aligned}M(fp, T(p,p,\ldots,p),a,t) &= M(S(p,p,\ldots,p),T(p,p,\ldots,p),a,t)\\&\geq \varphi\{M(fp,fp,a,t),M(fp,fp,a,t),\ldots,M(fp,fp,a,t)\}\\&\geq M(fp,fp,a,t)=1\end{aligned}$$

Thus

$$T(p,p,p,.....,p) = fp \tag{4.5}$$

Let (f,S) is $2k$-weakly compatible pair. Then we have

$$f(S(p,p,\ldots,p)) = S(fp,fp,\ldots,fp)$$

$$f^2p = f(fp) = f(S(p,p,\ldots,p)) = S(fp,fp,\ldots,fp)$$

Suppose that $fp \neq p$, then from eq.(4.3), we have

$$\begin{aligned}M(f^2p,fp,a,t) &= M(S(fp,fp,\ldots,fp),T(p,p,\ldots,p),a,t)\\&\geq \{M(f^2p,fp,a,t),M(f^2p,fp,a,t),\ldots,M(f^2p,fp,a,t)\}\\&\geq M(f^2p,fp,a,t)\end{aligned}$$

Which is a contradiction. Therefore our supposition is wrong. Hence $fp = p$
Now from eq.(4.4) and eq.(4.5), we have

$$fp = p = S(p,p,\ldots,p) = T(p,p,\ldots,p)$$

To prove the uniqueness of fixed point, let us assume that for some $q \in X$, $q \neq p$ such that

$$fq = q = S(q,q,\ldots,q) = T(q,q,\ldots,q)$$

Consider

$$\begin{aligned}M(fp,fq,a,t) &= M(S(p,p,\ldots,p),T(q,q,\ldots,q),a,t)\\&\geq \varphi\{M(fp,fq,a,t),M(fp,fq,a,t),\ldots,M(fp,fq,a,t)\}\\&\geq M(fp,fq,a,t)\end{aligned}$$

It is contradiction. Hence p is a unique fixed point in X, and this complete the proof of theorem. □

When $S = T$ and $2k$ is replaced by k in above theorem, we get the following

Corollary 4.1.2. Let $(X, M, *)$ be a fuzzy 2-metric space and $T : X^k \to X$, $f : X \to X$ be mappings satisfying for each positive integer k,

$$M(T(x_1, x_2, \ldots, x_{2k}), T(x_2, x_3, \ldots, x_{2k+1}), a, qt) \geq \varphi\{M(fx_i, fx_{i+1}, a, t) : 1 \leq i \leq 2k\} \tag{4.6}$$

for all $x_1, x_2, x_3, \ldots, x_{2k+1}, a \in X, 0 < q < \frac{1}{2}$, and $t \in [0, \infty)$ and

$$M(T(u, u, \ldots, u), T(v, v, \ldots, v), a, qt) > M(fu, fv, a, t) \quad \forall\, u, v, a \in X \text{and } u \neq v \tag{4.7}$$

Suppose then $f(X)$ is complete and (f, T) is k-weakly compatible pair. Then there exist a unique point p in X such that $fp = p = S(p, p, p, \ldots, p) = T(p, p, \ldots, p)$

To introduce the Presic type contraction in fuzzy 3-metric Space , we define the φ-function as follows.
Let a function $\varphi : [0, 1]^{2k} \to [0, 1]$ such that:

1. φ is an increasing function, i.e. $x_1 \leq y_1, \ldots, x_{2k} \leq y_{2k}, a \leq a', b \leq b'$ implies

$$\varphi(x_1, x_2, x_3, \ldots, x_{2k}, a, b) \leq \varphi(y_1, y_2, y_3, \ldots, y_{2k}, a', b')$$

2. $\varphi(t, t, t, \ldots) \geq t$, for all $t \in X$.
3. φ is continuous in each coordinate variable.

Theorem 4.1.3. Let $(X, M, *)$ be a fuzzy 3-metric space and $S, T : X^{2k} \to X$, $f : X \to X$ be mappings satisfying for each positive integer K,

$$M(S(x_1, x_2, \ldots, x_{2k}), T(x_2, x_3, \ldots, x_{2k+1}), a, b, qt) \geq \varphi\{M(fx_i, fx_{i+1}, a, b, t) : 1 \leq i \leq 2k\} \tag{4.8}$$

for all $x_1, x_2, x_3, \ldots, x_{2k+1}, a, b \in X, 0 < q < \frac{1}{2}$, and $t \in [0, \infty)$

$$M(T(y_1, y_2, \ldots, y_{2k}), S(y_2, y_3, \ldots, y_{2k+1}), a', b', qt) \geq \varphi\{M(fy_i, fy_{i+1}, a', b', t) : 1 \leq i \leq 2k\} \tag{4.9}$$

for all $y_1, y_2, y_3, \ldots, y_{2k+1}, a', b' \in X$, and

$$M(S(u, u, \ldots, u), T(v, v, \ldots, v), a, b, qt) > M(fu, fv, a, b, t) \quad \forall\, u, v, a, b \in X \text{and } u \neq v \tag{4.10}$$

Suppose then $f(X)$ is complete and either (f, S) or (f, T) is $2k$-weakly compatible pair. Then there exist a unique point p in X such that

$$fp = p = S(p, p, p, \ldots, p) = T(p, p, \ldots, p)$$

Proof. The proof of Theorem 4.1.5 is analogues as in Theorem 4.1.4 □

When $S = T$ and $2k$ is replaced by k in above theorem, we get the following corollary.

Corollary 4.1.4. Let $(X, M, *)$ be a fuzzy 3-metric space and $T : X^k \to X$, $f : X \to X$ be mappings satisfying for each positive integer k,

$$M(T(x_1, x_2, \ldots, x_{2k}), T(x_2, x_3, \ldots, x_{2k+1}), a, b, qt) \geq \varphi\{M(fx_i, fx_{i+1}, a, t) : 1 \leq i \leq 2k\} \tag{4.11}$$

for all $x_1, x_2, x_3, \ldots, x_{2k+1}, a, b \in X, 0 < q < \frac{1}{2}$, and $t \in [0, \infty)$ and

$$M(T(u, u, \ldots, u), T(v, v, \ldots, v), a, b, qt) > M(fu, fv, a, b, t) \quad \forall\ u, v, a \in X \text{and } u \neq v \tag{4.12}$$

Suppose then $f(X)$ is complete and (f, T) is k-weakly compatible pair. Then there exist a unique point p in X such that $fp = p = S(p, p, p, \ldots, p) = T(p, p, \ldots, p)$

4.2 Conclusion

In this thesis, we reviewed the fixed point theorem in [20] under the title " Some Fixed Point Theorem in Fuzzy 2 and Fuzzy 3-metric Spaces" They established fixed point theorem for Presic types contraction in Fuzzy metric spaces. To prove the existence of a common fixed point of 3-maps in Fuzzy metric space, they have used the non-decreasing continuous function φ defined in $[0, 1]^{2k}$.
After detailed review of their results, We extend the notion of Presic type contraction of $2k$-weakly compatible mapping on Fuzzy 2-Metric, Fuzzy 3-Metric Spaces and obtain fixed point results.

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